Completing the solution of the directed Oberwolfach problem with two tables

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Submitted: Aug 19, 2024; Accepted: Jan 23, 2026; Published: Feb 13, 2026

Abstract

We address the last outstanding case of the directed Oberwolfach problem with two tables of different lengths. Specifically, we show that the complete symmetric directed graph K_n^* admits a decomposition into spanning subdigraphs comprised of two vertex-disjoint directed cycles of length t_1 and t_2, respectively, where $t_1 \in \{4, 6\}$, t_2 is even, and $t_1 + t_2 \geqslant 14$. In conjunction with recent results of Kadri and Šajna, this gives a complete solution to the directed Oberwolfach problem with two tables of different lengths.

Mathematics Subject Classifications: 05B30

Keywords: Directed Oberwolfach problem; directed 2-factorization; complete symmetric directed graph.

1 Introduction

In this paper, we investigate a variation of the famous Oberwolfach Problem (OP). Introduced by Ringel [18] in 1967, the OP$(t_1, t_2, \dots, t_s)$ poses the following question: given $n = 2k+1$ people and s round tables that respectively seat $t_1, t_2, \dots, t_s$ people, where $t_1 + t_2 + \cdots + t_s = n$ and $t_i \geqslant 3$, does there exist a set of k seating arrangements such that each person sits beside every other person precisely once? This problem can be formulated as a graph-theoretic problem by considering the question of existence of a 2-factorization of the complete graph K_n such that each 2-factor is comprised of cycles of lengths $t_1, t_2, \dots, t_s$. In [22], the OP was adapted to consider the case where n is even. In that case, the existence of a 2-factorization of $K_n - I$ is considered, where

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$K_n - I$ is the complete graph with the edges of a 1-factor removed. Constructive solutions to the OP have been found in each of the following cases: cycles of uniform length [4, 5, 21, 22], two cycles [19, 30], any combination of cycles of even length [9, 20], and $n \leqslant 100$ [14, 15, 16, 26, 27]. Constructive solutions to the OP have also been found for several infinite families of cases in [3] and [10] and in the case where a single table is sufficiently large [29]. It has also been shown non-constructively that a solution to the OP exists for all sufficiently large n [17]. We refer the interested reader to [11] for a survey of known results on the Oberwolfach problem and related variants as of 2024.

The directed Oberwolfach problem ($\mathrm{OP}^*(t_1, t_2, \ldots, t_s)$) considers a similar scenario. This time, we let $t_i \geqslant 2$ and we seek $n-1$ seating arrangements with the added property that each guest is to be seated to the right of every other guest exactly once. If all s tables are of the same length t, we write $\mathrm{OP}^*(t; s)$. When n is odd and each $t_i \geqslant 3$, one can easily construct a solution to the $\mathrm{OP}^*(t_1, t_2, \ldots, t_s)$ from a solution to the $\mathrm{OP}(t_1, t_2, \ldots, t_s)$. Therefore it suffices to consider the $\mathrm{OP}^*(t_1, t_2, \ldots, t_s)$ for n even or, when n is odd, in those cases where a solution to the $\mathrm{OP}(t_1, t_2, \ldots, t_s)$ is unknown.

Recently, the last open case of the $\mathrm{OP}^*(t; s)$ was settled [24]. Thus we have a constructive proof of Theorem 1 below.

Theorem 1 ([1, 2, 6, 8, 12, 13, 24, 28])**.** *Let s and t be positive integers such that ts is even. The $\mathrm{OP}^*(t; s)$ has a solution if and only if $(s, t) \notin \{(1, 6), (1, 4), (2, 3)\}$.*

Naturally, the next step is to consider the case with cycles of varying length. The only result on this more general case of the OP^* when n is even can be found in [23] and [31]. In [31], Zhang and Du established the existence of solutions to $\mathrm{OP}^*(3^{m_1}, 4)$ and $\mathrm{OP}^*(3^{m_2}, 5)$ for all positive integers m_1 and m_2 such that $3m_1 + 4 \equiv 1 \pmod 3$ and $3m_2 + 5 \equiv 2 \pmod 3$. They do so by constructing resolvable Mendelsohn designs with parallel classes containing m_1 or m_2 blocks of size 3 and one block of size 4 or 5. Recently, Kadri and Šajna [23] used a recursive approach to obtain several infinite families of solutions. One of the key results of [23] is a near-complete constructive solution to the directed Oberwolfach problem with two cycles of varying lengths formulated in Theorem 2 below.

Theorem 2 ([23])**.** *Let t_1 and t_2 be integers such that $2 \leqslant t_1 < t_2$. Then the $\mathrm{OP}^*(t_1, t_2)$ has a solution if and only if $(t_1, t_2) \neq (3, 3)$ with a possible exception in the case where $t_1 \in \{4, 6\}$, t_2 is even, and $t_1 + t_2 \geqslant 14$.*

The recursive approach used to prove Theorem 2 relies on the existence of a solution to $\mathrm{OP}^*(t; 1)$. However, it is known from Theorem 1 that no such decomposition exists when $t_1 \in \{4, 6\}$. Therefore, the methods of [23] cannot be used to construct a solution to the $\mathrm{OP}^*(t_1, t_2)$ when $t_1 \in \{4, 6\}$ and t_2 is even.

Here we complement the results of Theorem 2 and complete the solution of the directed Oberwolfach problem with two tables.

Theorem 3. *Let t_1 and t_2 be positive even integers such that $t_1 \in \{4, 6\}$ and $t_1 + t_2 \geqslant 14$. Then the $\mathrm{OP}^*(t_1, t_2)$ has a solution.*

Theorems 2 and 3 jointly imply a complete constructive solution to the $\mathrm{OP}^*(t_1, t_2)$ stated below.

Theorem 4. *Let t_1 and t_2 be integers such that $2 \leqslant t_1 \leqslant t_2$. Then the $\mathrm{OP}^*(t_1, t_2)$ has a solution if and only if $(t_1, t_2) \neq (3, 3)$.*

This paper is structured as follows. In Section 2, we give key definitions. Then, in Section 3, we take a reduction step by showing that it suffices to find particular 2-factorizations of a class of sparser digraphs. Next, in Section 4, we describe the ingredients needed to obtain the desired 2-factorizations and prove that these indeed give rise to the appropriate solutions of the directed Oberwolfach problem. We conclude by constructing the desired set of ingredients required to form the directed 2-factorizations we need.

2 Key definitions

We make the standard assumption that all directed graphs (digraphs for short) are strict. This means that digraphs do not contain loops or parallel arcs. If G is a digraph (graph), we shall denote its vertex set as $V(G)$ and its arc set (edge set) as $A(G)$ ($E(G)$), respectively. For any graph G, let G^* denote the digraph with vertex set $V(G)$ and arc set $\{(x, y), (y, x) : \{x, y\} \in E(G)\}$. Let K_n^* denote the complete symmetric digraph on n vertices and let $\vec{C}_m$ denote the directed cycle on m vertices. Let E_m denote the undirected graph with m vertices and no edges.

The length of a directed path (dipath for short) or a directed cycle refers to the number of arcs it has. For a dipath P, we denote its length as $\mathrm{len}(P)$. Moreover, the *source* of a dipath P is the vertex with in-degree 0 and is denoted $s(P)$, while the *terminal* of P is the vertex with out-degree 0 and is denoted $t(P)$.

Let G be a digraph. A *decomposition* of a G is a set $\{H_1, H_2, \ldots, H_r\}$ of pairwise arc-disjoint subdigraphs of G such that $A(G) = A(H_1) \cup A(H_2) \cup \cdots \cup A(H_r)$. A *2-regular digraph* is a digraph comprised of disjoint directed cycles and a spanning subdigraph of G that is also a 2-regular digraph is a *directed 2-factor* of G. A $(\vec{C}_{t_1}, \vec{C}_{t_2}, \ldots, \vec{C}_{t_s})$*-factor* of G is a directed 2-factor that is the disjoint union of s directed cycles of lengths $t_1, t_2, \ldots, t_s$. A *bipartite 2-regular* digraph is a 2-regular digraph comprised of directed cycles of even lengths. If H is a spanning subdigraph of G and G admits a decomposition into subdigraphs isomorphic to H, then this decomposition is called an *H-factorization*. In particular, a $(\vec{C}_{t_1}, \vec{C}_{t_2}, \ldots, \vec{C}_{t_s})$*-factorization* of G is a decomposition of G into $(\vec{C}_{t_1}, \vec{C}_{t_2}, \ldots, \vec{C}_{t_s})$-factors. A *directed 2-factorization* of G is a decomposition of G into directed 2-factors. All these terms can be analogously defined for undirected graphs.

We now formulate the $\mathrm{OP}^*(t_1, t_2, \ldots, t_s)$ in graph-theoretic terms.

Problem 5 ($\mathbf{OP}^*(\boldsymbol{t_1, t_2, \ldots, t_s})$)**.** For integers $2 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$ such that $t_1 + t_2 + \cdots + t_s = n$, does K_n^* admit a $(\vec{C}_{t_1}, \vec{C}_{t_2}, \ldots, \vec{C}_{t_s})$-factorization?

To prove Theorem 3, we construct a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of K_n^* when $n = t_1 + t_2$, $t_1 \in \{4, 6\}$, t_2 is even, and $n \geqslant 14$.

We conclude this section with a pair of definitions that are used to construct the desired $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorizations of K_n^*.

Definition 6. For graphs G and H, the *wreath product of* G *with* H, denoted $G \wr H$, is the graph with vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Definition 7. For a subset S of $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the *circulant of order* n *with connection set* S, denoted $\mathrm{Circ}(n, S)$, is the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{i, i+s\} : i \in \mathbb{Z}_n, s \in S\}$ with addition performed modulo n.

3 Overall strategy

This section details the overall strategy we follow to prove Theorem 3. Our primary objective is to demonstrate that, in order to construct the desired $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of K_n^*, it suffices to construct a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of a sparser digraph that only requires seven or nine $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factors.

Crucial to our approach is the following immediate consequence of a lemma of Häggkvist, see [20].

Lemma 8 ([20])**.** *Let D be a bipartite 2-regular digraph of order $2m$ comprised of directed cycles of length at least 4. The digraph $(C_m \wr E_2)^*$ admits a D-factorization.*

Proof. Let F be the 2-factor obtained from D by replacing each arc (x, y) with an undirected edge $\{x, y\}$. By the first lemma of [20], commonly known as Häggkvist's Lemma, there is an F-factorization $\mathcal{F}$ of $C_m \wr E_2$. Thus there is an F^*-factorization $\mathcal{F}^*$ of $(C_m \wr E_2)^*$. Clearly, each copy of F^* in $\mathcal{F}^*$ can be decomposed into two copies of D and together, these copies of D form the desired directed 2-factorization of $(C_m \wr E_2)^*$. □

Lemma 8 does not apply to bipartite 2-regular digraphs containing at least one cycle of length 2. However, for our purpose, we do not need to consider this case.

Let D be a bipartite 2-regular digraph on $2m$ vertices comprised of directed cycles of length at least 4. Our overall strategy for finding D-factorizations of K_{2m}^* is first to decompose K_{2m}^* into copies of $(C_m \wr E_2)^*$ and a single copy of another graph that we call W_{2m}^*. It will then suffice to find a D-factorization of W_{2m}^* because we can form a D-factorization of K_{2m}^* by taking the union of this D-factorization with D-factorizations of the copies of $(C_m \wr E_2)^*$ provided by Lemma 8. In the remainder of this section, we define the graph W_{2m}^* and show that K_{2m}^* can indeed be decomposed into copies of $(C_m \wr E_2)^*$ and one copy of W_{2m}^*. This approach is inspired by the one used by Bryant and Danziger in [9].

Definition 9. If m is odd, we define W_{2m} to be $\mathrm{Circ}(m, \{1, 2\}) \wr K_2$ and if m is even, we define W_{2m} to be $\mathrm{Circ}(m, \{1, 3^{\mathrm{e}}\}) \wr K_2$, where $\mathrm{Circ}(m, \{1, 3^{\mathrm{e}}\})$ denotes the graph with vertex set $\mathbb{Z}_m$ and edge set

$$\{\{i, i+1\} : i \in \mathbb{Z}_m\} \cup \{\{i, i+3\} : i \in \mathbb{Z}_m \text{ is even}\}.$$

Note that W_{2m} is 9-regular if m is odd and is 7-regular if m is even. Also note that $\mathrm{Circ}(\{m, 1, 3^{\mathrm{e}}\})$ is not technically a circulant but we use this notation as we believe it is a useful mnemonic. See Figure 1 for an illustration of W_{20}.

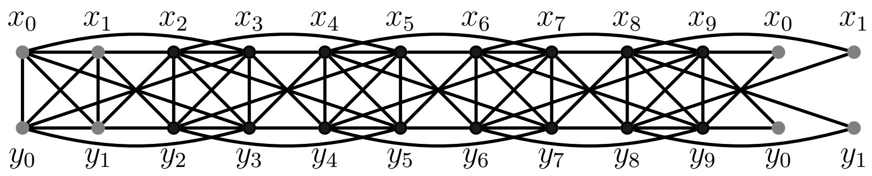

Figure 1: The graph W_{20}; the digraph W_{20}^* is obtained by replacing each edge with a pair of arcs oriented in opposite directions.

Our goal in this section is to prove the following result.

Lemma 10. *Let D be a bipartite 2-regular digraph of order $2m \geqslant 14$ comprised of directed cycles of length at least 4. There is a D-factorization of K_{2m}^* if there is a D-factorization of W_{2m}^*.*

When m is even, we will make use of the following result of Bryant and Danziger [9].

Lemma 11 ([9, Lemma 7])**.** *For each even $m \geqslant 8$, there is a factorization of K_m into $\frac{m-4}{2}$ copies of C_m and a copy of $\mathrm{Circ}(m, \{1, 3^{\mathrm{e}}\})$.*

We will require an analogue for Lemma 11 for the case where m is odd. To prove this, we state a lemma on decomposition of circulants into hamiltonian cycles. It asserts a special case of a result of Bermond, Favaron, and Mahéo [7] on 2-factorizations of Cayley graphs.

Lemma 12 ([7])**.** *Let m be an integer and let S be a subset of $\{1, \dots, \lfloor \frac{m-1}{2} \rfloor\}$. Then $\mathrm{Circ}(m, S)$ admits a C_m-factorization if $|S| = 2$ and $\gcd(S \cup \{m\}) = 1$.*

Lemma 13. *For each odd $m \geqslant 7$, the graph K_m admits a decomposition into $\frac{m-5}{2}$ copies of C_m and one copy of $\mathrm{Circ}(m, \{1, 2\})$.*

Proof. The graph $\mathrm{Circ}(m, \{1, \dots, \frac{m-1}{2}\})$ is a copy of K_m. If $m = 7$, then $\mathrm{Circ}(7, \{1, 2, 3\})$ has a decomposition $\mathcal{F} = \{\mathrm{Circ}(7, \{1, 2\}), \mathrm{Circ}(7, \{3\})\}$. If $m \geqslant 9$, then $\mathrm{Circ}(m, \{1, \dots, \frac{m-1}{2}\})$ has a decomposition $\mathcal{F}$ given by

$$\begin{cases} \{\mathrm{Circ}(m, S) : S \in \{\{1, 2\}, \{3, 4\}, \dots, \{\frac{m-3}{2}, \frac{m-1}{2}\}\}\} & \text{if } m \equiv 1 \pmod 4; \\ \{\mathrm{Circ}(m, S) : S \in \{\{1, 2\}, \{3, 5\}, \{4\}, \{6, 7\}, \{8, 9\}, \dots, \{\frac{m-3}{2}, \frac{m-1}{2}\}\}\} & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Clearly, $\mathrm{Circ}(m, \{4\})$ is a copy of C_m when m is odd and $\mathrm{Circ}(7, \{3\})$ is a copy of C_7. Therefore, in each case it can be seen that each subgraph in $\mathcal{F}$ other than $\mathrm{Circ}(m, \{1, 2\})$ admits a C_m-factorization by using Lemma 12. Taking the union of $\mathrm{Circ}(m, \{1, 2\})$ together with these C_m-factorizations completes the proof. □

Using Lemmas 11 and 13 we can complete our proof of Lemma 10.

Proof of Lemma 10. By Lemmas 11 and 13, there is a decomposition $\{G\} \cup \mathcal{C}$ of K_m where $\mathcal{C}$ is a set of directed cycles of length m, G is a copy of $\mathrm{Circ}(m, \{1, 3^{\mathrm{e}}\})$ if m is even, and G is a copy of $\mathrm{Circ}(m, \{1, 2\})$ if m is odd. Since $K_m \wr K_2$ is isomorphic to K_{2m}, we have that $\mathcal{F}$ is a decomposition of K_{2m} where

$$\mathcal{F} = \{G \wr K_2\} \cup \{C \wr E_2 : C \in \mathcal{C}\}.$$

Noting $(G \wr K_2)^*$ is a copy of W^*_{2m}, we see that $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$ is a decomposition of K^*_{2m} into copies of $(C_m \wr E_2)^*$ and one copy of W^*_{2m}. By Lemma 8, $(C \wr E_2)^*$ has a D-factorization $\mathcal{D}_C$ for each $C \in \mathcal{C}$. Thus, if W^*_{2m} has a D-factorization $\mathcal{D}'$, then $\mathcal{D}' \cup \{\mathcal{D}_C : C \in \mathcal{C}\}$ will be a D-factorization of K^*_{2m}. □

In summary Lemma 10 implies that, to prove Theorem 3, it suffices to construct a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of $W^*_{t_1+t_2}$ when $t_1 \in \{4, 6\}$, t_2 is even, and $t_1 + t_2 \geqslant 14$.

4 Main construction

Throughout this section we take t_1 and q to be fixed integers such that $t_1 \in \{4, 6\}$ and

$$q \in \begin{cases} \{10, 14, 16, 20\} & \text{if } t_1 = 4; \\ \{14, 16, 18, 20\} & \text{if } t_1 = 6. \end{cases} \tag{1}$$

Our goal will be to show that a $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factorization of $W^*_{t_1+q+8k}$ exists for each nonnegative integer k. Lemma 10 will establish our main theorem for all pairs (t_1, q) except those in $\{(4, 12), (6, 8), (6, 10), (6, 12)\}$. We will then deal with these special cases in Appendix A.3.

Notation 14. Throughout the remainder of the paper we shall assume that, in the definition of W^*_{2m} given in Definition 9, the copy of K_2 has vertex set $\{x, y\}$. Further, we will abbreviate vertices (a, x) to x_a and vertices (b, y) to y_b so that

$$V(W^*_{2m}) = \{x_a, y_b \;:\; a, b \in \mathbb{Z}_m\}.$$

In addition, we define the following permutations of $V(W^*_{2m})$.

Definition 15. For each even integer j, we will take ρ^j to be the permutation of $V(W^*_{2m})$ defined by $\rho^j(x_i) = x_{i+j}$ and $\rho^j(y_i) = y_{i+j}$, with subscript addition performed modulo m. For a dipath $P = v_0 v_1 \cdots v_t$ of W^*_{2m}, we let $\rho^j(P) = \rho^j(v_0)\rho^j(v_1)\rho^j(v_2)\cdots\rho^j(v_t)$. We refer to a dipath $\rho^j(P)$ as a *translation* of P, and note that $\rho^j(P)$ is also a dipath of W^*_{2m} since j is even.

Each of the factors in the directed 2-factorizations we desire will be created from what we call a *(t_1, q)-base tuple* (X, Q, R, S, T) where X is a directed t_1-cycle and Q, R, S and T are dipaths of various lengths. We will define (t_1, q)-base tuples formally in Definition 17 below, but first we give an informal overview of how they will be used. For a given nonnegative integer k, from each (t_1, q)-base tuple (X, Q, R, S, T), we will construct a $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factor which is a union of the following pieces:

- a directed t_1-cycle X;
- two dipaths I_0 and I_1 formed as the concatenation of k translations of S and T, respectively;
- two dipaths Q and R such that $s(Q) = t(I_1)$, $t(Q) = s(I_0)$, $s(R) = t(I_0)$, and $t(R) = s(I_1)$.

The union of Q, R, I_0, and I_1 will form a directed $(q + 8k)$-cycle that is disjoint from X. A schematic picture of this construction is given in Figure 2. Since each (t_1, q)-base tuple gives us factors of infinitely many orders, this approach will allow us to reduce our problem to finding only eight sets of (t_1, q)-base tuples (one for each possible choice of (t_1, q)).

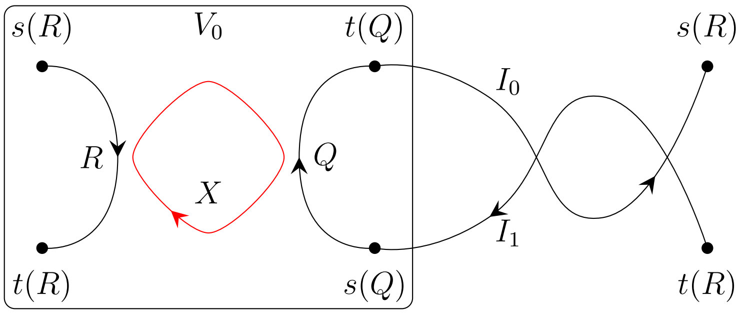

Figure 2: A schema of the construction of a directed 2-factor of W_{2m}^* from a (t_1, q)-base tuple (X, Q, R, S, T) with directed t_1-cycle X drawn in red.

Notation 16. For the remainder of this section, it will be useful to set $p = \frac{1}{2}(t_1 + q)$, so that $2p$ is the smallest of the orders of the directed 2-factorizations we desire. For each non-negative integer k, we also define certain subsets of the vertex set of W_{2p+4k}^* as follows:

$$\begin{aligned} V_0 &= \{x_i, y_i \ : \ 0 \leqslant i \leqslant p+1\}; \\ V_0^\dagger &= \{x_i, y_i \ : \ 2 \leqslant i \leqslant p-1\}; \\ V_j &= \{x_i, y_i \ : \ p+4j-4 \leqslant i \leqslant p+4j+1\} \text{ for each } j \in \{1, 2, \ldots, k\}. \end{aligned}$$

Observe that $V_0 = V(W_{2p+4k}^*)$ if $k = 0$, $|V_0| = 2(p + 2)$ otherwise, and $|V_j| = 12$ for each $j \in \{1, 2, \ldots, k\}$. Also, for all integers i and j with $0 \leqslant i < j \leqslant k$,

$$V_i \cap V_j = \begin{cases} \{x_{p-1}, x_p, y_{p-1}, y_p\} & \text{if } (i,j) = (0,1); \\ \{x_0, x_1, y_0, y_1\} & \text{if } (i,j) = (0,k); \\ \{x_{p+4i}, x_{p+4i+1}, y_{p+4i}, y_{p+4i+1}\} & \text{if } j = i+1 \text{ and } i \geqslant 1; \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

Let $k \geqslant 2$ be an integer and let $A = (P^1, \ldots, P^k)$ be a sequence of dipaths. We say that A *concatenates* if $t(P^i) = s(P^{i+1})$ for each $i \in \{1, \ldots, k-1\}$ and, aside from this, no vertex is in more than one dipath in the sequence. In this case we call the dipath $P^1 \cup \cdots \cup P^k$ the *concatenation* of A. Similarly we say that A *cyclically concatenates* if $t(P^k) = s(P^1)$, $t(P^i) = s(P^{i+1})$ for each $i \in \{1, \ldots, k-1\}$ and, aside from this, no vertex is in more than one dipath in the sequence. In this case we call the directed cycle $P^1 \cup \cdots \cup P^k$ the *cyclic concatenation* of A.

Definition 17. The 5-tuple (X, Q, R, S, T) is a (t_1, q)-*base tuple* if X is a directed t_1-cycle of $W^*_{t_1+q+24}$ and Q, R, S and T are dipaths of $W^*_{t_1+q+24}$ with the following properties.

- **B1** $V(X) \subseteq V_0^\dagger$; $V(Q) \subseteq V_0 \setminus \{x_0, y_0, x_1, y_1\}$; $V(R) \subseteq V_0 \setminus \{x_p, y_p, x_{p+1}, y_{p+1}\}$; $V(S)$, $V(T) \subseteq V_1$;
- **B2** $\mathrm{len}(Q) + \mathrm{len}(R) = q$ and $\mathrm{len}(S) + \mathrm{len}(T) = 8$;
- **B3** X, Q, and R are pairwise vertex-disjoint;
- **B4** $(Q, \rho^p(R))$ cyclically concatenates;
- **B5** (T, Q, S) and $(\rho^{-p-4}(S), R, \rho^{-p-4}(T))$ concatenate;
- **B6** $(S, \rho^4(S))$ and $(T, \rho^4(T))$ concatenate. Further, the concatenations of $(S, \rho^4(S))$ and $(T, \rho^4(T))$ are vertex-disjoint.

In Definition 17 above, we chose $t_1 + q + 24$ because it was large enough that it can be easily checked that the translations mentioned in B4, B5, and B6 do not contain vertices of $V^\dagger$. We could equivalently have chosen any other order large enough to ensure this property. Our next lemma describes how we can use a (t_1, q)-base tuple to obtain a directed 2-factor.

Lemma 18. *Let (X, Q, R, S, T) be a (t_1, q)-base tuple and let k be a nonnegative integer. In the host graph $W^*_{t_1+q+8k}$, denote $S^j = \rho^{4(j-1)}(S)$ and $T^j = \rho^{4(j-1)}(T)$ for each $j \in \{1, 2, \ldots, k\}$ and let*

$$A = (Q, S^1, S^2, \ldots, S^k, R, T^k, T^{k-1}, \ldots, T^1).$$

Then A cyclically concatenates and, furthermore, X and the cyclic concatenation of A form the cycles of a $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factor of $W^*_{t_1+q+8k}$.

Proof. If A cyclically concatenates, then its cyclic concatenation is a directed cycle of length $q + 8k$ by B2. So it suffices to prove that A does indeed cyclically concatenate and that its cyclic concatenation is vertex-disjoint from X.

Case 1. Suppose that $k = 0$. Then our host graph is $W^*_{t_1+q} = W^*_{2p}$ and $A = (Q, R)$. In W^*_{2p}, we have that ρ^p is the identity permutation and so B4 implies that A cyclically concatenates. Further, B3 implies that the cyclic concatenation of A is vertex-disjoint from X.

Case 2. Suppose that $k = 1$. Then our host graph is $W^*_{t_1+q+8} = W^*_{2(p+4)}$ and $A = (Q, S^1, R, T^1)$. In $W^*_{2(p+4)}$, we have that ρ^{-p-4} is the identity permutation and so B5 implies that (T^1, Q, S^1) and (S^1, R, T^1) concatenate. So, because Q and R are vertex-disjoint by B3, we have that A cyclically concatenates. By B3 Q and R are both vertex-disjoint from X. By B1 and (2), both S and T are vertex-disjoint from X. So the cyclic concatenation of A is vertex-disjoint from X.

Case 3. Suppose that $k \geqslant 2$. Our host graph is $W^*_{t_1+q+8k} = W^*_{2(p+4k)}$. In $W^*_{2(p+4k)}$, we have that $\rho^{4(k-1)} = \rho^{-p-4}$ and so B5 implies that (T^1, Q, S^1) and (S^k, R, T^k) concatenate. By B6, we have that (S^i, S^{i+1}) and (T^i, T^{i+1}) both concatenate for each $i \in \{1, \dots, k-1\}$. Also by B6, for each $i \in \{1, \dots, k-1\}$, we have that S^i is vertex-disjoint from T^i and T^{i+1} and that T^i is vertex-disjoint from S^i and S^{i+1}. Note that B1 implies that $V(S^i)$ and $V(T^i)$ are subsets of V_i for each $i \in \{1, \dots, k\}$. Thus we can conclude that A cyclically concatenates because B1 and (2) imply that all the remaining vertex-disjointness conditions are met. Further, using B3 together with B1 and (2), we have that the cyclic concatenation of A is vertex-disjoint from X. □

For digraphs G and H, we use the notation $G \cong H$ to indicate that G and H are isomorphic. Furthermore, for digraphs G_1, G_2, H_1 and H_2, we write $(G_1, G_2) \cong (H_1, H_2)$ to indicate that $G_1 \cong H_1$, $G_2 \cong H_2$, and $G_1 \cup G_2 \cong H_1 \cup H_2$. Note that if $(G_1, G_2) \cong (H_1, H_2)$ and H_1 is arc-disjoint from H_2, then we must have that G_1 is arc-disjoint from G_2 because

$$|A(G_1 \cup G_2)| = |A(H_1 \cup H_2)| = |A(H_1)| + |A(H_2)| = |A(G_1)| + |A(G_2)|.$$

The next lemma gives conditions under which the 2-factors arising from a number of (t_1, q)-base tuples form a directed 2-factorization.

Lemma 19. *Let $r = 9$ if $t_1 + q \equiv 2 \pmod 4$, and let $r = 7$ if $t_1 + q \equiv 0 \pmod 4$. For each $a \in \{0, \dots, r-1\}$ let $(X_a, Q_a, R_a, S_a, T_a)$ be a (t_1, q)-base tuple and let $\hat{F}_a$ be the corresponding $(\vec{C}_{t_1}, \vec{C}_{q+16})$-factor of $W^*_{t_1+q+16}$. If the digraphs in $\hat{\mathcal{F}} = \{\hat{F}_0, \dots, \hat{F}_{r-1}\}$ are pairwise arc-disjoint, then $W^*_{t_1+q+8k}$ admits a $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factorization for each positive integer k. If, in addition, each of the dipaths $Q_0, \dots, Q_{r-1}$ is arc-disjoint from each of the dipaths $\rho^p(R_0), \dots, \rho^p(R_{r-1})$, then $W^*_{t_1+q}$ admits a $(\vec{C}_{t_1}, \vec{C}_q)$-factorization.*

Proof. Fix a nonnegative integer k and suppose that the digraphs in $\hat{\mathcal{F}} = \{\hat{F}_0, \dots, \hat{F}_{r-1}\}$ are pairwise arc-disjoint and that, if $k = 0$, each of the paths $Q_0, \dots, Q_{r-1}$ is arc-disjoint from each of the paths $\rho^p(R_0), \dots, \rho^p(R_{r-1})$. For each $a \in \{0, \dots, r-1\}$, let F_a be the $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factor of $W^*_{t_1+q+8k}$ constructed using the (t_1, q)-base tuple $(X_a, Q_a, R_a, S_a, T_a)$. Using the notation of Lemma 18 relative to the host graph $W^*_{t_1+q+8k}$, $\hat{F}_a$ is the union of X_a and the cyclic concatenation of

$$(Q_a, S_a^1, S_a^2, S_a^2, \dots, S_a^k, R_a, T_a^k, T_a^{k-1}, \dots, T_a^1)$$

Again using the notation of Lemma 18, but adding hats to indicate that the notation is relative to the host graph $W^*_{t_1+q+16}$, we let $\hat{F}_a$ be the union of $\hat{X}_a$ and the cyclic concatenation of

$$(\hat{Q}_a, \hat{S}_a^1, \hat{S}_a^2, \hat{R}_a, \hat{T}_a^2, \hat{T}_a^1)$$

We must show that $\mathcal{F} = \{F_0, F_1, \dots, F_{r-1}\}$ is a $(\vec{C}_{t_1}, \vec{C}_{q+8k})$-factorization of $W^*_{t_1+q+8k}$. Observe that $\sum_{a=0}^{r-1} |A(F_a)| = |A(W^*_{t_1+q+8k})|$. Therefore, it suffices to show that the directed 2-factors in $\mathcal{F} = \{F_0, F_1, \dots, F_{r-1}\}$ are pairwise arc-disjoint. This follows immediately from the hypothesis if $k = 2$, so we can assume otherwise.

If $k = 1$ then, for each $a \in \{0, \dots, r-1\}$, F_a is obtained from $\hat{F}_a$ by associating the vertices $x_{p+4}, y_{p+4}, \dots, x_{p+7}, y_{p+7}$ with, respectively, the vertices $x_p, y_p, \dots, x_{p+3}, y_{p+3}$. In this association process, by their definitions, the subdigraphs $\hat{S}_a^1$ and $\hat{S}_a^2$ of $\hat{F}_a$ both map to the subdigraph S_a^1 of F_a, and the subdigraphs $\hat{T}_a^1$ and $\hat{T}_a^2$ of $\hat{F}_a$ both map to the subdigraph T_a^1 of F_a. Together with B1. This ensures that no two arcs in different factors of $\hat{\mathcal{F}}$ are mapped onto the same arc. Thus, because the factors in $\hat{\mathcal{F}}$ are pairwise arc-disjoint, the factors in $\mathcal{F}$ are pairwise arc-disjoint.

If $k = 0$ then, for each $a \in \{0, \dots, r-1\}$, F_a is obtained from $\hat{F}_a$ by deleting the arcs in $\hat{S}_a^1 \cup \hat{S}_a^2 \cup \hat{T}_a^1 \cup \hat{T}_a^2$, deleting the vertices $x_{p+2}, y_{p+2}, \dots, x_{p+7}, y_{p+7}$, and associating the vertices $x_p, y_p, x_{p+1}, y_{p+1}$ with, respectively, the vertices x_0, y_0, x_1, y_1. Our additional assumption in the case $k = 0$, together with B1, ensures that no two arcs in different factors of $\hat{\mathcal{F}}$ are mapped onto the same arc in this association process. Thus, because the factors in $\hat{\mathcal{F}}$ are pairwise arc-disjoint, the factors in $\mathcal{F}$ are pairwise arc-disjoint.

Lastly, we consider the case $k \geqslant 3$. For each $a \in \{0, \dots, r-1\}$, we let $Y_a = X_a \cup Q_a \cup R_a$ and, for each $i \in \{1, \dots, k\}$, $U_a^i = S_a^i \cup T_a^i$. Also, we let $\hat{Y} = \hat{X}_a \cup \hat{Q}_a \cup \hat{R}_a$ and, for $j \in \{1, 2\}$, we let $\hat{U}_a^j = \hat{S}_a^j \cup \hat{T}_a^j$. Let $\{V_0, \dots, V_k\}$ be the partition of $V(W^*_{t_1+q+8k})$ defined in Notation 16. By B1 and (2) we have, for each $a \in \{0, \dots, r-1\}$:

(i) $V(Y_a) \subseteq V_0$; and

(ii) $V(U_a^i) \subseteq V_i$ for each $i \in \{1, \dots, k\}$.

Let g and ℓ be distinct elements of $\{0, \dots, r-1\}$. We complete the proof by showing that F_ℓ is arc-disjoint from F_g. We do this by first showing that Y_ℓ is arc-disjoint from F_g. Then, for each $j \in \{1, \dots, k\}$, we show that U_ℓ^j is arc-disjoint from F_g.

Case 1: $\boldsymbol{Y_\ell}$. Using (i) and (ii) we have that Y_ℓ is vertex-disjoint from $\bigcup_{i=2}^{k-1} U_g^i$ because V_0 is vertex-disjoint from $V_2, \dots, V_{k-1}$. Now $Y_\ell \cup U_\ell^1$ is arc-disjoint from $Y_g \cup U_g^1$ because $(Y_\ell \cup U_\ell^1, Y_g \cup U_g^1) \cong (\hat{Y}_\ell \cup \hat{U}_\ell^1, \hat{Y}_g \cup \hat{U}_g^1)$ and $\hat{Y}_\ell \cup \hat{U}_\ell^1$ is arc-disjoint from $\hat{Y}_g \cup \hat{U}_g^1$. Similarly, $U_\ell^k \cup Y_\ell$ is arc-disjoint from $U_g^k \cup Y_g$ because $(U_\ell^k \cup Y_\ell, U_g^k \cup Y_g) \cong (\hat{U}_\ell^2 \cup \hat{Y}_\ell, \hat{U}_g^2 \cup \hat{Y}_g)$ and $\hat{U}_\ell^2 \cup \hat{Y}_\ell$ is arc-disjoint from $\hat{U}_g^2 \cup \hat{Y}_g$. So, in particular, Y_ℓ is arc-disjoint from $U_g^k \cup Y_g \cup U_g^1$ and hence from F_g.

Case 2: $\boldsymbol{U_\ell^j}$ where $\boldsymbol{j \in \{2, \dots, k-1\}}$. Let $\mathbb{I} = \{0, 1, \dots, j-2, j+2, j+3, \dots, k\}$. From (i) and (ii), the digraph U_ℓ^j is vertex-disjoint from $Y_g \cup \bigcup_{i \in \mathbb{I} \setminus \{0\}} U_g^i$ because V_j is vertex-disjoint from $\bigcup_{i \in \mathbb{I}} V_i$. Now, $U_\ell^{j-1} \cup U_\ell^j$ is arc-disjoint from $U_g^{j-1} \cup U_g^j$ because

$$(U_\ell^{j-1} \cup U_\ell^j, U_g^{j-1} \cup U_g^j) \cong (U_\ell^1 \cup U_\ell^2, U_g^1 \cup U_g^2) \cong (\hat{U}_\ell^1 \cup \hat{U}_\ell^2, \hat{U}_g^1 \cup \hat{U}_g^2)$$

and $\hat{U}_\ell^1 \cup \hat{U}_\ell^2$ is arc-disjoint from $\hat{U}_g^1 \cup \hat{U}_g^2$. Similarly, $U_\ell^j \cup U_\ell^{j+1}$ is arc-disjoint from $U_g^j \cup U_g^{j+1}$ because $(U_\ell^j \cup U_\ell^{j+1}, U_g^j \cup U_g^{j+1}) \cong (\hat{U}_\ell^1 \cup \hat{U}_\ell^2, \hat{U}_g^1 \cup \hat{U}_g^2)$. So, in particular, U_j is arc-disjoint from $U_g^{j-1} \cup U_g^j \cup U_g^{j+1}$ and hence from F_g.

Case 3: $\boldsymbol{U_\ell^1}$ and $\boldsymbol{U_\ell^k}$. From (i) and (ii), it follows that U_ℓ^1 is vertex-disjoint from $\bigcup_{i=3}^k U_g^i$ because V_1 is vertex-disjoint from $V_3, \dots, V_k$. Likewise, U_ℓ^k is arc-disjoint from $\bigcup_{i=1}^{k-2} U_g^i$ because V_k is vertex-disjoint from $V_1, \dots, V_{k-2}$. In Case 1, we saw that U_ℓ^1 and U_ℓ^k are both arc-disjoint from Y_g. In Case 2, (with $j = 2$) we saw that U_ℓ^1 is arc-disjoint from U_g^2. In Case 2, (with $j = k-1$) we also saw that U_ℓ^k is arc-disjoint from U_g^{k-1}. So both U_ℓ^1 and U_ℓ^k are arc-disjoint from F_g.

In summary, we have demonstrated that F_ℓ is arc-disjoint from F_g for distinct ℓ and g. Therefore, the given set of r (t_1, q)-base tuples gives rise to the desired directed 2-factorization of W_{2m}^*. □

We now conclude this section with the proof of this paper's main result, namely the proof of Theorem 3.

Proof of Theorem 3. We show that K_{2m}^* admits a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization when $t_1+t_2 = 2m$, $t_1 \in \{4, 6\}$, and $t_1 + t_2 \geqslant 14$. Lemma 10 implies that it suffices to find a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of W_{2m}^*. For the special cases where $(t_1, t_2) \in \{(4, 12), (6, 8), (6, 10), (6, 12)\}$ we give a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of W_{2m}^* in Appendix A.3. Otherwise, we have that $t_2 = q + 8k$ for some q satisfying (1) and nonnegative integer k. Let $r = 9$ if m is odd and $r = 7$ if m is even. To construct a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization of W_{2m}^*, it suffices to construct r (t_1, q)-base tuples satisfying the hypothesis of Lemma 19.

1. If m is odd, then $(t_1, q) \in \{(4, 10), (4, 14), (6, 16), (6, 20)\}$. Appendix A.1 gives a set of nine (t_1, q)-base tuples satisfying the hypothesis of Lemma 19 for each of these choices of (t_1, q).

2. If m is even, then $(t_1, q) \in \{(4, 16), (4, 20), (6, 14), (6, 18)\}$. Appendix A.2 gives a set of seven (t_1, q)-base tuples satisfying the hypothesis of Lemma 19 for each of these choices of (t_1, q).

In conclusion, the digraph W_{2m}^* admits a $(\vec{C}_{t_1}, \vec{C}_{t_2})$-factorization when $t_1 + t_2 = 2m$, $t_1 \in \{4, 6\}$, and $t_1+t_2 \geqslant 14$. It follows that the $\mathrm{OP}^*(t_1, t_2)$ has a solution for all applicable t_1 and t_2 values. □

The (t_1, q)-tuples presented in Appendices A.1 and A.2 were constructed by hand with the assistance of a computer. For example, in many cases, we first used a computer to obtain an exhaustive list of all possible sets of dipaths $\{S_0, T_0, \dots, S_{r-1}, T_{r-1}\}$ and $\{Q_0, R_0, \dots, Q_{r-1}, R_{r-1}\}$. The process of fitting these together, making adjustments if necessary, and completing the tuples such that they give rise to the desired directed 2-factorization, however, was largely accomplished by hand.

Acknowledgements

The first author was supported by the Australian Research Council (grants DP220102212 and DP240101048). The second author was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) Post Graduate Scholarship program and the NSERC Michael Smith Foreign Study Supplement program.

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A Supplementary material for the proof of Theorem 3

A.1 The case $t_1 + q \equiv 2 \pmod 4$

Case 1: $t_1 = 4$ and $q = 10$. See Figures 3–11 in Appendix B.1.

$$\begin{aligned}
X_0 &= x_4\,x_3\,y_5\,y_3\,x_4; \\
Q_0 &= y_7\,x_5\,y_6\,y_4\,x_6\,x_7; \\
R_0 &= x_0\,y_2\,x_1\,x_2\,y_1\,y_0; \\
S_0 &= x_7\,y_9\,x_9\,x_8\,y_{10}\,x_{11}; \\
T_0 &= y_{11}\,x_{10}\,y_8\,y_7;
\end{aligned}$$

$$\begin{aligned}
X_1 &= y_5\,x_3\,y_4\,x_5\,y_5; \\
Q_1 &= y_7\,x_6\,x_4\,y_6\,x_8; \\
R_1 &= x_1\,x_0\,x_2\,y_3\,y_1\,y_2\,y_0; \\
S_1 &= x_8\,x_7\,x_9\,x_{10}\,x_{12}; \\
T_1 &= y_{11}\,y_{10}\,y_8\,y_9\,y_7;
\end{aligned}$$

$$\begin{aligned}
X_2 &= y_5\,x_4\,x_2\,y_4\,y_5; \\
Q_2 &= x_8\,y_6\,x_6\,x_5\,y_7; \\
R_2 &= y_0\,y_2\,y_3\,x_3\,y_1\,x_0\,x_1; \\
S_2 &= y_7\,y_9\,y_8\,y_{10}\,y_{11}; \\
T_2 &= x_{12}\,x_{10}\,x_9\,x_7\,x_8;
\end{aligned}$$

$$\begin{aligned}
X_3 &= y_5\,y_4\,y_2\,x_4\,y_5; \\
Q_3 &= x_8\,y_7\,x_7\,x_5\,x_6\,y_6\,y_8; \\
R_3 &= y_1\,y_3\,x_2\,x_3\,x_1; \\
S_3 &= y_8\,x_9\,y_{11}\,x_{11}\,x_{10}\,y_{12}; \\
T_3 &= x_{12}\,y_{10}\,y_9\,x_8;
\end{aligned}$$

$$\begin{aligned}
X_4 &= x_5\,x_3\,x_4\,y_3\,x_5; \\
Q_4 &= x_7\,x_6\,y_4\,y_6\,y_5\,y_7; \\
R_4 &= y_0\,x_1\,y_1\,x_2\,y_2\,x_0; \\
S_4 &= y_7\,x_8\,y_9\,x_{10}\,y_{11}; \\
T_4 &= x_{11}\,y_{10}\,x_9\,y_8\,x_7;
\end{aligned}$$

$$\begin{aligned}
X_5 &= x_5\,y_3\,y_4\,x_4\,x_5; \\
Q_5 &= y_7\,y_6\,x_7\,y_5\,x_6\,y_8; \\
R_5 &= y_1\,x_1\,x_3\,y_2\,x_2\,y_0; \\
S_5 &= y_8\,x_8\,x_{10}\,y_{10}\,y_{12}; \\
T_5 &= y_{11}\,y_9\,x_{11}\,x_9\,y_7;
\end{aligned}$$

$$\begin{aligned}
X_6 &= x_4\,y_2\,y_4\,x_2\,x_4; \\
Q_6 &= x_8\,x_6\,y_7\,y_5\,y_6\,x_5\,x_7; \\
R_6 &= x_0\,y_1\,x_3\,y_3\,x_1; \\
S_6 &= x_7\,y_8\,x_{10}\,x_{11}; \\
T_6 &= x_{12}\,y_{11}\,x_9\,y_9\,y_{10}\,x_8;
\end{aligned}$$

$$\begin{aligned}
X_7 &= y_5\,x_5\,y_4\,y_3\,y_5; \\
Q_7 &= y_8\,y_6\,x_4\,x_6\,x_8; \\
R_7 &= x_1\,y_2\,x_3\,x_2\,x_0\,y_0\,y_1; \\
S_7 &= x_8\,x_9\,y_{10}\,x_{12}; \\
T_7 &= y_{12}\,x_{10}\,y_9\,x_7\,y_7\,y_8;
\end{aligned}$$

$$\begin{aligned}
X_8 &= x_5\,x_4\,y_4\,x_3\,x_5; \\
Q_8 &= y_8\,x_6\,y_5\,x_7\,y_6\,y_7; \\
R_8 &= y_0\,x_2\,x_1\,y_3\,y_2\,y_1; \\
S_8 &= y_7\,x_9\,x_{11}\,y_9\,y_{11}; \\
T_8 &= y_{12}\,y_{10}\,x_{10}\,x_8\,y_8.
\end{aligned}$$

Case 2: $t_1 = 4$ and $q = 14$. See Figures 12–20 in Appendix B.1.

$$\begin{aligned}
X_0 &= x_3\,y_3\,x_5\,y_4\,x_3; \\
Q_0 &= x_9\,x_8\,x_6\,y_5\,x_7\,y_6\,y_7\,y_8\,x_{10}; \\
R_0 &= x_1\,y_1\,y_0\,x_2\,x_4\,y_2\,x_0; \\
S_0 &= x_{10}\,y_{10}\,y_9\,x_{11}\,y_{12}\,x_{14}; \\
T_0 &= x_{13}\,x_{12}\,y_{11}\,x_9;
\end{aligned}$$

$$\begin{aligned}
X_1 &= x_3\,x_5\,x_4\,y_5\,x_3; \\
Q_1 &= x_{10}\,x_8\,y_{10}\,y_8\,x_6\,y_6\,x_7\,y_7\,y_9; \\
R_1 &= y_0\,x_0\,x_2\,y_4\,y_2\,y_3\,x_1; \\
S_1 &= y_9\,x_9\,x_{11}\,y_{11}\,y_{13}; \\
T_1 &= x_{14}\,x_{12}\,y_{14}\,y_{12}\,x_{10};
\end{aligned}$$

$$\begin{aligned}
X_2 &= y_3\,y_5\,x_4\,x_5\,y_3; \\
Q_2 &= y_9\,y_7\,y_6\,x_6\,x_7\,y_8\,y_{10}\,x_8\,x_{10}; \\
R_2 &= x_1\,x_3\,y_2\,y_4\,x_2\,x_0\,y_0; \\
S_2 &= x_{10}\,y_{12}\,y_{14}\,x_{12}\,x_{14}; \\
T_2 &= y_{13}\,y_{11}\,x_{11}\,x_9\,y_9;
\end{aligned}$$

$$\begin{aligned}
X_3 &= y_3\,x_4\,x_6\,y_4\,y_3; \\
Q_3 &= y_9\,x_7\,x_5\,y_5\,y_6\,x_8\,y_8\,y_7\,x_9; \\
R_3 &= x_0\,y_1\,y_2\,x_3\,x_1\,x_2\,y_0; \\
S_3 &= x_9\,y_{10}\,x_{12}\,y_{12}\,y_{11}\,x_{13}; \\
T_3 &= y_{13}\,x_{11}\,x_{10}\,y_9;
\end{aligned}$$

$X_4 = y_3\, y_4\, y_6\, x_4\, y_3$; $\quad S_4 = y_9\, x_{10}\, y_{11}\, x_{12}\, y_{13}$;
$Q_4 = x_9\, y_8\, x_7\, y_5\, y_7\, x_5\, x_6\, x_8\, y_9$; $\quad T_4 = x_{13}\, y_{12}\, x_{11}\, y_{10}\, x_9$;
$R_4 = y_0\, x_1\, y_2\, x_2\, x_3\, y_1\, x_0$;

$X_5 = y_4\, x_6\, x_4\, y_6\, y_4$; $\quad S_5 = y_9\, y_{10}\, x_{11}\, y_{13}$;
$Q_5 = x_{10}\, y_8\, x_8\, x_9\, y_7\, y_5\, x_5\, x_7\, y_9$; $\quad T_5 = x_{14}\, y_{12}\, x_{12}\, x_{13}\, y_{11}\, x_{10}$;
$R_5 = y_0\, y_1\, x_3\, x_2\, y_3\, y_2\, x_1$;

$X_6 = y_4\, x_5\, x_3\, x_4\, y_4$; $\quad S_6 = x_9\, x_{10}\, x_{12}\, x_{11}\, x_{13}$;
$Q_6 = y_9\, y_8\, y_6\, y_5\, x_6\, y_7\, x_8\, x_7\, x_9$; $\quad T_6 = y_{13}\, y_{12}\, y_{10}\, y_{11}\, y_9$;
$R_6 = x_0\, x_1\, y_3\, y_1\, x_2\, y_2\, y_0$;

$X_7 = y_4\, x_4\, x_3\, y_5\, y_4$; $\quad S_7 = y_9\, y_{11}\, y_{10}\, y_{12}\, y_{13}$;
$Q_7 = x_9\, x_7\, x_8\, y_6\, x_5\, y_7\, x_6\, y_8\, y_9$; $\quad T_7 = x_{13}\, x_{11}\, x_{12}\, x_{10}\, x_9$;
$R_7 = y_0\, y_2\, y_1\, y_3\, x_2\, x_1\, x_0$;

$X_8 = y_4\, y_5\, y_3\, x_3\, y_4$; $\quad S_8 = x_9\, y_{11}\, y_{12}\, x_{13}$;
$Q_8 = y_9\, x_8\, y_7\, x_7\, x_6\, x_5\, y_6\, y_8\, x_9$; $\quad T_8 = y_{13}\, x_{12}\, y_{10}\, x_{10}\, x_{11}\, y_9$.
$R_8 = x_0\, y_2\, x_4\, x_2\, y_1\, x_1\, y_0$;

Case 3: $t_1 = 6$ and $q = 16$. See Figures 21–29 in Appendix B.1.

$X_0 = y_5\, x_6\, y_8\, x_7\, x_8\, y_7\, y_5$; $\quad S_0 = y_{11}\, x_{13}\, y_{13}\, y_{12}\, x_{14}\, y_{15}$;
$Q_0 = x_{11}\, x_9\, y_9\, y_{10}\, x_{10}\, y_{11}$; $\quad T_0 = x_{15}\, y_{14}\, x_{12}\, x_{11}$;
$R_0 = y_0\, x_2\, y_2\, y_1\, x_3\, x_4\, y_6\, y_4\, x_5\, y_3\, x_1\, x_0$;

$X_1 = x_5\, y_5\, x_7\, y_6\, y_8\, y_7\, x_5$; $\quad S_1 = y_{12}\, x_{12}\, x_{14}\, y_{14}\, y_{16}$;
$Q_1 = y_{11}\, y_9\, x_9\, x_8\, x_{10}\, y_{10}\, y_{12}$; $\quad T_1 = y_{15}\, x_{13}\, x_{11}\, y_{13}\, y_{11}$;
$R_1 = y_1\, x_1\, x_3\, y_3\, x_4\, x_6\, y_4\, x_2\, x_0\, y_2\, y_0$;

$X_2 = y_5\, x_5\, y_7\, x_8\, x_7\, x_6\, y_5$; $\quad S_2 = y_{11}\, y_{13}\, x_{11}\, x_{13}\, y_{15}$;
$Q_2 = y_{12}\, y_{10}\, x_9\, x_{10}\, y_8\, y_9\, y_{11}$; $\quad T_2 = y_{16}\, y_{14}\, x_{14}\, x_{12}\, y_{12}$;
$R_2 = y_0\, y_2\, x_0\, x_2\, y_4\, y_6\, x_4\, y_3\, x_3\, x_1\, y_1$;

$X_3 = x_5\, x_6\, x_8\, y_6\, y_7\, x_7\, x_5$; $\quad S_3 = x_{12}\, y_{13}\, x_{15}\, y_{15}\, y_{14}\, x_{16}$;
$Q_3 = y_{12}\, x_{10}\, y_9\, y_8\, x_9\, x_{11}\, y_{11}\, y_{10}\, x_{12}$; $\quad T_3 = y_{16}\, x_{14}\, x_{13}\, y_{12}$;
$R_3 = x_1\, y_2\, x_4\, y_4\, y_3\, y_5\, x_3\, x_2\, y_1$;

$X_4 = x_6\, x_7\, y_9\, y_7\, y_6\, x_8\, x_6$; $\quad S_4 = x_{11}\, y_{12}\, x_{13}\, y_{14}\, x_{15}$;
$Q_4 = y_{11}\, x_{10}\, x_9\, y_8\, y_{10}\, x_{11}$; $\quad T_4 = y_{15}\, x_{14}\, y_{13}\, x_{12}\, y_{11}$;
$R_4 = x_0\, y_1\, x_2\, y_3\, y_4\, y_5\, x_4\, x_5\, x_3\, y_2\, x_1\, y_0$;

$X_5 = y_6\, x_5\, x_7\, x_9\, y_7\, x_6;$ $\quad S_5 = y_{12}\, y_{13}\, x_{14}\, y_{16};$
$Q_5 = x_{12}\, y_{10}\, y_9\, x_8\, y_8\, x_{10}\, y_{12};$ $\quad T_5 = x_{16}\, y_{14}\, x_{13}\, y_{11}\, x_{11}\, x_{12};$
$R_5 = y_1\, y_2\, x_3\, y_4\, x_4\, y_5\, y_3\, x_2\, y_0\, x_0\, x_1;$

$X_6 = y_6\, x_7\, y_7\, y_8\, x_6\, x_5\, y_6;$ $\quad S_6 = x_{12}\, x_{13}\, x_{15}\, x_{14}\, x_{16};$
$Q_6 = y_{11}\, x_9\, y_{10}\, x_8\, y_9\, x_{11}\, x_{10}\, x_{12};$ $\quad T_6 = y_{15}\, y_{13}\, y_{14}\, y_{12}\, y_{11};$
$R_6 = x_1\, x_2\, x_4\, x_3\, y_5\, y_4\, y_2\, y_3\, y_1\, y_0;$

$X_7 = y_5\, y_6\, x_6\, y_7\, y_9\, x_7\, y_5;$ $\quad S_7 = y_{11}\, y_{12}\, y_{14}\, y_{13}\, y_{15};$
$Q_7 = x_{12}\, x_{10}\, x_{11}\, y_{10}\, y_8\, x_8\, x_9\, y_{11};$ $\quad T_7 = x_{16}\, x_{14}\, x_{15}\, x_{13}\, x_{12};$
$R_7 = y_0\, y_1\, y_3\, y_2\, y_4\, x_3\, x_5\, x_4\, x_2\, x_1;$

$X_8 = y_5\, y_7\, x_9\, x_7\, y_8\, y_6\, y_5;$ $\quad S_8 = y_{11}\, x_{12}\, y_{14}\, y_{15};$
$Q_8 = x_{11}\, y_9\, x_{10}\, x_8\, y_{10}\, y_{11};$ $\quad T_8 = x_{15}\, y_{13}\, x_{13}\, x_{14}\, y_{12}\, x_{11}.$
$R_8 = y_0\, x_1\, y_3\, x_5\, y_4\, x_6\, x_4\, y_2\, x_2\, x_3\, y_1\, x_0;$

Case 4: $t_1 = 6$ and $q = 20$. See Figures 30–38 in Appendix B.1.

$X_0 = y_3\, y_5\, x_7\, x_5\, y_6\, y_4\, y_3;$ $\quad S_0 = y_{13}\, x_{14}\, y_{16}\, y_{17};$
$Q_0 = x_{13}\, y_{11}\, x_{11}\, x_{12}\, y_{10}\, x_9\, y_7\, x_8\, x_6\, y_8\, y_9\, x_{10}\, y_{12}\, y_{13};$ $\quad T_0 = x_{17}\, y_{15}\, x_{15}\, x_{16}\, y_{14}\, x_{13};$
$R_0 = y_0\, x_1\, x_3\, y_2\, x_4\, x_2\, y_1\, x_0;$

$X_1 = y_3\, y_4\, x_4\, y_5\, y_7\, x_5\, y_3;$ $\quad S_1 = y_{13}\, y_{14}\, y_{16}\, y_{15}\, y_{17};$
$Q_1 = x_{14}\, x_{12}\, x_{13}\, x_{11}\, x_{10}\, x_8\, x_9\, y_8\, y_6\, x_6\, x_7\, y_9\, y_{10}\, y_{12}\, y_{11}\, y_{13};$ $\quad T_1 = x_{18}\, x_{16}\, x_{17}\, x_{15}\, x_{14};$
$R_1 = y_0\, y_1\, x_3\, x_2\, y_2\, x_1;$

$X_2 = y_4\, x_5\, y_5\, y_6\, x_4\, x_3\, y_4;$ $\quad S_2 = x_{14}\, x_{15}\, x_{17}\, x_{16}\, x_{18};$
$Q_2 = y_{13}\, y_{11}\, y_{12}\, y_{10}\, y_9\, x_7\, y_8\, x_6\, y_7\, x_9\, x_8\, x_{10}\, x_{11}\, x_{13}\, x_{12}\, x_{14};$ $\quad T_2 = y_{17}\, y_{15}\, y_{16}\, y_{14}\, y_{13};$
$R_2 = x_1\, y_2\, x_2\, y_3\, y_1\, y_0;$

$X_3 = y_3\, x_5\, y_4\, y_6\, y_5\, x_3 y_3;$
$Q_3 = x_{14}\, y_{12}\, x_{11}\, y_9\, x_9\, x_{10}\, y_8\, y_7\, x_7\, x_6\, x_8\, y_{10}\, y_{11}\, x_{12}\, y_{14};$ $\quad S_3 = y_{14}\, y_{15}\, x_{16}\, y_{18};$
$R_3 = y_1\, x_2\, x_4\, y_2\, y_0\, x_0\, x_1;$ $\quad T_3 = x_{18}\, y_{16}\, x_{15}\, y_{13}\, x_{13}\, x_{14};$

$X_4 = y_3\, x_4\, y_6\, x_5\, x_6\, y_5\, y_3;$ $\quad S_4 = x_{13}\, y_{14}\, x_{15}\, y_{16}\, x_{17};$
$Q_4 = y_{13}\, x_{12}\, y_{11}\, x_{10}\, y_9\, x_8\, x_7\, y_7\, y_8\, x_9\, y_{10}\, x_{11}\, y_{12}\, x_{13};$ $\quad T_4 = y_{17}\, x_{16}\, y_{15}\, x_{14}\, y_{13};$
$R_4 = x_0\, y_1\, y_2\, y_4\, x_2\, x_3\, x_1\, y_0;$

$X_5 = y_3\, x_3\, y_5\, x_6\, x_5\, x_4 y_3;$ $\quad S_5 = x_{14}\, y_{15}\, x_{17}\, y_{17}\, y_{16}\, x_{18};$
$Q_5 = y_{14}\, x_{12}\, x_{11}\, y_{10}\, x_8\, y_7\, y_6\, x_7\, x_9\, y_9\, y_8\, x_{10}\, y_{11}\, x_{13}\, y_{13}\, y_{12}\, x_{14};$ $\quad T_5 = y_{18}\, x_{16}\, x_{15}\, y_{14};$
$R_5 = x_1\, x_2\, y_4\, y_2\, y_1;$

$X_6 = x_3\, x_4\, x_6\, y_4\, y_5\, x_5\, x_3$;
$Q_6 = y_{14}\, y_{12}\, x_{12}\, x_{10}\, y_{10}\, y_8\, x_7\, x_8\, y_6\, y_7\, y_9\, y_{11}\, x_9\, x_{11}\, y_{13}$;
$R_6 = y_0\, x_2\, x_0\, y_2\, y_3\, x_1\, y_1$;
$S_6 = y_{13}\, y_{15}\, x_{13}\, x_{15}\, y_{17}$;
$T_6 = y_{18}\, y_{16}\, x_{16}\, x_{14}\, y_{14}$;

$X_7 = x_3\, x_5\, x_7\, y_5\, x_4\, y_4\, x_3$;
$Q_7 = y_{13}\, x_{11}\, x_9\, y_{11}\, y_9\, y_7\, x_6\, y_6\, x_8\, y_8\, y_{10}\, x_{10}\, x_{12}\, y_{12}\, y_{14}$;
$R_7 = y_1\, x_1\, y_3\, y_2\, x_0\, x_2\, y_0$;
$S_7 = y_{14}\, x_{14}\, x_{16}\, y_{16}\, y_{18}$;
$T_7 = y_{17}\, x_{15}\, x_{13}\, y_{15}\, y_{13}$;

$X_8 = x_4\, x_5\, y_7\, y_5\, y_4\, x_6\, x_4$;
$Q_8 = x_{13}\, y_{12}\, x_{10}\, x_9\, x_7\, y_6\, y_8\, x_8\, y_9\, x_{11}\, y_{11}\, y_{10}\, x_{12}\, y_{13}$;
$R_8 = y_0\, y_2\, x_3\, y_1\, y_3\, x_2\, x_1\, x_0$;
$S_8 = y_{13}\, x_{15}\, y_{15}\, y_{14}\, x_{16}\, y_{17}$;
$T_8 = x_{17}\, y_{16}\, x_{14}\, x_{13}$.

A.2 The case $t_1 + q \equiv 0 \pmod 4$

Case 1: $t_1 = 4$ and $q = 16$. See Figures 39–45 in Appendix B.2.

$X_0 = x_7\, y_8\, y_7\, x_8\, x_7$;
$Q_0 = x_{11}\, x_{10}\, y_9\, x_9\, y_{10}$;
$R_0 = y_0\, x_3\, x_4\, y_4\, y_3\, y_2\, x_5\, y_6\, x_6\, y_5\, x_2\, y_1\, x_1$;
$S_0 = y_{10}\, x_{13}\, y_{13}\, x_{14}\, y_{14}$;
$T_0 = x_{15}\, x_{12}\, y_{12}\, y_{11}\, x_{11}$;

$X_1 = x_6\, x_9\, y_6\, y_9\, x_6$;
$Q_1 = x_{10}\, y_{10}\, x_{11}\, x_8\, y_8\, y_{11}$;
$R_1 = y_1\, x_2\, x_5\, y_5\, x_4\, y_7\, x_7\, y_4\, x_3\, y_2\, y_3\, x_0$;
$S_1 = y_{11}\, x_{12}\, x_{13}\, y_{12}\, y_{15}$;
$T_1 = x_{14}\, x_{15}\, y_{14}\, y_{13}\, x_{10}$;

$X_2 = x_7\, x_6\, y_7\, y_8\, x_7$;
$Q_2 = x_{10}\, y_{11}\, x_8\, x_9\, y_9\, y_{10}$;
$R_2 = y_0\, y_3\, x_2\, x_1\, y_2\, y_5\, y_6\, x_5\, y_4\, x_4\, x_3\, x_0$;
$S_2 = y_{10}\, y_{13}\, y_{12}\, x_{11}\, x_{12}\, y_{15}\, y_{14}$;
$T_2 = x_{14}\, x_{13}\, x_{10}$;

$X_3 = x_7\, x_8\, y_7\, y_6\, x_7$;
$Q_3 = y_{10}\, y_9\, x_{10}\, x_9\, y_8\, x_{11}$;
$R_3 = x_1\, y_1\, y_2\, x_2\, x_3\, y_4\, y_5\, x_6\, x_5\, x_4\, y_3\, y_0$;
$S_3 = x_{11}\, y_{11}\, y_{12}\, x_{12}\, x_{15}$;
$T_3 = y_{14}\, x_{14}\, y_{13}\, x_{13}\, y_{10}$;

$X_4 = x_9\, x_8\, y_9\, y_8\, x_9$;
$Q_4 = y_{10}\, y_{11}\, x_{10}\, x_{11}$;
$R_4 = x_1\, x_2\, y_3\, x_4\, x_7\, y_7\, y_4\, x_5\, x_6\, y_6\, y_5\, y_2\, x_3\, y_0$;
$S_4 = x_{11}\, y_{12}\, x_{15}$;
$T_4 = y_{14}\, x_{13}\, x_{14}\, y_{15}\, x_{12}\, y_{13}\, y_{10}$;

$X_5 = x_6\, x_7\, y_6\, x_9\, x_6$;
$Q_5 = x_{11}\, y_8\, y_9\, x_8\, y_{11}\, y_{10}\, x_{10}$;
$R_5 = x_0\, y_3\, x_3\, x_2\, y_5\, y_4\, y_7\, x_4\, x_5\, y_2\, x_1$;
$S_5 = x_{10}\, x_{13}\, y_{14}\, y_{15}\, x_{14}$;
$T_5 = x_{15}\, y_{12}\, y_{13}\, x_{12}\, x_{11}$;

$X_6 = x_6\, y_9\, y_6\, y_7\, x_6$;
$Q_6 = y_{11}\, y_8\, x_8\, x_{11}\, y_{10}\, x_9\, x_{10}$;
$R_6 = x_0\, x_3\, y_3\, y_4\, x_7\, x_4\, y_5\, x_5\, x_2\, y_2\, y_1$;
$S_6 = x_{10}\, y_{13}\, y_{14}\, x_{15}\, x_{14}$;
$T_6 = y_{15}\, y_{12}\, x_{13}\, x_{12}\, y_{11}$.

Case 2: $t_1 = 4$ and $q = 20$. See Figures 46–52 in Appendix B.2.

$$X_0 = x_{10}\, x_{11}\, y_{10}\, y_{11}\, x_{10}; \quad S_0 = y_{12}\, y_{15}\, x_{15}\, x_{14}\, x_{17}\, x_{16}\, y_{16};$$
$$Q_0 = y_{13}\, x_{12}\, x_{13}\, y_{12}; \quad T_0 = y_{17}\, y_{14}\, y_{13};$$
$$R_0 = y_0\, y_3\, y_4\, x_3\, x_4\, y_7\, y_8\, x_8\, x_7\, x_6\, x_9\, y_9\, y_6\, x_5\, y_5\, x_2\, y_2\, y_1;$$

$$X_1 = x_8\, x_{11}\, y_8\, y_{11}\, x_8; \quad S_1 = y_{13}\, x_{14}\, y_{14}\, y_{17};$$
$$Q_1 = x_{12}\, y_{12}\, x_{13}\, x_{10}\, y_{10}\, y_{13}; \quad T_1 = x_{16}\, x_{17}\, y_{16}\, x_{15}\, y_{15}\, x_{12};$$
$$R_1 = y_1\, y_2\, y_5\, x_6\, y_9\, x_9\, y_6\, y_7\, x_7\, y_4\, x_5\, x_4\, x_3\, x_2\, y_3\, x_0;$$

$$X_2 = x_8\, y_9\, x_{10}\, x_9\, x_8; \quad S_2 = x_{13}\, x_{14}\, y_{15}\, y_{14}\, x_{17};$$
$$Q_2 = y_{12}\, y_{11}\, x_{11}\, x_{12}\, y_{13}\, y_{10}\, x_{13}; \quad T_2 = y_{16}\, y_{17}\, x_{16}\, x_{15}\, y_{12};$$
$$R_2 = x_1\, y_2\, x_2\, y_5\, x_4\, x_5\, y_6\, x_6\, x_7\, y_8\, y_7\, y_4\, y_3\, x_3\, y_0;$$

$$X_3 = y_8\, x_9\, y_{10}\, y_9\, y_8; \quad S_3 = y_{13}\, y_{14}\, x_{15}\, y_{16}\, y_{15}\, x_{16}\, y_{17};$$
$$Q_3 = x_{13}\, x_{12}\, x_{11}\, x_{10}\, y_{11}\, y_{12}\, y_{13}; \quad T_3 = x_{17}\, x_{14}\, x_{13};$$
$$R_3 = y_1\, x_2\, x_3\, y_3\, x_4\, y_4\, y_5\, y_6\, x_7\, x_8\, y_7\, x_6\, x_5\, y_2\, x_1;$$

$$X_4 = x_8\, x_9\, y_8\, x_{11}\, x_8; \quad S_4 = y_{13}\, x_{13}\, y_{14}\, x_{14}\, y_{17};$$
$$Q_4 = x_{12}\, y_{11}\, y_{10}\, x_{10}\, y_{13}; \quad T_4 = x_{16}\, y_{15}\, y_{12}\, x_{15}\, x_{12};$$
$$R_4 = y_1\, x_1\, x_2\, x_5\, y_4\, x_7\, y_6\, y_9\, x_6\, y_7\, x_4\, y_5\, y_2\, y_3\, y_0\, x_3\, x_0;$$

$$X_5 = x_8\, y_{11}\, y_8\, y_9\, x_8; \quad S_5 = x_{12}\, x_{15}\, x_{16};$$
$$Q_5 = y_{13}\, x_{10}\, x_{13}\, y_{10}\, x_{11}\, y_{12}\, x_{12}; \quad T_5 = y_{17}\, y_{16}\, x_{17}\, y_{14}\, y_{15}\, x_{14}\, y_{13};$$
$$R_5 = x_0\, y_3\, y_2\, x_3\, y_4\, x_4\, x_7\, y_7\, y_6\, x_9\, x_6\, y_5\, x_5\, x_2\, y_1;$$

$$X_6 = x_9\, x_{10}\, y_9\, y_{10}\, x_9; \quad S_6 = x_{12}\, y_{15}\, y_{16}\, x_{16};$$
$$Q_6 = y_{13}\, y_{12}\, x_{11}\, y_{11}\, x_{12}; \quad T_6 = y_{17}\, x_{14}\, x_{15}\, y_{14}\, x_{13}\, y_{13}.$$
$$R_6 = x_0\, x_3\, y_2\, x_5\, x_6\, y_6\, y_5\, y_4\, y_7\, x_8\, y_8\, x_7\, x_4\, y_3\, x_2\, x_1\, y_1;$$

Case 3: $t_1 = 6$ and $q = 14$. See Figures 53–59 in Appendix B.2.

$$X_0 = x_4\, y_7\, x_6\, y_5\, y_6\, x_7\, x_4; \quad S_0 = y_{11}\, x_{10}\, x_{11}\, y_{12}\, x_{12}\, y_{15};$$
$$Q_0 = y_{10}\, y_9\, x_9\, y_8\, x_8\, y_{11}; \quad T_0 = y_{14}\, y_{13}\, x_{13}\, y_{10};$$
$$R_0 = y_1\, x_0\, x_1\, x_2\, x_5\, y_2\, x_3\, y_4\, y_3\, y_0;$$

$$X_1 = x_4\, x_7\, y_7\, y_4\, y_5\, x_5\, x_4; \quad S_1 = x_{11}\, y_{10}\, y_{13}\, y_{12}\, x_{15};$$
$$Q_1 = y_{11}\, y_8\, y_9\, y_6\, x_6\, x_9\, x_8\, x_{11}; \quad T_1 = y_{15}\, x_{12}\, x_{13}\, x_{10}\, y_{11};$$
$$R_1 = x_1\, y_0\, y_3\, x_0\, x_3\, x_2\, y_2\, y_1;$$

$X_2 = y_4\, x_5\, y_5\, x_6\, y_7\, x_7\, y_4;$ $\quad S_2 = y_{11}\, x_{11}\, x_{12}\, y_{12}\, y_{15};$
$Q_2 = x_{10}\, y_{10}\, x_9\, y_6\, y_9\, x_8\, y_8\, y_{11};$ $\quad T_2 = x_{14}\, y_{14}\, x_{13}\, y_{13}\, x_{10};$
$R_2 = y_1\, x_1\, y_2\, x_2\, y_3\, x_4\, x_3\, x_0;$

$X_3 = x_4\, x_5\, y_4\, x_7\, y_8\, y_7\, x_4;$ $\quad S_3 = y_{10}\, x_{13}\, y_{14};$
$Q_3 = y_{11}\, x_8\, y_9\, x_6\, y_6\, x_9\, y_{10};$ $\quad T_3 = y_{15}\, y_{12}\, x_{11}\, x_{10}\, y_{13}\, x_{12}\, y_{11};$
$R_3 = y_0\, x_1\, x_0\, y_3\, x_3\, y_2\, y_5\, x_2\, y_1;$

$X_4 = x_5\, y_6\, y_7\, x_8\, x_7\, x_6\, x_5;$ $\quad S_4 = x_{11}\, y_{11}\, x_{12}\, x_{15};$
$Q_4 = y_{10}\, x_{10}\, x_9\, y_9\, y_8\, x_{11};$ $\quad T_4 = y_{14}\, x_{14}\, x_{13}\, y_{12}\, y_{13}\, y_{10};$
$R_4 = x_1\, y_1\, y_2\, y_3\, x_2\, y_5\, x_4\, y_4\, x_3\, y_0;$

$X_5 = x_5\, x_6\, x_7\, x_8\, y_7\, y_6\, x_5;$ $\quad S_5 = y_{10}\, y_{11}\, y_{12}\, x_{13}\, x_{14}\, y_{13}\, y_{14};$
$Q_5 = x_{11}\, y_8\, x_9\, x_{10}\, y_9\, y_{10};$ $\quad T_5 = x_{15}\, x_{12}\, x_{11};$
$R_5 = y_0\, y_1\, x_2\, x_3\, y_3\, y_4\, x_4\, y_5\, y_2\, x_1;$

$X_6 = y_5\, y_4\, y_7\, y_8\, x_7\, y_6\, y_5;$ $\quad S_6 = x_{10}\, x_{13}\, x_{12}\, y_{13}\, x_{14};$
$Q_6 = x_{11}\, x_8\, x_9\, x_6\, y_9\, x_{10};$ $\quad T_6 = x_{15}\, y_{12}\, y_{11}\, y_{10}\, x_{11}.$
$R_6 = x_0\, y_1\, y_0\, x_3\, x_4\, y_3\, y_2\, x_5\, x_2\, x_1;$

Case 4: $t_1 = 6$ and $q = 18$. See Figures 60–66 in Appendix B.2.

$X_0 = x_6\, y_9\, x_8\, y_7\, y_8\, x_9\, x_6;$ $\quad S_0 = x_{13}\, y_{12}\, y_{13}\, x_{14}\, y_{14}\, x_{17};$
$Q_0 = x_{12}\, y_{11}\, x_{11}\, y_{10}\, x_{10}\, x_{13};$ $\quad T_0 = x_{16}\, x_{15}\, y_{15}\, x_{12};$
$R_0 = x_1\, y_0\, y_1\, x_2\, y_5\, y_6\, x_5\, x_4\, x_7\, y_4\, x_3\, y_2\, y_3\, x_0;$

$X_1 = x_7\, x_6\, x_9\, y_9\, y_6\, y_7\, x_7;$ $\quad S_1 = y_{13}\, x_{13}\, x_{14}\, y_{17};$
$Q_1 = y_{12}\, x_{12}\, x_{11}\, x_8\, y_8\, y_{11}\, x_{10}\, y_{10}\, y_{13};$ $\quad T_1 = y_{16}\, x_{16}\, y_{15}\, y_{14}\, x_{15}\, y_{12};$
$R_1 = y_1\, x_1\, x_2\, y_2\, y_5\, x_4\, x_5\, y_4\, y_3\, x_3\, y_0;$

$X_2 = x_7\, y_7\, x_8\, y_9\, x_9\, y_6\, x_7;$ $\quad S_2 = y_{13}\, x_{12}\, y_{15}\, y_{12}\, x_{15}\, y_{14}\, y_{17};$
$Q_2 = x_{13}\, y_{10}\, y_{11}\, y_8\, x_{11}\, x_{10}\, y_{13};$ $\quad T_2 = x_{17}\, x_{14}\, x_{13};$
$R_2 = y_1\, y_2\, x_2\, x_5\, x_6\, y_5\, y_4\, x_4\, y_3\, y_0\, x_3\, x_0\, x_1;$

$X_3 = x_6\, x_7\, y_6\, x_9\, y_{10}\, y_9\, x_6;$ $\quad S_3 = y_{12}\, y_{15}\, x_{14}\, x_{15}\, y_{16};$
$Q_3 = y_{13}\, x_{10}\, x_{11}\, y_8\, x_8\, y_{11}\, y_{12};$ $\quad T_3 = y_{17}\, y_{14}\, x_{13}\, x_{12}\, y_{13};$
$R_3 = y_0\, y_3\, x_2\, x_3\, x_4\, y_7\, y_4\, x_5\, y_5\, y_2\, x_1\, x_0\, y_1;$

$$X_4 = x_7\, y_8\, y_9\, x_{10}\, x_9\, x_8\, x_7; \qquad S_4 = x_{12}\, x_{13}\, y_{14}\, y_{15}\, y_{16}\, x_{15}\, x_{16};$$
$$Q_4 = y_{13}\, y_{10}\, x_{11}\, y_{12}\, y_{11}\, x_{12}; \qquad T_4 = y_{17}\, x_{14}\, y_{13};$$
$$R_4 = x_0\, y_3\, x_4\, y_4\, y_7\, x_6\, y_6\, y_5\, x_5\, y_2\, x_3\, x_2\, x_1\, y_1;$$

$$X_5 = x_7\, x_8\, x_9\, x_{10}\, y_9\, y_8\, x_7; \qquad S_5 = x_{13}\, y_{13}\, y_{14}\, x_{14}\, x_{17};$$
$$Q_5 = x_{12}\, y_{12}\, x_{11}\, y_{11}\, y_{10}\, x_{13}; \qquad T_5 = x_{16}\, y_{16}\, y_{15}\, x_{15}\, x_{12};$$
$$R_5 = x_1\, y_2\, x_5\, y_6\, x_6\, y_7\, x_4\, x_3\, y_3\, y_4\, y_5\, x_2\, y_1\, x_0;$$

$$X_6 = y_7\, y_6\, y_9\, y_{10}\, x_9\, y_8\, y_7; \qquad S_6 = x_{12}\, x_{15}\, x_{14}\, y_{15}\, x_{16};$$
$$Q_6 = x_{13}\, x_{10}\, y_{11}\, x_8\, x_{11}\, x_{12}; \qquad T_6 = x_{17}\, y_{14}\, y_{13}\, y_{12}\, x_{13}.$$
$$R_6 = x_0\, x_3\, y_4\, x_7\, x_4\, y_5\, x_6\, x_5\, x_2\, y_3\, y_2\, y_1\, y_0\, x_1;$$

A.3 Special cases

A $(\vec{C}_4, \vec{C}_{12})$-factorization of W^*_{16}:

$$F_0 = \{\, x_5\, x_4\, y_3\, y_4\, x_5,\;\; x_0\, x_3\, x_2\, y_2\, y_5\, y_6\, x_1\, x_6\, y_1\, y_0\, y_7\, x_7\, x_0\};$$
$$F_1 = \{\, x_6\, y_6\, x_5\, y_5\, x_6,\;\; x_0\, y_1\, y_2\, x_2\, x_1\, y_0\, x_7\, y_4\, x_3\, y_3\, x_4\, y_7\, x_0\};$$
$$F_2 = \{\, x_5\, x_6\, y_7\, x_4\, x_5,\;\; x_0\, x_1\, y_6\, y_5\, y_2\, y_1\, x_2\, x_3\, y_4\, x_7\, y_0\, y_3\, x_0\};$$
$$F_3 = \{\, y_6\, x_6\, x_7\, y_7\, y_6,\;\; x_0\, y_0\, y_1\, x_1\, x_2\, y_3\, y_2\, x_5\, y_4\, y_5\, x_4\, x_3\, x_0\};$$
$$F_4 = \{\, x_2\, y_5\, y_4\, y_3\, x_2,\;\; x_0\, y_7\, y_0\, x_1\, y_2\, x_3\, x_4\, x_7\, x_6\, x_5\, y_6\, y_1\, x_0\};$$
$$F_5 = \{\, y_3\, y_0\, x_3\, y_2\, y_3,\;\; x_0\, x_7\, y_6\, y_7\, y_4\, x_4\, y_5\, x_5\, x_2\, y_1\, x_6\, x_1\, x_0\};$$
$$F_6 = \{\, x_0\, y_3\, x_3\, y_0\, x_0,\;\; x_1\, y_1\, y_6\, x_7\, x_4\, y_4\, y_7\, x_6\, y_5\, x_2\, x_5\, y_2\, x_1\}.$$

A $(\vec{C}_6, \vec{C}_8)$-factors of W^*_{14}:

$$F_0 = \{x_0\, y_1\, y_3\, x_5\, y_6\, y_0\, x_0,\; x_1\, x_2\, y_2\, x_3\, y_4\, x_4\, y_5\, x_6\, x_1\};$$
$$F_1 = \{y_0\, y_1\, x_3\, y_3\, x_4\, x_5\, y_0,\; x_0\, x_1\, y_2\, x_2\, y_4\, y_5\, y_6\, x_6\, x_0\};$$
$$F_2 = \{x_5\, y_5\, y_0\, x_1\, y_3\, x_3\, x_5,\; x_0\, x_2\, x_4\, x_6\, y_1\, y_2\, y_4\, y_6\, x_0\};$$
$$F_3 = \{y_2\, y_3\, y_5\, x_5\, x_0\, y_0\, y_2,\; x_1\, y_1\, x_2\, x_3\, x_4\, y_4\, x_6\, y_6\, x_1\};$$
$$F_4 = \{y_3\, y_4\, x_5\, x_6\, y_0\, x_2\, y_3,\; x_0\, y_2\, x_4\, y_6\, y_1\, x_1\, x_3\, y_5\, x_0\};$$
$$F_5 = \{y_6\, y_5\, y_4\, x_3\, y_2\, y_0\, y_6,\; x_0\, x_5\, y_3\, y_1\, x_6\, x_4\, x_2\, x_1\, x_0\};$$
$$F_6 = \{x_2\, x_0\, x_6\, x_5\, y_4\, y_3\, x_2,\; y_0\, y_5\, x_3\, x_1\, y_6\, x_4\, y_2\, y_1\, y_0\};$$
$$F_7 = \{y_6\, x_5\, x_4\, x_3\, x_2\, y_1\, y_6,\; x_0\, y_5\, y_3\, x_1\, y_0\, x_6\, y_4\, y_2\, x_0\};$$
$$F_8 = \{x_6\, y_5\, x_4\, y_3\, y_2\, x_1\, x_6,\; x_0\, y_6\, y_4\, x_2\, y_0\, x_5\, x_3\, y_1\, x_0\}.$$

A $(\vec{C}_6, \vec{C}_{10})$-factorization of W^*_{16}:

$$F_0 = \{\, y_1\, x_0\, x_1\, y_6\, y_5\, y_2\, y_1,\;\; y_0\, y_7\, x_4\, y_4\, y_3\, x_3\, x_2\, x_5\, x_6\, x_7\, y_0\};$$
$$F_1 = \{\, x_4\, x_7\, x_0\, y_1\, y_0\, y_3\, x_4,\;\; x_1\, x_2\, x_3\, y_4\, y_7\, y_6\, x_5\, y_2\, y_5\, x_6\, x_1\};$$
$$F_2 = \{\, y_1\, x_2\, y_2\, x_5\, y_6\, x_6\, y_1,\;\; x_0\, y_7\, x_7\, x_4\, y_5\, y_4\, x_3\, y_3\, y_0\, x_1\, x_0\};$$
$$F_3 = \{\, x_0\, y_3\, y_2\, x_1\, y_0\, x_3\, x_0,\;\; y_1\, y_6\, x_7\, y_4\, x_5\, x_4\, y_7\, x_6\, y_5\, x_2\, y_1\};$$
$$F_4 = \{\, y_0\, y_1\, x_6\, y_7\, x_0\, x_3\, y_0,\;\; x_1\, y_2\, y_3\, x_2\, y_5\, x_4\, x_5\, y_4\, x_7\, y_6\, x_1\};$$
$$F_5 = \{\, x_1\, x_6\, x_5\, y_5\, y_6\, y_1\, x_1,\;\; x_0\, y_0\, x_7\, y_7\, y_4\, x_4\, x_3\, y_2\, x_2\, y_3\, x_0\};$$
$$F_6 = \{\, y_7\, y_0\, x_0\, x_7\, x_6\, y_6\, y_7,\;\; x_1\, y_1\, y_2\, x_3\, x_4\, y_3\, y_4\, y_5\, x_5\, x_2\, x_1\}.$$

A $(\vec{C}_6, \vec{C}_{12})$-factorization of W^*_{18}:

$$
\begin{aligned}
F_0 &= \{y_1\, y_8\, x_8\, x_6\, x_5\, y_3\, y_1,\ \ x_0\, x_1\, y_0\, x_7\, y_5\, x_3\, y_4\, x_2\, y_2\, x_4\, y_6\, y_7\, x_0\};\\
F_1 &= \{x_2\, y_0\, y_2\, x_3\, x_5\, x_4\, x_2,\ \ x_0\, x_7\, y_8\, x_1\, y_3\, y_4\, y_6\, x_6\, y_5\, y_7\, x_8\, y_1\, x_0\};\\
F_2 &= \{y_0\, x_0\, y_8\, x_6\, x_8\, y_7\, y_0,\ \ x_1\, x_3\, y_3\, x_5\, y_4\, y_5\, x_7\, y_6\, x_4\, y_2\, y_1\, x_2\, x_1\};\\
F_3 &= \{x_5\, y_6\, y_4\, x_4\, x_6\, x_7\, x_5,\ \ x_0\, x_8\, x_1\, x_2\, y_1\, x_3\, y_5\, y_3\, y_2\, y_0\, y_7\, y_8\, x_0\};\\
F_4 &= \{x_1\, y_2\, y_3\, x_4\, x_3\, y_1\, x_1,\ \ x_0\, y_0\, x_2\, y_4\, x_6\, y_8\, x_7\, y_7\, x_5\, y_5\, y_6\, x_8\, x_0\};\\
F_5 &= \{x_2\, y_3\, x_3\, x_4\, y_4\, y_2\, x_2,\ \ x_0\, y_1\, x_8\, y_8\, y_7\, y_5\, x_5\, x_6\, y_6\, x_7\, y_0\, x_1\, x_0\};\\
F_6 &= \{x_0\, y_2\, y_4\, x_5\, x_3\, x_2\, x_0,\ \ y_0\, x_8\, x_7\, x_6\, y_7\, y_6\, y_5\, x_4\, y_3\, x_1\, y_8\, y_1\, y_0\};\\
F_7 &= \{y_0\, y_8\, y_6\, x_5\, x_7\, x_8\, y_0,\ \ x_0\, y_7\, x_6\, x_4\, y_5\, y_4\, y_3\, x_2\, x_3\, x_1\, y_1\, y_2\, x_0\};\\
F_8 &= \{x_0\, x_2\, x_4\, x_5\, y_7\, x_7\, x_0,\ \ y_0\, y_1\, y_3\, y_5\, x_6\, y_4\, x_3\, y_2\, x_1\, x_8\, y_6\, y_8\, y_0\}.
\end{aligned}
$$

B Illustration of all (t_1, q)-base tuples given in Appendix A

B.1 The case $t_1 + q \equiv 2 \pmod 4$

Case 1: $t_1 = 4$ and $q = 10$

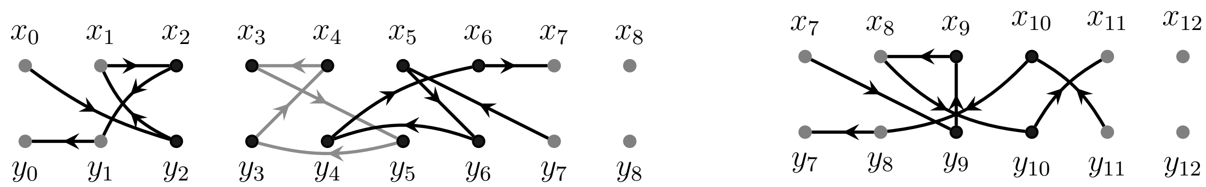

Figure 3: The $(4, 10)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

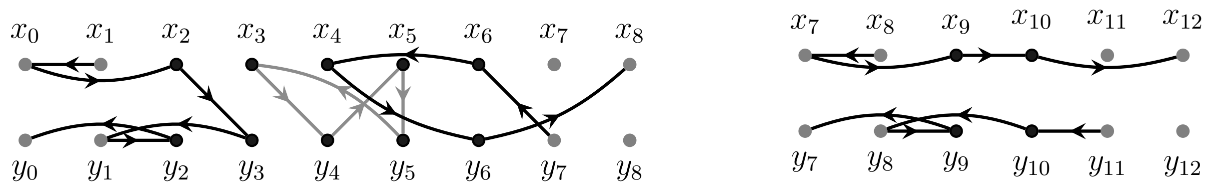

Figure 4: The $(4, 10)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

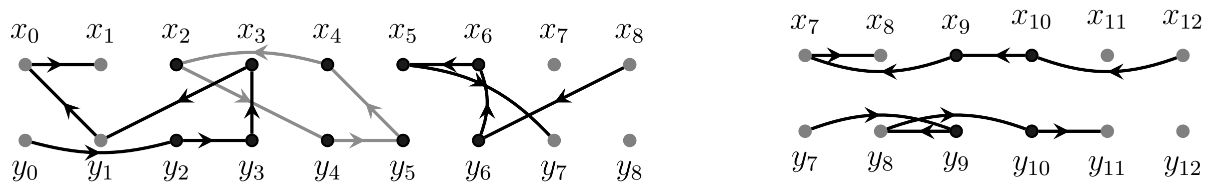

Figure 5: The $(4, 10)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

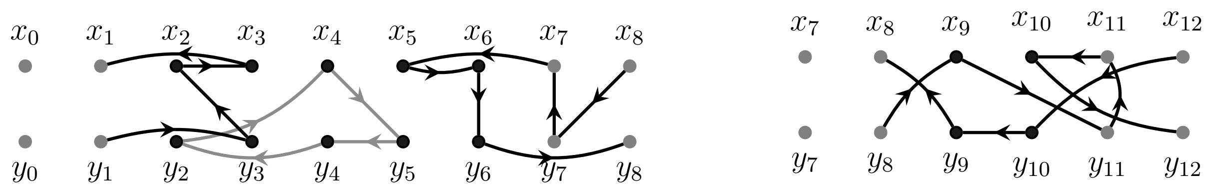

Figure 6: The $(4, 10)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

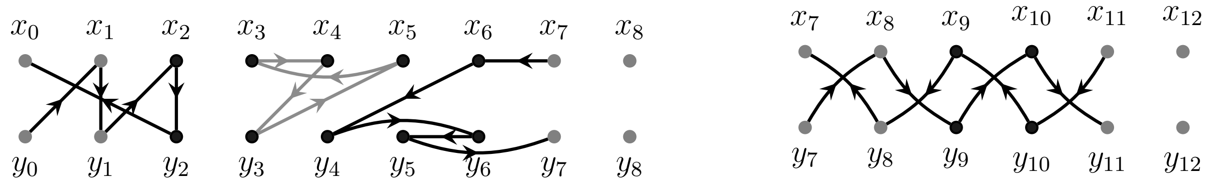

Figure 7: The $(4, 10)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

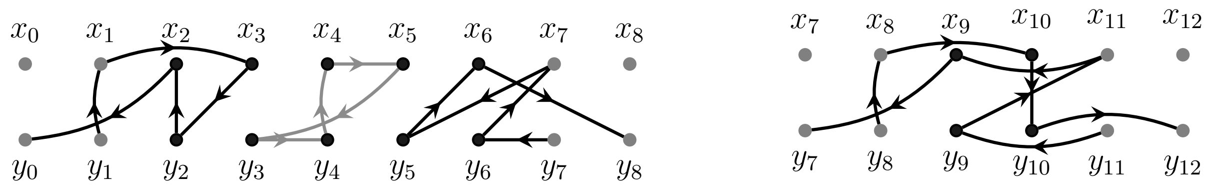

Figure 8: The $(4, 10)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

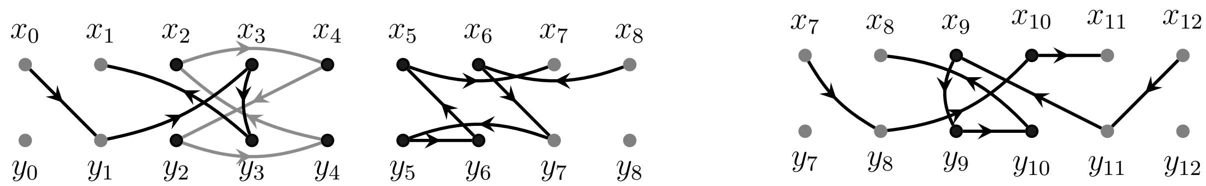

Figure 9: The $(4, 10)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

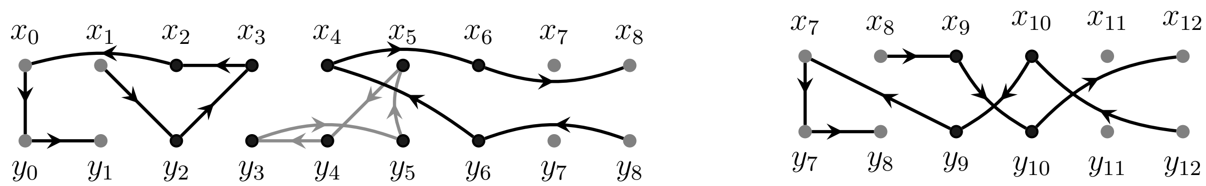

Figure 10: The $(4, 10)$-base tuple $(X_7, Q_7, R_7, S_7, T_7)$.

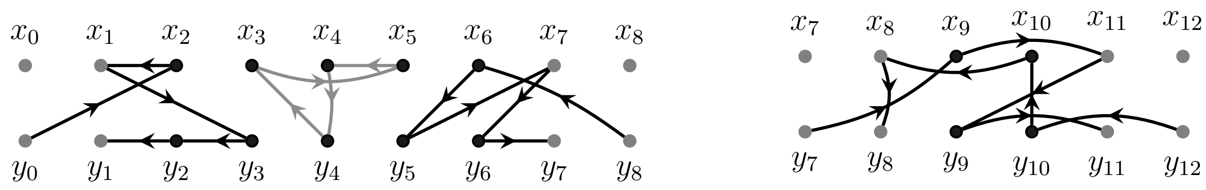

Figure 11: The $(4, 10)$-base tuple $(X_8, Q_8, R_8, S_8, T_8)$.

Case 2: $t_1 = 4$ and $q = 14$

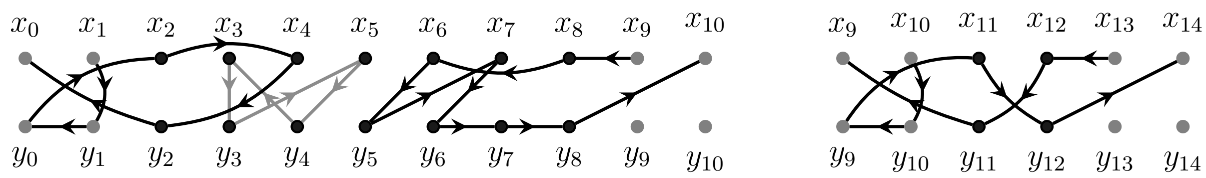

Figure 12: The $(4, 14)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

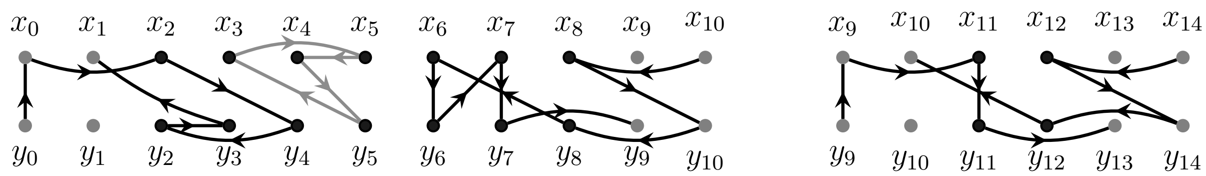

Figure 13: The $(4, 14)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

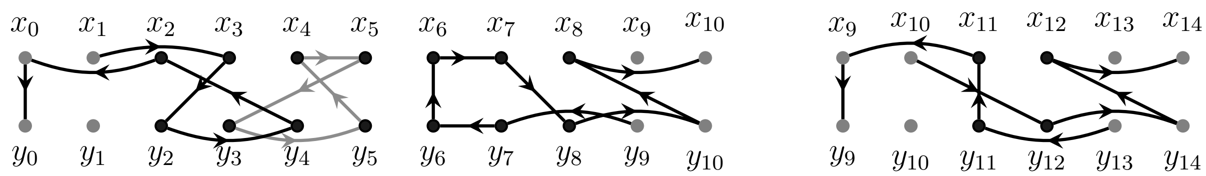

Figure 14: The $(4, 14)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

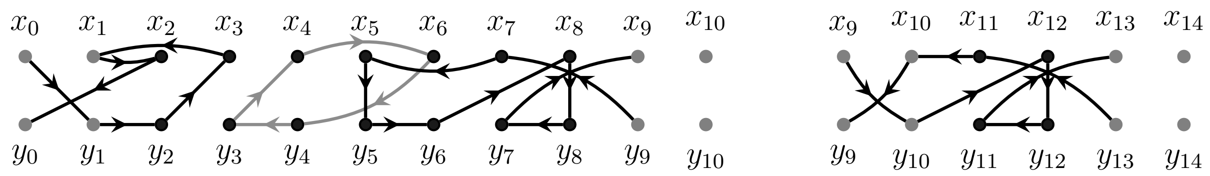

Figure 15: The $(4, 14)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

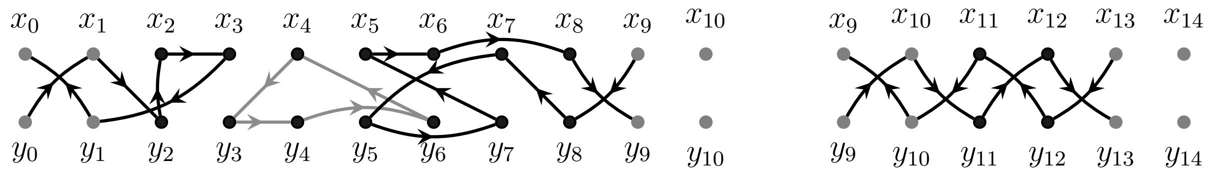

Figure 16: The $(4, 14)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

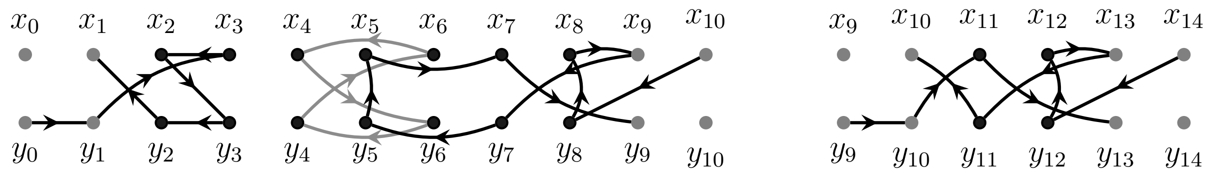

Figure 17: The $(4, 14)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

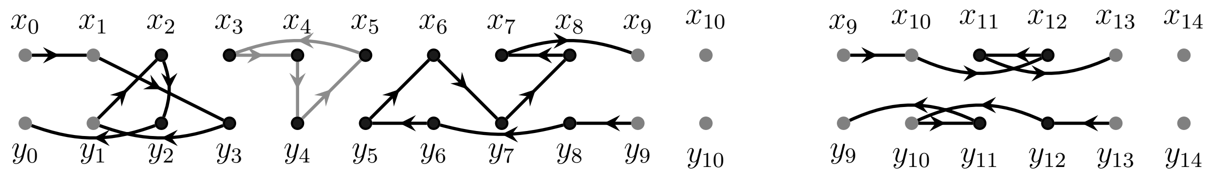

Figure 18: The $(4, 14)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

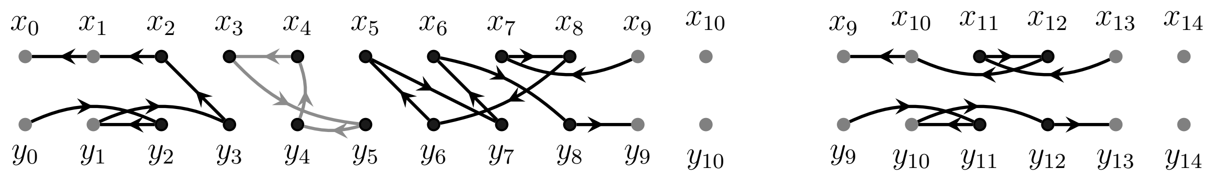

Figure 19: The (4, 14)-base tuple $(X_7, Q_7, R_7, S_7, T_7)$.

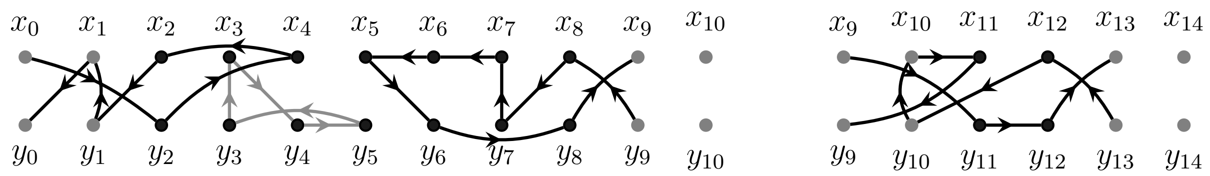

Figure 20: The (4, 14)-base tuple $(X_8, Q_8, R_8, S_8, T_8)$.

Case 3: $t_1 = 6$ and $q = 16$

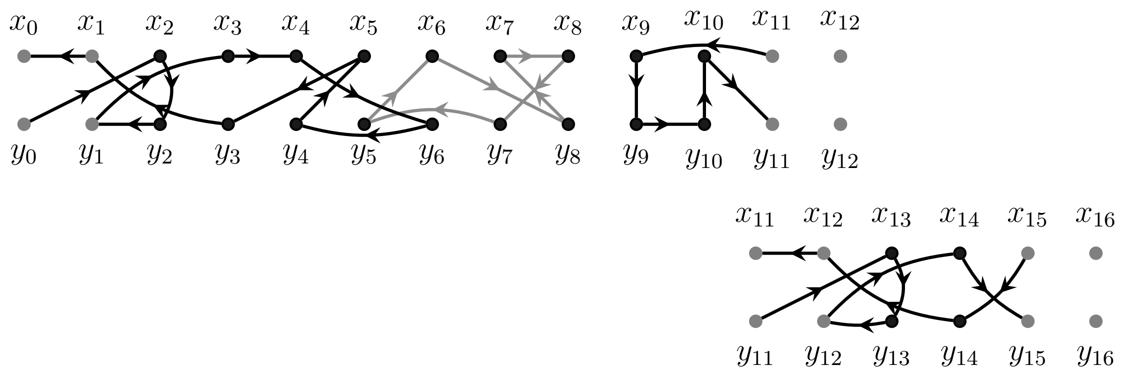

Figure 21: The (6, 16)-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

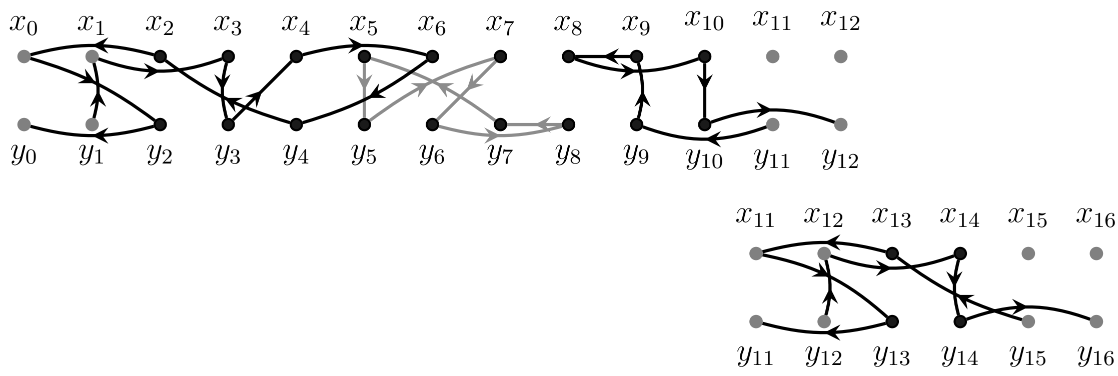

Figure 22: The $(6, 16)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

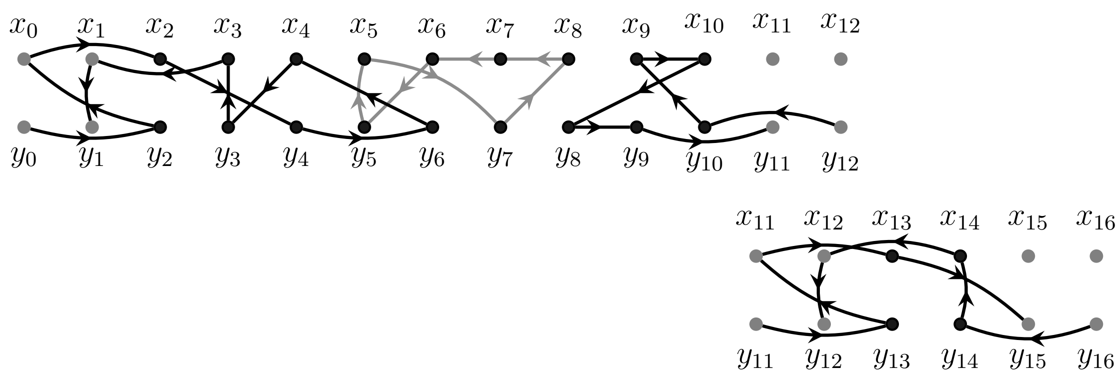

Figure 23: The $(6, 16)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

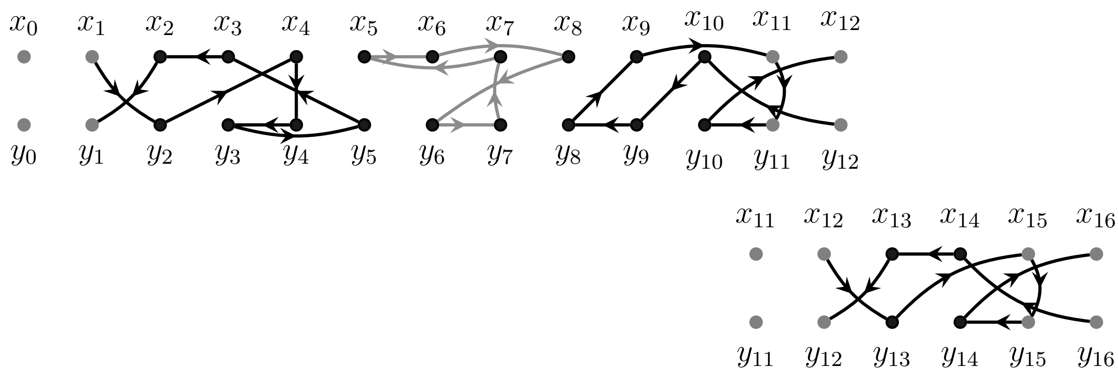

Figure 24: The $(6, 16)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

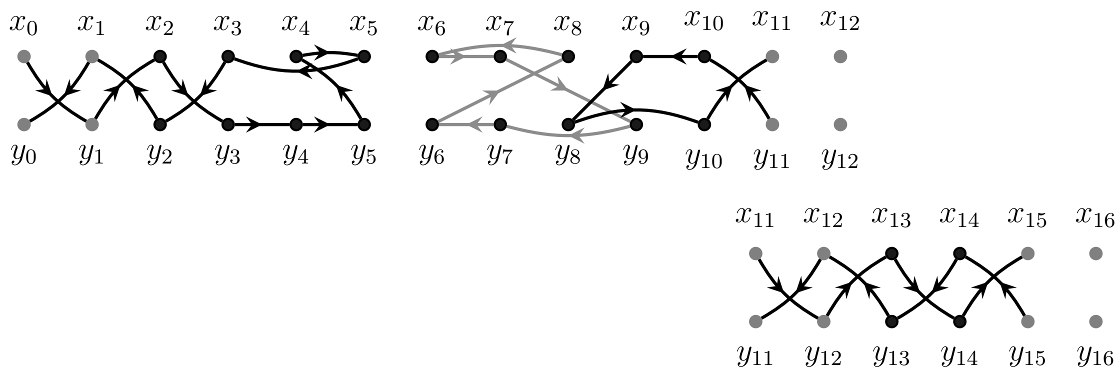

Figure 25: The $(6, 16)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

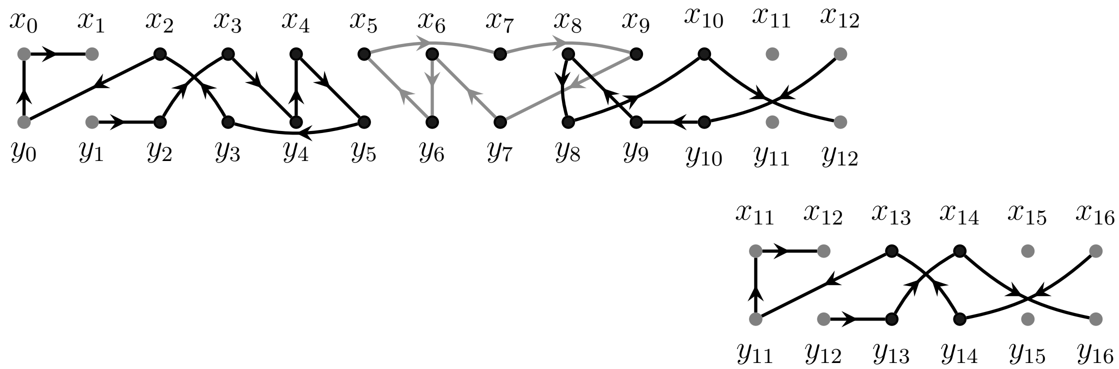

Figure 26: The $(6, 16)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

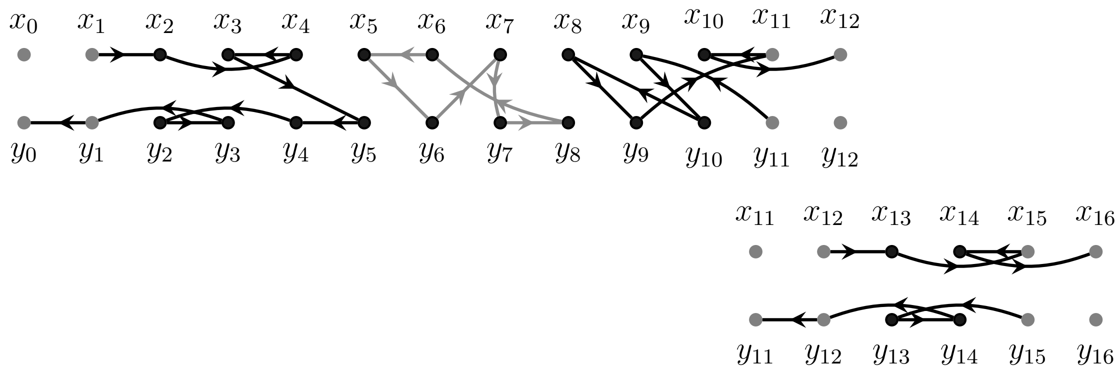

Figure 27: The $(6, 16)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

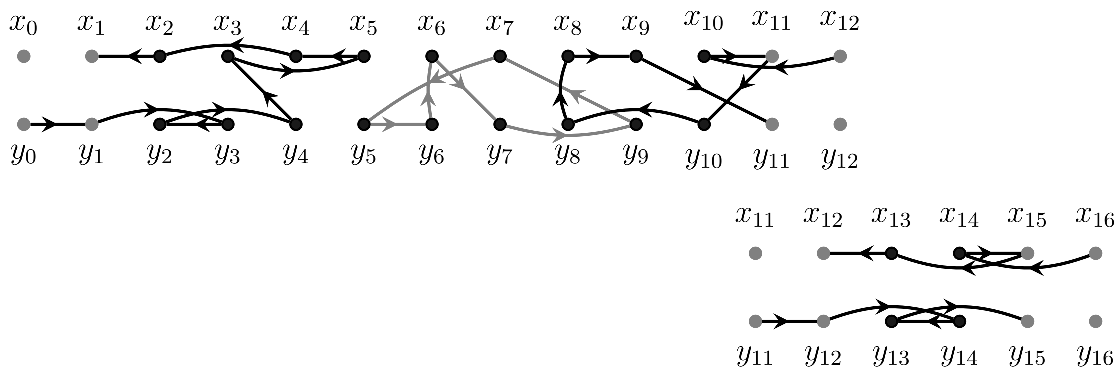

Figure 28: The $(6, 16)$-base tuple $(X_7, Q_7, R_7, S_7, T_7)$.

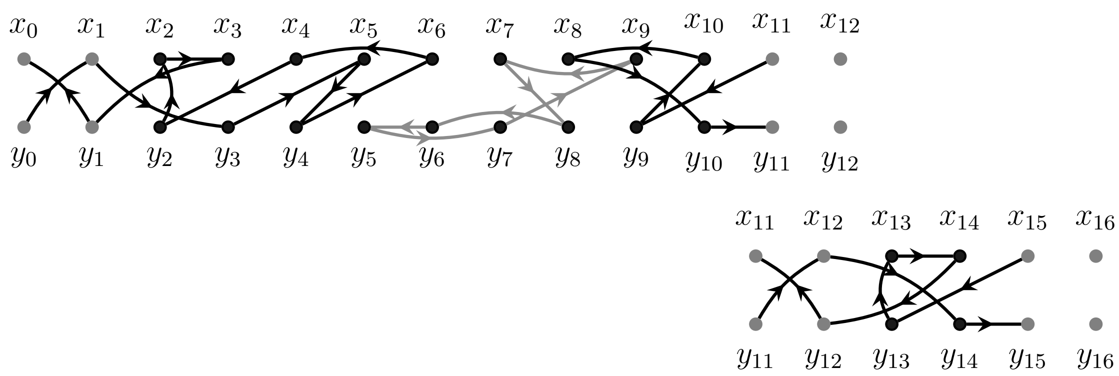

Figure 29: The $(6, 16)$-base tuple $(X_8, Q_8, R_8, S_8, T_8)$.

Case 4: $t_1 = 6$ and $q = 20$

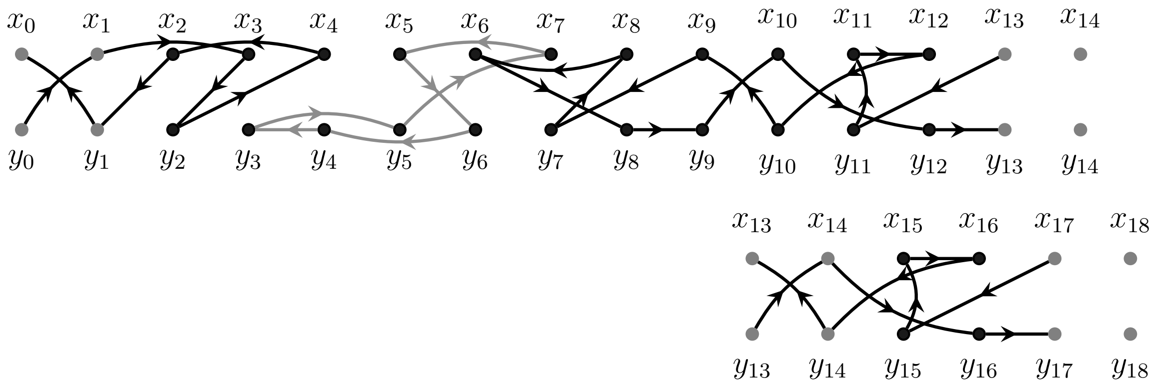

Figure 30: The $(6, 20)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

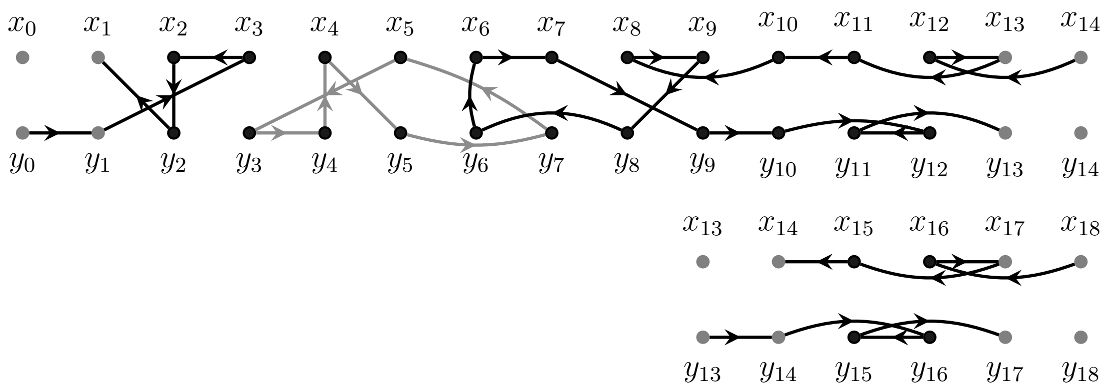

Figure 31: The $(6, 20)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

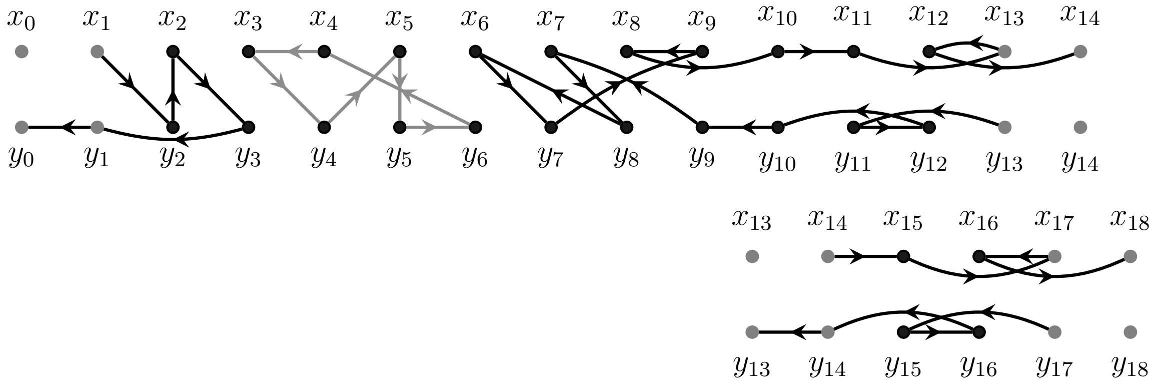

Figure 32: The $(6, 20)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

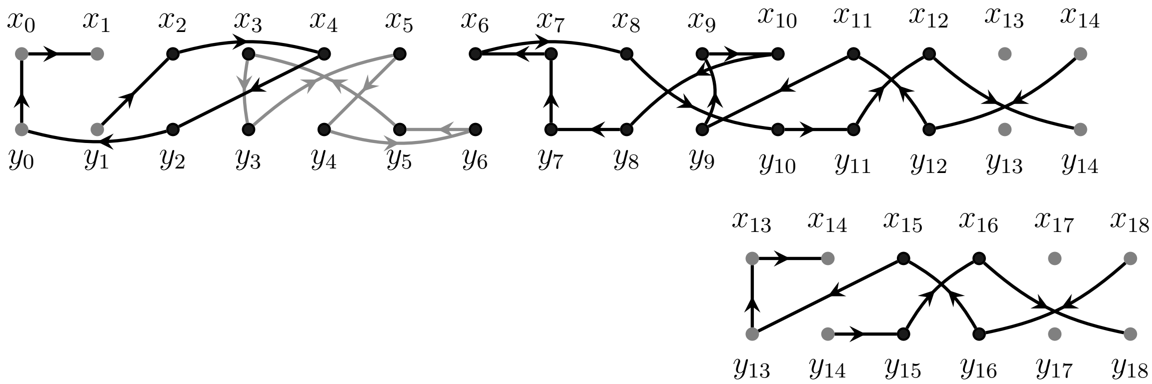

Figure 33: The $(6, 20)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

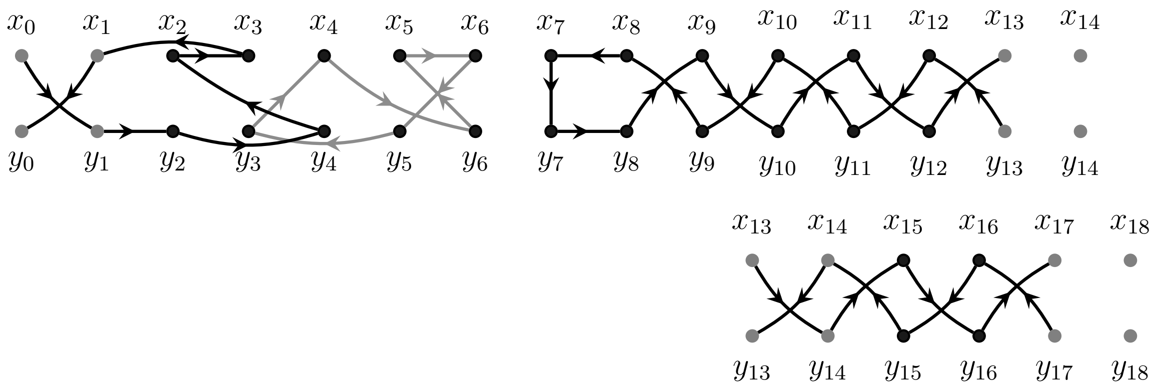

Figure 34: The $(6, 20)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

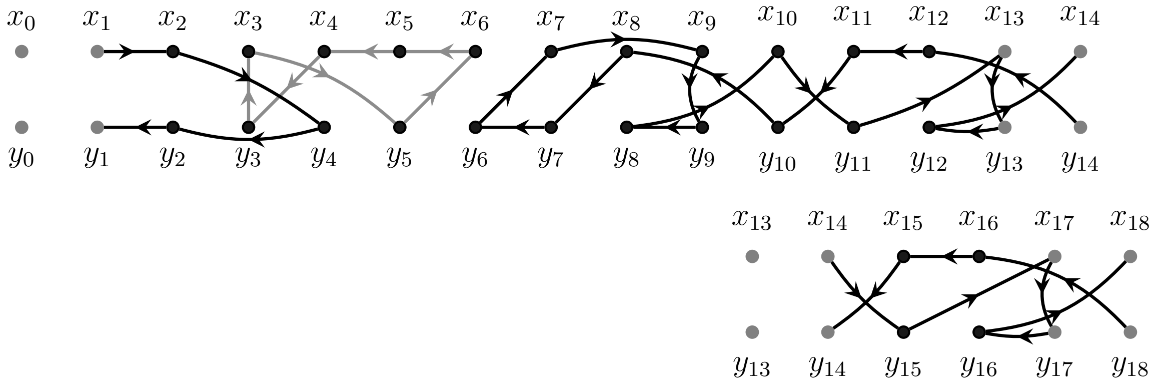

Figure 35: The $(6, 20)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

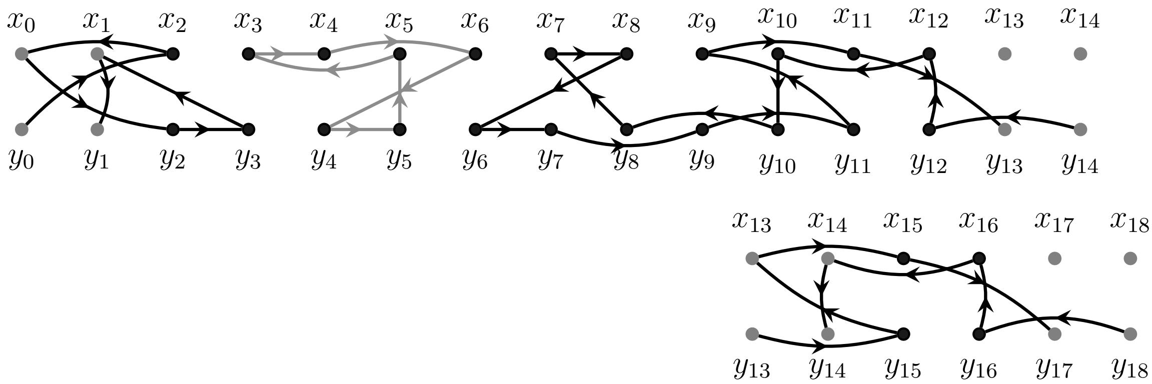

Figure 36: The $(6, 20)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

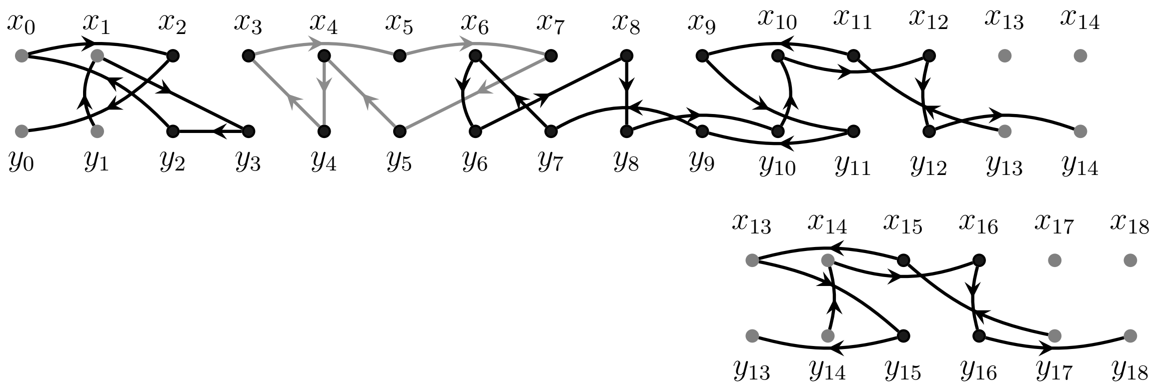

Figure 37: The $(6, 20)$-base tuple $(X_7, Q_7, R_7, S_7, T_7)$.

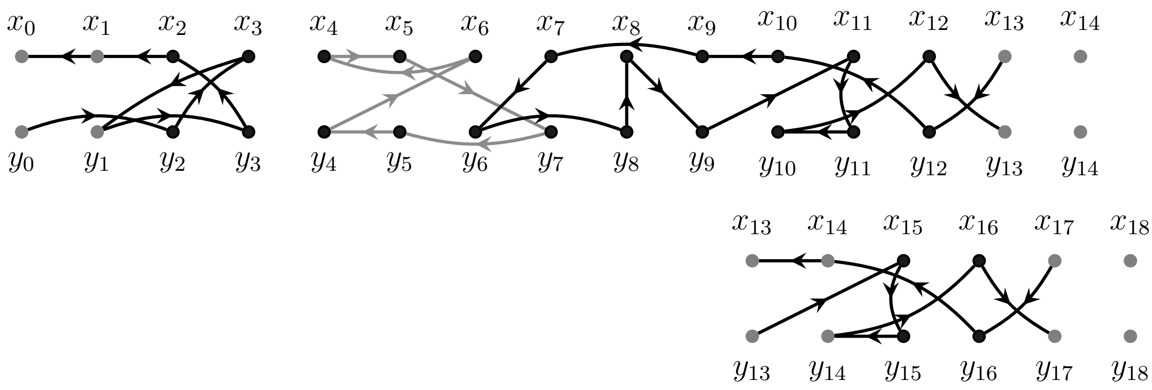

Figure 38: The $(6, 20)$-base tuple $(X_8, Q_8, R_8, S_8, T_8)$.

B.2 The case $t_1 + q \equiv 0 \pmod 4$

Case 1: $t_1 = 4$ and $q = 16$

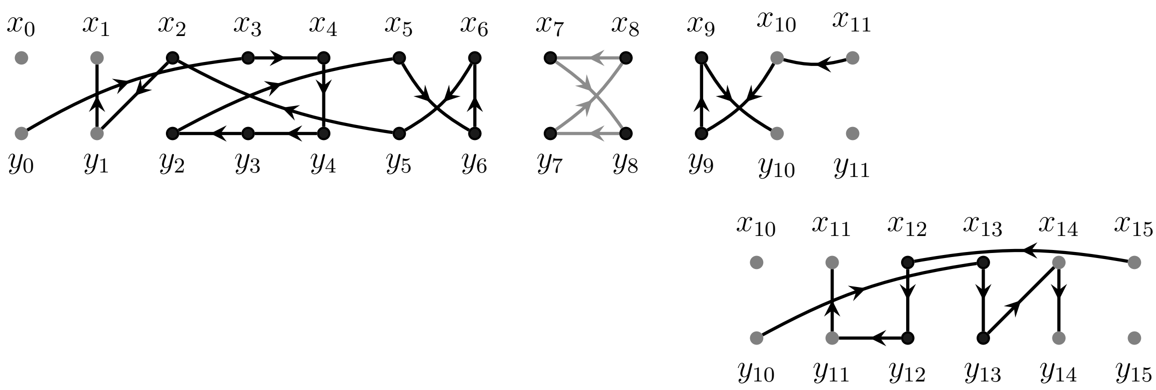

Figure 39: The $(4, 16)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

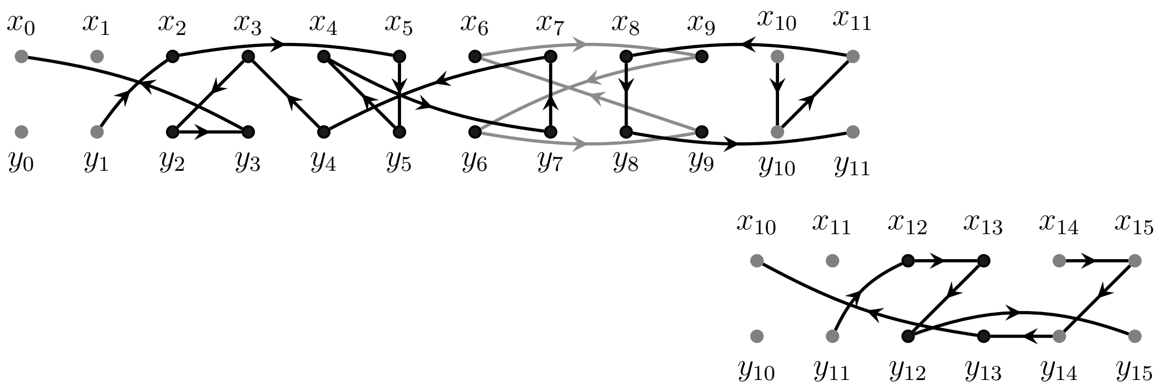

Figure 40: The $(4, 16)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

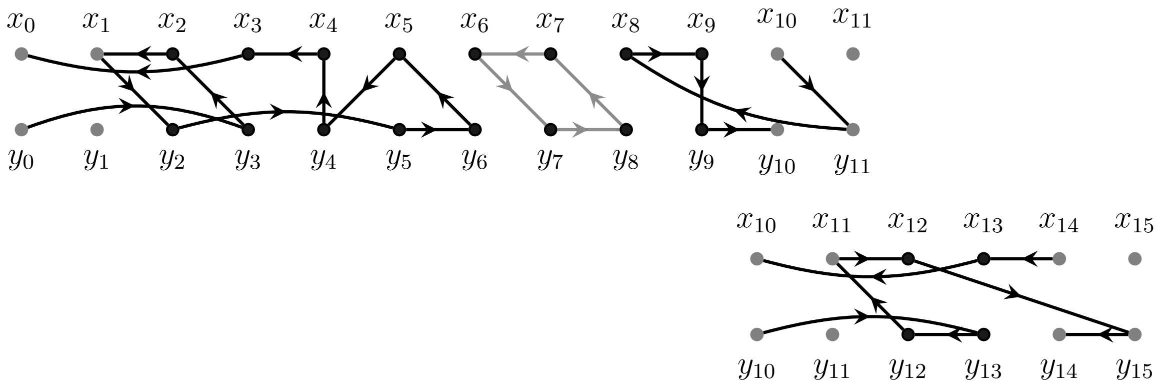

Figure 41: The $(4, 16)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

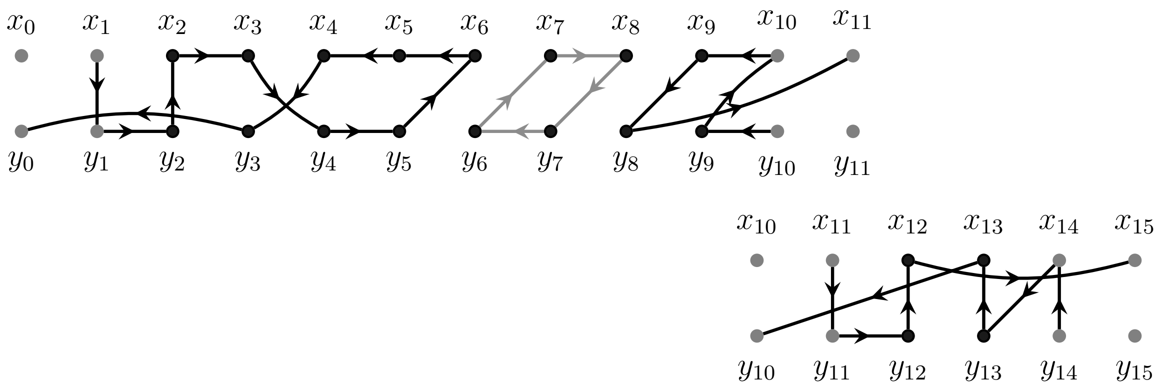

Figure 42: The $(4, 16)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

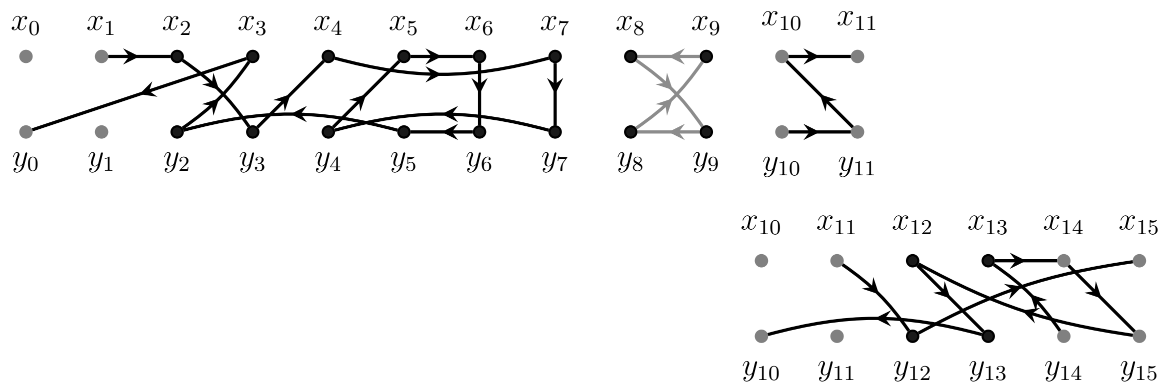

Figure 43: The $(4, 16)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

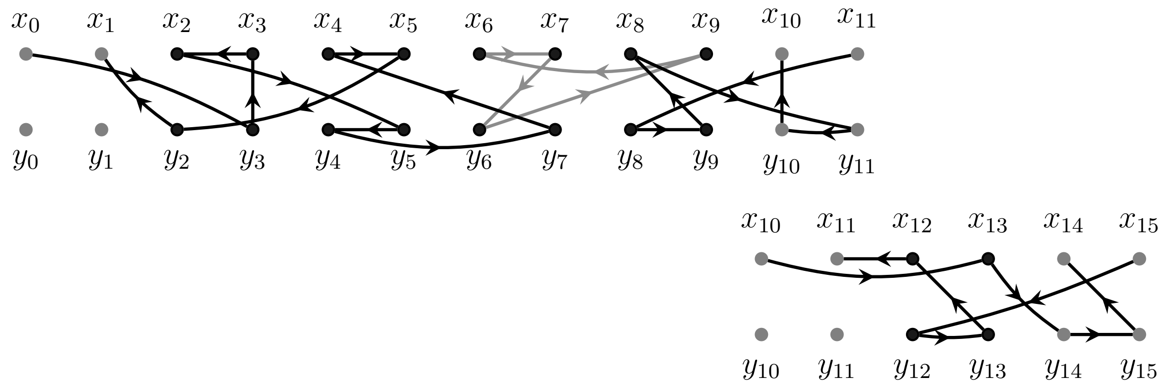

Figure 44: The $(4, 16)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

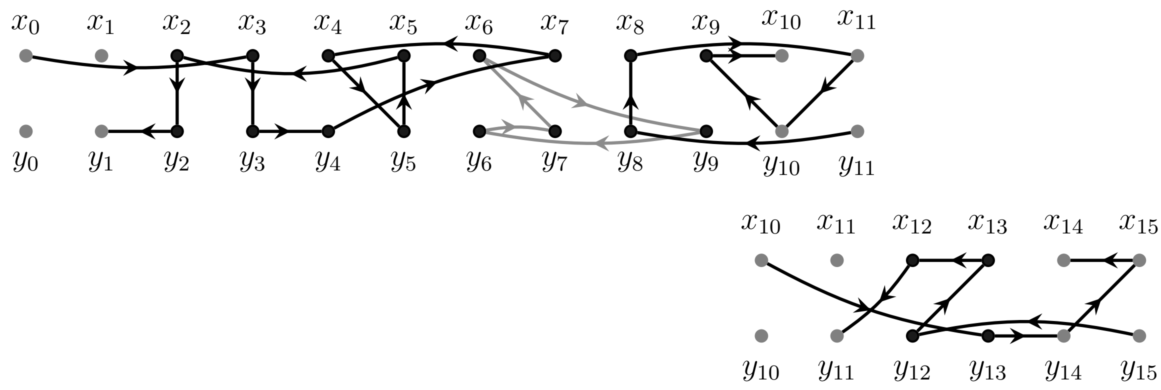

Figure 45: The $(4, 16)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

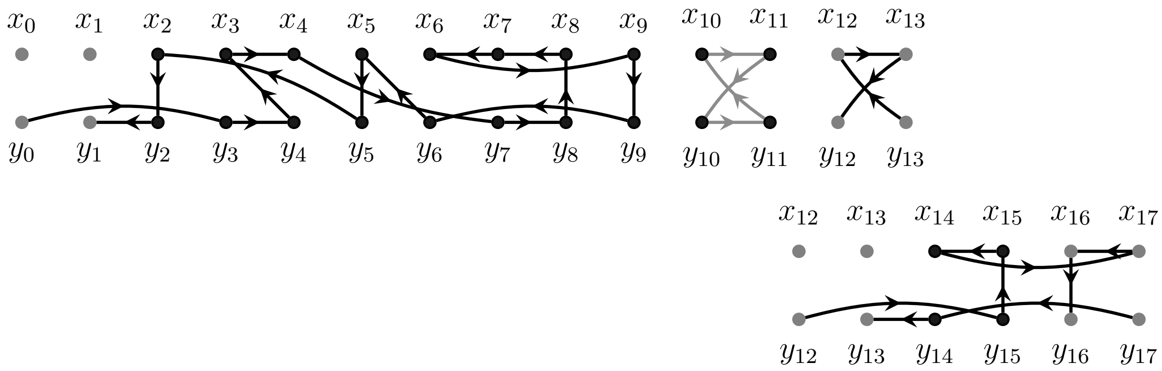

Figure 46: The $(4, 20)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

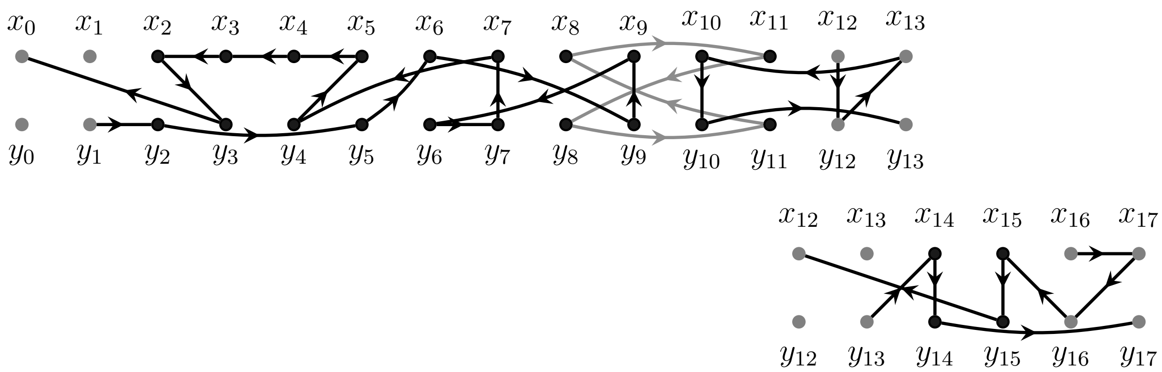

Figure 47: The $(4, 20)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

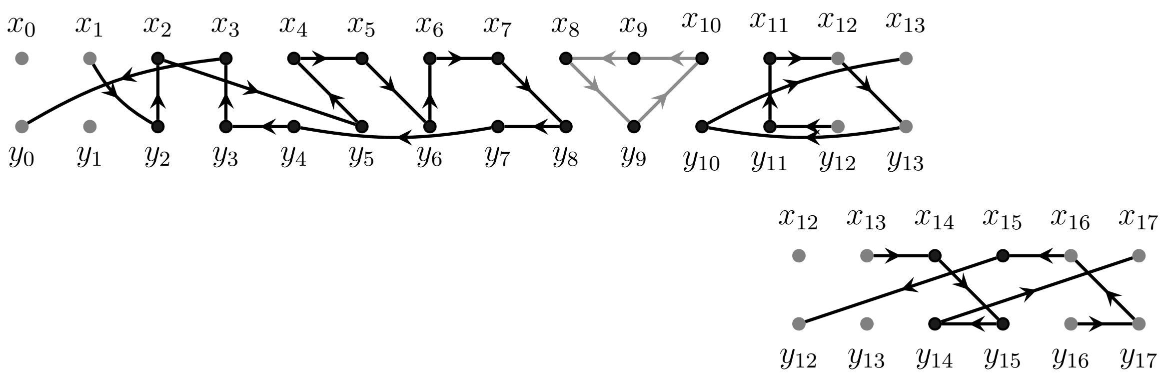

Figure 48: The $(4, 20)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

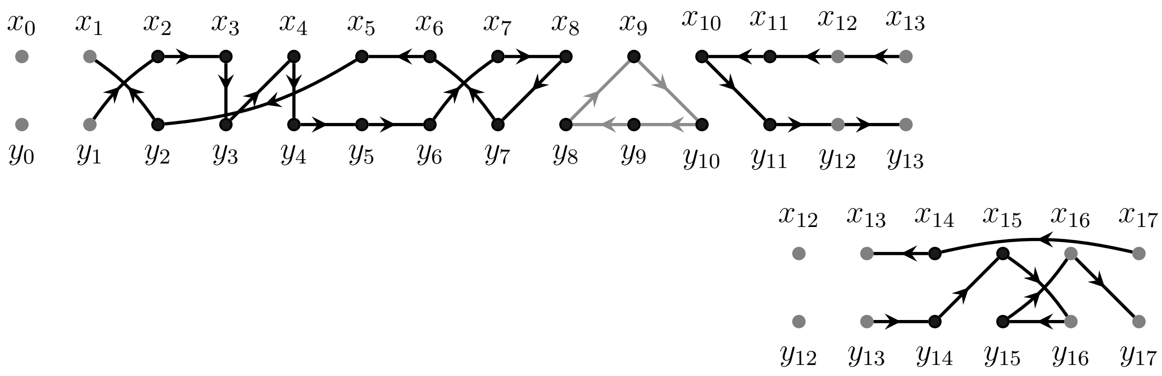

Figure 49: The $(4, 20)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

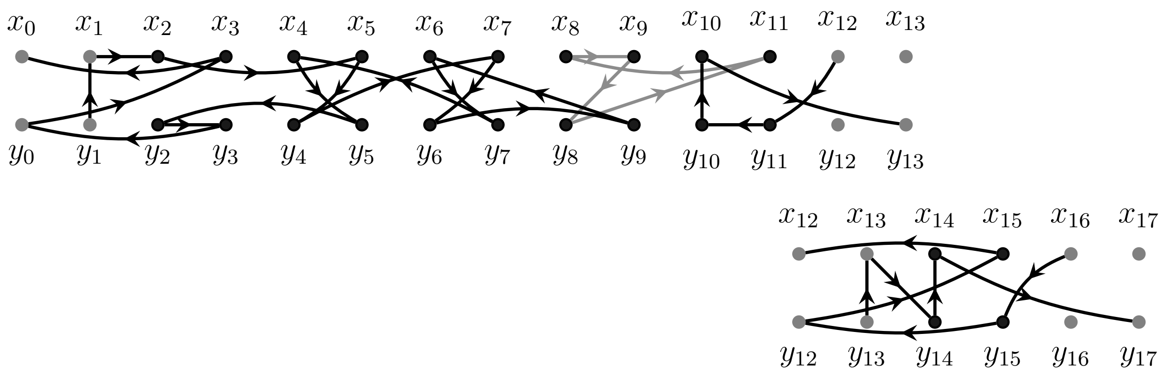

Figure 50: The $(4, 20)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

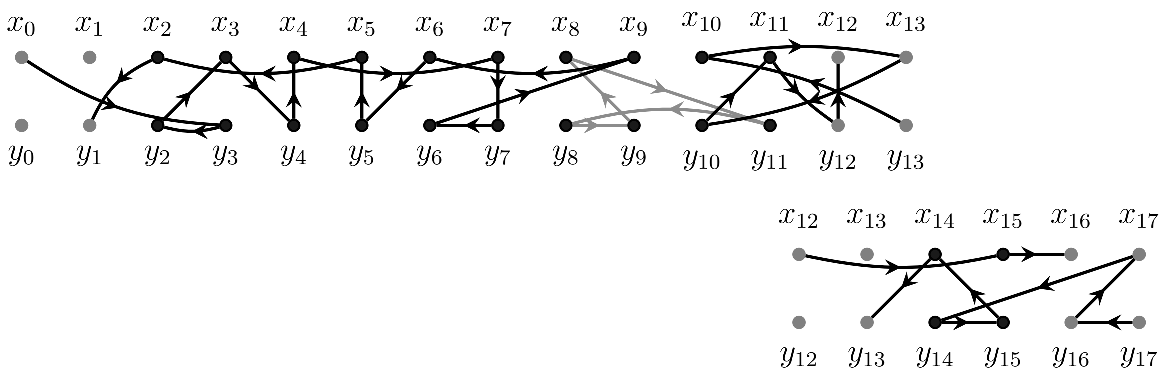

Figure 51: The $(4, 20)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

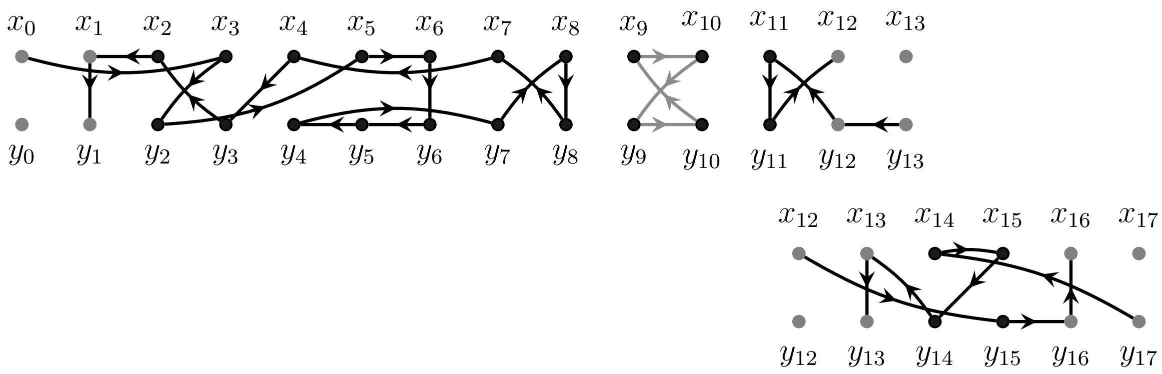

Figure 52: The $(4, 20)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

Case 3: $t_1 = 6$ and $q = 14$

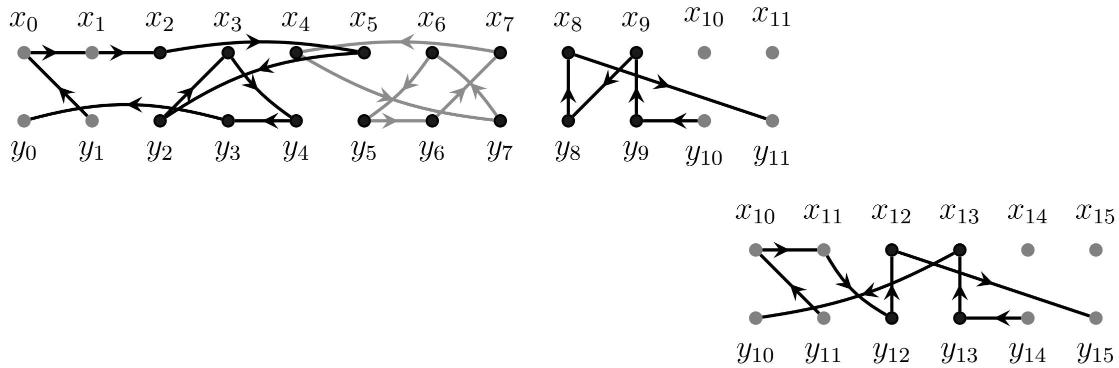

Figure 53: The $(6, 14)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

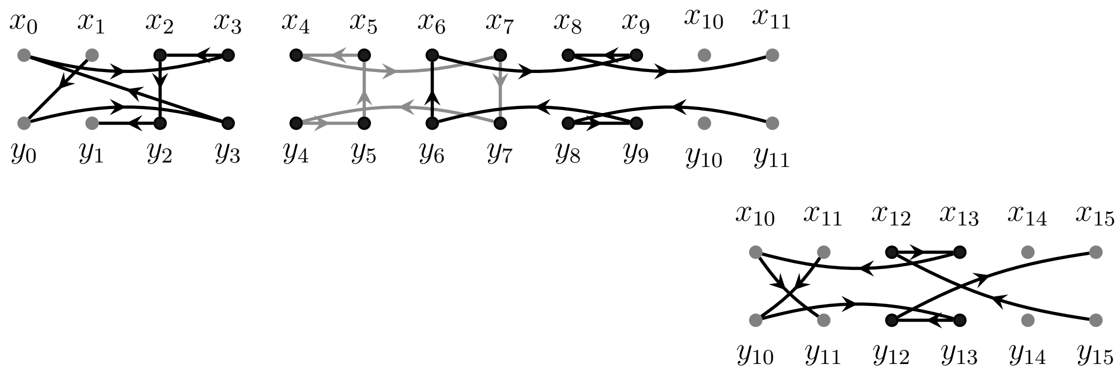

Figure 54: The $(6, 14)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

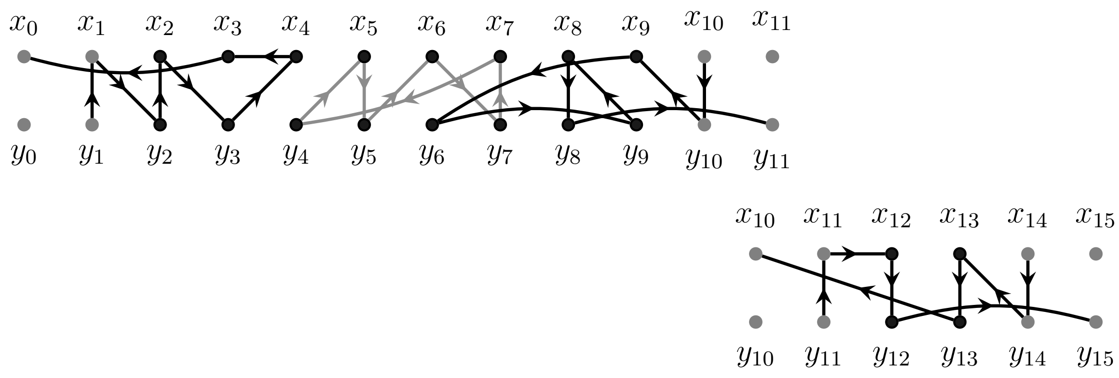

Figure 55: The $(6,14)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

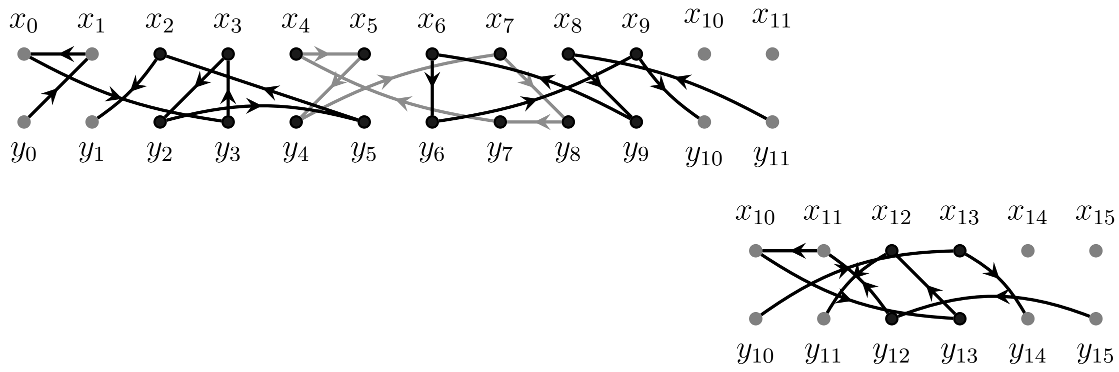

Figure 56: The $(6,14)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

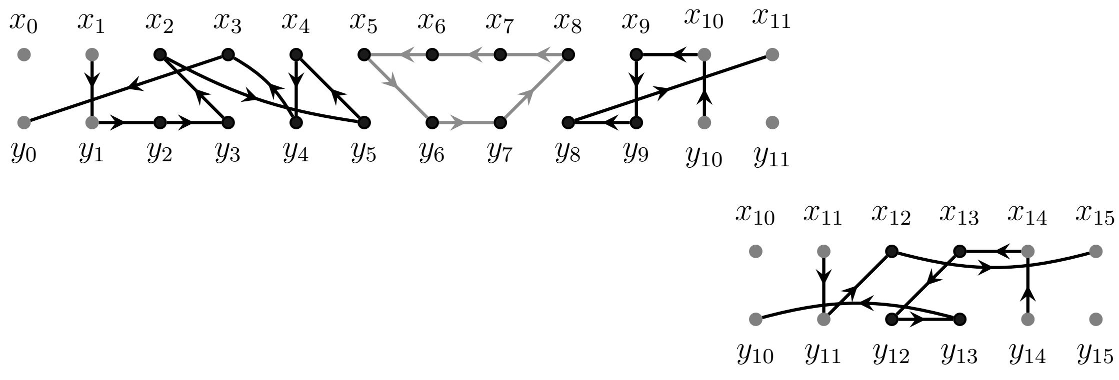

Figure 57: The $(6,14)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

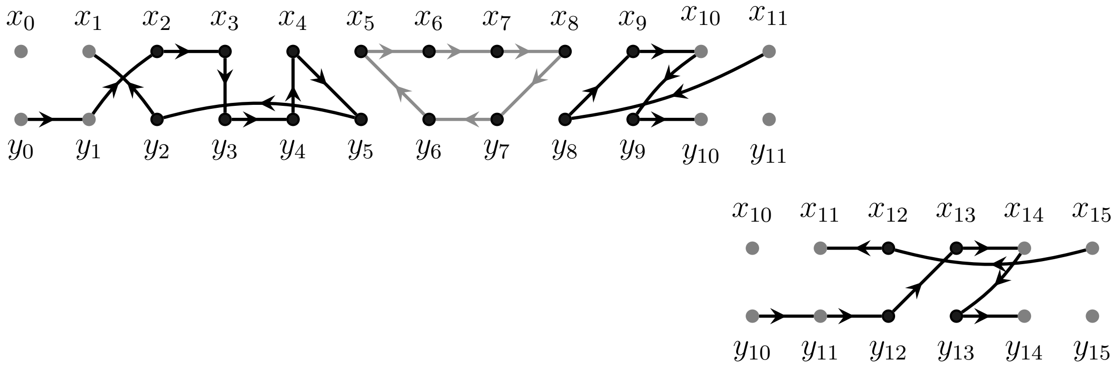

Figure 58: The (6, 14)-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

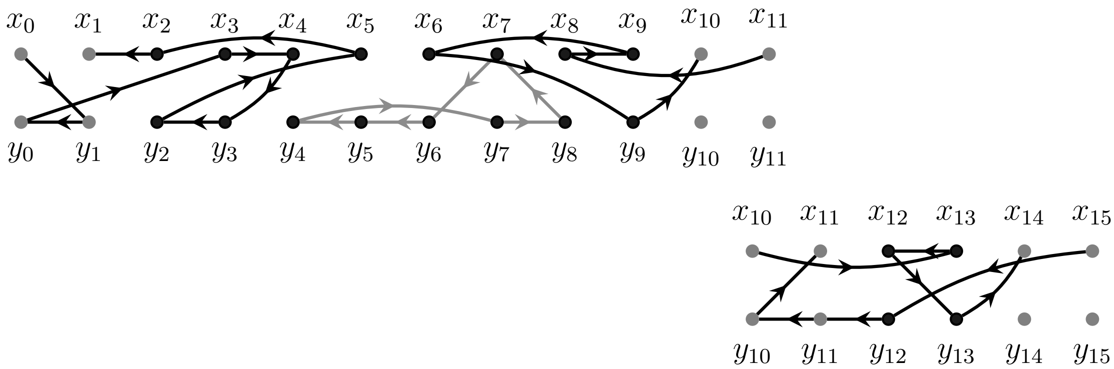

Figure 59: The (6, 14)-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.

Case 4: $t_1 = 6$ and $q = 18$

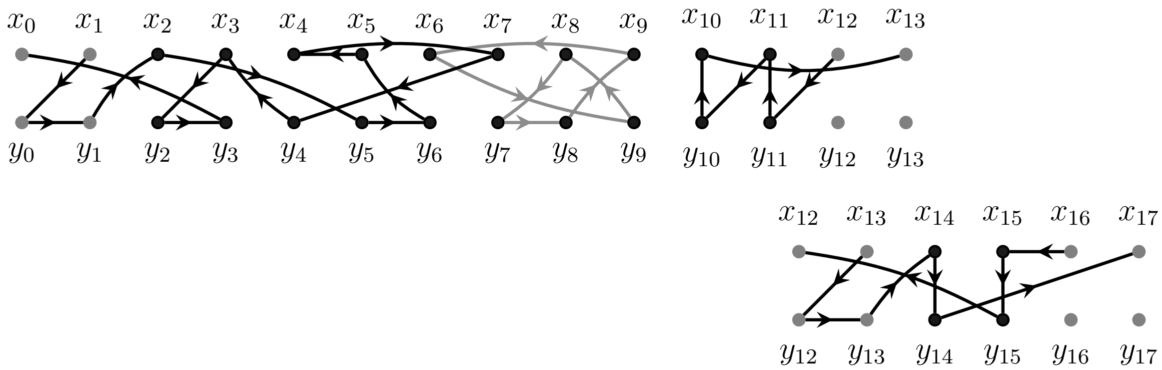

Figure 60: The $(6, 18)$-base tuple $(X_0, Q_0, R_0, S_0, T_0)$.

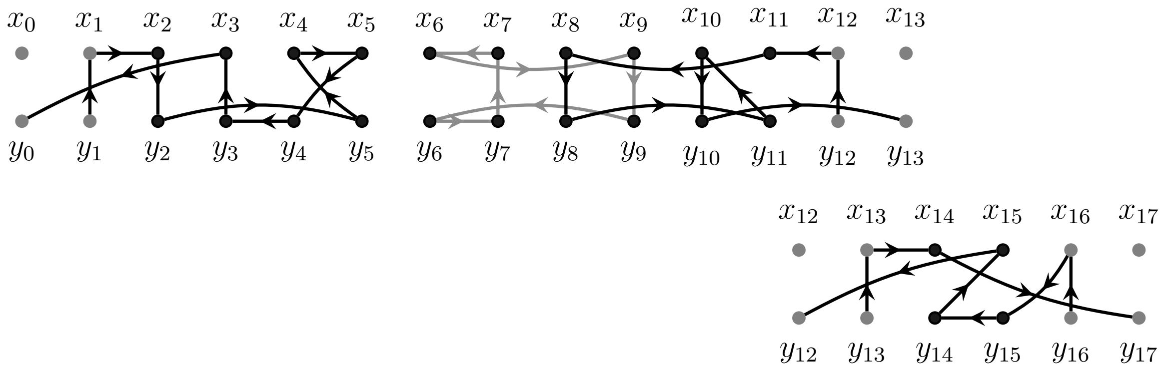

Figure 61: The $(6, 18)$-base tuple $(X_1, Q_1, R_1, S_1, T_1)$.

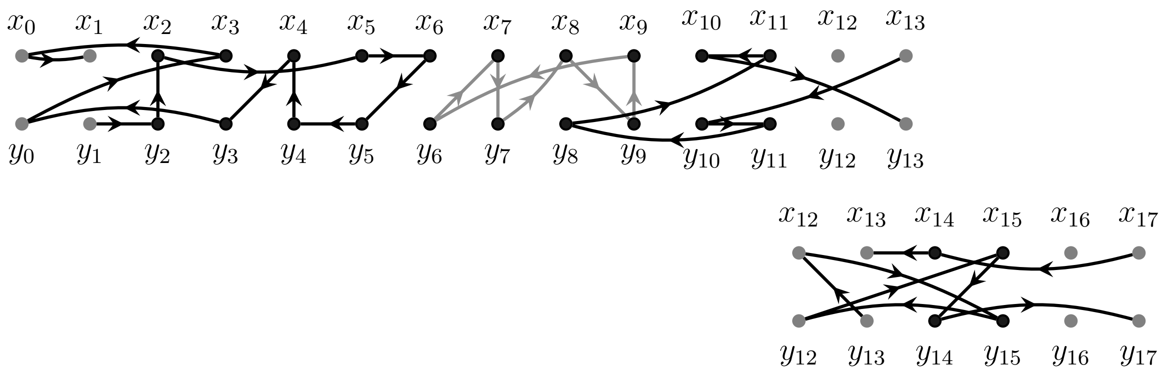

Figure 62: The $(6, 18)$-base tuple $(X_2, Q_2, R_2, S_2, T_2)$.

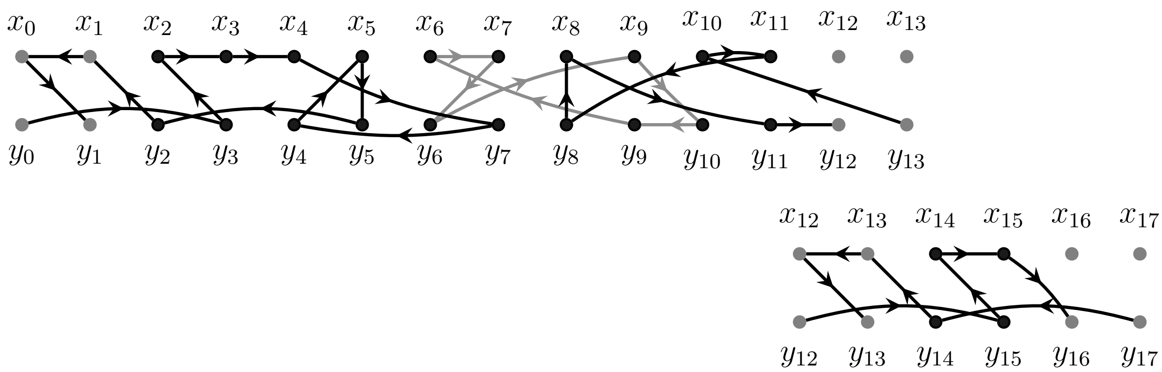

Figure 63: The $(6, 18)$-base tuple $(X_3, Q_3, R_3, S_3, T_3)$.

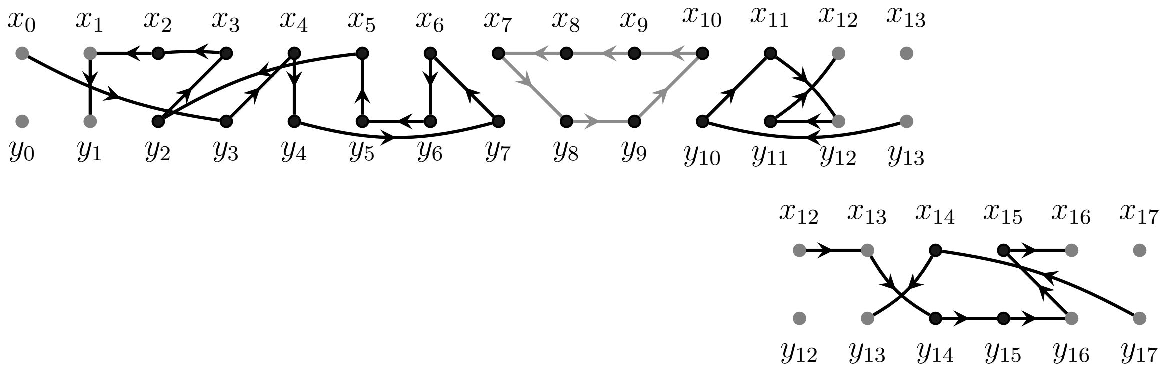

Figure 64: The $(6, 18)$-base tuple $(X_4, Q_4, R_4, S_4, T_4)$.

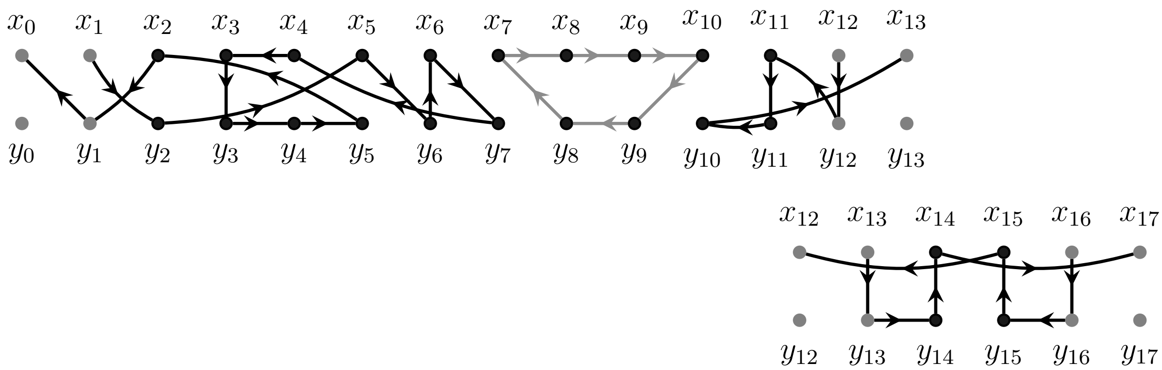

Figure 65: The $(6, 18)$-base tuple $(X_5, Q_5, R_5, S_5, T_5)$.

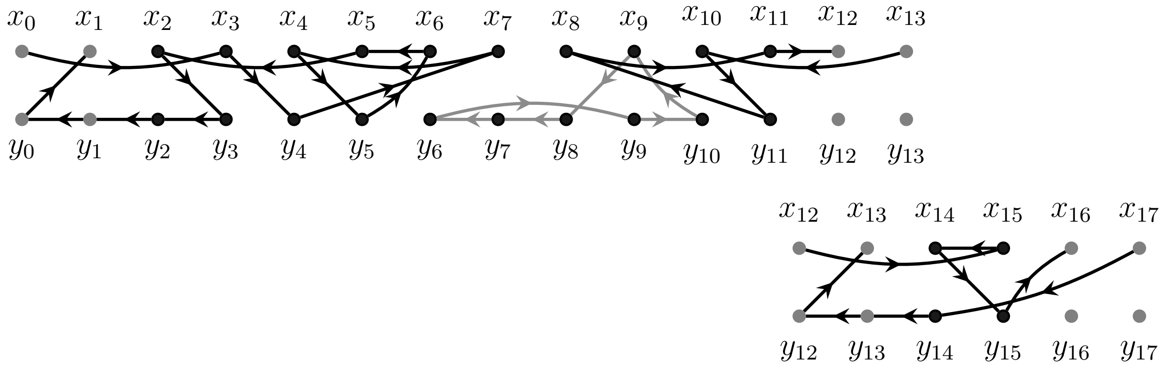

Figure 66: The $(6, 18)$-base tuple $(X_6, Q_6, R_6, S_6, T_6)$.