Sabotaging Mantel's Theorem

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Abstract

One of the earliest results in extremal graph theory, Mantel's theorem, states that the maximum number of edges in a triangle-free graph G on n vertices is $\lfloor n^2/4 \rfloor$. We investigate how this extremal bound is affected when G is additionally required to contain a prescribed graph H as a subgraph. We establish general upper and lower bounds for this problem, which are tight in the exponent for random triangle-free graphs and graphs generated by the triangle-free process, when the size of H lies within certain ranges.

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1 Introduction

Mantel's theorem [11], one of the earliest results in extremal graph theory, states that the number of edges in every triangle-free graph on n vertices is at most $\lfloor n^2/4 \rfloor$, with equality attained uniquely by the balanced complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

We consider the following variation of this classical result: Given a vertex set V of size n and a prescribed triangle-free graph H on V, what is the maximum number of edges in another triangle-free graph G on V that contains H as a subgraph? In other words, we are interested in the function

$$\mathrm{ex}_H(n, K_3) \coloneqq \max \left\{ e(G) \colon G \subseteq \binom{V}{2} \text{ is } K_3\text{-free and } H \subseteq G \right\}.$$

Here, we identify a graph G with its edge set and use $e(G)$ to denote the number of edges in it. It is clear that if H is a subgraph of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, then $\mathrm{ex}_H(n, K_3)$ equals $\mathrm{ex}(n, K_3)$, and is strictly smaller otherwise (due to the fact that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is uniquely extremal). Informally speaking, we will focus in this paper on the case where H is 'large'.

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One of the simplest examples is perhaps when $H = S_m$, the star with m edges. Note that for $m > \lceil n/2 \rceil$, the graph S_m is not a subgraph of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, so it follows from the discussion above that $\mathrm{ex}_{S_m}(n, K_3) < \mathrm{ex}(n, K_3)$. The exact value of $\mathrm{ex}_{S_m}(n, K_3)$ follows from a theorem[1] of Balister–Bollobás–Riordan–Schelp in [1], which shows that

$$\mathrm{ex}_{S_m}(n, K_3) = m(n-m) \quad \text{for every } m > \lceil n/2 \rceil.$$

The value of $\mathrm{ex}_H(n, K_3)$ for a general triangle-free graph H does not appear to have been systematically studied in the literature before. In this short note, we aim to establish general lower and upper bounds for $\mathrm{ex}_H(n, K_3)$. Before stating the main result, we introduce several related parameters.

Given a graph H, let $d(H)$ denote its *average degree* and let $\Delta(H)$ denote its *maximum degree.* Let $\mathrm{N}(S_2, H)$ denote the number of (unlabelled) copies of S_2 in H, noting that

$$\begin{aligned} e(H) + \mathrm{N}(S_2, H) &= \frac{1}{2} \sum_{v \in V(H)} d_H(v) + \sum_{v \in V(H)} \binom{d_H(v)}{2} \\ &= \frac{1}{2} \sum_{v \in V(H)} d_H(v)^2 \qquad (1) \\ &\leqslant \frac{1}{2} \sum_{v \in V(H)} d_H(v) \Delta(H) = e(H)\Delta(H). \qquad (2) \end{aligned}$$

For a graph H satisfying $e(H) + \mathrm{N}(S_2, H) = \frac{1}{2} \sum_{v \in V(H)} d_H(v)^2 < \lfloor n^2/4 \rfloor$, we define the following function:

$$\gamma(H) \coloneqq \frac{n(n-2)}{\lfloor n^2/4 \rfloor - e(H) - \mathrm{N}(S_2, H)} = \frac{n(n-2)}{\lfloor n^2/4 \rfloor - \frac{1}{2} \sum_{v \in V(H)} d_H(v)^2}.$$

The function γ may seem mysterious to the reader, and its exact statement arises simply as an artifact of our proof.

Recall that a vertex set $I \subseteq V(H)$ is an *independent set* of H if it does not contain any edge of H. The *independence number* $\alpha(H)$ of H is the size of the largest independent set in H. Let $\psi \colon [0, \infty) \to (0, 1]$ be the decreasing continuous function defined by

$$\psi(d) \coloneqq \begin{cases} \frac{d \ln d - d + 1}{(d-1)^2}, & \text{if } d \notin \{0, 1\} \\ 1, & \text{if } d = 0, \\ \frac{1}{2}, & \text{if } d = 1. \end{cases}$$

The function ψ comes from the following classical result of Shearer, which gives a lower bound on the independence number of triangle-free graphs.

Theorem 1 ([14]). *Suppose that G is a triangle-free graph on n vertices with average degree d. Then $\alpha(G) \geqslant n \cdot \psi(d)$.*

[1] In fact, they showed that the same conclusion holds for large n when K_3 is replaced by any cycle of odd length.

Our main result is the following theorem. We use the notation $[n] := \{1, 2, \ldots, n\}$.

Theorem 2. *Let H be a K_3-free graph on vertex set $[n]$. Then the following statements hold.*

(i) We have $\mathrm{ex}_H(n, K_3) \leqslant n\alpha(H)/2$.

(ii) Suppose that $\frac{1}{2}\sum_{v\in V(H)} d_H(v)^2 < \lfloor n^2/4 \rfloor$. *Then*

$$\mathrm{ex}_H(n, K_3) \geqslant \left(\lfloor n^2/4 \rfloor - \frac{1}{2} \sum_{v \in V(H)} d_H(v)^2 \right) \cdot \psi\big(\gamma(H) d(H)\big).$$

For positive real numbers β_1 and β_2, we say a graph H on $[n]$ is (β_1, β_2)-*constrained* (or simply *constrained* when the constants β_1 and β_2 are not essential) if

$$\alpha(H) \leqslant \frac{\beta_1 n \ln d(H)}{d(H)} \quad \text{and} \quad e(H)\Delta(H) \leqslant \left(\frac{1}{4} - \beta_2\right) n^2.$$

Note that for the first condition in the definition to hold, we implicitly require that $d(H) > 1$.

The following corollary is an immediate consequence of Theorem 2 and Inequality (2).

Corollary 3. *Let $\beta_1 > 0$ and $\beta_2 > 0$ be two constants. Suppose that H is a (β_1, β_2)-constrained triangle-free graph on $[n]$ with average degree d. Then*

$$\mathrm{ex}_H(n, K_3) = \Theta_n\left(\frac{n^2 \ln d}{d}\right).$$

A proof of Theorem 2 as well as the calculations giving Corollary 3 are provided in full in Section 2.

The definition of constrained graphs is motivated by properties of the Erdős–Rényi random graph $G(n, d/n)$, where each pair of vertices forms an edge independently with probability d/n. We consider this model for $d = d(n)$ be a fixed function of n and we say that an event occurs with high probability (*w.h.p.* for short) if it occurs with probability tending to 1 as n tends to infinity. It is a well-known result in random graph theory (see e.g. [8] and [7, Theorem 3.4]) that *w.h.p.*,

$$\alpha(G(n, d/n)) \leqslant \frac{(2 + o(1)) n \ln d}{d} \quad \text{and} \quad \Delta(G(n, d/n)) \leqslant (1 + o(1)) \max\left\{d, \ \frac{\ln n}{\ln \ln n}\right\}.$$

Thus $G(n, d/n)$ is $(2 + o(1), \varepsilon)$-constrained *w.h.p.* whenever $d \leqslant \sqrt{(1/2 - \varepsilon)n}$. Unfortunately, $G(n, d/n)$ is not triangle-free *w.h.p.* when $d \gg 1$ (see e.g. [7, Theorem 1.12]). Below, we examine three random models that are triangle-free and share certain properties with the Erdős–Rényi random graph.

Let $\mathcal{T}(n,d)$ denote the family of all triangle-free graphs with vertex set $[n]$ and average degree d. Let $T(n,d)$ be a graph chosen uniformly at random from $\mathcal{T}(n,d)$. It was shown in [12, Lemma 3] that there exists $c > 0$ such that for $d \in [4, cn^{1/4}\sqrt{\ln n}]$, *w.h.p.*,

$$\alpha(T(n,d)) \leqslant \frac{4n \ln d}{d}.$$

By combining the Chernoff inequality (see e.g. [7, Theorem 27.6]) with [12, Lemma 4], it is straightforward to verify that there exist constants $c', C > 0$ such that for every $d \in [4, c'n^{1/2}]$, *w.h.p.*,

$$\Delta(T(n,d)) \leqslant Cd^3.$$

Thus we obtain the following corollary for the case $H = T(n,d)$.

Corollary 4. *There exists a constant $c_4 > 0$ such that for every $d \in [4, c_4 n^{1/4}]$, w.h.p.*

$$\mathrm{ex}_{T(n,d)}(n, K_3) = \Theta_n\left(\frac{n^2 \ln d}{d}\right).$$

Another important model of random triangle-free graphs, which plays a significant role in constructions related to Ramsey theory for triangles (see, e.g., [2, 6, 3]), is the triangle-free process. This process begins with the empty graph $G(0)$ on n vertices. For each $i \geqslant 1$, the graph $G(i)$ is obtained by adding an edge e_i to $G(i-1)$, where e_i is chosen uniformly at random from the set of non-edges of $G(i-1)$ that do not form a triangle in $G(i)$.

It was shown in works such as [2, 6, 3] that the triangle-free process terminates *w.h.p.* after $\Theta(n^{3/2}\sqrt{\ln n})$ steps. Heuristically, the graph $G(i)$ closely resembles the Erdős–Rényi random graph $G(n,p)$ with $p = i/\binom{n}{2}$, except for the triangle-freeness. In particular, the maximum degree of $G(i)$ was shown in e.g. [3, Section 3.2] to have *w.h.p.* the same order as those $G(n,p)$ with $p = i/\binom{n}{2}$, when i is not too large. The same coupling argument can be repeated to show that this also holds for the independence number[2].

Specifically, there exists a constant $c > 0$ such that for $i \leqslant cn^{3/2}$, *w.h.p.*, and with $p = i/\binom{n}{2}$,

$$\Delta(G(i)) = O\left(\Delta(G(n,p))\right) \quad \text{and} \quad \alpha(G(i)) = O\left(\alpha(G(n,p))\right).$$

These properties, together with Corollary 3, yield the following result for the case $H = G(i)$.

Corollary 5. *There exists a constant $c_5 > 0$ such that for every $i \in [n, c_5 n^{3/2}]$ and with $d = 2i/n$, w.h.p.*

$$\mathrm{ex}_{G(i)}(n, K_3) = \Theta_n\left(\frac{n^2 \ln d}{d}\right).$$

[2] Personal communication with Peter Keevash.

2 Proof of Theorem 2

We prove Theorem 2 and Corollary 3 in this section.

Proof of Theorem 2. Fix a K_3-free graph H on $[n]$. We say a graph G on $[n]$ is H-admissible if it is K_3-free and contains H as a subgraph.

First, we prove Theorem 2 (i). Suppose that G is a H-admissible graph. Let $v \in [n]$ be a vertex of maximum degree in G. Since G is K_3-free, the neighborhood $N_G(v)$ of v in G must be an independent set in G, and in particular, is independent in H. It follows that $\Delta(G) \leqslant \alpha(H)$, and hence,

$$e(G) \leqslant \frac{n \cdot \Delta(G)}{2} \leqslant \frac{n \cdot \alpha(H)}{2},$$

which proves Theorem 2 (i).

Next, we prove Theorem 2 (ii). Consider the auxiliary 3-graph $\mathcal{H}$ whose vertex set is $\binom{[n]}{2}$ (i.e. the edge set of the complete graph K_n), where $\{e_1, e_2, e_3\} \subseteq \binom{[n]}{2}$ forms an edge in $\mathcal{H}$ iff $\{e_1, e_2, e_3\}$ spans a copy of K_3 in K_n. Define

$$\mathcal{B}_1 := \left\{ e_1 \in \binom{[n]}{2} : \text{there exist } e_2, e_3 \in H \text{ such that } \{e_1, e_2, e_3\} \in \mathcal{H} \right\} \quad \text{and}$$

$$\mathcal{B}_2 := \left\{ \{e_1, e_2\} \subseteq \binom{[n]}{2} : \text{there exists } e_3 \in H \text{ such that } \{e_1, e_2, e_3\} \in \mathcal{H} \right\},$$

where we think of $\mathcal{B}_2$ as a graph on vertex set $\binom{[n]}{2}$.

Let $\mathcal{B}_3$ denote the induced subgraph of $\mathcal{H}$ on $\binom{[n]}{2} \setminus H$.

We say that a set $I \subseteq V(\mathcal{H})$ is independent in $\mathcal{H}$ if no edge of $\mathcal{H}$ is entirely contained in I. Note that every K_3-free graph on $[n]$ corresponds to an independent set in $\mathcal{H}$, and vice versa. In particular, H is an independent set in $\mathcal{H}$, and a H-admissible graph is simply an independent set in $\mathcal{H}$ that contains H.

The following claim follows directly from the definitions.

Claim 6. *A graph G is H-admissible iff $G = H \cup I$ for some $I \subseteq \binom{[n]}{2} \setminus H$ satisfying all of the following conditions:*

(i) $I \cap \mathcal{B}_1 = \emptyset$,

(ii) I is an independent set in $\mathcal{B}_2$, and

(iii) I is an independent set in $\mathcal{B}_3$.

Therefore, to prove the lower bound for $\mathrm{ex}_H(n, K_3)$, it suffices to provide a lower bound for $I \subseteq \binom{[n]}{2} \setminus H$ satisfying Claim 6 (i), (ii), and (iii).

Claim 7. *We have $e(\mathcal{B}_1) \leqslant \mathrm{N}(S_2, H)$.*

Proof of Claim 7. It follows from the definition of $\mathcal{B}_1$ that a pair $\{u,v\} \in \binom{[n]}{2}$ is contained in $\mathcal{B}_1$ iff there exists a vertex $w \in [n] \setminus \{u,v\}$ such that $\{w,u\} \in H$ and $\{w,v\} \in H$. In other words, $\{u,v\}$ is contained in the neighborhood $N_H(w)$ of some vertex w. Therefore,

$$e(\mathcal{B}_1) \leqslant \sum_{w\in[n]} \binom{d_H(w)}{2} = \mathrm{N}(S_2, H),$$

which proves Claim 7. □

Claim 8. *We have* $e(\mathcal{B}_2) = e(H)(n-2)$.

Proof of Claim 8. It follows from the definition that $\{e_1, e_2\} \subseteq \binom{[n]}{2}$ is an edge in $\mathcal{B}_2$ iff there exists three vertices $u, v, w \in [n]$ such that $e_1 = \{u,v\}$, $e_2 = \{u,w\}$, and $\{v,w\} \in H$. Therefore, each edge in H contributes exactly $n-2$ edges to $\mathcal{B}_2$. It follows that $e(\mathcal{B}_2) = e(H)(n-2)$. □

Claim 9. *The graph* $\mathcal{B}_2$ *is* K_3*-free.*

Proof of Claim 9. Suppose to the contrary that there exist three pairs $e_1, e_2, e_3 \in \binom{[n]}{2}$ that span a copy of K_3 in $\mathcal{B}_2$, i.e. $\big\{\{e_1,e_2\},\{e_1,e_3\},\{e_2,e_3\}\big\} \subseteq \mathcal{B}_2$. By the definition of $\mathcal{B}_2$, for each pair of indices $\{i,j\} \in \binom{[3]}{2}$, there exists $e_{ij} \in H$ such that $\{e_i, e_j, e_{ij}\}$ forms a triangle in K_n, i.e. $|e_i \cap e_j| = 1$ and $e_{ij} = e_i \triangle e_j$, the symmetric difference of e_i and e_j.

Since $|e_1 \cap e_2| = |e_1 \cap e_3| = |e_2 \cap e_3| = 1$, the set $\{e_1, e_2, e_3\}$ must span either a 3-edge star or a triangle in K_n. In both cases, the corresponding set $\{e_{12}, e_{13}, e_{23}\} \subseteq H$ forms a triangle, contradicting the K_3-freeness of H. Therefore, $\mathcal{B}_2$ is K_3-free. □

We are now ready to construct the set $I \subseteq \binom{[n]}{2} \setminus H$ that satisfies Claim 6 (i), (ii), and (iii). We begin with an independent set $S \subseteq \binom{[n]}{2}$ in $\mathcal{H}$ of size $\lfloor n^2/4 \rfloor$. Note that as this corresponds to a K_3-free graph on $[n]$ of size $\lfloor n^2/4 \rfloor$, it is uniquely realized by choosing a balanced complete bipartite graph on $[n]$.

Define $S' \coloneqq S \setminus (H \cup \mathcal{B}_1)$, and let I be a maximum independent set of the induced subgraph $\mathcal{B}_2[S']$. Note that I satisfies Claim 6 (i), (ii), and (iii).

Claim 10. *We have* $|I| \geqslant (\lfloor n^2/4 \rfloor - e(H) - \mathrm{N}(S_2, H)) \cdot \psi\big(\gamma(H)d(H)\big)$.

Proof of Claim 10. Let d denote the average degree of the induced subgraph $\mathcal{B}_2[S']$. It follows from Claims 8 and 7 that

$$\begin{aligned} d = \frac{2e(\mathcal{B}_2[S'])}{|S'|} \leqslant \frac{2e(\mathcal{B}_2)}{|S'|} &\leqslant \frac{2e(H)(n-2)}{\lfloor n^2/4 \rfloor - e(H) - e(\mathcal{B}_1)} \\ &\leqslant \frac{2e(H)(n-2)}{\lfloor n^2/4 \rfloor - e(H) - \mathrm{N}(S_2, H)} = \gamma(H)d(H). \end{aligned}$$

Since, by Claim 9, $\mathcal{B}_2[S']$ is K_3-free, it follows from Theorem 1 that

$$|I| = \alpha(\mathcal{B}_2[S']) \geqslant |S'| \cdot \psi(d) \geqslant \left(\left\lfloor \frac{n^2}{4} \right\rfloor - e(H) - \mathrm{N}(S_2, H) \right) \cdot \psi\big(\gamma(H)d(H)\big),$$

which proves Claim 10. □

Claim 10 and Equality (1) complete the proof of Theorem 2 (ii). □

Now we will provide a full proof of the Corollary.

Proof of Corollary 3. Let H be a (β_1, β_2)-constrained triangle-free graph on $[n]$ with average degree d. By Theorem 2 (i), we have

$$\mathrm{ex}_H(n, K_3) \leqslant n\alpha(H)/2 \leqslant \frac{\beta_1}{2} \cdot \frac{n^2 \ln d}{d}.$$

Applying Inequality (2), we see that $\frac{1}{2}\sum_{v\in V(H)} d_H(v)^2 \leqslant e(H)\Delta(H) \leqslant \left(\frac{1}{4} - \beta_2\right) n^2$. As a result, clearly $\beta_2 < 1/4$ and

$$\gamma(H) = \frac{n(n-2)}{\lfloor n^2/4 \rfloor - \frac{1}{2}\sum_{v\in V(H)} d_H(v)^2} \leqslant \frac{n^2}{\lfloor n^2/4 \rfloor - \left(\frac{1}{4} - \beta_2\right) n^2} \leqslant \frac{2}{\beta_2}.$$

As ψ is decreasing, this implies that

$$\psi\big(\gamma(H)d\big) \geqslant \frac{\frac{2}{\beta_2} d \ln\left(\frac{2}{\beta_2} d\right) - \frac{2}{\beta_2} d + 1}{\left(\frac{2}{\beta_2} d - 1\right)^2} \geqslant \frac{\frac{2}{\beta_2} d \ln(d) + \frac{2}{\beta_2} d \left(\ln(\frac{2}{\beta_2}) - 1\right)}{\left(\frac{2}{\beta_2} d\right)^2} \geqslant \frac{\beta_2}{2} \frac{\ln d}{d}.$$

Therefore, by Theorem 2 (ii), we have

$$\begin{aligned}
\mathrm{ex}_H(n, K_3) &\geqslant \left(\lfloor n^2/4 \rfloor - \frac{1}{2}\sum_{v\in V(H)} d_H(v)^2\right) \cdot \psi\big(\gamma(H)d\big) \\
&\geqslant \left(\lfloor n^2/4 \rfloor - \left(\frac{1}{4} - \beta_2\right) n^2\right) \cdot \frac{\beta_2}{2} \frac{\ln d}{d} \\
&\geqslant \frac{\beta_2^2}{4} \cdot \frac{n^2 \ln d}{d}.
\end{aligned}$$

This completes the proof. □

3 Concluding remarks

1. The study of the function $\mathrm{ex}_H(n, K_3)$ was partially motivated by results in [4, Section 5], where the authors needed to upper bound the number of edges in graphs satisfying certain properties and containing a specific subgraph. Both of these problems fall under the scope of the following more general meta-question: What happens to an extremal problem when we require the presence of a prescribed subgraph/subset? We hope that our results could inspire further research in this direction.

2. Determining the asymptotic behavior of $\mathrm{ex}_{T(n,d)}(n, K_3)$ and $\mathrm{ex}_{G(i)}(n, K_3)$ for all feasible values of d and i seems to be an interesting open problem. It is well known

in random graph theory that when $d = o(1)$, the components of the random graph $T(n, d)$ (which is essentially equivalent to the Erdős–Rényi random graph $G(n, d/n)$ in this regime) are, *w.h.p.*, trees of size $o(\ln n)$ (see [7, Theorems 2.1 and 2.9]). As a result, $T(n, d)$ admits a bipartition of its vertex set in which the part sizes differ by $o(\ln n)$. Therefore, *w.h.p.*,

$$\mathrm{ex}_{T(n,d)}(n, K_3) = (1/4 - o(1))n^2 \quad \text{if} \quad d = o(1).$$

For $d \geqslant \left(\sqrt{3}/2 + o(1)\right)\sqrt{n \ln n}$, a classical theorem of Osthus–Prömel–Taraz [13] shows that, *w.h.p.*, $T(n, d)$ is bipartite. Moreover, their proof implies that the bipartition has nearly equal part sizes, that is, the part sizes differ by $o(n)$. Therefore, *w.h.p.*

$$\mathrm{ex}_{T(n,d)}(n, K_3) = (1/4 - o(1))n^2 \quad \text{if} \quad d \geqslant \left(\sqrt{3}/2 + o(1)\right)\sqrt{n \ln n}.$$

The cases not covered by Corollary 4 and the discussion above remain open for $\mathrm{ex}_{T(n,d)}(n, K_3)$. It is worth noting that work by Jenssen–Perkins–Potukuchi [10] appears to be helpful for the case when d is near the threshold $\left(\sqrt{3}/2 + o(1)\right)\sqrt{n \ln n}$. Similarly, the range not addressed by Corollary 5 remains open for $\mathrm{ex}_{G(i)}(n, K_3)$.

3. In [9], Guo–Warnke introduced a refined alteration model in which one constructs an F-free graph by deleting all edges lying in copies of F in $G(n, p)$, rather than removing just a single edge from each copy as in classical alteration arguments. They showed that, for appropriate values of p, this full-deletion approach does not substantially affect key parameters such as the independence number (see Theorem 1 and Lemma 6 therein), and often leads to cleaner analyses in applications such as online Ramsey games. One can apply Theorem 2 to their model and obtain straightforwardly an analogous result to that of Corollary 5.

   Warnke also pointed out to us that the structural properties needed for Corollary 5 follow from standard random-graph estimates and do not require more sophisticated analysis of the triangle-free process.

4. A natural extension of Theorem 2 is to replace K_3 with a general family of graphs $\mathcal{F}$. Given a family $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if it does not contain any member of $\mathcal{F}$ as a subgraph. Analogously, given an $\mathcal{F}$-free graph H on $[n]$, one may ask for the value of the following function:

$$\mathrm{ex}_H(n, \mathcal{F}) := \left\{ e(G) \colon G \subseteq \binom{[n]}{2} \text{ is } \mathcal{F}\text{-free and } H \subseteq G \right\}.$$

5. Note that Corollaries 4 and 5 can be viewed as results concerning $\mathrm{ex}_H(n, K_3)$ in the average case. One can also consider the worst-case scenario. Let $m, n \geqslant 1$ be integers. Define

$$\mathrm{ex}_m(n, \mathcal{F}) = \min \left\{ \mathrm{ex}_H(n, \mathcal{F}) \colon H \text{ is } \mathcal{F}\text{-free and } e(H) \leqslant m \right\}.$$

The function $\mathrm{ex}_m(n, \mathcal{F})$ is closely related to a classical problem introduced by Erdős–Hajnal–Moon [5]. Given a family $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$*-saturated* if G is $\mathcal{F}$-free but the addition of any new edge to G creates a subgraph that belongs to $\mathcal{F}$. The *saturation number* $\mathrm{sat}(n, \mathcal{F})$ is defined as

$$\mathrm{sat}(n, \mathcal{F}) \coloneqq \min \{e(G) \colon v(G) = n \text{ and } G \text{ is } \mathcal{F}\text{-saturated}\} .$$

Note that for $m \geqslant \mathrm{sat}(n, \mathcal{F})$, the function $\mathrm{ex}_m(n, \mathcal{F})$ reduces to $\mathrm{sat}(n, \mathcal{F})$.

For m close to $\mathrm{sat}(n, K_3)$ (i.e. $n-1$), it appears that S_m is the worst graph, that is, $\mathrm{ex}_m(n, K_3) = \mathrm{ex}_{S_m}(n, K_3) = m(n-m)$. However, for smaller values of m, the disjoint union of 5-cycles is worse than S_m. We leave it as an open problem to determine the behavior of $\mathrm{ex}_m(n, \mathcal{F})$, in particular $\mathrm{ex}_m(n, K_3)$, as m increases from 1 to $\mathrm{sat}(n, \mathcal{F})$.

6. The definition of the function $\mathrm{ex}_H(n, \mathcal{F})$ can be extended even to the case when H is not necessarily $\mathcal{F}$-free: Given a graph H on a vertex set V of size n and a family of graphs $\mathcal{F}$, what is the maximum number of edges in a graph G on V that contains H as a subgraph and satisfies $\mathrm{N}(F, G) = \mathrm{N}(F, H)$ for every $F \in \mathcal{F}$? (For graphs G_1 and G_2, the notation $\mathrm{N}(G_1, G_2)$ denotes the number of copies of G_1 in G_2.) Here, we are interested in the function

$$\mathrm{ex}_H(n, \mathcal{F}) \coloneqq \max \left\{e(G) \colon H \subseteq G \subseteq \binom{V}{2} \text{ and } \mathrm{N}(F, G) = \mathrm{N}(F, H) \text{ for every } F \in \mathcal{F}\right\} .$$

For the general function $\mathrm{ex}_H(n, K_3)$ when H is not necessarily triangle-free, Corollary 3 continues to hold, which then applies for the random graph models, where triangle-freeness is not required. A slight modification of our arguments can be used to prove Corollary 3 in this more general setting. We omit the details and leave these general problems to future investigation.

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