A note on the minimum size of Turán systems

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Abstract

For positive integers $n \geqslant s > r$, a *Turán* (n,s,r)*-system* is an n-vertex r-graph in which every set of s vertices contains at least one edge. Let $T(n,s,r)$ denote the the minimum size of a Turán (n,s,r)-system.

Upper bounds on $T(n,s,r)$ were established by Sidorenko [20] for the case $s-r = \Omega(r/\ln r)$, based on a construction of Frankl–Rödl [7], and by a number of authors in the case $s-r = O(1)$. Motivated by these results and recent work [16] of the second author, we investigate in this note the intermediate regime where $s = s(r)$ satisfies both $s-r = \Omega(1)$ and $s-r = O(r/\ln r)$, and establish upper bounds for $T(n,s,r)$ in this range as $r \to \infty$.

Mathematics Subject Classifications: 05D05

1 Introduction

Given an integer $r \geqslant 2$, an ***r*-uniform hypergraph** (henceforth an ***r*-graph**) $\mathcal{H}$ is a collection of r-subsets of some set V. We call V the **vertex set** of $\mathcal{H}$ and denote it by $V(\mathcal{H})$. When V is understood, we usually identify a hypergraph $\mathcal{H}$ with its set of edges.

For positive integers $n \geqslant s > r$, a **Turán (n,s,r)-system** is an r-graph $\mathcal{H}$ on an n-set V such that every s-subset $S \subseteq V$ contains at least one edge from $\mathcal{H}$. Denote by $T(n,s,r)$ the smallest **size** (i.e. the number of edges) of a Turán (n,s,r)-system. Observe that $T(n,s,r) = \binom{n}{r} - \mathrm{ex}(n, K_s^r)$, where $\mathrm{ex}(n, K_s^r)$ denotes the Turán number of the complete r-graph on s vertices K_s^r. A simple averaging argument shows that $\mathrm{ex}(n, K_s^r)/\binom{n}{r}$ is non-increasing (see e.g. [10]), and hence the following limit exists:

$$t(s,r) \coloneqq \lim_{n\to\infty} \frac{T(n,s,r)}{\binom{n}{r}}. \qquad (1)$$

Determining the value of $t(s,r)$ is a central topic in Extremal Combinatorics. The seminal paper of Turán [22] established that $t(s,2) = \frac{1}{s-1}$ for all $s \geqslant 3$ (with the case $s = 3$

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solved earlier by Mantel [14]). Erdős [6] offered \$500 for the determination of $t(s,r)$ for a single pair s,t with $s > r \geqslant 3$. This prize is still unclaimed, despite decades of active attempts. Turán and other researchers conjectured that $t(s,3) = \frac{4}{(s-1)^2}$ for $s \geqslant 4$. Various constructions achieving this bound are known (see e.g. [19]). For $r \geqslant 4$, there is no general conjectured value for $t(s,r)$, except for the case $(s,r) = (5,4)$ (see [9, 15]). For further related results, we refer the reader to surveys such as [8, 5, 19, 11].

In this note, we focus on the case where $r \to \infty$, and all asymptotics are taken with respect to r unless otherwise specified. The trivial lower bound is $t(s,r) \geqslant 1/\binom{s}{r}$, which follows from the monotonicity of the ratio in (1). For convenience, let us define

$$\mu(s,r) \coloneqq t(s,r) \cdot \binom{s}{r}.$$

Note that $\mu(s,r)$ is always at least 1.

The best-known general lower bound, $t(s,r) \geqslant 1/\binom{s-1}{r-1}$ (i.e. $\mu(s,r) \geqslant s/r$) is due to de Caen [3]. In particular, $t(r+1,r) \geqslant 1/r$, a result that was independently proved by de Caen [4], Sidorenko [18], and Tazawa–Shirakura [21]. Further improvements on $t(r+1,r)$ in lower order terms were made by Giraud (unpublished, see [5, Page 189]), Chung–Lu [2], and Lu–Zhao [13].

Improving the previous upper bounds established by Sidorenko [17], Kim–Roush [12], Frankl–Rödl [7], and Sidorenko [20], the second author established the following upper bound, which disproved the \$500 conjecture of de Caen [5] that $r \cdot t(r+1,r) \to \infty$.

Theorem 1 ([16])**.** *For every integer $R \geqslant 1$, it holds that*

$$\mu(r+R,r) \leqslant \alpha + o(1), \quad \text{as } r \to \infty,$$

where $\alpha \coloneqq (c_0+1)^{R+1}/c_0^R$ with $c_0 = c_0(R)$ being the largest real root of the equation $\mathrm{e}^x = (x+1)^{R+1}$. In particular, $\mu(r+1,r) \leqslant 4.911$ for all sufficiently large r.

An immediate corollary of Theorem 1 (for derivation see [16, Corollary 1.3]) is that, for every sufficiently large R,

$$\limsup_{r\to\infty} \mu(r+R,r) \leqslant R\ln R + 3R\ln\ln R = (1+o(1))R\ln R. \tag{2}$$

This improves asymptotically the previous bound by Frankl–Rödl [7] which states that, for any fixed $R \geqslant 1$, we have

$$\mu(r+R,r) \leqslant (1+o(1))R(R+4)\ln r, \quad \text{as } r \to \infty.$$

For the case $R \geqslant \frac{r}{\log_2 r}$, Sidorenko [20] (by analyzing the construction of Frankl–Rödl [7] in this regime) proved that

$$\mu(r+R,r) \leqslant (1+o(1))R\ln\binom{r+R}{R}. \tag{3}$$

Although Turán systems were actively studied, it seems that no general upper bounds on $t(r+R,r)$ have been published in the intermediate regime $1 \ll R \leqslant r/\log_2 r$. This is the case we address in this note. In brief, we show that Sidorenko's bound in (3) applies in the whole range as long as R is sufficiently large, while the asymptotic bound in (2) can be extended from constant R to any $R = o(\sqrt{r})$.

Theorem 2. *For every $\varepsilon > 0$, there exists r_0 such that the following statements hold for all r, R satisfying $R \geqslant r_0$.*

(i) It holds that $\mu(r+R,r) \leqslant (1+\varepsilon)R\ln\binom{r+R}{R}$.

(ii) Suppose that $R \leqslant \sqrt{18r\ln r}$. Then

$$\mu(r+R,r) \leqslant \mathrm{e}^{18R^2/r}\cdot(1+\varepsilon)R\ln R.$$

Our construction for Theorem 2 (i) will be a straightforward modification of the random construction used by Frankl and Rödl [7, Theorem 3], while the construction for Theorem 2 (ii) will be an extension of the recursive construction of the second author [16].

2 Proofs

In this section, we prove Theorem 2. We will use the following notation. For an integer $n \geqslant 1$ and a set X, we denote $[n] := \{1,\dots,n\}$ and $\binom{X}{n} := \{Y \subseteq X : |Y| = n\}$.

Let us begin with the proof of Theorem 2 (i).

Proof of Theorem 2 (i). Given $\varepsilon > 0$, let r_0 be sufficiently large. Take any r, R with $R \geqslant r_0$. By (3), we can assume that $R \leqslant \frac{r}{\ln r}$. Let $s := r+R$. Define

$$N := \left\lfloor r(r-1)\binom{s}{R}/(2R)\right\rfloor \quad \text{and} \quad \ell := \left\lfloor \binom{s}{R}\Big/\ln\left(\binom{s}{R}^2\binom{N-s}{R}\right)\right\rfloor.$$

In the rest of the proof, we repeat the construction of Frankl and Rödl [7, Theorem 3], arguing that our choices of N and ℓ give the stated bound.

Consider a random colouring $c : \binom{[N]}{r} \to [\ell]$. For each s-set $A \in \binom{[N]}{s}$ we have a bad event that some colour is not present in $\binom{A}{r}$. Its probability p is at most $\ell(1-1/\ell)^{\binom{s}{r}} \leqslant \ell\mathrm{e}^{-\binom{s}{R}/\ell}$ (note that $\binom{s}{r} = \binom{s}{s-r} = \binom{s}{R}$). Define the obvious dependency graph D (with loops) on $\binom{[N]}{s}$ where $A \sim B$ if $|A \cap B| \geqslant r$. It is regular of degree $\Delta := \sum_{i=r}^{s}\binom{s}{i}\binom{N-s}{s-i}$.

Our goal is to use the Lovász Local Lemma to deduce the existence of a colouring in which none of the bad events occur. In order to apply the variant of the Lovász Local Lemma given in [1, Corollary 5.1.2], it suffices to check that $\mathrm{e}p\Delta < 1$, for which it suffices to prove

$$\mathrm{e}^{\binom{s}{R}/\ell} > \mathrm{e}\ell\Delta. \tag{4}$$

First, we estimate Δ. Consider the ratio of two consecutive terms:

$$\frac{\binom{s}{i+1}\binom{N-s}{s-i-1}}{\binom{s}{i}\binom{N-s}{s-i}} = \frac{(s-i)^2}{(i+1)(N-2s+i+1)}.$$

Since $3 \leqslant R \leqslant s/2$ and $N \geqslant \binom{s}{R} \geqslant \binom{s}{3}$, this ratio is at most $1/2$ for every $i \in [r,s]$, provided r_0 is sufficiently large. Thus, we can bound

$$\Delta \leqslant 2\binom{s}{r}\binom{N-s}{s-r} = 2\binom{s}{R}\binom{N-s}{R}. \tag{5}$$

Therefore, by the choice of ℓ, we have

$$\mathrm{e}^{\binom{s}{R}/\ell} \geqslant \binom{s}{R}^2\binom{N-s}{R} \geqslant \mathrm{e}\cdot\frac{\binom{s}{R}}{2\ln\binom{s}{R}}\cdot 2\binom{s}{R}\binom{N-s}{R} \geqslant \mathrm{e}\ell\Delta,$$

as desired. (In fact, our definition of ℓ comes from (4), given N.)

Thus by the Lovász Local Lemma, there exists a colouring c with no bad events, meaning that each colour gives a Turán (N,s,r)-system on $[N]$. Let $\mathcal{A} \subseteq \binom{[N]}{r}$ be the r-graph formed by the least frequent colour. We have

$$|\mathcal{A}| \leqslant \frac{1}{\ell}\binom{N}{r}. \tag{6}$$

Now let $n := mN$ with integer $m \to \infty$. Our Turán (n,s,r)-system $\mathcal{B}$ on $[n]$ is made of a blowup of $\mathcal{A}$ plus all r-sets that intersect at least one part in more than one vertex. We have

$$|\mathcal{B}| \leqslant m^r|\mathcal{A}| + N\binom{m}{2}\binom{mN-2}{r-2} \leqslant m^r\cdot\frac{1}{\ell}\binom{N}{r} + \frac{r(r-1)}{2N}\binom{mN}{r} \leqslant \binom{mN}{r}f,$$

where $f := \frac{1}{\ell}+\frac{r(r-1)}{2N}$. Since f does not depend on m, it gives an upper bound on $t(r+R,r)$ and thus $f\cdot\binom{s}{R} \geqslant \mu(s,r)$. Note that we have $N \leqslant \binom{s}{R}^2$ and $\ln N \leqslant 2R\ln s$. Therefore,

$$\ell \geqslant \frac{\binom{s}{R}}{2\ln\binom{s}{R} + R\ln N} - 1$$

can be forced to be arbitrarily large if r_0 was sufficiently large. Thus the rounding in the definition of ℓ gives only a multiplicative term that is arbitrarily close to 1 and we have

$$\begin{aligned}\binom{s}{R}f &\leqslant (1+\varepsilon/4)\left(2\ln\binom{s}{R} + R\ln N + \frac{r(r-1)\binom{s}{R}}{2N}\right)\\ &\leqslant (1+\varepsilon/2)\left(2\ln\binom{s}{R} + R\left(2\ln r + \ln\binom{s}{R}\right) + R\right) \leqslant (1+\varepsilon)R\ln\binom{s}{R},\end{aligned}$$

proving Theorem 2 (i). Note that our choice of N was determined by using the fact that the function $\ln x + c/x$, with fixed $c > 0$, is minimized on the interval $(0,\infty)$ at $x = c$. $\square$

Next, we prove Theorem 2 (ii). Before doing so, let us present some preliminary results. The following lemma is derived from the proof of [16, Lemma 2.3].

Lemma 3 ([16])**.** *For all integers $r, R \geqslant 1$, $k \in [R, r-1]$, and a real number $c \in \left[0, \binom{k}{R}\right]$, it holds that*

$$\mu(r+R, r) \leqslant \binom{r+R}{R}\left(\frac{c}{\binom{k}{R}} + \frac{\mu(r-k+R, r-k)}{\mathrm{e}^c \cdot \binom{r-k+R}{R}}\right). \tag{7}$$

Sketch of Proof. Let S be a random subset of $\binom{[n]}{k-R}$ where each $(k-R)$-set is included into S with probability $p := c/\binom{k}{R}$. Let $S^* \subseteq \binom{[n]}{r}$ consist of those r-sets $\{x_1 < \cdots < x_r\}$ such that $\{x_1, \ldots, x_{k-R}\} \in S$, that is, we include an r-set into S^* if its initial $(k-R)$-segment is in S. Let $T \subseteq \binom{[n]}{k}$ consist of those k-sets X such that $\binom{X}{k-R} \cap S = \emptyset$, that is, X does not contain any element of S as a subset. For each $y \in [n]$, take a minimum Turán $(n-y, r-k+R, r-k)$-system F_y on $\{y+1, \ldots, n\}$. Let $T^* \subseteq \binom{[n]}{r}$ be be union over $Y \in T$ of the r-graphs $\{Y \cup Z : Z \in F_{\max Y}\}$. Informally speaking, we extend every $Y \in T$ by a minimum Turán system to the right of Y.

It is easy to check that $G := S^* \cup T^*$ is a Turán $(n, r+R, r)$-system, regardless of the choice of S. By taking S such that $|G|$ is at most its expected value, it is routine to see that

$$\begin{aligned}
T(n, r+R, r) &\leqslant \mathbb{E}|S^*| + \mathbb{E}|T^*| \\
&= p\binom{n}{r} + \sum_{y=k}^{n} (1-p)^{\binom{k}{R}} \binom{y-1}{k-1} \cdot T(n-y, r-k+R, r-k) \\
&\leqslant \left(\frac{c}{\binom{k}{R}} + \frac{\mathrm{e}^{-c}}{\binom{r-k+R}{R}}\mu(r-k+R, r-k)\right)\binom{n}{r},
\end{aligned}$$

giving the required bound. □

Fact 4. *For any integers $r_1 \geqslant r_2 > R$, we have*

$$\binom{r_1}{R}\Big/\binom{r_2}{R} = \prod_{i=0}^{R-1} \frac{r_1 - i}{r_2 - i} \leqslant \left(\frac{r_1 - R}{r_2 - R}\right)^R.$$

Fact 5. *Let $r \geqslant 1, R \geqslant 1$ be integers and δ be a real number satisfying $18R^2/r \leqslant \delta \leqslant R$. Let $k := \left\lceil \frac{Rr}{R+\delta} \right\rceil + R$. Then*

$$k \leqslant r-1, \quad \frac{r}{k-R} \leqslant 1 + \frac{\delta}{R}, \quad \textit{and} \quad \frac{r}{r-k} \leqslant \frac{3R}{\delta}.$$

Proof of Fact 5. Since $\delta \geqslant \frac{18R^2}{r}$, straightforward calculations show that

$$\frac{\delta r}{R+\delta} - R - 2 = \frac{\delta r - (R+\delta)(R+2)}{R+\delta} \geqslant \frac{18R^2 - (R+R)(R+2)}{R+\delta} \geqslant 0 \quad \text{and}$$
$$\frac{Rr}{R+\delta/2} - \frac{Rr}{R+\delta} = \frac{\delta r R}{(R+\delta)(2R+\delta)} \geqslant \frac{18R^3}{(R+R)(2R+R)} \geqslant R+1.$$

It follows that

$$k \leqslant \frac{Rr}{R+\delta} + 1 + R = r - 1 - \left(\frac{\delta r}{R+\delta} - R - 2\right) \leqslant r - 1,$$

$$\frac{r}{k-R} \leqslant \frac{r}{\frac{Rr}{R+\delta} + R - R} = \frac{R+\delta}{R} = 1 + \frac{\delta}{R}, \quad \text{and}$$

$$\frac{r}{r-k} \leqslant \frac{r}{r - \frac{Rr}{R+\delta} - 1 - R} \leqslant \frac{r}{r - \frac{Rr}{R+\delta/2}} = \frac{2R+\delta}{\delta} \leqslant \frac{3R}{\delta},$$

which proves Fact 5. □

We are now ready to present the proof of Theorem 2 (ii).

Proof of Theorem 2 (ii). Given $\varepsilon > 0$, choose a sufficiently small real number $\varepsilon_1 > 0$ and then a sufficiently large integer r_0. Take any integers r, R such that $R \geqslant r_0$ and $R \leqslant \sqrt{18r \ln r}$.

Case 1. Suppose that $R \geqslant \ln r$.

We define

$$\delta := \max\left\{\varepsilon_1,\ \frac{18R^2}{r}\right\}, \quad k := \left\lceil \frac{Rr}{R+\delta} \right\rceil + R, \quad \text{and} \quad c := R \ln\left(\frac{3R}{\delta}\right) + \ln(2R^3).$$

Clearly, $c \leqslant \binom{k}{R}$. Since R is large (which can be ensured by choosing r_0 sufficiently large) and $\ln r \leqslant R$, it follows from Theorem 2 (i) that

$$\mu(r-k+R, r-k) \leqslant 2\,R \ln\binom{r-k+R}{R} \leqslant 2\,R^2 \ln r \leqslant 2R^3.$$

Combining this with Lemma 3, Facts 4 and 5, we obtain

$$\begin{aligned}\mu(r+R, r) &\leqslant \binom{r+R}{R}\left(\frac{c}{\binom{k}{R}} + \frac{2R^3}{\mathrm{e}^c \cdot \binom{r-k+R}{R}}\right)\\ &\leqslant \left(\frac{r}{k-R}\right)^R \cdot c + \left(\frac{r}{r-k}\right)^R \cdot \frac{2R^3}{\mathrm{e}^c}\\ &\leqslant \left(1+\frac{\delta}{R}\right)^R \cdot c + \left(\frac{3R}{\delta}\right)^R \cdot \frac{2R^3}{\mathrm{e}^c} \leqslant \mathrm{e}^\delta \cdot c + \mathrm{e}^{R\ln\left(\frac{3R}{\delta}\right) - c} \cdot 2R^3 \leqslant \mathrm{e}^\delta \cdot c + 1.\end{aligned}$$

If $\frac{18R^2}{r} \leqslant \varepsilon_1$, that is, $R \leqslant \sqrt{\varepsilon_1 r/18}$, then

$$\begin{aligned}\mu(r+R, r) &\leqslant \mathrm{e}^{\varepsilon_1} \cdot c + 1 \leqslant (1+2\varepsilon_1)\left(R\ln\left(\frac{3R}{\varepsilon_1}\right) + \ln(2R^3)\right) + 1\\ &= (1+2\varepsilon_1)\left(R\left(\ln R + \ln\left(\frac{3}{\varepsilon_1}\right)\right) + 3\ln R + \ln 2\right) + 1\\ &\leqslant (1+\varepsilon)R\ln R,\end{aligned}$$

as desired. Note that the last inequality holds since $R \geqslant r_0$ (and we choose r_0 sufficiently large depending on ε_1) and $\varepsilon_1 \ll \varepsilon$.

If $\frac{18R^2}{r} > \varepsilon_1$, that is, $r < \frac{18R^2}{\varepsilon_1}$, then

$$\begin{aligned}\mu(r+R,r) &\leqslant \mathrm{e}^{18R^2/r}\cdot c+1 \leqslant \mathrm{e}^{18R^2/r}\left(R\ln\left(\frac{r}{6R}\right)+\ln(2R^3)\right)+1\\ &\leqslant \mathrm{e}^{18R^2/r}\left(R\ln\left(\frac{3R}{\varepsilon_1}\right)+\ln(2R^3)\right)+1\\ &\leqslant \mathrm{e}^{18R^2/r}(1+\varepsilon)R\ln R,\end{aligned}$$

also as desired. Note that, as in the previous calculation, the last inequality holds since $R \geqslant r_0 \gg 1/\varepsilon_1$ and $\varepsilon_1 \ll \varepsilon$.

Case 2. Suppose that $R < \ln r$.

Let $r_1 \coloneqq r$ and, inductively for $i = 1, 2, \ldots$, define

$$k_i \coloneqq \left\lceil \frac{Rr_i}{R+\varepsilon_1}\right\rceil + R, \quad \text{and} \quad r_{i+1} \coloneqq r_i - k_i;$$

if $r_{i+1} < 18R^2/\varepsilon_1$ then let $t := i$ and stop. Since r_i decreases each time, this process terminates.

We prove by backward induction on $i \in [t]$ that

$$\mu(r_i + R, r_i) \leqslant (1+\varepsilon)R\ln R. \tag{8}$$

First, consider the base case $i = t$. Note that, for $i \leqslant t$, we have by $r_i \geqslant 18R^2/\varepsilon_1$ that

$$r_{i+1} = r_i - k_i \geqslant r_i - \frac{Rr_i}{R+\varepsilon_1} - 1 - R = \frac{\varepsilon_1 r_i}{R+\varepsilon_1} - (R+1) \geqslant \frac{\varepsilon_1 r_i}{2(R+\varepsilon_1)}.$$

In particular this holds for $i = t$, giving that $r_t \leqslant \frac{18R^2}{\varepsilon_1}\cdot\frac{2(R+\varepsilon_1)}{\varepsilon_1}$, which is clearly at most e^R. Thus $R \geqslant \ln r_t$ and the desired conclusion follows by Case 1.

Now consider the inductive step for some $i \in [t-1]$. Let

$$c \coloneqq R\ln(3R/\varepsilon_1) + \ln(2R\ln R) \leqslant \binom{k_i}{R}.$$

It follows from Lemma 3, Facts 4 and 5 (note that $\varepsilon_1 \geqslant 18R^2/r_t \geqslant 18R^2/r_i$), and the inductive hypothesis that

$$\begin{aligned}\mu(r_i+R,r_i) &\leqslant \binom{r_i+R}{R}\left(\frac{c}{\binom{k_i}{R}} + \frac{\mu(r_{i+1}+R,r_{i+1})}{\mathrm{e}^c\cdot\binom{r_{i+1}+R}{R}}\right)\\ &\leqslant \left(\frac{r_i}{k_i-R}\right)^R\cdot c + \left(\frac{r_i}{r_{i+1}}\right)^R\cdot\frac{(1+\varepsilon)R\ln R}{\mathrm{e}^c}\\ &\leqslant \left(1+\frac{\varepsilon_1}{R}\right)^R\cdot c + \left(\frac{3R}{\varepsilon_1}\right)^R\cdot\frac{(1+\varepsilon)R\ln R}{\mathrm{e}^c}\end{aligned}$$

$$\leqslant e^{\varepsilon_1} \cdot c + e^{R\ln\left(\frac{3R}{\varepsilon_1}\right)-c} \cdot 2R\ln R$$
$$\leqslant (1+2\varepsilon_1)\left(R\ln\left(\frac{3R}{\varepsilon_1}\right) + \ln(2R\ln R)\right) + 1 \leqslant (1+\varepsilon)R\ln R,$$

as desired.

This completes the proof of Theorem 2. □

Remark 6. We did not optimise the bound in Theorem 2 (ii) when $R = \Omega(\sqrt{r})$, since our main aim was to extend the inequality $\mu(r+R,r) \leqslant (1+o(1))R\ln R$ for constant R from [16] to as large as possible range of functions $R(r)$.

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