On Ribbon Graphs that Admit a Partial Dual of Euler Genus at Most Two

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Abstract

Moffatt in his paper *Excluded minors and the ribbon graphs of knots* (Journal of Graph Theory, 2016), conjectures that every ribbon graph minor-closed family can be characterized by a finite set of excluded ribbon graph minors. He supports this conjecture in several papers, particularly in *Ribbon graph minors and low-genus partial duals* (Annals of Combinatorics, 2016), by giving a finite list of excluded minors that characterizes the class of ribbon graphs with a partial dual of Euler genus at most one. In this paper, we give a finite list of excluded minors that characterizes ribbon graphs with a partial dual of Euler genus at most two, subject to the condition that any bouquet related by partial duality to the ribbon graph satisfies that the intersection graph of the induced subgraph of its non-orientable loops and the complement of the intersection graph of the induced subgraph of its orientable loops are both 3-cycle free.

Mathematics Subject Classifications: 05C83, 05C10

1 Introduction

A landmark result by Robertson and Seymour proves that every minor-closed family of graphs can be characterized by a finite set of excluded minors [20]. Moffatt [16] conjectures that the same applies to minor-closed families of ribbon graphs. He supports this conjecture in a series of papers by giving explicit lists of excluded minors for several classes of ribbon graphs, particularly those that admit partial duals of low genus [14, 16, 17]. The past few decades have seen an increase in the study of ribbon graphs with much interest arising from their connections to knot theory, topological graph theory, graph polynomials, matroid theory, and quantum field theory, among other fields

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[3, 4, 6, 8, 13, 14, 16, 22]. There is a fundamental difference between graph minors and ribbon graph minors: in the latter, one may need to contract loops, which can create additional vertices and components [1, 3, 16]. The partial dual of a ribbon graph is the ribbon graph obtained by forming its geometric dual but only with respect to a subset of its edges [8, 12], for details see Section 2. Our main result is a characterization by a finite list of excluded minors for ribbon graphs with a partial dual of Euler genus at most two. This characterization assumes that any bouquet related by partial duality to the ribbon graph satisfies the following conditions: the intersection graph of the induced subgraph of its non-orientable loops and the complement of the intersection graph of the induced subgraph of its orientable loops are both 3-cycle free. We denote the set of all bouquets that satisfy these conditions by $\mathcal{E}$.

In Section 2, we introduce the main concepts and terminology used throughout the paper. Also, we state some important results needed for our proofs. In Section 3, we study some basic properties of orientable and non-orientable bouquets. We begin by giving the list of minor minimal ribbon graphs whose partial duals all have Euler genus at least three. We continue with several propositions arranged according to whether they consider orientable or non-orientable bouquets. An important result is Theorem 15, which characterizes by excluded minors those non-orientable bouquets without orientable loops that have a partial dual of Euler genus at most two. In Section 4, we prove the main result. To begin, we establish two lemmas that form the core of its proof. We characterize by excluded minors the bouquets of $\mathcal{E}$ that have a partial dual of Euler genus at most two. Lemma 18 covers the orientable case and Lemma 19 the non-orientable case. We then prove Lemma 20, which states that a bouquet in $\mathcal{E}$ has no partial dual of Euler genus less than or equal to 2 if and only if it has a minor equivalent to one of the excluded bouquet minors. Using Lemma 20, we prove our main result Theorem 21. In Section 5, we describe ongoing work, particularly an analogous characterization for delta-matroids. Finally, the Appendix exhibits all the bouquets in each class of excluded ribbon graph minors.

2 Preliminaries

Throughout the paper, we use concepts and results found in Moffatt's papers: [14, 15, 16, 17]. A *ribbon graph* $G = (V, E)$ is a surface with boundary, represented as the union of two sets of discs, a set V of vertices and a set E of edges such that (1) the vertices and edges intersect in disjoint line segments; (2) each such line segment lies on the boundary of precisely one vertex and precisely one edge; and (3) every edge contains exactly two such line segments. A *bouquet* B is a ribbon graph with precisely one vertex, whose edges are called *loops*. Two loops e and f are *interlaced* if their ends are met in the cyclic order $e\,f\,e\,f$ when traveling round the boundary of the unique vertex. A loop is non-orientable if together with its incident vertex it forms a Möbius band, and is orientable otherwise. A ribbon graph is non-orientable if it contains a cycle C homeomorphic to a Möbius band. It is well-known that ribbon graphs are equivalent to cellularly embedded graphs in surfaces [8].

Following [16], let G be a ribbon graph, $e \in E(G)$, and $v \in V(G)$. Then $G \setminus e$ denotes the ribbon graph obtained from G by deleting the edge e, and $G \setminus v$ denotes the ribbon graph obtained from G by deleting the vertex v together with all its incident edges. If u_1 and u_2 are the (not necessarily distinct) vertices incident to e, then the *contraction* of e in G, denoted G/e, is the ribbon graph obtained from G by the following process: (1) Consider the boundary components of $e \cup u_1 \cup u_2$ as curves in G. (2) For each resulting curve, attach a disc (which will form a vertex of G/e) by identifying its boundary component with the curve. (3) Delete e, u_1 and u_2 to get the ribbon graph G/e, see Figure 1 which is based on Table I of [16]. If multiple edges of a ribbon graph are contracted and/or deleted, the resulting ribbon graph does not depend on the order of the operations. We say that a ribbon graph H is a *minor* of a ribbon graph G if H is obtained from G by a sequence of edge deletions, vertex deletions, or edge contractions. In addition, we say that G has a H-ribbon graph minor if it has a minor equivalent to H. A set S of ribbon graphs is said to be *minor-closed* if, for each $G \in S$, every minor of G is also in S.

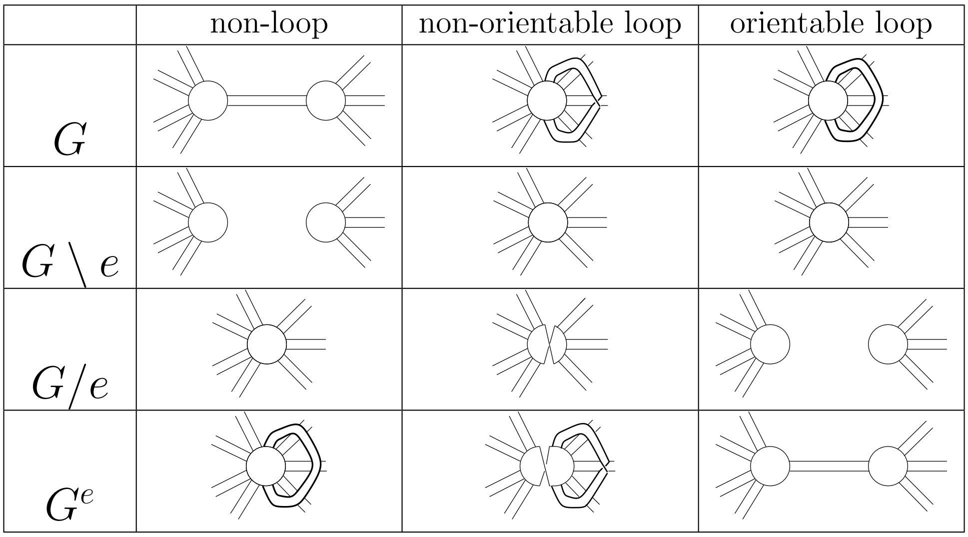


Figure 1: Deletion, contraction and partial duality of an edge in a ribbon graph.

Let G be a ribbon graph, its *geometric dual* G^* is formed by regarding the boundary components of the ribbon graph as curves on the surface of G. Gluing a disc to G along each of these curves by identifying the boundary of the disc with the curve, and removing the interior of all the vertices of G. In [3], Chmutov introduces an extension of duality obtained by forming the geometric dual with respect to a subset A of the edges. The resulting ribbon graph is called the *partial dual* of G with respect to A and is denoted by G^A, see Figure 1 and Figure 2. Observe from Figure 1 that $G/e = G^e \setminus e$. Partial duality has further been developed and generalized in several papers [3, 7, 8, 11, 12, 21].

In [16], the *genus* $g(G)$ of a ribbon graph G is defined as its genus when viewed as a surface. Its *Euler genus*, $\gamma(G)$, is defined as $2g(G)$ if G is orientable and $g(G)$ if G is non-orientable. Two ribbon graphs are *equivalent* if they describe equivalent cellular embedded

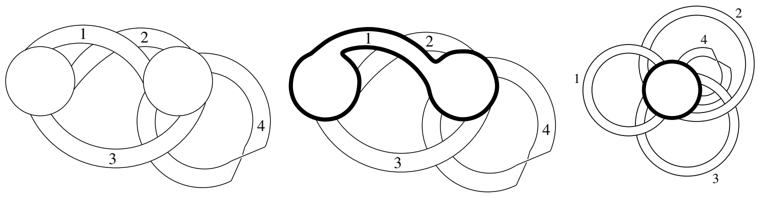


Figure 2: A ribbon graph G, the boundary component of $G|_A$ where $A = \{1\}$, and the partial dual G^A.

graphs. Therefore, we consider ribbon graphs up to equivalence. So, two ribbon graphs are equivalent if a homeomorphism takes one to the other, preserving the vertex-edge structure and the cyclic order of the half-edges at each vertex. For non-connected ribbon graphs, the Euler genus is defined as the sum of the Euler genus of its components [17]. The relation between the Euler characteristic $\chi(G)$ of a ribbon graph G and its Euler genus $\gamma(G)$ is $\chi(G) = v(G)-e(G)+f(G) = 2k(G)-\gamma(G)$, where $v(G)$ is the number of vertices, $e(G)$ is the number of edges, $f(G)$ is the number of boundary components and $k(G)$ is the number of connected components of G. In particular, for a ribbon graph $G = (V, E)$ and a subset A of its edges, we denote by $k(A)$ the number of connected components of the ribbon subgraph (V, A) and by $f(A)$ the number of boundary components of (V, A). A *quasi-tree* is a ribbon subgraph that has exactly one boundary component, see the middle graph in Figure 2.

Following [17], the *intersection graph* $\mathcal{I}(B)$ of a bouquet B is the vertex-weighted simple graph whose vertex set is $E(B)$ and two vertices e and f of $\mathcal{I}(B)$ are adjacent if and only if the loops e and f interlace in B. A vertex e of $\mathcal{I}(B)$ has weight $+$ if e is an orientable loop in B and has weight $-$ if it is non-orientable. The *complementary graph* $\overline{\mathcal{I}}(B)$ is defined on the same vertex set of $\mathcal{I}(B)$, where two distinct vertices are adjacent if and only if they are not adjacent in $\mathcal{I}(B)$. For examples of intersection and complementary graphs asociated to a bouquet B, see Tables 1 to 7 in the Appendix. For a vertex v of a simple graph G, let $G \setminus v$ denote the graph obtained by deleting the vertex v and all edges incident to it. An *independent set* in G is a subset of vertices in which no two vertices are adjacent.

A *chord diagram* consists of a circle in the plane and some line segments, called chords, whose endpoints lie on the circle. The endpoints of chords are taken to be distinct. A graph is a circle graph if it is the intersection graph of a chord diagram. Moffatt [18] gives a natural way to associate a chord diagram with an orientable bouquet B: take the boundary of the vertex as the circle and place a chord between the two ends of each loop of B. By forming the intersection graph of this chord diagram, we obtain a natural way to associate a circle graph with a bouquet. In Theorem 4 and Corollary 1 of [19], Moffatt and Oh show that two chord diagrams are related by mutation and isomorphism if and only if they have isomorphic intersection graphs. As a corollary, they show that two orientable bouquets are related by mutation and isomorphism if and only if they have isomorphic intersection graphs. Therefore, we always consider intersection graphs up to isomorphism. For the definition of mutation see Section 3 of [19].

Remark 1. Let B be an orientable bouquet and C_i denote a chordless cycle of length i for $i \in \mathbb{N}$. If the graph $\overline{\mathcal{I}}(B)$ contains an induced subgraph isomorphic either to: C_6, or C_5 together with an isolated vertex or C_4 with three pendant edges attached to three of its four vertices, then the intersection graph $\mathcal{I}(B)$ cannot have a ribbon graph associated with it. If such a ribbon graph existed, then the intersection graph $\mathcal{I}(B)$ would have a pivot-minor isomorphic to W_5 (the wheel with 5 spokes), which is not possible since this graph is an obstruction for the class of circle graphs, see [2], Theorem 9.1 and Theorem 9.2 of [18].

Following [15], we say that G is a 1-*sum* of G_1 and G_2, written $G = G_1 \oplus G_2$, if $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$, see Figure 3. The 1-sum is said to occur at the vertex v. We can also view a 1-sum as a way to construct a ribbon graph G out of two connected ribbon graphs P and Q. Given P, Q, consider a vertex v_P in P and a vertex v_Q in Q. Then identify v_P with v_Q in a way such that the edges incident to v_P and to v_Q do not intersect. We obtain a ribbon graph G that has the property that $G = P \oplus Q$, with the 1-sum occurring at v. Let G be a ribbon graph and $A \subseteq E(G)$. We denote by $G|_A$ the *restriction* of G to A, the ribbon subgraph with edge set A and vertices incident to edges in A. We denote by A^c the edge set $E(G) \setminus A$.

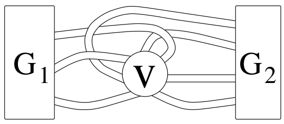


Figure 3: A 1-sum.

The following lemma shows that the partial duals of the ribbon graph minors of G are the ribbon graph minors of the partial duals of G.

Lemma 2. *(Moffatt [16] Lemma 3.2). Let G be a ribbon graph and $A \subseteq E(G)$. Then*

$$\{J^A : J \text{ is a ribbon graph minor of } G\} = \{H : H \text{ is a ribbon graph minor of } G^A\}$$

The next lemma addresses minor-closed properties with respect to the genus of ribbon graphs.

Lemma 3. *(Moffatt [16] Lemma 3.3). Let G be a ribbon graph*

1. *If H is a ribbon graph minor of G, then $g(H) \leqslant g(G)$ (respectively, $\gamma(H) \leqslant \gamma(G)$). In particular, for each $k \in \mathbb{N}_0$, the set of ribbon graphs of genus (respectively, Euler genus) at most k is ribbon graph minor-closed.*

2. *For each $k \in \mathbb{N}_0$ the set of all ribbon graphs that have a partial dual of genus (or Euler genus) at most k is ribbon graph minor-closed.*

The next theorem provides a characterization of ribbon graphs that have a partial dual of Euler genus at most one.

Theorem 4. *(Moffatt [17] Theorem* 1.1*). Let X_1, X_2, X_3 be the ribbon graphs in Figure 4. Then, a ribbon graph has a partial dual of Euler genus at most* 1 *if and only if it has no ribbon graph minor equivalent to X_1, X_2, or X_3.*

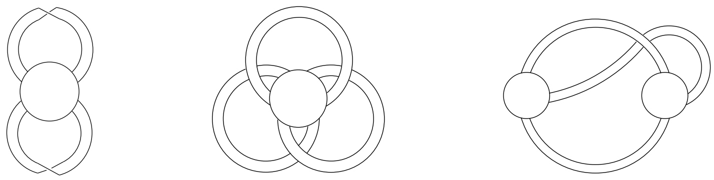


Figure 4: The excluded ribbon graph minors X_1, X_2, and X_3.

We make use of the following results from Gross, Mansour and Tucker [10].

Corollary 5. *(Gross, Mansour and Tucker [10] Corollary* 2.2*). For an arbitrary ribbon graph G and $A \subseteq E(G)$, we have $\gamma(G^A) = \gamma(A)+\gamma(A^c)+2(k(G)-k(A)-k(A^c))+2(v(G))$.*

If the ribbon graph is a bouquet, then the following result is obtained.

Corollary 6. *(Gross, Mansour and Tucker [10] Corollary* 2.3*). Let B be a bouquet and $A \subseteq E(B)$. Then $\gamma(B^A) = \gamma(A) + \gamma(A^c)$.*

3 Orientable and Non-orientable Bouquets

The 44 bouquets appearing in Tables 1, 2, 3, 4, 5, 6 and 7 of the Appendix, have been partitioned by the partial duality relation into nine families $\mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 9$. We denote by $\mathrm{B}_{[i](j)}$ the bouquet belonging to the family i and whose label within the family is j. We write $B = B_O \sqcup B_N$ where B_O is the ribbon subgraph of B consisting of its orientable loops and B_N the ribbon subgraph of B consisting of its non-orientable loops. The $B \in \mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 4$ are non-orientable bouquets and $B \in \mathcal{B}_{[i]}$ for $5 \leqslant i \leqslant 9$ are orientable bouquets. The 184 ribbon graphs obtained by partial duality starting with the bouquets in Tables 1, 2, 3, 4, 5, 6 and 7 can be partitioned by the partial duality relation into nine families $\mathcal{G}_{[i]}$ for $1 \leqslant i \leqslant 9$. We have that $\mathcal{B}_{[i]} \subseteq \mathcal{G}_{[i]}$ for $1 \leqslant i \leqslant 9$.

Let $\mathcal{E}$ denote the bouquets B such that $B = B_O \sqcup B_N$ satisfies that $\mathcal{I}(B_N)$ and $\overline{\mathcal{I}}(B_O)$ are both 3-cycle free. Any bouquet $B \in \mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 8$ belongs to $\mathcal{E}$ as can be examined in Tables 1, 2, 3 and the first three rows in Table 4. On the other hand, no bouquet $B \in \mathcal{B}_{[9]}$ belongs to $\mathcal{E}$, as can be verified in Tables 4, 5, 6 and 7.

Lemma 7. *The ribbon graphs $G \in \mathcal{G}_{[i]}$ (bouquets $B \in \mathcal{B}_{[i]}$) for $1 \leqslant i \leqslant 9$ are minor minimal for the family of ribbon graphs (bouquets) that do not have a partial dual of Euler genus less than or equal to* 2*.*

Proof. By computer we can verify that any ribbon graph $G \in \mathcal{G}_{[i]}$ (bouquet $B \in \mathcal{B}_{[i]}$) for $1 \leqslant i \leqslant 9$ and any partial dual of G (B) have Euler genus greater than 2. However, each proper minor of any $G \in \mathcal{G}_{[i]}$ ($B \in \mathcal{B}_{[i]}$) for $1 \leqslant i \leqslant 9$ has a partial dual with Euler genus less than or equal to 2. Lemmas 2 and 3, imply that any $G \in \mathcal{G}_{[i]}$ ($B \in \mathcal{B}_{[i]}$) for $1 \leqslant i \leqslant 9$ cannot be a minor of a ribbon graph that has a partial dual with Euler genus less than or equal to 2. □

Henceforth we will refer to the ribbon graphs $G \in \mathcal{G}_{[i]}$ (bouquets $B \in \mathcal{B}_{[i]}$) for $1 \leqslant i \leqslant 9$ as *excluded minors*, that is, ribbon graphs (bouquets) that do not have a partial dual with Euler genus less than or equal to 2 but whose proper minors all have a partial dual of Euler genus less than or equal to 2.

3.1 Orientable bouquets

We begin by studying orientable bouquets.

Lemma 8. *Let B be an orientable bouquet with $|E(B)| \leqslant 5$. Then, either B has a partial dual of Euler genus at most 2, or B is equivalent to $\mathrm{B}_{[5](1)}$.*

Proof. Consider the intersection graph of B. Since $|E(B)| \leqslant 5$, then $|V(\mathcal{I}(B))| \leqslant 5$. If there are two vertices v_x, $v_y \in \mathcal{I}(B)$ that are not adjacent, then there are two orientable loops e_x, e_y associated with v_x, v_y that do not interlace in B. Let $A = \{e_x, e_y\}$, then $B|_A$ is planar and by Corollary 6 we have $\gamma(B^A) = \gamma(B|_A) + \gamma(B|_{A^c}) = \gamma(B|_{A^c})$. As $|A^c| = 3$, then $\gamma(B|_{A^c}) \leqslant 2$. Therefore B has a partial dual B^A of Euler genus at most 2. Otherwise, if all loops are pairwise interlaced, then B is equivalent to $\mathrm{B}_{[5](1)}$. □

Lemma 9. *Let B be an orientable bouquet. If there exists a pair of adjacent vertices $\{a, b\} \in \overline{\mathcal{I}}(B)$ such that $\overline{\mathcal{I}}(B) \setminus \{a, b\}$ induces a ribbon subgraph H of B with Euler genus at most two, then B has a partial dual B^A of Euler genus at most two, where $A = \{a, b\}$.*

Proof. Let $B|_A$ be the ribbon subgraph associated to the pair of vertices $\{a, b\}$. By hypothesis, the ribbon subgraph $H = B|_{A^c}$ has Euler genus at most two. Given that $\gamma(B|_A) = 0$ by Corollary 6, B has a partial dual B^A of Euler genus at most two. □

Lemma 10. *Let B be an orientable bouquet whose complementary graph contains an independent set with at least 5 vertices, then B does not have a partial dual of Euler genus at most two.*

Proof. Suppose that $\overline{\mathcal{I}}(B)$ has an independent set of size greater than or equal to 5. Consider the ribbon subgraph H induced by any five vertices from the independent set. Then, B has a ribbon subgraph H equivalent to $\mathrm{B}_{[5](1)}$. By Lemma 8, B has no partial dual of Euler genus at most two. □

Let B be a bouquet. We define a *smoothing* at the pair of adjacent interlaced loops g and h met in the cyclic order $\ldots g \ldots h\, g \ldots h \ldots$, as the operation of sliding g or h so that the interlace of g and h is removed and they are met in the cyclic order $\ldots g \ldots g\, h \ldots h \ldots$; followed by interlacing a new loop e only with each of g and h to obtain $\ldots g \ldots e\, g\, h\, e \ldots h \ldots$ (see Figure 5). The smoothing operation is obtained as the effect on B of subdividing an edge in its intersection graph $\mathcal{I}(B)$.

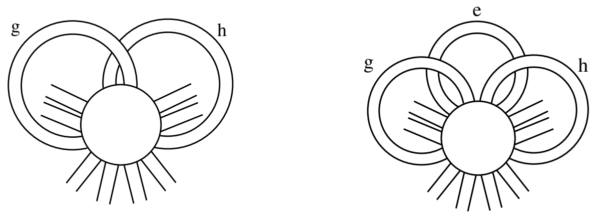


Figure 5: Two interlaced loops and the effect after a smoothing.

Lemma 11. *The bouquet* $\mathrm{B}_{[9](1)}$ *can be obtained by successive application of smoothings to the interlaced loops associated to the vertices of a cycle* C_5 *in the intersection graph of the bouquet* $\mathrm{B}_{[5](1)}$.

Proof. By subdividing the edges of a cycle C_5 in the intersection graph of $\mathrm{B}_{[5](1)}$ we obtain the intersection graph of $\mathrm{B}_{[9](1)}$ and by the relation between the subdivision of an edge in the intersection graph with the smoothing applied to the loops involved, the result follows. □

Let C_n^2 be the graph obtained from a cycle of size n adding the edges between vertices at distance 2 in the cycle. The following remark concerns properties obtained through smoothings on a ribbon graph whose intersection graph is isomorphic to C_n^2.

Remark 12. Let B_{C_n} denote the orientable bouquet with $2n$ edges, given by $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, with $n \in \mathbb{N}$. We assume that these edges are met in the cyclic order $e_1e_3f_1f_2e_2e_4f_2f_3e_3e_5f_3f_4 \ldots e_nf_{n-2}f_{n-1}e_{n-1}e_1f_{n-1}f_ne_ne_2f_nf_1$. Let $\widehat{C}$ be obtained by subdividing the edges of one n-cycle of C_n^2, see Figure 6 for the case $n = 5$. Note that $\widehat{C} = \mathcal{I}(B_{C_n})$. Consider $\mathcal{F} = \{B_{C_{2k+1}} \mid 2 \leqslant k \in \mathbb{N}\}$. Then, by contracting the loops e_n, e_{n-1}, f_n, f_{n-1} in $B_{C_{2k+1}}$ we obtain $B_{C_{2k-1}}$ as a minor, for $3 \leqslant k \in \mathbb{N}$, see Figure 7. Furthermore, $B_{C_{2k}}$ has a partial dual of Euler genus at most 2, for $2 \leqslant k$.

Proof. The contraction sequences of the edges in $A = A_1 \cup A_2$, where $A_1 = \{e_{n-1}, e_n\}$ and $A_2 = \{f_{n-1}, f_n\}$ is as follows:
Step 1. Identify the edges of A_1.
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}\underline{\mathbf{e_{n-1}}}\overline{f_{n-3}f_{n-2}e_{n-2}}\underline{\mathbf{e_n}}f_{n-2}f_{n-1}\underline{\mathbf{e_{n-1}}}\overline{e_1f_{n-1}f_n}\underline{\mathbf{e_n}}e_2f_nf_1.$
Step 2. Take the partial dual with respect to A_1.
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}\underline{\mathbf{e_{n-1}}}\overline{e_1f_{n-1}f_n}\underline{\mathbf{e_n}}f_{n-2}f_{n-1}\underline{\mathbf{e_{n-1}}}\overline{f_{n-3}f_{n-2}e_{n-2}}\underline{\mathbf{e_n}}e_2f_nf_1.$
Step 3. Given that $B_{C_n}/A_1 = B_{C_n}^{A_1} \setminus A_1$, results in:
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}e_1f_{n-1}f_nf_{n-2}f_{n-1}f_{n-3}f_{n-2}e_{n-2}e_2f_nf_1.$
Step 4. Identify the edges of A_2.
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}e_1\underline{\mathbf{f_{n-1}}}\underline{\mathbf{f_n}}f_{n-2}\underline{\mathbf{f_{n-1}}}\overline{f_{n-3}f_{n-2}e_{n-2}e_2}\underline{\mathbf{f_n}}f_1.$
Step 5. Take the partial dual with respect to A_2.
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}e_1\underline{\mathbf{f_{n-1}}}\overline{f_{n-3}f_{n-2}e_{n-2}e_2}\underline{\mathbf{f_n}}f_{n-2}\underline{\mathbf{f_{n-1}}}\underline{\mathbf{f_n}}f_1.$
Step 6. Finally, given that $(B_{C_n}^{A_1} \setminus A_1)^{A_2} \setminus A_2 = B_{C_n}^{A} \setminus A = B_{C_n}/A$, results in:
$e_1e_3f_1f_2 \ldots f_{n-4}f_{n-3}e_{n-3}e_1f_{n-3}f_{n-2}e_{n-2}e_2f_{n-2}f_1.$
Therefore, $B_{C_{n-2}}$ is a minor of B_{C_n}. Hence, every ribbon graph in $\mathcal{F}$ has the ribbon graph

$B_{[9](1)} = B_{C_5}$ as a minor. Furthermore, $B_{C_{2k}}$ has a partial dual of Euler genus at most 2, for $2 \leqslant k$, since it can be seen as the 1-sum of two planar ribbon subgraphs. One is obtained as $A = A_1 \cup A_2$, where $A_1 = \{e_i \in E(B) \mid i \text{ is an odd number less than } 2k\}$, $A_2 = \{f_i \in E(B) \mid i \text{ is an even number less than or equal to } 2k\}$ and the other one is given by A^c. Note that $B|_A$ and $B|_{A^c}$ are planar. Hence, by Corollary 6, we obtain the result. □

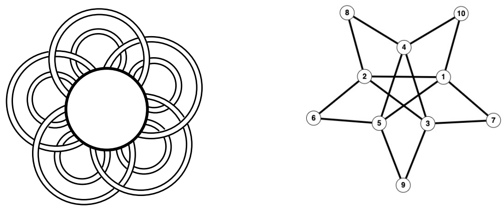
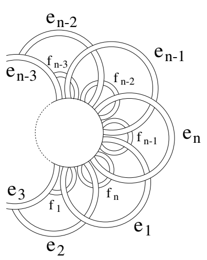
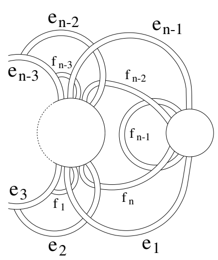


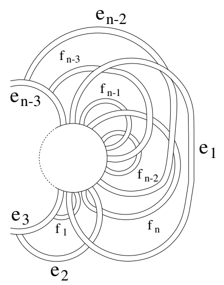
Figure 6: The bouquet B_{C_5} and $\widehat{C} = \mathcal{I}(B_{C_5})$.



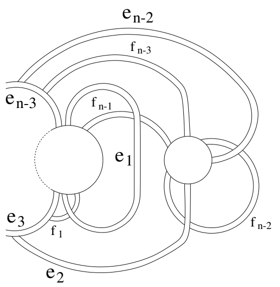
B_{C_n}



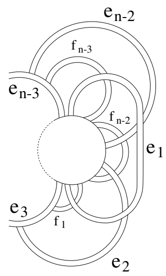
B_{C_n}/e_n



$B_{C_n}/\{e_n, e_{n-1}\}$



$B_{C_n}/\{e_n, e_{n-1}, f_n\}$



$B_{C_n}/\{e_n, e_{n-1}, f_n, f_{n-1}\} = B_{C_{n-4}}$

Figure 7: $B_{C_{n-4}}$ is a minor of B_{C_n}.

As an example, consider the graph C_7^2 which is isomorphic to $\mathcal{I}(B_{[8](6)})$ (see row 3 of Table 4). Let $e_1, e_2, \ldots, e_7$ be the loops of $B_{[8](6)}$ such that their ends are met in the cyclic order $e_2\ e_7\ e_3\ e_1\ e_4\ e_2\ e_5\ e_3\ e_6\ e_4\ e_7\ e_5\ e_1\ e_6$. Applying seven smoothings at the pairs of loops $\{e_1\ e_3,\ e_2\ e_4,\ e_3\ e_5,\ e_4\ e_6,\ e_5\ e_7,\ e_1\ e_6,\ e_2\ e_7\}$ in $B_{[8](6)}$, we obtain $e_2f_7f_1e_1e_3f_1f_2e_2$ $e_4f_2f_3e_3e_5f_3f_4e_4$ $e_6f_4f_5e_5e_7f_5f_6e_6$ $e_1f_6f_7e_7$ which is equivalent to B_{C_7}. By Remark 12, it follows that B_{C_7} has $B_{C_5} = B_{[9](1)}$ as a minor.

3.2 Non-orientable bouquets

We now study non-orientable bouquets. For $n \in \mathbb{N}$ let $\widetilde{B}_n$ denote a bouquet with n non-orientable pairwise interlaced loops. The edges $e_1, \dots, e_n$ are met in the cyclic order $e_1e_2\dots e_ne_1e_2\dots e_n$ when traveling round the boundary of the unique vertex.

Lemma 13. *Let B be a bouquet such that $B_O = \emptyset$ and $B_N \neq \emptyset$. If $B = \widetilde{B}_P \oplus \widetilde{B}_Q$ with $P + Q = |E(B)|$, then B has a partial dual of Euler genus two.*

Proof. The fact that $\widetilde{B}_n$ has Euler genus equal to 1 for all $n \in \mathbb{N}$ follows by observing that $\widetilde{B}_n$ has 1 vertex, n edges and n faces, hence $\chi(\widetilde{B}_n) = 1$. Thus $\gamma(\widetilde{B}_n) = 1$ for all $n \in \mathbb{N}$. Now, by hypothesis we have $B = \widetilde{B}_P \oplus \widetilde{B}_Q$, with $P + Q = |E(B)|$ that together with Corollary 6 applied to $A = E(\widetilde{B}_P)$ or $A = E(\widetilde{B}_Q)$ gives $\gamma(B^A) = \gamma(B^{A^c}) = \gamma(B|_A) + \gamma(B|_{A^c}) = 1 + 1 = 2$. Therefore, B has a partial dual of Euler genus two. □

Let $\widehat{B}_{2k+1}$ be the bouquet with $2k+1$ non-orientable loops such that the complementary graph $\overline{\mathcal{I}}(\widehat{B}_{2k+1})$ is an odd cycle C_{2k+1}.

Lemma 14. *If $B \in \{\widehat{B}_{2k+1} \mid k \in \mathbb{N}\}$, then B has a minor equivalent to* $\mathrm{B}_{[1](1)}$ *or* $\mathrm{B}_{[4](2)}$.

Proof. Let $B, B' \in \{\widehat{B}_{2k+1} \mid k \in \mathbb{N}\}$ such that $\overline{\mathcal{I}}(B) = C_{2k+1}$ and $\overline{\mathcal{I}}(B') = C_{2(k+2)+1}$. It is straightforward to verify that B is a minor of B' for each $k \in \mathbb{N}$ by contracting four loops of B' corresponding to four consecutive vertices in the odd cycle C_{2k+5}. Consider $1, \dots, n$ as the vertices of C_n, and $e_1, \dots, e_n$ as the edges of $\widehat{B}_n$ associated with them, such that the edges appear in the cyclic order:
$e_2e_1e_4e_3\dots e_{n-6}e_{n-3}e_{n-4}e_{n-1}e_{n-2}e_1e_ne_3e_2e_5\dots e_{n-4}e_{n-5}e_{n-2}e_{n-3}e_ne_{n-1}$.
Consider the contraction sequence of the edges $\{e_{n-1}, e_{n-3}, e_{n-2}, e_n\}$ as follows:
Step 1. Identify the edge e_{n-1}.
$e_2e_1e_4e_3\dots e_{n-6}e_{n-3}e_{n-4}\underline{\mathbf{e_{n-1}}}\overline{e_{n-2}e_1e_ne_3e_2e_5\dots e_{n-4}e_{n-5}e_{n-2}e_{n-3}e_n}\underline{\mathbf{e_{n-1}}}$.
Step 2. Take the partial dual with respect to e_{n-1}.
$e_2e_1e_4e_3\dots e_{n-6}e_{n-3}e_{n-4}\underline{\mathbf{e_{n-1}}}\overline{e_ne_{n-3}e_{n-2}e_{n-5}e_{n-4}\dots e_5e_2e_3e_ne_1e_{n-2}}\underline{\mathbf{e_{n-1}}}$.
Step 3. Given that $\widehat{B}_n/e_{n-1} = \widehat{B}_n^{e_{n-1}} \setminus e_{n-1}$, results in
$e_2e_1e_4e_3\dots e_{n-6}e_{n-3}e_{n-4}e_ne_{n-3}e_{n-2}e_{n-5}e_{n-4}\dots e_5e_2e_3e_ne_1e_{n-2}$.
Step 4. Repeat for the other three edges.
Identify the edge e_{n-3}, take the partial dual with respect to e_{n-3}. Delete the edge e_{n-3},
$e_2e_1e_4e_3\dots e_{n-6}e_ne_{n-4}e_{n-2}e_{n-5}e_{n-4}\dots e_5e_2e_3e_ne_1e_{n-2}$.
Identify the edge e_{n-2}, take the partial dual with respect to e_{n-2}. Delete the edge e_{n-2},
$e_2e_1e_4e_3\dots e_{n-6}e_ne_{n-4}e_1e_ne_3e_2e_5\dots e_{n-4}e_{n-5}$.
Identify the edge e_n, take the partial dual with respect to e_n. Delete the edge e_n, to finally obtain $e_2e_1e_4e_3\dots e_{n-6}e_1e_{n-4}e_3e_2e_5\dots e_{n-4}e_{n-5}$. Which is equivalent to $\widehat{B}_{n-4}$. Therefore, $\widehat{B}_{n-4}$ is a minor of $\widehat{B}_n$. Recursively, we conclude that $\overline{\mathcal{I}}(B')$ induces a minor isomorphic to either an odd cycle C_3 or C_5. Therefore, B has a minor equivalent to $\mathrm{B}_{[1](1)}$ or $\mathrm{B}_{[4](2)}$. □

The following result is a characterization of the pure non-orientable bouquets, in the sense that the bouquet B is assumed to be free of orientable loops and no other conditions hold.

Theorem 15. *If B is a bouquet such that $B_O = \emptyset$ and $B_N \neq \emptyset$, then B has a partial dual of Euler genus at most 2 if and only if B has no minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $i = 1$ or 4.*

Proof. By Lemma 7 any $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $i = 1$ or 4 is an excluded minor for the property of having a partial dual of Euler genus less than or equal to 2.
For the converse, consider the complementary graph of B. Suppose $\overline{\mathcal{I}}(B)$ is bipartite, then each part induces a clique in $\mathcal{I}(B)$. So B is the 1-sum of two bouquets of Euler genus 1. By Lemma 13, B has a partial dual of Euler genus 2, which forms a contradiction. Hence, $\overline{\mathcal{I}}(B)$ is not bipartite. Thus, $\overline{\mathcal{I}}(B)$ has a minimal odd cycle C of size p. Let H be the ribbon subgraph induced by the vertices of C. If $p = 3$, then H is equivalent to $\mathrm{B}_{[1](1)}$. If $p = 5$, then H is equivalent to $\mathrm{B}_{[4](2)}$. Finally, if $p \geqslant 7$, then the odd cycle corresponds to a ribbon subgraph of H that is equivalent to $\widehat{B}_{2k+1}$ for some $k \in \mathbb{N}$. By Lemma 14, H has a minor equivalent to $\mathrm{B}_{[1](1)}$ or $\mathrm{B}_{[4](2)}$. This completes the proof. □

Lemma 16. *Let B be a non-orientable bouquet such that $B_O \neq \emptyset$ and $B_N \neq \emptyset$. If $\mathcal{I}(B_O)$ is not bipartite, then B has a minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ $(1 \leqslant i \leqslant 3)$.*

Proof. Suppose that $\mathcal{I}(B_O)$ is not bipartite. Therefore, there is an odd cycle C. Let J be the ribbon subgraph associated to C. Let $e \in B_N$. Consider H the ribbon subgraph $J \cup e$. By a sequence of contractions done on interlaced loops of J one obtains a minor of J equivalent to X_2. Hence, H has a minor equivalent to $X_2 \cup e$. We have four cases: If e does not interlace with any element of X_2, then H has a minor equivalent to $\mathrm{B}_{[2](1)}$. If e interlaces with one element of X_2, then H induces a minor equivalent to $\mathrm{B}_{[3](1)}$. If e interlaces with two elements of X_2, then H induces a minor equivalent to $\mathrm{B}_{[3](3)}$. If e interlaces with all elements of X_2, then by contracting the loop e, H has a minor equivalent to $\mathrm{B}_{[1](1)}$. This completes the proof. □

Lemma 17. *Let B be a non-orientable bouquet. If $\mathcal{I}(B_N)$ contains an independent set with at least 3 vertices, then B does not have a partial dual of Euler genus at most two.*

Proof. Suppose that $\mathcal{I}(B_N)$ has an independent set with at least 3 vertices. Let J be the ribbon subgraph consisting of the vertices in the independent set. Then J has a minor equivalent to $\mathrm{B}_{[1](1)}$, obtained by deleting all loops except the three loops associated to these vertices. Hence, B has a minor equivalent to $\mathrm{B}_{[1](1)}$. □

4 Main Result

Recall that $\mathcal{E}$ denotes the bouquets B such that $B = B_O \sqcup B_N$ satisfies that $\mathcal{I}(B_N)$ and $\overline{\mathcal{I}}(B_O)$ are both 3-cycle free. We prove two lemmas that are necessary for the proof of the main result. These two lemmas characterize by excluded minors the bouquets in $\mathcal{E}$ that have a partial dual of Euler genus at most two. Lemma 18 covers the orientable case and Lemma 19 the non-orientable case. We then prove Lemma 20, which shows that a bouquet in $\mathcal{E}$ has no partial dual of Euler genus less than or equal to 2 if and only if it has a minor equivalent to one of the excluded bouquet minors. Using this result, we prove

Theorem 21, which states that a ribbon graph G such that all the bouquets obtained as partial duals of G are in $\mathcal{E}$, has no partial dual of Euler genus at most 2 if and only if G has a minor equivalent to one of the excluded ribbon graph minors obtained as partial duals of the bouquets in the Tables 1 to 7 in the Appendix.

Orientable Lemma

Lemma 18. *Let B be an orientable bouquet such that $B \in \mathcal{E}$, then B has a partial dual of Euler genus at most* 2 *if and only if B has no minor equivalent to* $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ $(5 \leqslant i \leqslant 9)$.

Proof. By Lemma 7 any $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $5 \leqslant i \leqslant 9$ is an excluded minor for the property of having a partial dual of Euler genus less than or equal to 2.

For the converse, we show that if B does not have a partial dual of Euler genus at most two, then B has a minor $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $5 \leqslant i \leqslant 9$. Suppose that $\overline{\mathcal{I}}(B)$ is bipartite. Denote by P and Q the two classes of the partition, by Lemma 10, $|V(P)|$ or $|V(Q)|$ cannot be greater or equal to 5, otherwise B would have a ribbon subgraph H equivalent to $\mathrm{B}_{[5](1)}$. It follows that each of P and Q have size less than or equal to four. On the other hand, if a bouquet B has at most 5 loops then by Lemma 8, B has a partial dual of Euler genus at most two, or it has $\mathrm{B}_{[5](1)}$ as a minor. Hence, $6 \leqslant |V(\overline{\mathcal{I}}(B))| \leqslant 8$. Now, consider $\overline{\mathcal{I}}(B)$ as the complete bipartite graph $K_{i,j}$ for $i = 2, 3, 4$ and $j = 3, 4$ together with all subgraphs obtained by deleting a set X of edges such that $0 \leqslant |X| \leqslant ij$ and $6 \leqslant i + j \leqslant 8$. We show that for each case $K_{i,j} \setminus X$, the associated ribbon graph has a minor $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $i = 5, \ldots, 9$.

Let $\widetilde{K} = K_{i,j} \setminus X$ such that $0 \leqslant |X| \leqslant ij$ and $6 \leqslant i + j \leqslant 8$. It is straightforward to verify the following statements:

(a) If there exists an induced subgraph $K_{3,3} \subseteq \widetilde{K}$, then the ribbon subgraph associated to $K_{3,3}$ is equivalent to $\mathrm{B}_{[6](1)}$.

(b) If there exists an induced subgraph $K_{3,3} \setminus \{e\} \subseteq \widetilde{K}$ for $e \in X$, then the ribbon subgraph associated to $K_{3,3} \setminus \{e\}$ is equivalent to $\mathrm{B}_{[7](4)}$.

(c) Let e_1, $e_2 \in X$. If there exists an induced subgraph $K_{3,3} \setminus \{e_1, e_2\} \subseteq \widetilde{K}$ for e_1 incident to e_2, then the ribbon subgraph associated to $K_{3,3} \setminus \{e_1, e_2\}$ is equivalent to $\mathrm{B}_{[7](1)}$.

(d) If there exists an induced subgraph $K_{3,3} \setminus \{C_4\} \subseteq \widetilde{K}$, then the ribbon subgraph associated to $K_{3,3} \setminus \{C_4\}$ is equivalent to $\mathrm{B}_{[7](2)}$.

(e) If there exists an induced subgraph $H \subseteq \widetilde{K}$ isomorphic to $K_{2,3} \cup \{v_1\}$, with v_1 an isolated vertex, then contracting the loop corresponding to the vertex v_1 results in a minor equivalent to a partial dual of $\mathrm{B}_{[5](1)}$.

(f) If there exists an induced subgraph $H \subseteq \widetilde{K}$ isomorphic to two disjoint $K_{1,2}$, then the ribbon subgraph associated to H is equivalent to $\mathrm{B}_{[7](3)}$.

(g) If $\widetilde{K}$ contains an independent set with at least 5 vertices, then by Lemma 10, B contains a minor equivalent to $\mathrm{B}_{[5](1)}$.

(h) Let O_1 be the simple graph in Figure 8. If there exists an induced ribbon subgraph H whose complement graph is isomorphic to O_1, then H is equivalent to $\mathrm{B}_{[8](3)}$.

(i) Let O_2 be the simple graph in Figure 8. If there exists an induced ribbon subgraph H whose complement graph is isomorphic to O_2, then H is equivalent to $\mathrm{B}_{[8](4)}$.

(j) Let O_3 be the simple graph in Figure 8. If there exists an induced ribbon subgraph H whose complement graph is isomorphic to O_3, then H/e is equivalent to $\mathrm{G} \in \mathcal{G}_{[7]}$ ($\mathrm{B}_{[i](j)} \in \mathcal{B}_{[7]}$), where $e \in E(H)$ and the vertex associated with e has degree three.

(k) If there exists an induced subgraph $H \subseteq \widetilde{K}$ isomorphic to C_6, then we obtain a contradiction by Remark 1.

(l) Let O_4 be the simple graph in Figure 8. By Remark 1, there does not exist a ribbon graph whose complement graph is isomorphic to O_4.

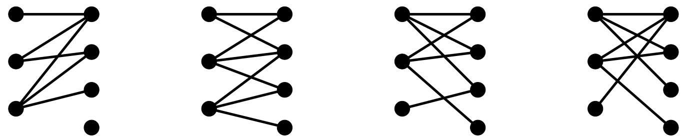


Figure 8: The simple graphs O_1, O_2, O_3 and O_4.

For the case $\widetilde{K} = K_{2,4} \setminus X$. Suppose $|X| = 0$, then for $\widetilde{K} = K_{2,4}$, consider the adjacent vertices v_1, v_2 of $\widetilde{K}$. In the intersection graph v_1, v_2 form an independent set, therefore e_1 and e_2 the loops in B associated with them do not interlace. Let $A = \{e_1, e_2\}$, hence $B|_A$ has Euler genus 0. On other hand, the graph $\widetilde{K} \setminus \{v_1, v_2\}$ is isomorphic to $K_{1,3}$. Which seen in the intersection graph forms a cycle of size 3 together with an isolated vertex. The associated ribbon subgraph has Euler genus at most 2. By Lemma 9, B has a partial dual B^A of Euler genus at most 2, which leads to a contradiction. If $|X| = 1$, then by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 2$, then we have case (e) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 3$, then by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 4$, then we have cases: (f), (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $5 \leqslant |X| \leqslant 6$, then we have case (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $7 \leqslant |X| \leqslant 8$, then we have case (g).

For the case $\widetilde{K} = K_{3,3} \setminus X$. If $|X| = 0$, then we have case (a). If $|X| = 1$, then we have case (b). If $|X| = 2$, then we have case (c) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 3$, then we have cases: (c), (k) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 4$, then we have case (d) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 5$, then we have case (f) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $6 \leqslant |X| \leqslant 7$, then we have case (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $8 \leqslant |X| \leqslant 9$, then we have case (g).

For the case $\widetilde{K} = K_{3,4} \setminus X$. If $0 \leqslant |X| \leqslant 1$, then we have case (a). If $|X| = 2$, then we have cases: (a) and (b). If $|X| = 3$, then we have cases: (a), (b), (c) and (k). If $|X| = 4$, then we have cases: (b), (c), (f),(g), (i) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 5$. then we have cases: (c), (d), (e), (f), (g) and (j). If $\widetilde{K}$ is isomorphic to O_4, then by (l), $\widetilde{K}$ is not associated with any ribbon graph. Otherwise by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 6$, then we have cases: (d), (f), (g), (h), (k) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 7$, then we have cases: (d), (f), (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 8$, then we have cases: (f), (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 9$, then we have case (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $10 \leqslant |X| \leqslant 12$, then we have case (g).

For the case $\widetilde{K} = K_{4,4} \setminus X$. If $0 \leqslant |X| \leqslant 1$, then we have case (a). If $2 \leqslant |X| \leqslant 3$, then we have cases (a) and (b). If $|X| = 4$, then we have cases: (a), (b), (f), (g) or (k). If $|X| = 5$, then we have cases: (a), (b), (c), (f), (g) or (k). If $|X| = 6$, then we have cases: (a), (b), (c), (d), (f), (g), (i), (k) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 7$, then we have cases: (b), (c), (d), (f), (g), (i), (k) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 8$, then we have cases: (c), (d), (f), (g), (h), (k). If $\widetilde{K}$ is isomorphic to $C_4 \cup \{v_1, v_2, v_3, v_4\}$, where v_i is adjacent only to $x_i \in C_4$ for $i = 1, \ldots, 4$ and $x_i \neq x_j$ for $i \neq j$, then deleting any vertex v_i, for $i = 1, \ldots, 4$, results in a graph isomorphic to O_4. Thus, we obtain case (l), which forms a contradiction. Otherwise by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 9$, then we have cases: (d), (f), (g), (h), (k) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 10$, then we have cases: (d), (f), (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 11$, then we have cases: (f), (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $|X| = 12$, then we have case (g) or by Lemma 9, B has a partial dual of Euler genus at most 2, which forms a contradiction. If $13 \leqslant |X| \leqslant 16$, then we have case (g).

Now, suppose that $\overline{\mathcal{I}}(B)$ is not bipartite, then there exists an induced odd cycle C of minimal size p with $|p| > 3$. Suppose the vertices of C are $1, 2, \ldots, p$ in that order. Consider the ribbon subgraph H of B corresponding to C. Since C is minimal, H consists of p orientable loops, which we label $1, \ldots, p$. If $p \geqslant 11$, then there exists an independent set with at least 5 vertices and by Lemma 10, B has a minor equivalent to $\mathrm{B}_{[5](1)}$. If $p = 9$, then by deleting in H the loops $\{1, 2, 6\}$, we obtain a minor equivalent to $\mathrm{B}_{[7](3)}$. If $p = 7$, then H induces a minor equivalent to $\mathrm{B}_{[8](6)}$. For $p = 5$. Let D be the set of vertices in $\overline{\mathcal{I}}(B) \setminus C$. If $D = \emptyset$, then by Lemma 8, we obtain a contradiction. Since, $\overline{\mathcal{I}}(B)$ is isomorphic to a cycle C_5 corresponding to the five loops of B. Thus, B is not equivalent to $\mathrm{B}_{[5](1)}$. Hence, B has a partial dual of Euler genus at most two. Consider $v \in D$. If v is not adjacent to any element of C, then we can check that $\mathcal{I}(C \cup \{v\})$ is isomorphic to W_5. This is a contradiction, by Remark 1. Therefore v must be adjacent to at least one element of C. If v is adjacent to two consecutive elements of C we obtain a contradiction to the minimality of C. So without loss of generality we have that v is adjacent to only one vertex $i \in C$, or v is adjacent to the vertices i and $i + 2 \in C$. Note that if $|D| = 1$, we obtain a contradiction since the bouquet B associated to $\overline{\mathcal{I}}(B)$ would be the $1-$sum of a ribbon subgraph of Euler genus two with a planar subgraph, which by Corollary 6, implies that B has a partial dual of Euler genus at most two. Therefore $|D| \geqslant 2$. Let $u, v \in D$ and H the ribbon subgraph associated to $C \cup \{u, v\}$. We have two cases:

Case 1. Suppose u and v are not adjacent in $\overline{\mathcal{I}}(H)$.

Case 1.1. Let u be adjacent only to vertex $i \in C$ and v only adjacent to vertex $j \in C$ for $i, j = 1, \ldots, 5$. If $j = i$, then by contracting the loop associated to the vertex $i+1$ ($i-1$ resp.) we obtain a minor $H/i+1$ ($H/i-1$ resp.) which is equivalent to $\mathrm{G} \in \mathcal{G}_{[7]}$. For example, the first ribbon graph in Figure 9 corresponds to H and the last corresponds to $(H/2)^5 = H^{\{2,5\}} \setminus 2$ which is equivalent to $\mathrm{B}_{[7](4)}$. Now if $j \neq i$, we have two cases: if $j = i+1$ ($j = i+4$ resp.), then by deleting the loop associated to the vertex $i+3$ ($i+2$ resp.) we obtain a minor $H \setminus i+3$ ($H \setminus i+2$ resp.) equivalent to $\mathrm{B}_{[7](2)}$. If $j = i+2$ ($j = i+3$ resp.), then H is equivalent to $\mathrm{B}_{[8](2)}$.

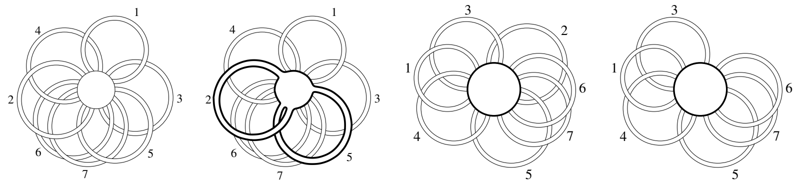


Figure 9: A ribbon graph, boundary component of $G|_A$ where $A = \{2, 5\}$, the partial dual G^A and $\mathrm{B}_{[7](2)}$.

Case 1.2. Suppose that u is adjacent only to a vertex $i \in C$ and v is adjacent to j and $j+2$. If $j = i$ ($j = i+3$ resp.), then by contracting the loop associated to the vertex $i+4$ ($i+1$ resp.), we obtain a minor $H/i+4$ ($H/i+1$ resp.) equivalent to $\mathrm{G} \in \mathcal{G}_{[7]}$. If $j \neq i$, we have two cases: if $j = i+1$ ($j = i+2$ resp.), then by deleting the loop associated to the vertex $i+3$ ($i+2$ resp.) we obtain a minor $H \setminus i+3$ ($H \setminus i+2$ resp.) equivalent to $\mathrm{B}_{[7](2)}$. If $j = i+4$, then the intersection graph associated to $C \cup \{u, v\} \setminus i$ is W_5, which by Remark 1, gives rise to a contradiction.

Case 1.3. Suppose that u is adjacent to $i, i+2 \in C$ and v is adjacent to j, $j+2 \in C$. If $j = i$, then by contracting the loop associated to the vertices $i, i+2$ we obtain a minor $H/\{i,\, i+2\}$ equivalent to $\mathrm{B}_{[5](1)}$. When $i \neq j$, we have two cases: if $j = i+1$ ($j = i+4$ resp.), then H is equivalent to $\mathrm{B}_{[8](5)}$. If $j = i+2$ ($j = i+3$ resp.), then by deleting the loop associated to the vertex $i+2$ (i resp.) we obtain a minor $H \setminus i+2$ ($H \setminus i$ resp.) equivalent to $\mathrm{B}_{[7](2)}$.

Case 2. Suppose u is adjacent to v in $\overline{\mathcal{I}}(H)$.

Case 2.1. Suppose that u is only adjacent to $i \in C$ and v is only adjacent to $j \in C$ for $i, j = 1, \ldots, 5$. When $j = i$, results in a contradiction to the minimality of C. When $i \neq j$, we have two cases: if $j = i+1$ ($j = i+4$ resp.), then H is equivalent to $\mathrm{B}_{[8](1)}$. If $j = i+2$ ($j = i+3$ resp.), then by deleting the vertex $i+1$ ($i+4$ resp.) we obtain a C_6, which by Remark 1, gives rise to a contradiction.

Case 2.2. Suppose that u is only adjacent to $i \in C$ and v is adjacent to j and $j+2$ for $i, j = 1, \ldots, 5$. If $j = i$ ($j = i+3$ resp.), then the minimality of C is contradicted. If

$i \neq j$ and $j \neq i+3$, then H is equivalent to $\mathrm{B}_{[8](5)}$.

Case 2.3. Suppose that u is adjacent to i and $i+2$ and v is adjacent to j and $j+2$ for $i, j = 1, \dots, 5$. If $j = i$, then the minimality of C is contradicted. When $i \neq j$, we have two cases: if $j = i+1$ ($j = i+4$ resp.), then by deleting the loop associated to the vertex $i+3$ ($i+2$ resp.) we obtain a minor $H \setminus i+3$ ($H \setminus i+2$ resp.) equivalent to $\mathrm{B}_{[7](1)}$. If $j = i+2$ ($j = i+3$ resp.), then the minimality of C is contradicted.

Thus we have verified all cases for $p = 5$. Hence, it follows that $B \in \mathcal{E}$ has a minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ $(5 \leqslant i \leqslant 9)$. □

Non-orientable Lemma

Lemma 19. *If $B \in \mathcal{E}$ is a non-orientable bouquet such that $B_O \neq \emptyset$, $B_N \neq \emptyset$, then B has a partial dual of Euler genus at most 2 if and only if B has no minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ $(1 \leqslant i \leqslant 4)$.*

Proof. By Lemma 7 any $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 4$ is an excluded minor for the property of having a partial dual of Euler genus less than or equal to 2.

Suppose that B does not have a partial dual of Euler genus at most 2. We prove that B has a minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 4$. Suppose $\mathcal{I}(B)$ is bipartite. Let P and Q be the two classes of the partition, such that $P = P_O \sqcup P_N$ and $Q = Q_O \sqcup Q_N$, where P_O and Q_O are the sets of positive vertices (vertices associated to orientable loops). Also, denote by P_N and Q_N the sets of negative vertices (vertices associated with the non-orientable loops). If $|P_N| \geqslant 3$ or $|Q_N| \geqslant 3$, then $\mathcal{I}(B)$ contains an independent set with at least 3 negative vertices. Hence, by Lemma 17 B has a minor equivalent to $\mathrm{B}_{[1](1)}$. Let H_P and H_Q the set of loops associated to the vertices of P and Q, respectively.

Since $\mathcal{I}(B)$ is bipartite, we have the following cases:

[1] If $|P_N| = |Q_N| = 0$, then $B_N = \emptyset$, which forms a contradiction to the assumption that $B_N \neq \emptyset$.

[2] If $|P_N| = 0$ and $|Q_N| = 1$, then by Corollary 6, B has a partial dual $B^{E(H_P)}$ of Euler genus $\gamma(B^{E(H_P)}) = \gamma(H_P) + \gamma(H_Q) = 0 + 1 = 1$, which forms a contradiction.

[3] If $|P_N| = 0$ and $|Q_N| = 2$, then by Corollary 6, B has a partial dual $B^{E(H_P)}$ of Euler genus $\gamma(B^{E(H_P)}) = \gamma(H_P) + \gamma(H_Q) = 0 + 2 = 2$, which forms a contradiction.

[4] Suppose that $|P_N| = |Q_N| = 1$. By Corollary 6, B has a partial dual $B^{E(H_P)}$ of Euler genus $\gamma(B^{E(H_P)}) = \gamma(H_P) + \gamma(H_Q) = 1 + 1 = 2$, which forms a contradiction.

[5] Consider $|P_N| = 1$ and $|Q_N| = 2$ (analogously $|P_N| = 2$ and $|Q_N| = 1$).

Case 5.1. If $\mathcal{I}(B)$ contains an independent set with at least 3 negative vertices, then by Lemma 17 B has a minor equivalent to $\mathrm{B}_{[1](1)}$.

Case 5.2. Suppose there exits a minimal chordless cycle C of size at most 6 in $\mathcal{I}(B)$ that contains the 3 negative vertices. We distinguish two cases $|C| = 4$ or $|C| = 6$. If $|C| = 4$, then the ribbon subgraph associated with C is equivalent to $\mathrm{B}_{[3](4)}$. If $|C| = 6$, by parity at least two positive adjacent vertices exist. Contracting the associated loops results in a smaller even cycle of size 4, whose ribbon subgraph is equivalent to $\mathrm{B}_{[3](4)}$.

Case 5.3. Suppose there exits a minimal chordless cycle C in $\mathcal{I}(B)$ with exactly two negative vertices. If the third negative vertex is not adjacent to any vertex of C, then contracting all the loops associated to the positive vertices of C, results in a minor equivalent to $\mathrm{B}_{[1](1)}$. If the third negative vertex is adjacent to any positive vertex of C, then contracting the loops associated to all positive vertices except one of C and deleting the loops associated to the positive vertices not in C, we obtain a ribbon subgraph equivalent to $\mathrm{B}_{[3](4)}$.

Case 5.4. If there exits a cycle C' in $\mathcal{I}(B)$ such that only one vertex is positive, then the subribbon graph induced by $H_{C'}$ is equivalent to $\mathrm{B}_{[3](4)}$.

For any other case different from the ones given above, there exists a subset of loops $A \subseteq E(B)$ such that $\gamma(B|_A) = \gamma(B|_{A^c}) = 1$ or $\gamma(B|_A) = 0$ and $\gamma(B|_{A^c}) = 2$. By Corollary 6, B has a partial dual B^A of Euler genus at most 2, which forms a contradiction.

[6] Finally, the case $|P_N| = 2$ and $|Q_N| = 2$ is done similar to the case (5) above.

Now, suppose that $\mathcal{I}(B)$ is not bipartite, therefore it contains an odd cycle C. Let p be the size of C and let $1, 2, \ldots, p$ be the vertices of C. We have the following cases:

Case 1. If every element of C is positive, then by Lemma 16, B has a minor equivalent to $\mathrm{B}_{[1](1)}$ or $\mathrm{B}_{[2](1)}$ or $\mathrm{B}_{[3](1)}$ or $\mathrm{B}_{[3](3)}$.

Case 2. Suppose every element of C is negative. Let H be the ribbon subgraph induced by the vertices of C. If $p \geqslant 7$, then H has a minor equivalent to $\mathrm{B}_{[1](1)}$, hence also B. If $p = 5$, then H is equivalent to $\mathrm{B}_{[4](2)}$, hence B has a minor equivalent to $\mathrm{B}_{[4](2)}$. If $p = 3$, then the vertices of C induce an odd cycle of size 3 in $\mathcal{I}(B)$, which forms a contradiction.

Case 3. Suppose C contains both positive and negative vertices. Let $k \in \mathbb{N}$ and the sets $A_1, A_2, \ldots, A_k$ be disjoint paths of negative vertices in C. If $k \geqslant 3$, then there exists an independent set of at least 3 negative vertices, by Lemma 17 B has a ribbon subgraph equivalent to $\mathrm{B}_{[1](1)}$. Therefore, $k \leqslant 2$. If $|A_k| \geqslant 5$ for $k = 1, 2$, then there is an independent set consisting of 3 negative vertices, that by Lemma 17 implies that B has a minor equivalent to $\mathrm{B}_{[1](1)}$. Therefore, $|A_k| \leqslant 4$ for $k = 1, 2$.

From the fourteen possible cases, seven contain an independent set with at least three negative vertices in C. In all these cases, B has a minor equivalent to $\mathrm{B}_{[1](1)}$. For the remaining seven cases, we consider $D = \{v \mid v \in \mathcal{I}(B) \setminus C\}$, such that $D = D_O \sqcup D_N$ with D_O and D_N the subsets of vertices associated to $\mathcal{I}(B_O) \setminus C$ and $\mathcal{I}(B_N) \setminus C$, respectively.

Case 3.1. First, assume that $|A_1| \neq 0$ and $|A_2| = 0$. We have the following cases:

Case 3.1.1. Suppose $|A_1| = 1$. Let $1 \in A_1$ and $2, 3, \ldots, p \in C \cap \mathcal{I}(B_O)$. Observe that if $p \geqslant 7$, then there exists a subset of three independent vertices in $\mathcal{I}(B_O) \cap C$, that induce in $\overline{\mathcal{I}}(B_O)$ a cycle of size three, which forms a contradiction. Hence, $p \leqslant 5$. For $p = 5$, observe that for each $v \in D$, v is adjacent to at most two vertices of C. Any other case gives rise to a contradiction on the minimality of C. By Lemma 16, if $\mathcal{I}(B_O)$ is not bipartite, then B has an excluded minor. Thus, $\mathcal{I}(B_O)$ is bipartite. Hence, the maximum number of vertices in $\mathcal{I}(B_O)$ is 4. Thus, D_O must be empty. We show that D_N has at most one vertex. Let $t \in D_N$, observe that if t is not adjacent to any vertex of C, then the ribbon subgraph H induced by $C \cup t$ has a minor $H/\{1, 2, 3\}$ equivalent to $\mathrm{B}_{[1](1)}$. Suppose t is adjacent to one vertex of C, say to the vertex 1, then H has a minor $H/\{2, 3\}$ equivalent to $\mathrm{B}_{[3](5)}$. If t is adjacent to the vertex 3 or 4, then H has a minor $H/\{2, 3\}$ or $H/\{4, 5\}$ in each case equivalent to $\mathrm{B}_{[3](5)}$. Thus, t is only adjacent to the vertex 2 or 5. Now, suppose that t is adjacent to two vertices of C, say $\{1, 3\}$ or $\{1, 4\}$, then H has a minor $H/\{1, 2, 3\}$ or $H/\{1, 4, 5\}$ equivalent to $\mathrm{B}_{[1](1)}$. If t is adjacent to the vertices $\{2, 4\}$ or $\{3, 5\}$, then H has a minor $H \setminus 3/t$ or $H \setminus 4/t$ both equivalent to $\mathrm{B}_{[3](4)}$. Hence, every element of D_N is adjacent to the vertices 2 and 5 or exactly one of them. Let $t_1, t_2 \in D_N$. As seen above, t_1 and t_2 must be adjacent to the vertices 2 and 5 or each exactly to one of them. If t_1 is not adjacent to t_2, then the set of vertices $\{1, t_1, t_2\}$ induce an independent set consisting of 3 negative vertices, that by Lemma 17 implies B has a minor equivalent to $\mathrm{B}_{[1](1)}$. If t_1 is adjacent to t_2, then t_1 and t_2 cannot be adjacent to the same vertices in C, since this would contradict the minimality of C. Hence, the ribbon subgraph J induced by the vertices $C \cup \{t_1, t_2\}$ has a minor $J/\{3, 4\}$ equivalent to $\mathrm{B}_{[4](1)}$. Therefore, D_N has at most one vertex. This vertex is adjacent to the vertices 2 and 5 or to precisely one of them. In both cases, there exists a subset of vertices $\{1, 2, 4, t_1\}$ that induces a ribbon subgraph of Euler genus 2 and the subset of vertices $\{3, 5\}$ induce a ribbon subgraph of Euler genus 0. By Lemma 6, it follows that B has a partial dual of Euler genus at most two, which forms a contradiction. Now, suppose that $p = 3$. If there exists $t \in D_N$ such that it is not adjacent to any vertex of C, then the ribbon subgraph H induced by $C \cup t$ has a minor $H/\{1\}$ equivalent to $\mathrm{B}_{[1](1)}$. If there exists $t \in D_N$ such that it is only adjacent to vertex $1 \in C$, then the ribbon subgraph H induced by $C \cup t$ is equivalent to $\mathrm{B}_{[3](5)}$. Let $t_1, t_2 \in D_N$. Assume that t_i is adjacent to only one or both positive vertices of C, for $i = 1, 2$. If t_1 is not adjacent to t_2, then B has a minor $B|_{\{1, t_1, t_2\}}$ equivalent to $\mathrm{B}_{[1](1)}$. If t_1 is adjacent to t_2, then we have two cases: without loss of generality, suppose that t_1 is only adjacent to 2 and t_2 is only adjacent to 3, then the ribbon subgraph H induced by $C \cup \{t_1, t_2\}$ is equivalent to $\mathrm{B}_{[3](5)}$. Now, assume that t_i is adjacent to both positive vertices of C. We can see that B has a partial dual of Euler genus at most two, which forms a contradiction. Similarly for the case that one of t_i is adjacent to one vertex of C and other t_i is adjacent to two vertices of C. In the case $|D_N| \geqslant 3$ and any $v \in D_N$ is not adjacent to $1 \in C$, we can see that B has a minor equivalent to $\mathrm{B}_{[1](1)}$ or $\mathrm{B}_{[3](4)}$ or $\mathrm{B}_{[3](5)}$ or B has a partial dual of Euler genus at most two, which forms a contradiction. Suppose that t_i is adjacent to one positive vertex and one negative vertex of C, for $i = 1, 2$. For the case where t_1 is not adjacent to t_2, we have two cases: if t_1 and t_2 are both adjacent to the same positive vertex of C, then the ribbon subgraph H induced by $C \cup \{t_1, t_2\}$ has a minor $H \setminus \{3\}$ equivalent to $\mathrm{B}_{[3](2)}$. If t_1 and t_2 are not adjacent to the same positive vertex of C, then the ribbon subgraph H induced by $C \cup \{t_1, t_2\}$ is equivalent to $\mathrm{B}_{[4](3)}$. If t_1 is adjacent to t_2, then $\mathcal{I}(B)$ has a C_3 in D_N, which forms a contradiction.

Suppose that t_i is adjacent to three vertices of C, for $i = 1, 2$. For the case where t_1 is not adjacent to t_2, the ribbon subgraph H induced by $C \cup \{t_1, t_2\}$ has a minor $H \setminus \{3\}$ equivalent to $\mathrm{B}_{[3](2)}$. If t_1 is adjacent to t_2, then $\mathcal{I}(B)$ has a C_3 in D_N, which forms a contradiction.

All remaining cases are treated in a similar manner.

Case 3.1.2. Assume $|A_1| = 2$. Suppose that $1, 2 \in A_1$ and $3, 4, \ldots, p \in C \cap \mathcal{I}(B_O)$. Observe that if $p \geqslant 7$, then there exists a subset of three independent vertices in $\mathcal{I}(B_O) \cap C$, that induce in $\overline{\mathcal{I}}(B_O)$ a cycle of size three, which forms a contradiction. Hence, $p \leqslant 5$. For

$p = 5$, observe that for each $v \in D$, v is adjacent to at most two vertices of C. If not we obtain a contradiction to the minimality of C. If $s_i \in D_O$ such that s_i is not adjacent to a vertex of C, then the vertices $\{3, 5, s_i\}$ form an independent set in $\mathcal{I}(B_O)$, which forms a contradiction. Hence, s_i is adjacent to at least one vertex or at most two vertices of C. Therefore, s_i is adjacent to the vertices 3 and 5 or to exactly one of them. In any other case, we obtain a contradiction to the fact that $B \in \mathcal{E}$. By Lemma 16, we have that $\mathcal{I}(B_O)$ is bipartite. Hence, $|D_O| \leqslant 1$. Let $t_1 \in D_N$. If t_1 is adjacent to both vertices $\{1, 3\}$, then the ribbon subgraph H induced by $C \cup \{t_1\}$ has a minor $H \setminus \{4, 5\}$ equivalent to $\mathrm{B}_{[3](4)}$. Otherwise, B has a partial dual $B^{\{3,5\}}$ or $B^{\{3,5,t_1\}}$ of Euler genus 2. Let t_1, $t_2 \in D_N$ and H the ribbon subgraph induced by $C \cup \{t_1, t_2\}$. First suppose that t_1 is not adjacent to t_2, then H has a minor $H|_{\{1,t_1,t_2\}}$ or $H|_{\{2,t_1,t_2\}}$ or $H/\{4, 5, t_1\}$ or $H/\{3, 4\} \setminus \{1, 5\}$ or $H/\{3, 4\} \setminus \{1\}$ or $H \setminus \{4, 5, t_2\}$, equivalent to $\mathrm{B}_{[1](1)}$ or $\mathrm{B}_{[3](2)}$ or $\mathrm{B}_{[3](4)}$ or $\mathrm{B}_{[3](5)}$. Hence, t_1 is adjacent to t_2. In this case H has a minor $H/\{4, 5\} \setminus \{2\}$ or $H \setminus \{2, 3, 4\}$ or $H \setminus \{1, 4, 5\}$ or $H \setminus \{4, 5, t_2\}$ or $H \setminus \{3, 4, t_1\}$ or $H/\{1, 5\} \setminus \{2\}$ or $H/\{2\} \setminus \{1, 5\}$ equivalent to $\mathrm{B}_{[3](4)}$. In any other case, H has a partial dual $H^{\{3,5\}}$ or $H^{\{3,5,t_1\}}$ of Euler genus at most 2. Hence, any two vertices taken in D_N must be adjacent. By hypothesis, H has no cycles of size 3, so D_N has at most 2 elements. Since, $|D_O| \leqslant 1$, B has a partial dual obtained by taking $B^{\{3,5\}}$ or $B^{\{3,5,t_1,t_2\}}$ of Euler genus at most 2. Otherwise, B has a minor obtained by taking $B/\{5, t_1\} \setminus \{1, 2\}$ or $B/\{2\} \setminus \{1, 4, 5\}$, equivalent to $\mathrm{B}_{[3](5)}$ or $\mathrm{B}_{[3](4)}$. This concludes the case $p = 5$. For $p = 3$ the proof is similar to the Case 3.1.1.

Case 3.1.3. Assume $|A_1| = 3$. Suppose that 1, 2, $3 \in A_1$ and 4, $5, \ldots, p \in C \cap \mathcal{I}(B_O)$. Note that for $p \geqslant 9$, there always exists an independent set of 3 vertices in $\mathcal{I}(B_O) \cap C$, such that, considered in $\overline{\mathcal{I}}(B_O)$ induces a cycle of size three, which forms a contradiction. On the other hand, we can see that $|p|$ is not three, thus $5 \leqslant p \leqslant 7$. If $D = \emptyset$, it easily follows that B has a partial dual $B^{\{3,5\}}$ or $B^{\{3,5,7\}}$ of Euler genus at most two, which forms a contradiction. Now, observe that for all $v \in D$, v is adjacent to at most two vertices of C, any other case gives rise to a contradiction on the minimality of C. Now, for both cases $p = 5$, 7, let $t_1 \in D_N$. If t_1 is not adjacent to the vertices $1, 3 \in C$, then $\{1, 3, t_1\}$ is an independent set in $\mathcal{I}(B_N)$. It follows that the ribbon subgraph induced by these vertices is equivalent to $\mathrm{B}_{[1](1)}$. Hence, t_1 is adjacent either to 1 or to 3 or to both these vertices in C. Let H be the ribbon subgraph induced by $C \cup \{t_1\}$. If t_1 is adjacent to the vertices $v_1 \in \mathcal{I}(B_O) \cap C$ and $v_2 \in \mathcal{I}(B_N) \cap C$, then H has a minor $H/\{t_1\} \setminus \{2\}$ or $H/\{t_1, 4, 5\} \setminus \{2\}$ or $H/\{t_1, 6, 7\} \setminus \{2\}$ equivalent to $\mathrm{B}_{[3](5)}$. We show than D_N has at most one vertex. Let t_1, $t_2 \in D_N$, we have that if t_1 is not adjacent to t_2, then the ribbon subgraph induced by the vertices $\{2, t_1, t_2\}$ is equivalent to $\mathrm{B}_{[1](1)}$. Now, if t_1 is adjacent to t_2, then we have two cases: If both t_1 and t_2 are adjacent to the same vertex i for $i = 1$, 3, then we obtain a contradiction to the minimality of C. If t_1 is adjacent to i and t_2 is adjacent to j for $i \neq j$ and $i, j = 1, 3$, then the ribbon subgraph induced by the vertices $\{t_1, t_2, i, j, 2\}$ is equivalent to $\mathrm{B}_{[4](2)}$. Thus, $|D_N| \leqslant 1$. Now, if $p = 7$, then by Lemma 16, we have that $\mathcal{I}(B_O)$ is bipartite and the bipartite classes have at most two elements, thus $|D_O| = 0$. Given that $|D_N| = 1$, then B has a partial dual $B^{\{2,3,5,7\}}$ or $B^{\{1,2,4,6\}}$ of Euler genus at most two, which forms a contradiction. Now, suppose that $p = 5$. If $|D| = 0$, then B has a partial dual $B^{\{2,3,5\}}$ of Euler genus at most two, which forms a contradiction. Let

$s_1 \in D_O$. If s_1 is adjacent to the vertices $1, 3 \in C$, then the ribbon subgraph induced by the vertices $\{1,\, 2,\, 3,\, s_1\}$ results in a minor equivalent to $\mathrm{B}_{[3](4)}$. If s_1 is adjacent to the vertices $1, 5 \in C$, then the ribbon subgraph induced by the vertices $\{1,\, 2,\, 5,\, s_1\}$ results in a minor equivalent to $\mathrm{B}_{[3](5)}$. By Lemma 16, implies that if $\mathcal{I}(B_O)$ is not bipartite, then B has a minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ $(1 \leqslant i \leqslant 3)$. Thus $\mathcal{I}(B_O)$ is bipartite. Hence $|D_O| \leqslant 2$. Now, let $s_1,\, s_2 \in D_O$ and $t_1 \in D_N$. In this case, one can verify that B always has a partial dual of Euler genus at most two or an excluded minor.

Case 3.1.4. Assume $|A_1| = 4$ and suppose that, $1,\, 2,\, 3,\, 4 \in A_1$. Let H be the ribbon subgraph induced by the vertices of C. By contracting the loops associated to the vertices in $C \setminus \{1,\, 2,\, 3, p\}$ we obtain a minor equivalent to $\mathrm{B}_{[3](4)}$.

Case 3.2. Assume $|A_1| = |A_2| = 1$. Let $e_1 \in A_1$, $e_2 \in A_2$ and $e_1,\, e_2,\, 1,\, 2,\, \dots,\, p-2 \in C$. Without loss of generality, suppose that e_1 is adjacent to both 1 and $p-2$. Let H be the ribbon subgraph induced by C. Now, $H/\{C\backslash\{e_1, 1, 2, p-2\}\}$ or $H/\{C\backslash\{e_1, 1, p-2,\, p-3\}\}$ is equivalent to $\mathrm{B}_{[3](4)}$.

Case 3.3. Assume $|A_1| = 1$ and $|A_2| = 2$. Suppose that, $1 \in A_1$, $j,\, j+1 \in A_2$ for $j = 3,\, \dots,\, p-2$. Let $L_1,\, L_2$ be disjoint paths of positive vertices in C, such that $L_i \neq \emptyset$ for $i = 1,\, 2$. If $|L_i| \geqslant 3$ for any $i = 1,\, 2$, then there exists an independent set of positive vertices in C of size 3 such that this set induces a cycle of size 3 in $\overline{\mathcal{I}}(B)$, which forms a contradiction. Thus, $1 \leqslant |L_i| \leqslant 2$ for $i = 1,\, 2$. Therefore, the size p of the cycle C is in the range $5 \leqslant p \leqslant 7$. Now, every vertex $v \in D$ can be adjacent to at most two vertices of C, otherwise, the minimality of C is contradicted. Suppose $t \in D_N$, we can see that: For $p = 5$, if t is adjacent to both the vertices 1 and j or 1 and $j+1$ in C, then the ribbon subgraph H associated with $C \cup \{t\}$ has a minor equivalent to $\mathrm{B}_{[3](4)}$, obtained by taking $H \setminus L_2 \cup j+1$ (similarly for $j+1$). For $p = 5$ or $p = 7$, if t is not adjacent to both the vertices 1 and j or 1 and $j+1$ in C, then there exists an independent set with at least three negative vertices in $C \cup \{t\}$, and the associated ribbon subgraphs $B|_{\{1,j,t_1\}}$ or $B|_{\{1,j+1,t_1\}}$ induce a minor equivalent to $\mathrm{B}_{[1](1)}$. Hence, t is adjacent to only one of the vertices 1, or j, or $j+1$. Now, we show than D_N has at most one vertex. Let $t_1, t_2 \in D_N$. For any $t \in D_N$, t is adjacent to only one vertex 1, or j, or $j+1$. Suppose that t_1 is not adjacent to t_2. Then the ribbon subgraph induced by the vertices $\{j,\, t_1,\, t_2\}$ or $\{j+1,\, t_1,\, t_2\}$ or $\{1,\, j,\, t_2\}$ or $\{1,\, j+1,\, t_2\}$ is equivalent to $\mathrm{B}_{[1](1)}$. If t_1 is adjacent to both $1,\, j$ and t_2 is adjacent to both $1,\, j+1$, then the ribbon subgraph induced by the vertices $1,\, j,\, j+1,\, t_1,\, t_2$ is equivalent to $\mathrm{B}_{[4](2)}$. Suppose that t_1 is adjacent to t_2. For $p = 5$, the ribbon subgraph H associated to $C \cup t_1, t_2$ has a minor $H|_{\{1,j+1,t_2\}}$ or $H|_{\{1,j,t_1\}}$ equivalent to $\mathrm{B}_{[1](1)}$ or there is a cycle of size smaller than C, which forms a contradiction. For $p = 7$, H has a minor $H \setminus L_2/\{L_1, j+1\}$ or $H \setminus L_1/\{L_2, j\}$ equivalent to $\mathrm{B}_{[3](4)}$ or there is a cycle of size smaller than C, which forms a contradiction. Therefore, $0 \leqslant |D_N| \leqslant 1$.

On the other hand, suppose $s \in D_O$. For $p = 5$, we have: *a*) If s is not adjacent to any vertex of $\mathcal{I}(B_O) \cap C$, then there exists an independent set of three positive vertices in $\mathcal{I}(B)$ such that they induce a cycle of size 3 in $\overline{\mathcal{I}}(B)$, which forms a contradiction. Hence, every vertex $s \in D_O$ is adjacent to at least one positive vertex of C. *b*) If s is adjacent to v_1 and v_2 such that $v_1 \in \mathcal{I}(B_O) \cap C$ and $v_2 \in \mathcal{I}(B_N) \cap C$, then the ribbon subgraph J associated to $C \cup s$ has a minor $J \setminus 5/1$ or $J \setminus 2/1$ equivalent to $\mathrm{B}_{[3](4)}$. By Lemma 16, we have that $\mathcal{I}(B_O)$ is bipartite and the color classes have at most two elements, thus $|D_O| \leqslant 2$. If $|D_O| = 0$, then B has a partial dual of Euler genus at least 2, which forms a contradiction. If $|D_O| = 1$, then s is adjacent to $v_1 \in \mathcal{I}(B_O) \cap C$ or s is adjacent to $v_1, v_3 \in \mathcal{I}(B_O) \cap C$. Since, $0 \leqslant |D_N| \leqslant 1$, then B has an excluded minor or B has a partial dual of Euler genus at most 2, which forms a contradiction. Since $0 \leqslant |D_N| \leqslant 1$, if $|D_O| = 2$, then B has an excluded minor or there exists an independent set of positive vertices in $\mathcal{I}(B)$ such that these vertices induce a cycle of size 3 in $\overline{\mathcal{I}}(B)$ or B has a partial dual of Euler genus at most 2, which forms a contradiction. For $p = 7$, by Lemma 16, we have that $D_O = \emptyset$, since there exists an independent set of size at least three in $\mathcal{I}(B)$. Since D_N has at most one element, we have that B has a partial dual of Euler genus at most 2, which forms a contradiction.

Case 3.4. Suppose $|A_1| = 2$ and $|A_2| = 2$. Let $i \in C$ denote the element in position i of C. Suppose that e_1, $e_2 \in A_2$ are in positions j and $j+1$ of C. Let H be the ribbon subgraph induced by C. Then, $H/\{C \setminus \{j-2, j-1, j, j+1\}\}$ (where $j-1$ and $j-2$ may belong to A_1) or $H/\{C \setminus \{j, j+1, j+2, j+3\}\}$ (where $j+2$ and $j+3$ may belong to A_1) are equivalent to $\mathrm{B}_{[3](4)}$. □

Main Theorem

We begin proving:

Lemma 20. *If $B \in \mathcal{E}$, then B has a partial dual of Euler genus at most 2 if and only if B has no minor equivalent to $\mathrm{B}_{[i](j)}$ in $\mathcal{B}_{[i]}$ where $1 \leqslant i \leqslant 9$.*

Proof. By Lemma 7 any bouquet in $\mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 9$ is an excluded minor for the property of having a partial dual of Euler genus less than or equal to 2. For the converse, suppose that B has no partial dual of Euler genus less than or equal to 2, then we need to show that B has a minor equivalent to one of the bouquets $\mathrm{B}_{[i](j)}$ for $1 \leqslant i \leqslant 9$. Recall that we consider any bouquet B expressed as $B = B_O \sqcup B_N$, where B_O is the set of its orientable loops and B_N the set of its non-orientable loops.

I. Orientable case. Suppose that $B_O \neq \emptyset$ and $B_N = \emptyset$, if B does not have a partial dual of Euler genus at most two, then by Lemma 18, B has a minor isomorphic to an element $\mathrm{B}_{[i](j)}$ in $\mathcal{B}_{[i]}$ where $i = 5, \dots, 9$.

II. Non-orientable case. We have two cases:

II(a) Suppose that $B_O = \emptyset$ and $B_N \neq \emptyset$, if B does not have a partial dual of Euler genus at most two, then by Theorem 15, B has a minor equivalent to $\mathrm{B}_{[i](j)} \in \mathcal{B}_{[i]}$ for $i = 1, 4$.

II(b) Suppose that both B_O and B_N are nonempty. By Lemma 19, B has a minor equivalent to an element $\mathcal{B}_{[i]}$ for $1 \leqslant i \leqslant 4$. This completes the proof. □

Now, we prove the main theorem.

Theorem 21. *(Main Theorem) Let G be a ribbon graph such that for all quasi-trees T of G we have that $G^T \in \mathcal{E}$, then G has a partial dual of Euler genus at most 2 if and only if G has no minor equivalent to a ribbon graph in $\mathcal{G}_{[i]}$ where $1 \leqslant i \leqslant 9$.*

Proof. By Lemma 7, any ribbon graph in $\mathcal{G}_{[i]}$ for $1 \leqslant i \leqslant 9$ is an excluded minor for the property of having a partial dual of Euler genus less than or equal to 2. For the converse, let T be a quasi-tree of G, a spanning ribbon subgraph with one face, then G^T is a bouquet. By hypothesis, $G^T \in \mathcal{E}$. If G does not have a partial dual of Euler genus at most 2, then by Lemma 20, G^T has a minor in $\mathcal{B}_{[i]}$ where $i = 1, \ldots, 9$. By Lemma 2, $G = (G^T)^T$ has B^T as a minor, but $B^T \in \mathcal{G}_{[i]}$ for $1 \leqslant i \leqslant 9$. This completes the proof. □

5 Future work

A natural compatibility exists between ribbon graphs (graphs embedded in a surface) and determined delta-matroids, see [4, 5, 18]. Ribbon graphs are related to delta-matroids, in a manner analogous to the relationship between graphs and matroids. A delta-matroid is related with a ribbon graph by associating generating quasi-trees of the ribbon graph with feasible sets of the delta-matroid. The width of a delta-matroid is the difference between the largest and smallest cardinalities of its feasible sets, and can be viewed as the analogue of the Euler genus of a ribbon graph. The twisting operation in delta-matroids is the analogue of partial duality. Thus, characterising ribbon graphs having a partial dual of Euler genus less than or equal to two is the analogue of characterising certain delta-matroids with a twist of width at most two. In this setting we have obtained a list of excluded minors characterising these delta-matroids.

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Appendix

Table 1: Non-orientable excluded minors.

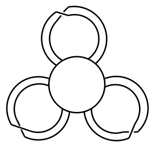
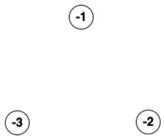
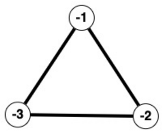
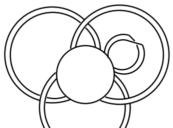
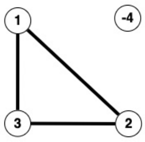
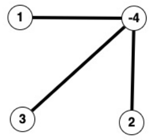
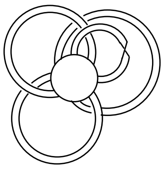
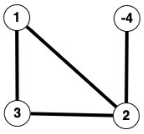
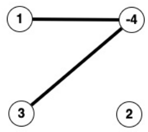
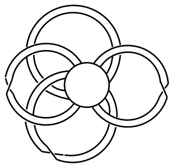
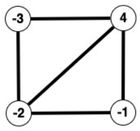
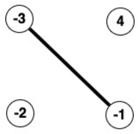
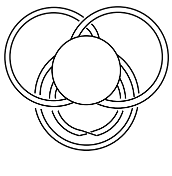
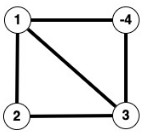
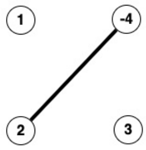
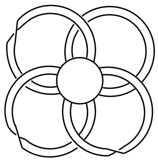
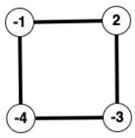
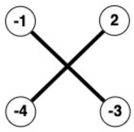
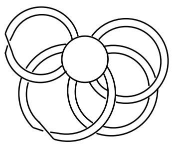
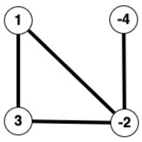
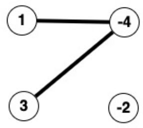
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[1](1)}$				3
$B_{[2](1)}$				3
$B_{[3](1)}$				4
$B_{[3](2)}$				4
$B_{[3](3)}$				3
$B_{[3](4)}$				3
$B_{[3](5)}$				3

Table 2: Non-orientable and orientable excluded minors.

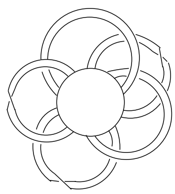
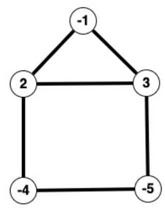
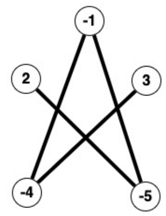
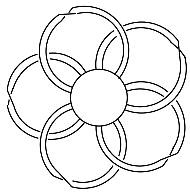
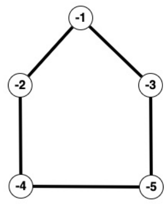
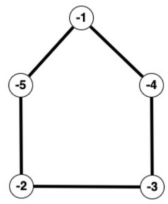
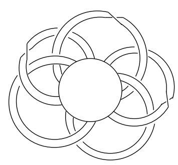
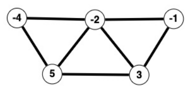
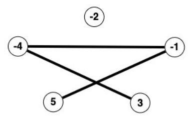
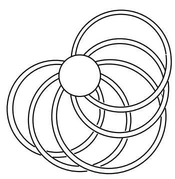
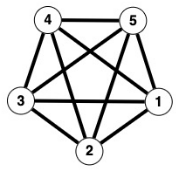
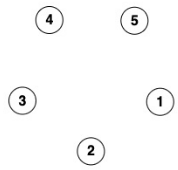
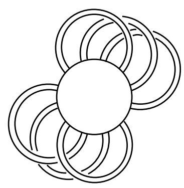
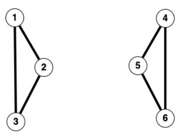
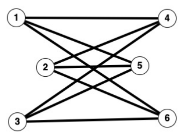
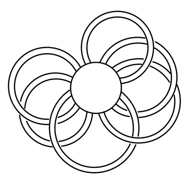
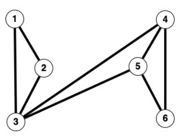
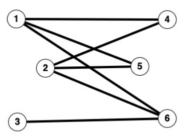
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[4](1)}$				5
$B_{[4](2)}$				5
$B_{[4](3)}$				4
$B_{[5](1)}$				4
$B_{[6](1)}$				4
$B_{[7](1)}$				4

Table 3: Orientable excluded minors.

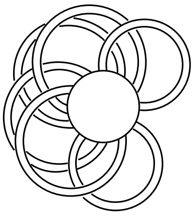
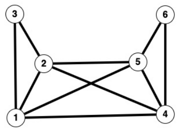
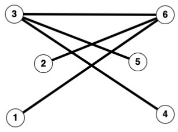
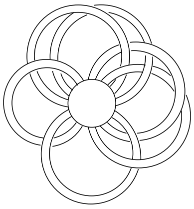
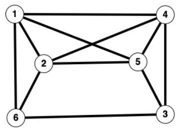
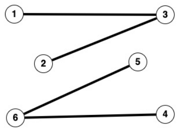
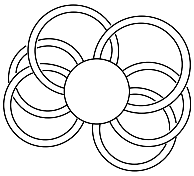
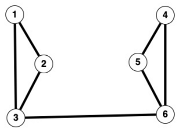
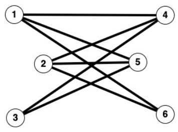
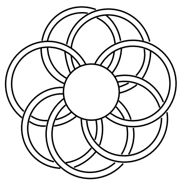
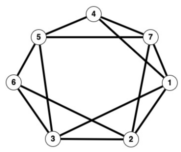
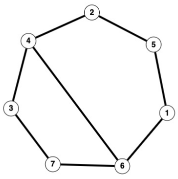
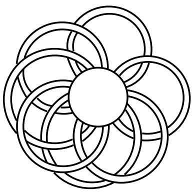
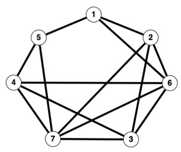
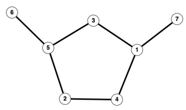
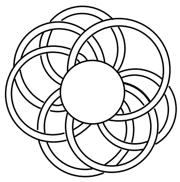
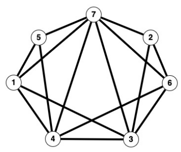
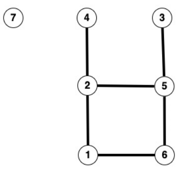
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[7](2)}$				4
$B_{[7](3)}$				6
$B_{[7](4)}$				6
$B_{[8](1)}$				6
$B_{[8](2)}$				6
$B_{[8](3)}$				6

Table 4: Orientable excluded minors.

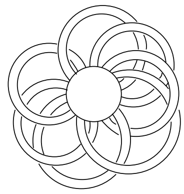
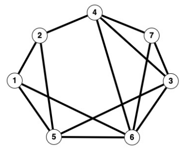
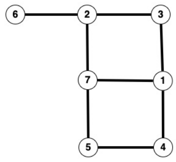
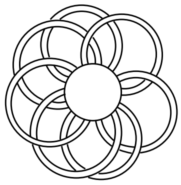
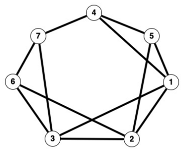
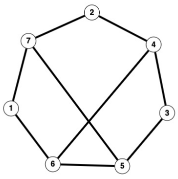
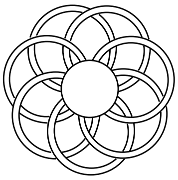
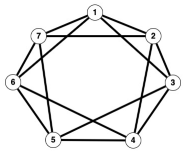
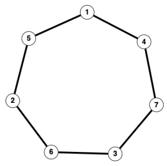
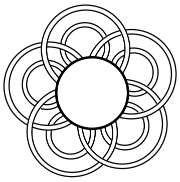
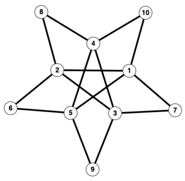
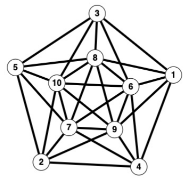
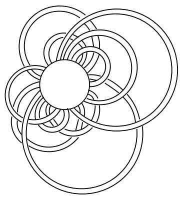
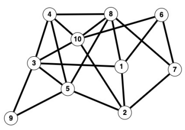
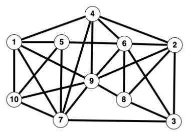
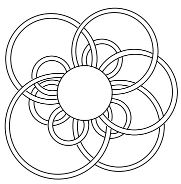
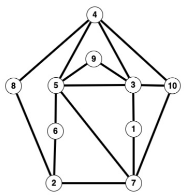
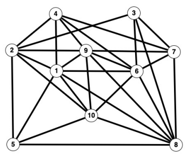
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[8](4)}$				6
$B_{[8](5)}$				6
$B_{[8](6)}$				6
$B_{[9](1)}$				8
$B_{[9](2)}$				8
$B_{[9](3)}$				10

Table 5: Orientable excluded minors.

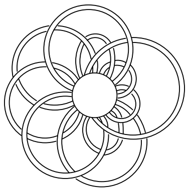
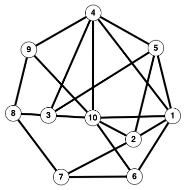
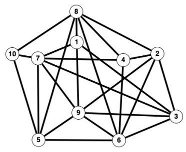
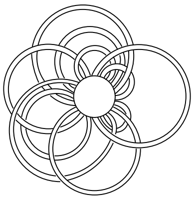
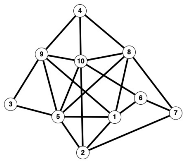
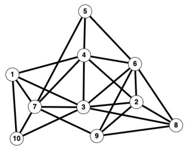
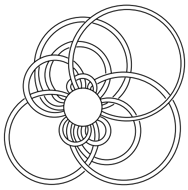
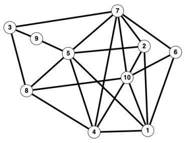
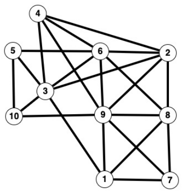
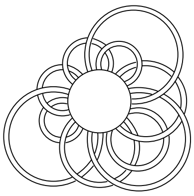
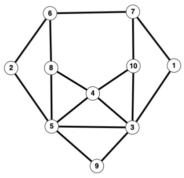
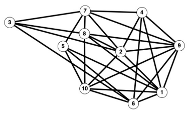
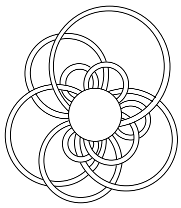
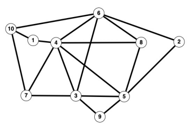
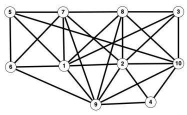
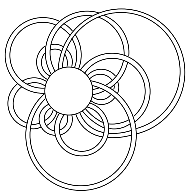
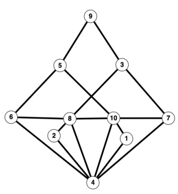
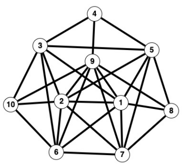
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[9](4)}$				6
$B_{[9](5)}$				8
$B_{[9](6)}$				8
$B_{[9](7)}$				8
$B_{[9](8)}$				10
$B_{[9](9)}$				8

Table 6: Orientable excluded minors.

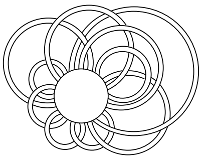
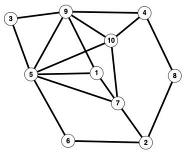
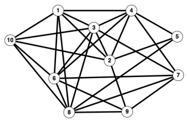
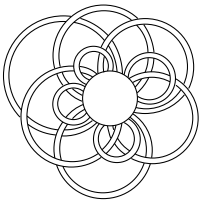
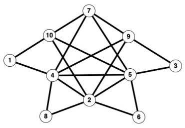
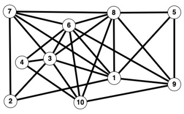
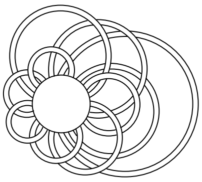
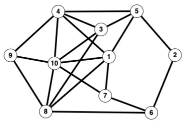
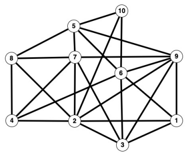
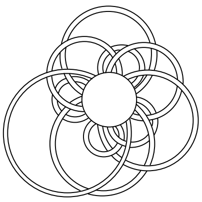
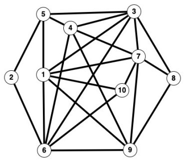
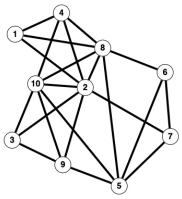
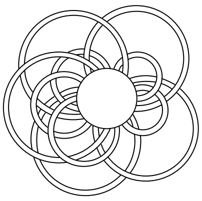
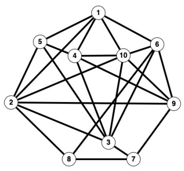
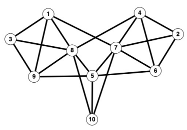
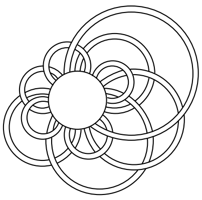
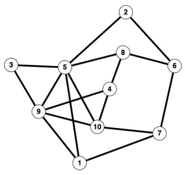
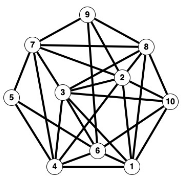
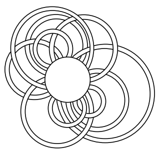
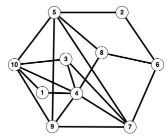
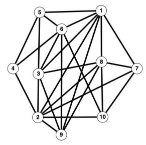
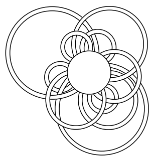
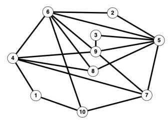
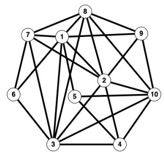
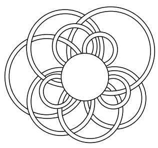
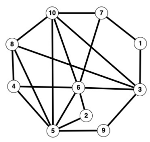
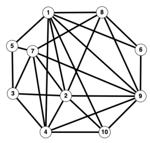
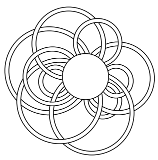
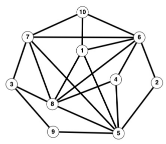
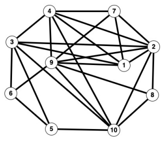
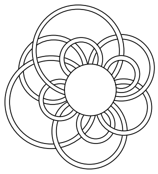
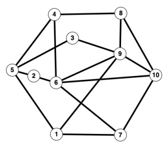
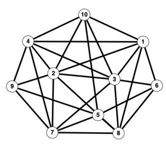
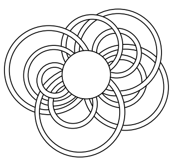
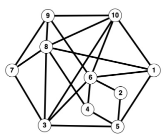
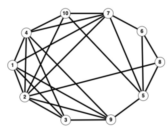
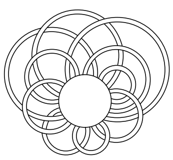
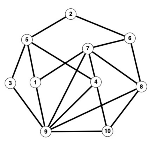
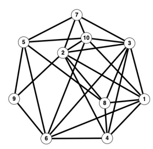
Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[9](10)}$				10
$B_{[9](11)}$				8
$B_{[9](12)}$				8
$B_{[9](13)}$				8
$B_{[9](14)}$				8
$B_{[9](15)}$				8

Table 7: Orientable excluded minors.

Graph	B	$\mathcal{I}(B)$	$\overline{\mathcal{I}}(B)$	$\gamma(B)$
$B_{[9](16)}$				10
$B_{[9](17)}$				8
$B_{[9](18)}$				8
$B_{[9](19)}$				10
$B_{[9](20)}$				8
$B_{[9](21)}$				10
$B_{[9](22)}$				8