A Canonical Ramsey Theorem with List Constraints in Random (Hyper-)graphs

José D. Alvarado[a] Yoshiharu Kohayakawa[b] Patrick Morris[c]
Guilherme O. Mota[b]

Submitted: Jan 14, 2025; Accepted: Dec 19, 2025; Published: Feb 27, 2026

Abstract

The celebrated canonical Ramsey theorem of Erdős and Rado implies that for a given k-uniform hypergraph (or k-graph) H, if n is sufficiently large then any colouring of the edges of the complete k-graph $K_n^{(k)}$ gives rise to copies of H that exhibit certain colour patterns. We are interested in sparse random versions of this result and the thresholds at which the random k-graph $\mathbf{G}^{(k)}(n,p)$ inherits the canonical Ramsey properties of $K_n^{(k)}$. Our main result here pins down this threshold when we focus on colourings that are constrained by some prefixed lists. This result is applied in an accompanying work of the authors on the threshold for the canonical Ramsey property (with no list constraints) in the case that H is a (2-uniform) even cycle.

Mathematics Subject Classifications: 05C80, 05C55

1 Introduction

We begin by focusing on (2-uniform) graphs.[1] For $r \in \mathbb{N}$ and graphs G and H, we say G has the *r-Ramsey property with respect to H*, denoted $G \xrightarrow{r} H$, if every colouring of the edges of G with r colours results in a *monochromatic* copy of H, that is, a copy of H with all its edges in the same colour. The classical theorem of Ramsey [19], from which the term *Ramsey theory* stems, states that if n is large enough in terms of r and H,

[a] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia (jose.alvarado@fmf.uni-lj.si)
[b] Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–090 São Paulo, Brazil ({yoshi | mota}@ime.usp.br)
[c] Departament de Matemàtiques, Universitat Politècnica de Catalunya (UPC), Carrer de Pau Gargallo 14, 08028 Barcelona, Spain (pmorrismaths@gmail.com)

[1] An extended abstract introducing our result here (focusing just on the case of graphs) appeared in the proceedings of the XII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS 2023) [1]

then $K_n \xrightarrow{r} H$. In a highly influential work, Erdős and Rado [8] explored which "colour patterns" are guaranteed when one colours a graph with no restriction on the number of colours. Clearly, monochromatic copies of H are no longer guaranteed as one can colour each edge of K_n with a unique colour. In such a case, any copy of H in K_n is said to be *rainbow*. If H contains a cycle, it is also not the case that every colouring of K_n induces either a monochromatic or rainbow copy of H. Indeed, one can associate a unique colour $c(i)$ to each vertex i in $[n] = V(K_n)$ and colour each edge $ij \in E(K_n)$ by $c(\min\{i,j\})$. Then any copy of H in K_n is neither monochromatic nor rainbow but is coloured *lexicographically*.

Definition 1 (Lexicographic colouring)**.** Let H be a graph, σ an ordering of $V(H)$ and $\chi : E(H) \to \mathbb{N}$ an edge colouring of H. We say that the pair (H, χ) is *lexicographic with respect to* σ if there exists an injective assignment of colours $\phi : V(H) \to \mathbb{N}$ such that for every edge $e = uv \in E(H)$ with $u <_\sigma v$, we have that $\chi(e) = \phi(u)$. If χ is clear from the context, we simply say that H is lexicographic with respect to σ.

The celebrated *canonical Ramsey theorem* of Erdős and Rado [8] implies that if n is large enough in terms of $m \in \mathbb{N}$, then any colouring of K_n results in a copy of K_m that is either monochromatic, rainbow or lexicographic. This theorem serves as a beautiful example of the popular Ramsey theory maxim that there is an inevitable order amongst chaos. Applying the canonical Ramsey theorem with $m = v(H)$ implies the existence of copies of H with certain colourings. The following definition captures this behaviour.

Definition 2 (The canonical Ramsey property)**.** Given a graph H and an ordering σ of $V(H)$, an edge-coloured copy of H is *canonical with respect to* σ if it is monochromatic, rainbow or lexicographic with respect to σ. A graph G has the *H-canonical Ramsey property*, denoted $G \xrightarrow{\text{can}} H$, if for every edge colouring $\chi : E(G) \to \mathbb{N}$ and every ordering σ of $V(H)$, there is a copy of H which is canonical with respect to σ.

Note that in the case that there are neither monochromatic nor rainbow copies of H, our definition requires copies of H with *all* possible lexicographic colourings (that is, for every σ, we require a lexicographic colouring with respect to σ). This notion of a canonical Ramsey property is therefore as strong as possible: there is no other *colour pattern*[2] that one can guarantee when avoiding monochromatic and rainbow copies of H. Indeed, as observed by Jamison and West [12], the canonical Ramsey theorem implies that for any H, a set $\mathcal{C}$ of colour patterns of H is *unavoidable* in all colourings of large enough cliques[3] if and only if $\mathcal{C}$ contains the monochromatic pattern, the rainbow pattern and at least one lexicographic pattern. Alternative definitions of canonical Ramsey properties are discussed in Section 1.5.

[2] A colour pattern of a graph H is a partition of its edge set.

[3] That is, large enough cliques are such that any colouring of their edges admits a copy of H whose edges are partitioned according to one of the given patterns.

1.1 Sparse Ramsey theory and random graphs

Returning to the setting of colourings with a bounded number of colours, a prominent theme in Ramsey theory has been to explore the existence of *sparse* graphs G that are r-Ramsey with respect to H; see for example [18] and the references therein. One famous example is the work of Frankl and Rödl [9], who used a random graph to construct a K_4-free graph G such that $G \xrightarrow{2} K_3$. This prompted Łuczak, Ruciński and Voigt [15] to initiate the study of thresholds for Ramsey properties in random graphs, which has since become a prominent theme in probabilistic combinatorics. It turns out that the threshold for $\mathbf{G}(n,p)$ having the Ramsey property with respect to a graph H is governed by the following parameter of H.

Definition 3. Given a graph H with at least two edges, the *maximum* 2*-density* of H is defined by

$$m_2(H) := \max\left\{\frac{e(F)-1}{v(F)-2} : F \subseteq H,\, v(F) > 2\right\}.$$

In a seminal series of papers, Rödl and Ruciński [20, 21, 22] established the threshold for the Ramsey property when the random graph is coloured with a bounded number of colours. Here and throughout, we say that a function $\hat{p} = \hat{p}(n)$ is the *threshold* for a monotone increasing graph property $\mathcal{P}$ if

$$\lim_{n\to\infty} \Pr(\mathbf{G}(n,p) \text{ satisfies } \mathcal{P}) = \begin{cases} 0 & \text{if } p = o(\hat{p}), \\ 1 & \text{if } p = \omega(\hat{p}). \end{cases}$$

We refer to $\hat{p}$ as *the* threshold for $\mathcal{P}$, although it is only defined up to order of magnitude. We now state an abridged form of the Rödl–Ruciński theorem.

Theorem 4 (Rödl–Ruciński [20, 21, 22])**.** *Let $r \geqslant 2$ be an integer and H be a graph that is not a star forest. Then $n^{-1/m_2(H)}$ is the threshold for the property that $\mathbf{G}(n,p) \xrightarrow{r} H$.*

1.2 Canonical Ramsey properties of random graphs

The motivation for the current work is to establish the threshold for the canonical Ramsey property with respect to a given graph H. Note that for any H as in Theorem 4 that is not a triangle, the threshold for $\mathbf{G}(n,p) \xrightarrow{\text{can}} H$ is at least $n^{-1/m_2(H)}$. Indeed for any such H, there is an ordering σ of its vertices such that the lexicographic colouring of H with respect to σ uses at least 3 colours. Using Theorem 4, we have that when $p = o\left(n^{-1/m_2(H)}\right)$, asymptotically almost surely (a.a.s. from now on) there is a 2-colouring of $\mathbf{G}(n,p)$ that avoids monochromatic copies of H. Moreover, such a colouring avoids rainbow and lexicographic copies of H with respect to σ, simply because there are not enough colours available for such colour patterns. This shows that $\mathbf{G}(n,p)$ does not have the canonical Ramsey property for H for such p.

We believe that this lower bound is in fact the correct threshold for the canonical Ramsey property and that when $p = \omega\left(n^{-1/m_2(H)}\right)$, a.a.s. $\mathbf{G}(n,p) \xrightarrow{\text{can}} H$. Here, we provide evidence for this by focusing on colourings that are constrained to be compatible with a given list assignment.

Definition 5 (Colourings with list constraints)**.** Let $1 \leqslant r \in \mathbb{N}$ and $\mathcal{L} : E(K_n) \to \mathbb{N}^r$ be an assignment of lists of colours to the edges of K_n (note that we allow lists to have repeated colours). We say that a colouring $\chi : E(G) \to \mathbb{N}$ of an n-vertex graph $G \subseteq K_n$, is *compatible* with $\mathcal{L}$ if for all $e \in E(G)$, we have that $\chi(e) \in \mathcal{L}(e)$.

Our main theorem shows that for any assignment $\mathcal{L}$ of bounded lists to the edges of K_n, the threshold for the canonical Ramsey property with respect to H when considering colourings that are compatible with $\mathcal{L}$ is at most $n^{-1/m_2(H)}$. Let us write $G \xrightarrow[\mathcal{L}]{\text{can}} H$ if any edge colouring χ of G that is compatible with $\mathcal{L}$ contains a canonical copy of H with respect to σ for all orderings σ of $V(H)$.

Theorem 6. *Let H be a graph with at least two edges. Let $1 \leqslant r \in \mathbb{N}$ and let $\mathcal{L}$: $E(K_n) \to \mathbb{N}^r$ be a list assignment of colours. If $\mathbf{G} \sim \mathbf{G}(n,p)$ with $p = \omega\left(n^{-1/m_2(H)}\right)$, then $\mathbf{G} \xrightarrow[\mathcal{L}]{\text{can}} H$ a.a.s.*

Note that for many assignments of lists $\mathcal{L}$ this theorem establishes the threshold for the canonical Ramsey property restricted to colourings compatible with $\mathcal{L}$. Indeed, by the same reasoning discussed above, this is the case whenever there are 2 colours that feature on all lists. We will deduce Theorem 6 from a more general result, namely Theorem 18, which deals with graphs of the form $\Gamma \cap \mathbf{G}(n,p)$ where Γ is a "locally dense graph" (see Section 2.2 for the relevant definitions).

1.3 An application for even cycles

We believe Theorem 6 provides a natural stepping stone towards establishing $n^{-1/m_2(H)}$ as the threshold for the canonical Ramsey property in random graphs for all graphs H that are not forests and are different from the triangle. In fact, the theorem arose naturally in the work of the authors proving a 1-statement for the canonical Ramsey property with respect to even cycles. Indeed, Theorem 6 (or rather its stronger version, Theorem 18) is a key component of the proof of the following theorem, which is given in an accompanying paper [2].

Theorem 7. *Let $k \geqslant 2$ be an integer. If $p = \omega\left(n^{-(2k-2)/(2k-1)} \log n\right)$, then a.a.s.*

$$\mathbf{G}(n,p) \xrightarrow{\text{can}} C_{2k}.$$

As we mentioned before, $n^{-1/m_2(H)}$ is a lower bound for the threshold for the canonical Ramsey property with respect to even cycles. Thus, since $m_2(C_{2k}) = (2k-1)/(2k-2)$, Theorem 7 establishes the threshold for the canonical Ramsey property with respect to even cycles, up to the log factor.

1.4 Hypergraphs

With our methods, we can also prove an analogue of Theorem 6 for hypergraphs. In order to introduce this, we need to adapt our definitions appropriately. For $k \geqslant 2$, we refer to k-uniform hypergraphs as k-graphs, use the notation $K_n^{(k)}$ to denote the complete k-graph on n vertices and let $\mathbf{G}^{(k)}(n,p)$ denote the binomial random k-graph obtained by keeping each edge of $K_n^{(k)}$ independently with probability $p = p(n)$.

Firstly, it is far from clear how to generalise the canonical Ramsey property (Definition 2) to k-graphs. This was done already by Erdős and Rado [8]. To start with, we make the following definition.

Definition 8 (Projection maps)**.** Let $2 \leqslant k \in \mathbb{N}$ and let V be a set of size $v \geqslant k$ and σ an ordering of V. Then for a (possibly empty) set $S \subseteq [k]$, we define the *S-projection map* π_S (with respect to σ) to be a function $\pi_S : \binom{V}{k} \to \binom{V}{|S|}$ such that

$$\pi_S(T) = \{v \in T : \ v \text{ is the } i\text{-th element of } T \text{ in the ordering } \sigma \text{ for some } i \in S\}.$$

In words, the map π_S pulls out the elements of an ordered k-set that occupy the positions dictated by S. Note that $\pi_\emptyset(T) = \emptyset$ whilst $\pi_{[k]}(T) = T$ for all $T \in \binom{V}{k}$. We also need the following definition.

Definition 9 (The reversal involution)**.** For any set $S \subseteq [k]$, let

$$\iota(S) = \{k - x + 1 : x \in S\}$$

be the *reverse* of S.

The map $\iota : 2^{[k]} \to 2^{[k]}$ partitions $2^{[k]}$ into blocks $\{S, \iota(S)\}$ (some of which are pairs of sets and others are singletons). We say a collection $\mathcal{T} \subseteq 2^{[k]}$ is a *transversal* for the reversal involution if $\mathcal{T}$ contains one set from each block. We can now describe the collection of canonical colourings of a hypergraph H.

Definition 10 (Canonical copies)**.** Let $2 \leqslant k \in \mathbb{N}$ and $\mathcal{T} \subseteq 2^{[k]}$ a transversal for the reversal involution on $[k]$. Further, let H be a k-graph, σ an ordering of $V(H)$ and $\chi : E(H) \to \mathbb{N}$ an edge colouring of H. We say (H, χ) is *canonical* with respect to σ and $\mathcal{T}$ if there is $S \in \mathcal{T}$ and an *injective* assignment of colours $\phi : \binom{V(H)}{|S|} \to \mathbb{N}$ such that for every $e \in E(H)$, we have that $\chi(e) = \phi(\pi_S(e))$, where π_S is the S-projection map with respect to σ. If χ is clear from the context, we simply say that H is canonical with respect to σ and $\mathcal{T}$.

The canonical copies given by Definition 10 give *all* the colour patterns that we are interested in. Indeed, if a copy of H is canonical because $S = \emptyset$, then this copy is monochromatic and for $S = [k]$, this copy is rainbow. The intermediate S cover all the other colour patterns. For example, when $k = 2$, the set $S = \{1\}$ corresponds to a copy that is lexicographic with respect to the ordering σ (Definition 1). In fact, in Definition 2 (where $k = 2$), we define canonical copies with respect to σ with the convention of choosing $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$. If the transversal $\mathcal{T}$ rather contains the set $S = \{2\}$, this corresponds to a copy of H that is lexicographic with respect to the ordering on $V(H)$ which is reverse to σ. Similarly, for larger k, when considering all orderings σ of $V(H)$ and transversals $\mathcal{T}$, certain colour patterns of H will appear multiple times.

As with the graph case, the canonical Ramsey theorem of Erdős and Rado [8] implies that for any $2 \leqslant k \in \mathbb{N}$ and any k-graph H, if n is sufficiently large, then $K_n^{(k)}$ has the *H-canonical Ramsey property*, that is, for any edge colouring of $E(K_n^{(k)})$, any ordering σ of $V(H)$ and any transversal $\mathcal{T}$ for the reverse involution on $[k]$, there is a canonical copy of H with respect to σ and $\mathcal{T}$.

Similarly, in the study of Ramsey properties of random graphs the situation is considerably more involved for hypergraphs of uniformity greater than 2. Indeed, such properties are not fully understood to this date. The analogue of Definition 3 for $3 \leqslant k \in \mathbb{N}$ is given by the *maximal k-density* of H defined by

$$m_k(H) := \max\left\{\frac{e(F)-1}{v(F)-k} : F \subseteq H,\, v(F) > k\right\}, \tag{1}$$

where we assume that H has at least two edges. The generalisation of the 1-statement of Theorem 4 to hypergraphs, by which we mean that $\mathbf{G}^{(k)}(n,p)$ a.a.s. has the H-Ramsey property (for any number of colours $r \geqslant 2$) when $p = \omega(n^{-1/m_k(H)})$, has been established for all H. However, this took much longer than the graph case, eventually being resolved by Friedgut, Rödl and Schacht [10] and, independently, by Conlon and Gowers [6]. A corresponding 0-statement has also been proven for various hypergraphs, in particular for cliques [16, 24]. The natural expectation that $n^{-1/m_k(H)}$ should indeed be the threshold for the H-Ramsey property in $\mathbf{G}^{(k)}(n,p)$ (as in Theorem 4), except for some simple cases, was shattered by Gugelmann, Nenadov, Person, Škorić and Steger [11], who presented a family of exceptions richer than in the graph case. Nevertheless, this expectation has been proved to hold for 'most' k-graphs H by Bowtell, Hancock and Hyde [5].

As with Theorem 6, we show that when considering colourings that are compatible with some prefixed lists, taking $p = \omega(n^{-1/m_k(H)})$ suffices to find canonical copies. The definition of $G \xrightarrow[\mathcal{L}]{\text{can}} H$ in hypergraphs is completely analogous to the graph case except that one now insists that we find canonical copies with respect to all orderings σ of $V(H)$ *and* all transversals $\mathcal{T}$ of the reverse involution on $[k]$.

Theorem 11. *Let $2 \leqslant k \in \mathbb{N}$ and H be a k-graph with at least two edges. Further, let $1 \leqslant r \in \mathbb{N}$ and $\mathcal{L} : E(K_n^{(k)}) \to \mathbb{N}^r$ be a list assignment of colours. If $\mathbf{G} \sim \mathbf{G}^{(k)}(n,p)$ with $p = \omega\left(n^{-1/m_k(H)}\right)$, then $\mathbf{G} \xrightarrow[\mathcal{L}]{\text{can}} H$ a.a.s.*

1.5 Alternative definitions for the canonical Ramsey property

The original canonical Ramsey theorem of Erdős and Rado [8] was stated using slightly different notions to those used here. Indeed, they consider *ordered* copies of $K_m^{(k)}$ and $K_n^{(k)}$ on vertex sets $[m]$ and $[n]$, respectively. They prove that if n is large enough in terms of m, then for any colouring $\chi : E(K_n^{(k)}) \to \mathbb{N}$, one can find an *ordered embedding*[4] ψ of $K_m^{(k)}$ in $K_n^{(k)}$ which is *canonical* with respect to the ordering on $V(K_m^{(k)}) = [m]$. In more detail, they show that there is some $S \subseteq [k]$ and an injective assignment of colours $\phi : \binom{[m]}{|S|} \to \mathbb{N}$ such that for every $e \in E(K_m^{(k)})$, we have that $\chi(\psi(e)) = \phi(\pi_S(e))$, with π_S being the S-projection map with respect to the standard ordering on $[m]$.

Using this, we can see that for any H with $m = |V(H)|$, if n is sufficiently large, then $K_n^{(k)} \xrightarrow{\text{can}} H$, that is, $K_n^{(k)}$ has the H-canonical Ramsey property. Indeed, fix some edge colouring χ of $E(K_n^{(k)})$, some ordering σ of $V(H)$ and a transversal $\mathcal{T} \subseteq 2^{[k]}$ for the reversal involution on $[k]$. Labelling the vertices of $K_n^{(k)}$ by $[n]$ arbitrarily, the theorem of Erdős and Rado [8] gives some ordered embedding $\psi : [m] \to [n]$, a set $S \subseteq [k]$ and an injective map $\phi : \binom{[m]}{|S|} \to \mathbb{N}$ such that $\chi(\psi(e)) = \phi(\pi_S(e))$ for all $e \in \binom{[m]}{k}$. Now if $S \in \mathcal{T}$, then we can embed H by mapping the i^{th} vertex of $V(H)$ according to σ, to the vertex $\psi(i) \in [n]$ for $i \in [m]$. If, on the other hand $S \notin \mathcal{T}$, then we must have $\iota(S) \in \mathcal{T}$ and we can embed H by mapping the i^{th} vertex of $V(H)$ according to σ to the vertex $\psi(m - i + 1) \in [n]$. In either case, the resulting copy of H is canonical with respect to σ and $\mathcal{T}$.

The argument above shows that our notion of H-canonical Ramsey property makes sense, but the reader may wonder if there are alternative ways to define such a property. Indeed, one alternative which has more in common with the original theorem of Erdős and Rado [8] is to look at *ordered copies* of H and $K_n^{(k)}$ with a completely analogous notion of an ordered embedding of H being canonical as in the definition in the theorem of Erdős and Rado [8] when $H = K_m^{(k)}$. We remark that it would certainly be possible to work with this definition and prove our results here in the context of this ordered notion of canonical Ramsey properties.

There are several reasons for our choice to use the alternative characterisation of the canonical Ramsey property as in Definition 2 (see also Definition 10). Firstly, working with ordered graphs and ordered embeddings diverges from the classical Ramsey setting in unordered (hyper-)graphs. Secondly (and more importantly) when dealing with H that are not complete, the ordered notion of the canonical Ramsey property is in fact weaker. This is already apparent in the setting of graphs where the ordered notion will find an ordered copy of H that is either monochromatic, rainbow, min-coloured (each edge inherits its colour from its smaller endpoint) or max-coloured (each edge inherits its colour from its larger endpoint). Now consider H to be a triangle with a pendent edge. By taking an ordering σ of the vertices of H in which the (unique) vertex of degree 3 comes first, our notion in Definition 2 guarantees that if there is no copy of H which is monochromatic or rainbow, then one can find a copy of H which has all vertices incident to the degree 3 vertex in one colour and the remaining edge in a distinct colour. On the other hand, such a colouring is not forced when considering the ordered notion of the canonical Ramsey property. Indeed, if one uses only that in the absence of monochromatic and rainbow H, one finds min- or max- coloured copies of all orderings of H, then one

[4]By an ordered embedding, we mean that $\psi(i) < \psi(j)$ in $[n]$ if and only if $i < j$ in $[m]$.

can adversarially choose min- or max- for each order in such a way that the considered colouring is never output.

This is the principal reason for why we have opted to state our results here without looking for ordered embeddings, so as to capture all sets of unavoidable colour patterns for graphs, as in [3, 12]. The definition in hypergraphs is slightly more cumbersome (having to vary over all transversals $\mathcal{T}$ of the reverse involution on $[k]$) but also captures more colour patterns than the canonical Ramsey notion via ordered hypergraphs.

1.6 Related work

Until very recently, to our knowledge, there have been no results on canonical Ramsey properties in random graphs. However, simultaneously and independently to our work here and in [2], Kamčev and Schacht [13, 14] obtained a remarkable result, completely resolving the problem for the case when $H = K_m$. Indeed, they show that for $4 \leqslant m \in \mathbb{N}$ and $p = \omega(n^{-2/(m+1)})$, a.a.s. $\mathbf{G}(n,p) \xrightarrow{\text{can}} K_m$. As $m_2(K_m) = (m+1)/2$, this establishes the threshold for the canonical Ramsey property with respect to complete (2-uniform) graphs K_m. In contrast to our proofs, which appeal to the method of hypergraph containers, their proof relies on the transference principle of Conlon and Gowers [6]. Their methods also allow them to obtain partial results in the case where H is a *strictly balanced* graph, that is when $m_2(H)$ is achieved only by H itself. For such a graph H and for $p = \omega(n^{-1/m_2(H)})$, they can prove the existence of monochromatic, rainbow or *some* lexicographic copies of H in any colouring of $\mathbf{G} \sim \mathbf{G}(n,p)$. Intriguingly, their results do not generalise to the hypergraph setting, where it seems new ideas are necessary.

1.7 Organisation and conventions

As the setting of graphs (Theorem 6) is notationally simpler than the general result for k-graphs (Theorem 11), we first restrict our attention to Theorem 6 and then simply discuss the proof of Theorem 11, which is almost identical, in Section 5. Also, for graphs, we actually prove a stronger statement, Theorem 18, which we need in our application in [2] and from which Theorem 6 will follow. Before proving Theorem 18 in Section 4, we provide some necessary tools in Section 2 and an overview of our proof in Section 3.

As usual, we omit floor and ceiling symbols whenever they are not essential. Finally, note that, in our main results, we may assume without loss of generality that our target graph H has no isolated vertices. Assuming this will occasionally simplify our exposition.

2 Preliminaries

In this section, we collect the necessary theory and tools needed in our proof. We will introduce the method of containers in Section 2.1 and the theory of locally dense graphs in Section 2.2. Before all of this though, we collect the relevant notation.

For simplicity, given a graph H we use $v(H)$ and $e(H)$, respectively, for $|V(H)|$ and $|E(H)|$. For a k-uniform hypergraph $\mathcal{H}$ and for $U \subseteq V(\mathcal{H})$, we let $\mathcal{H}[U]$ denote the hypergraph induced by $\mathcal{H}$ on U. Furthermore, for any vertex subset $T \subseteq V(\mathcal{H})$, let $d_{\mathcal{H}}(T)$ denote the number of edges of $\mathcal{H}$ containing T and, for $0 \leqslant j \leqslant k$, let $\Delta_j(\mathcal{H}) := \max\{d_{\mathcal{H}}(T) : T \subseteq V(\mathcal{H}), |T| = j\}$ denote the maximum degree of a vertex set of size j in $\mathcal{H}$.

The binomial random graph $\mathbf{G}(n,p)$ refers to the probability distribution of graphs on vertex set $[n]$ obtained by taking every possible edge independently with probability $p = p(n)$. We say an event happens asymptotically almost surely (a.a.s. for short) in $\mathbf{G} \sim \mathbf{G}(n,p)$ if the probability that it happens tends to 1 as n tends to infinity. We will also use standard asymptotic notation throughout, with asymptotics always being taken as the number of vertices n tends to infinity. Finally, we use the notation $a = b \pm c$ to denote a number a between $b - c$ and $b + c$ and we omit floors and ceilings throughout, so as not to clutter the arguments.

2.1 The method of containers

We will appeal to the method of hypergraph containers, developed by Balogh, Morris and Samotij [4], and independently, by Saxton and Thomason [23]. The key idea underlying this method is that if a uniform hypergraph has an edge set that is evenly distributed, then one can group the independent sets of the hypergraph into a well-behaved collection of *containers*. In more detail, these containers are vertex subsets that are almost independent (in that they induce few edges of the hypergraph), every independent set of the hypergraph lies in some container and, crucially, we have a bound on the number of containers. As there are far fewer containers than independent sets in the hypergraph, reasoning about containers rather than independent sets leads to more efficient arguments and this technique has proven to be extremely powerful. Indeed, the setting of independent sets in hypergraphs can be used to encode a wide range of problems in combinatorics and the method of hypergraph containers has been successfully exploited in a multitude of different settings. Particularly relevant to our work here are the applications of the method in sparse Ramsey theory, a program which was initiated by Nenadov and Steger [17], who reproved the 1-statement of Theorem 4 using containers.

Below, we state the container lemma in the form given in [4, Theorem 2.2]. Before doing so, we need to establish some terminology and definitions.

Definition 12 (($\mathcal{H}, \varepsilon$)-abundant set families)**.** Let $\mathcal{H} = (V, E)$ be a hypergraph and let $0 < \varepsilon \leqslant 1$. We say a family $\mathcal{F} \subseteq 2^V$ is $(\mathcal{H}, \varepsilon)$-*abundant* if the following hold:

(1) $\mathcal{F}$ is *increasing*: for all $A, B \subseteq V$ with $A \in \mathcal{F}$ and $A \subseteq B$, we have that $B \in \mathcal{F}$;

(2) $\mathcal{F}$ contains only large vertex sets: for all $A \in \mathcal{F}$, we have that $|A| \geqslant \varepsilon v(\mathcal{H})$;

(3) $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-*dense*: for all $A \in \mathcal{F}$, we have that $e(\mathcal{H}[A]) \geqslant \varepsilon e(\mathcal{H})$.

For a hypergraph $\mathcal{H} = (V, E)$, we also define $\mathcal{I}(\mathcal{H}) \subseteq 2^V$ to be the collection of independent vertex sets in $\mathcal{H}$. We now state the container theorem [4, Theorem 2.2] and we refer to the discussion in [4] for motivation and context.

Theorem 13 (Hypergraph Container Theorem)**.** *For every $k \in \mathbb{N}$ and $\varepsilon, D_0 > 0$, there exists $D > 0$ such that the following holds for k-uniform hypergraphs $\mathcal{H}$. If $\mathcal{F} \subseteq 2^{V(\mathcal{H})}$ is an $(\mathcal{H}, \varepsilon)$-abundant set family and $q \in (0,1)$ is such that for each $j \in [k]$ we have $\Delta_j(\mathcal{H}) \leqslant D_0 q^{j-1} e(\mathcal{H})/v(\mathcal{H})$, then there exists a family $\mathcal{S} \subseteq 2^{V(\mathcal{H})}$ of 'fingerprints' and two functions $f : \mathcal{S} \to 2^{V(\mathcal{H})} \setminus \mathcal{F}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that:*

(a) (*'small' fingerprints*) *for each $S \in \mathcal{S}$, we have that $|S| \leqslant Dqv(\mathcal{H})$;*

(b) (*containment*) *for each $I \in \mathcal{I}(\mathcal{H})$, we have that $g(I) \subseteq I$ and $I \setminus g(I) \subseteq f(g(I))$.*

In applications of Theorem 13, the set $\mathcal{C} := \{f(S) \cup S : S \in \mathcal{S}\}$ is usually referred to as the set of *containers* for the hypergraph $\mathcal{H}$. Note that property (a) can be used to bound the size of $\mathcal{C}$ whilst property (b) shows that for every independent set $I \in \mathcal{I}(\mathcal{H})$ there is some $C \in \mathcal{C}$ such that $I \subseteq C$.

2.2 Locally dense graphs

We will establish our main theorem in the context of random sparsifications of locally dense graphs, a more general setting than $\mathbf{G}(n, p)$ which will be useful for applications. Here we collect some key properties of locally dense graphs.

Definition 14 ((ρ, d)-dense graphs)**.** For $\rho, d \in (0, 1]$, we say a graph Γ is *(ρ, d)-dense* if the following holds: for every $S \subseteq V(\Gamma)$ with $|S| \geqslant \rho n$, the induced graph $\Gamma[S]$ has at least $d\binom{|S|}{2}$ edges.

Remark 15. In [22], the authors observed that in order to establish that a graph Γ is (ρ, d)-dense it suffices to deal with subsets $S \subseteq V(\Gamma)$ of cardinality exactly ρn.

Graphs with the (ρ, d)-denseness property for $\rho = o(1)$ are often called *locally dense* graphs and this property can be viewed as a weak quasirandomness property. The following result appears in [22, Lemma 2] and can be proven by induction.

Proposition 16. *For every $m \geqslant 2$ and $d > 0$ there exist $\rho, c_0 > 0$ such that if Γ is an n-vertex (ρ, d)-dense graph and n is sufficiently large, then Γ contains at least $c_0 n^m$ copies of K_m.*

We close this section with the following result about robustness of the locally denseness property in the sense that it is preserved after removing a small fraction of edges of a dense graph.

Proposition 17. *Suppose $\rho, d, \gamma > 0$. If Γ is a (ρ, d)-dense graph on n vertices with n sufficiently large, then every spanning subgraph Γ' of Γ with $e(\Gamma') \geqslant (1-\gamma)e(\Gamma)$ is a (ρ, d')-dense graph, with $d' := d - (2\gamma/\rho^2)$. In particular, if $\gamma \leqslant \rho^2 d/4$, then such Γ' is a $(\rho, d/2)$-dense graph.*

Proof. Let Γ be a (ρ, d)-dense graph on n vertices and let Γ' be a spanning subgraph of Γ with $e(\Gamma') \geqslant (1-\gamma)e(\Gamma)$ edges. If $S \subseteq V(\Gamma)$ has cardinality ρn, then

$$e_{\Gamma'}(S) \geqslant e_{\Gamma}(S) - \gamma e(\Gamma) \geqslant d\binom{|S|}{2} - \frac{\gamma}{2}n^2 = d\binom{|S|}{2} - \frac{\gamma}{\rho^2}\frac{|S|^2}{2} \geqslant d'\binom{|S|}{2},$$

using that n is sufficiently large in the last step and hence $|S|^2/2 \leqslant 2\binom{|S|}{2}$. Appealing to Remark 15, we have that Γ' is a (ρ, d')-dense graph on n vertices, as desired. □

3 Outline of the proof

Our proof follows the scheme of Nenadov and Steger [17] and appeals to the method of hypergraph containers (see Section 2.1). We start by giving a rough outline of the proof.

Suppose we want to find a monochromatic H in any r-colouring of $\mathbf{G} \sim \mathbf{G}(n,p)$. Our idea is to create an auxiliary hypergraph $\mathcal{H}$ whose vertex set consists of r copies of $E(K_n)$ (one copy for each colour) and whose edge set encodes the monochromatic copies of H in K_n. Thus, an edge in $\mathcal{H}$ corresponds to a copy of H entirely contained in one of the r copies of K_n; putting it another way, $\mathcal{H}$ is the disjoint union of r copies of the $|E(H)|$-uniform hypergraph on $E(K_n)$ that encodes all the copies of H in K_n. The key observation is the following: any colouring of $\mathbf{G}$ that avoids monochromatic copies of H can be identified with an independent set in $\mathcal{H}$. Using containers, one can efficiently group together these independent sets and identify a small set $\mathcal{C}$ of *containers* such that each independent set of $\mathcal{H}$ belongs to one such container. The proof then proceeds by showing that, for each *fixed* container $C \in \mathcal{C}$, it is very unlikely[5] that the graph $\mathbf{G}$ lies within C. By this, we mean that it is unlikely that there is a colouring of $\mathbf{G}$ that, when mapped in the obvious way to a vertex subset of $\mathcal{H}$, corresponds to a subset of C. The proof follows by performing a union bound over the choices for a container $C \in \mathcal{C}$.

This containers approach relies crucially on the fact that the hypergraph $\mathcal{H}$ that encodes the monochromatic copies of H has $v(\mathcal{H}) = O(n^2)$ and so satisfies the the degree constraints of Theorem 13 with $q = \Theta(n^{-1/m_2(H)})$. Having an unbounded number of copies of $E(K_n)$ in $V(\mathcal{H})$, which corresponds to an unbounded number of colours available, leads to an adjustment on the degree constraints and the parameter q in Theorem 13. This renders the upper bound on the number of containers useless and the proof falls apart. In our proof we avoid such a problem by creating a hypergraph whose vertex set is composed by r copies of $E(K_n)$ with each copy of an edge corresponding to a choice of colour of that edge according to the list of available colours. The edge set of the hypergraph then encodes the canonical copies of H. The fact that each list is bounded allows us to apply Theorem 13 to our hypergraph with the correct parameters. As previously discussed, we can in fact prove the following more applicable result, which is a strengthening of Theorem 6 and applies to random sparsifications of locally dense graphs.

Theorem 18 (Sparse canonical Ramsey theorem with list constraints)**.** *Let H be a graph with at least two edges and suppose $1 \leqslant r \in \mathbb{N}$ and $d > 0$. Then there exist ρ, $c > 0$ such*

[5]That is, it happens with probability $\exp(-\Omega(n^2 p))$.

that the following holds. Let σ be an ordering of $V(H)$, let Γ be an n-vertex (ρ, d)-dense graph and let $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$ be a list assignment. If $p = \omega(n^{-1/m_2(H)})$ and n is large enough, then with probability at least $1 - \exp(-cpn^2)$, any edge colouring χ of $\Gamma \cap \mathbf{G}(n,p)$ that is compatible with $\mathcal{L}$ contains a canonical copy of H with respect to σ.

Theorem 6 follows from Theorem 18 applied to $\Gamma = K_n$, which is $(\rho, 1)$-dense for all $\rho > 0$, and a union bound over the $v(H)!$ orderings σ of $V(H)$.

4 Proof

In this section we make the outline given in Section 3 precise, proving Theorem 18.

4.1 A hypergraph encoding canonical copies

With an eye to apply the hypergraph container theorem (Theorem 13), we first define an appropriate hypergraph that encodes the canonical copies of H with respect to a colouring that is compatible with some list assignment. We make the following definition.

Definition 19 (Canonical copy hypergraphs)**.** Given a graph H, an ordering σ of $V(H)$, an integer $r \geqslant 1$, an n-vertex graph Γ and a list assignment $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$, we define the *canonical copy hypergraph* $\mathcal{H} = \mathcal{H}_H^\sigma(\Gamma, \mathcal{L})$ as follows:

The hypergraph $\mathcal{H}$ is $e(H)$-uniform with $V(\mathcal{H}) = E(\Gamma) \times [r]$. A collection $\{(e_i, s_i) : 1 \leqslant i \leqslant e(H)\} \subseteq V(\mathcal{H})$ of vertices form an edge of $\mathcal{H}$ if and only if the collection $\{e_i : 1 \leqslant i \leqslant e(H)\} \subseteq E(\Gamma)$ form a copy of H in Γ that is canonical with respect to σ when each edge e_i is coloured by the s_i-th colour $\mathcal{L}(e_i)[s_i]$ of $\mathcal{L}(e_i)$.

Let $\mathcal{H} = \mathcal{H}_H^\sigma(\Gamma, \mathcal{L})$ be a canonical copy hypergraph as in Definition 19. For $W \subseteq V(\mathcal{H})$, let the *graph shadow* $G_W \subseteq \Gamma$ of W be the subgraph of Γ spanned by $E(G_W) := \{e \in \Gamma : (e, s) \in W \text{ for some } s \in [r]\}$. Note that the vertex set W is a set of pairs in $E(\Gamma) \times [r]$, and hence the graph shadow G_W is just the graph obtained by projecting the vertices of W onto their first coordinates.

We also need the following observation on the degrees of $\mathcal{H}$.

Lemma 20. *For any graph H, ordering σ of $V(H)$, integer $r \geqslant 1$, graph Γ with n vertices and list assignment $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$, the canonical copy hypergraph $\mathcal{H} = \mathcal{H}_H^\sigma(\Gamma, \mathcal{L})$ satisfies the following: for $1 \leqslant j \leqslant e(H)$, we have $\Delta_j(\mathcal{H}) \leqslant r^{e(H)} \left(n^{-1/m_2(H)}\right)^{j-1} n^{v(H)-2}$.*

Proof. Note that the result holds for $j = 1$, as any edge of H is in at most $r^{e(H)} n^{v(H)-2}$ canonical copies of H with respect to σ. Then, we assume that $2 \leqslant j \leqslant e(H)$ and fix an arbitrary j-element set $U = \{(e_i, s_i) : 1 \leqslant i \leqslant j\} \subseteq V(\mathcal{H})$. It suffices to show that the number $d_{\mathcal{H}}(U)$ of edges of $\mathcal{H}$ containing U is at most $r^{e(H)} \left(n^{-1/m_2(H)}\right)^{j-1} n^{v(H)-2}$.

Consider the graph $F = G_U \subseteq \Gamma$ induced by the edge set $E(G_U) = \{e_i : 1 \leqslant i \leqslant j\}$. If F is not a subgraph of some copy of H in Γ, then $d_{\mathcal{H}}(U) = 0$ and we are done. If F is a copy of a subgraph of H, then note that there are at most $n^{v(H)-v(F)}$ extensions of F

to a copy of H in K_n and at most $r^{e(H)-e(F)}$ choices of second coordinate for the vertices of $\mathcal{H}$ corresponding to the copy of H that extends F. Therefore,

$$d_{\mathcal{H}}(U) \leqslant r^{e(H)-e(F)} n^{v(H)-v(F)} \leqslant r^{e(H)} n^{-(v(F)-2)} n^{v(H)-2}.$$

By using the definition of $m_2(H)$, we conclude that

$$d_{\mathcal{H}}(U) \leqslant r^{e(H)} \left(n^{-\frac{v(F)-2}{j-1}} \right)^{j-1} n^{v(H)-2} \leqslant r^{e(H)} \left(n^{-1/m_2(H)} \right)^{j-1} n^{v(H)-2},$$

as required. □

4.2 Supersaturation for canonical copies

Next we prove a supersaturation-type result for canonical copies in locally dense graphs.

Lemma 21 (Canonical supersaturation)**.** *Let H be a graph and let $d \in (0,1]$. Then there exist ρ, $c_0 > 0$ such that for any n-vertex (ρ, d)-dense graph Γ with large enough n, any ordering σ of $V(H)$ and any colouring $\chi : E(\Gamma) \to \mathbb{N}$, one can find at least $c_0 n^{v(H)}$ copies of H in Γ coloured by χ in a canonical fashion with respect to σ.*

Proof. Fix a graph H and $d \in (0,1]$. It is well known by the classical canonical Ramsey theorem [8] that there is a positive integer R such that $K_R \xrightarrow{\text{can}} K_{v(H)}$. Now fix $\rho, c_0 > 0$ as output by Proposition 16 with input $m = R$ and $d > 0$. Then fix Γ, an ordering σ of $V(H)$ and χ as in the statement.

Note that in a canonical copy of $K_{v(H)}$ (with respect to any ordering), one can find a canonical copy of H with respect to σ. By Proposition 16, the graph Γ contains at least $c_0 n^R$ copies of K_R and, as $K_R \xrightarrow{\text{can}} K_{v(H)}$, each of these copies contains a canonical copy of $K_{v(H)}$ and hence a canonical copy of H with respect to σ. On the other hand, each copy of H in Γ is contained in at most $n^{R-v(H)}$ copies of K_R (in Γ). Hence, the number of canonical copies of H in Γ coloured by χ is at least $c_0 n^R / n^{R-v(H)} \geqslant c_0 n^{v(H)}$. □

Using Lemma 21, we can show that the family of sets of vertices of $V(\mathcal{H})$ with large graph shadows are abundant with respect to $\mathcal{H}$, in the sense of Definition 12.

Lemma 22. *For all $d > 0$ and $1 \leqslant r \in \mathbb{N}$, there exist ρ, γ, $\varepsilon > 0$ such that the following holds.*

Let Γ and H be graphs such that Γ is (ρ, d)-dense and $e(H) \geqslant 1$, let σ be an ordering of $V(H)$ and let $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$ be given. Furthermore, let $\mathcal{H} = \mathcal{H}_H^\sigma(\Gamma, \mathcal{L})$ and define $\mathcal{F} := \{W \subseteq V(\mathcal{H}) : e(G_W) \geqslant (1-\gamma)e(\Gamma)\} \subseteq 2^{V(\mathcal{H})}$. Suppose $n = v(\Gamma)$ is large enough. Then $\mathcal{F}$ is $(\mathcal{H}, \varepsilon)$-abundant.

Proof. Fix ρ, $c_0 > 0$ as output by Lemma 21 with input $d/2$ rather than d. Furthermore, fix $0 < \gamma \leqslant \rho^2 d/4$ and $0 < \varepsilon < c_0/(2r^{e(H)})$. We aim to show that $\mathcal{F}$ is $(\mathcal{H}, \varepsilon)$-abundant and so we need to establish conditions (1)–(3) of Definition 12. Note that (1) is immediate from the definition of $\mathcal{F}$ and condition (2) follows from our definition of ε and the fact that any $W \in \mathcal{F}$ must have $|W| \geqslant (1-\gamma)e(\Gamma) \geqslant e(\Gamma)/2 = v(\mathcal{H})/(2r)$.

Therefore it remains to prove condition (3). For this, fix $W \in \mathcal{F}$ and consider the graph shadow $G = G_W$. Moreover, define a colouring $\chi : E(G) \to \mathbb{N}$ so that for each edge $e \in E(G)$ we have that $\chi(e) = \mathcal{L}(e)[s]$ for some s with $(e, s) \in W$. Now note that each copy of H in G coloured by χ, which is canonical with respect to σ, corresponds to an edge in $\mathcal{H}[W]$. Therefore it suffices to prove the existence of $\varepsilon e(\mathcal{H})$ canonical copies of H in G coloured by χ. Let G' be the spanning subgraph of Γ with edge set $E(G)$ (that is, add to G the vertices in $V(\Gamma) \setminus V(G)$ as isolated vertices). By the choice of γ and Proposition 17, the graph G' is $(\rho, d/2)$-dense and so, by Lemma 21, contains at least $c_0 n^{v(H)}$ canonical copies of H with respect to σ. Our assumption that H has no isolated vertices implies that these copies are subgraphs of G. The lemma thus follows from our definition of ε and the fact that $e(\mathcal{H}) \leqslant n^{v(H)} r^{e(H)}$. □

4.3 Proof of Theorem 18

We are now in a position to prove our main result.

Proof of Theorem 18. Let H, $1 \leqslant r \in \mathbb{N}$ and $d > 0$ be as in the statement and let $\rho, \gamma, \varepsilon, c_0 > 0$ be small enough for us to be able to apply Lemmas 21 and 22. Further, fix $0 < c < \gamma d/16$, $D_0 = r^{e(H)+1}/c_0$, and let $D > 0$ be as output by Theorem 13 applied with $k = e(H)$.

Next we fix an ordering σ of $V(H)$, some n-vertex (ρ, d)-dense graph Γ with n large enough and some list assignment $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$. Let $\mathcal{H} = \mathcal{H}_H^\sigma(\Gamma, \mathcal{L})$ be the canonical copy hypergraph as in Definition 19 and $\mathcal{F} \subseteq 2^{V(\mathcal{H})}$ be as in Lemma 22. We claim that all the conditions of the Hypergraph Container Theorem (Theorem 13) are satisfied with $q := n^{-1/m_2(H)}$. Indeed the condition that $\mathcal{F}$ is $(\mathcal{H}, \varepsilon)$-abundant is given by Lemma 22 and the degree conditions on $\mathcal{H}$ follow from Lemma 20 and the facts that $v(\mathcal{H}) = r\binom{n}{2} \leqslant rn^2$ and $e(\mathcal{H}) \geqslant c_0 n^{v(H)}$, as can be seen by applying Lemma 21 to the colouring defined by each edge taking, say, the first colour in its list.

By Theorem 13 we thus get a collection of fingerprints $\mathcal{S} \subseteq 2^{V(\mathcal{H})}$ such that $|S| \leqslant Dqv(\mathcal{H})$ for all $S \in \mathcal{S}$. Moreover, there are two functions $f : \mathcal{S} \to 2^{V(\mathcal{H})} \setminus \mathcal{F}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for each $I \in \mathcal{I}(\mathcal{H})$, we have that $g(I) \subseteq I$ and $I \setminus g(I) \subseteq f(g(I))$. Now for each $S \in \mathcal{S}$, define $C(S) := f(S) \cup S \subseteq V(\mathcal{H})$. We claim that for all $S \in \mathcal{S}$, we have that

$$e\big(\Gamma \setminus G_{C(S)}\big) \geqslant \frac{\gamma}{2} e(\Gamma) \geqslant \frac{\gamma d}{8} n^2. \tag{2}$$

Indeed this follows because $f(S) \notin \mathcal{F}$ and so $e(G_{f(S)}) < (1-\gamma)e(\Gamma)$ and as n is sufficiently large, $e(G_S) \leqslant |S| \leqslant Dqv(\mathcal{H}) \leqslant \gamma e(\Gamma)/2$.

For an n-vertex graph $G \subseteq \Gamma$ and a colouring $\chi : E(G) \to \mathbb{N}$ compatible with $\mathcal{L}$, let $W(G, \chi) \subseteq V(\mathcal{H})$ be the set of vertices $(e, s) \in V(\mathcal{H})$ such that $e \in E(G)$ and $\chi(e) = \mathcal{L}(e)[s]$. Note that if the graph G, when coloured by χ, contains no canonical copies of H with respect to σ, then $W(G, \chi) \in \mathcal{I}(\mathcal{H})$, that is, $W(G, \chi)$ is an independent set. Therefore, by property (b) of Theorem 13, the event that $\mathbf{G} \sim \mathbf{G}(n, p)$ is such that there is a colouring of $\mathbf{G} \cap \Gamma$ compatible with $\mathcal{L}$ that avoids canonical copies of H is contained in the event $\bigcup_{S \in \mathcal{S}} \Phi(S)$, where

$$\Phi(S) := \{\exists \mathcal{L}\text{-compatible } \chi : E(\mathbf{G} \cap \Gamma) \to \mathbb{N} \text{ such that } S \subseteq W(\mathbf{G}, \chi) \subseteq C(S)\}.$$

Now note that if $\Phi(S)$ occurs then $G_S \subseteq \mathbf{G}$ and $\mathbf{G} \setminus G_S \subseteq G_{C(S)}$, and these events are independent. Hence for a fixed $S \in \mathcal{S}$, we have

$$\begin{aligned}
\Pr[\Phi(S)] &\leqslant \Pr[G_S \subseteq \mathbf{G}] \cdot \Pr[\mathbf{G} \setminus G_S \subseteq G_{C(S)}] \\
&\leqslant p^{e(G_S)} \cdot \Pr[\mathbf{G} \cap (\Gamma \setminus G_{C(S)}) = \emptyset] \\
&\leqslant p^{e(G_S)} \cdot (1-p)^{\gamma d n^2/8} \\
&\leqslant p^{e(G_S)} e^{-\gamma d n^2 p/8},
\end{aligned}$$

using (2) in the penultimate inequality. Therefore, the probability that there is an $\mathcal{L}$-compatible colouring of $\mathbf{G} \cap \Gamma$ avoiding canonical copies of H with respect to σ is at most

$$\sum_{S \in \mathcal{S}} \Pr[\Phi(S)] \leqslant e^{-\gamma d n^2 p/8} \sum_{S \in \mathcal{S}} p^{e(G_S)}. \tag{3}$$

For a given $0 \leqslant t \leqslant e(\Gamma)$, the number of sets $S \subseteq V(\mathcal{H})$ such that $e(G_S) = t$ is at most $\binom{e(\Gamma)}{t}(2^r - 1)^t \leqslant \binom{n^2}{t} 2^{rt}$. Theorem 13(a) tells us that $e(G_S) \leqslant T$ for every $S \in \mathcal{S}$, where $T = rDqn^2$. Hence, the summation on the right-hand side of (3) is at most

$$\sum_{0 \leqslant t \leqslant T} \binom{n^2}{t} 2^{rt} p^t \leqslant 2 \binom{n^2}{T} 2^{rT} p^T \leqslant 2 \left(\frac{e 2^r n^2 p}{T} \right)^T, \tag{4}$$

where the first inequality follows by comparing the sum with a geometric series, assuming that $p \geqslant 2^{1-r} rDq$. Thus, the right-hand side of (3) is at most

$$\begin{aligned}
2e^{-\gamma d n^2 p/8} \left(\frac{e 2^r n^2 p}{T} \right)^T &\leqslant 2e^{-\gamma d n^2 p/8} \left(\frac{e 2^r n^2 p}{rDqn^2} \right)^{rDqn^2} \\
&\leqslant 2 \exp \left\{ n^2 p \left(-\frac{1}{8}\gamma d + rD\frac{q}{p} \log \frac{e 2^r p}{rDq} \right) \right\} \\
&\leqslant e^{-\gamma d n^2 p/10},
\end{aligned}$$

as long as p/q is larger than some constant that depends only on r, D, γ and d. This completes the proof. □

5 Hypergraphs

In this section, we discuss the proof of Theorem 11, which is very similar to our proof of Theorem 6 with essentially only notational differences. Indeed, it is well known that one of the key advantages of the method of hypergraph containers is that proofs in the graph case easily generalise to hypergraphs. In any case, for completeness, we discuss some of the details.

Firstly, note that we concentrate solely on the random subgraph $\mathbf{G}^{(k)}(n,p)$ and so do not need to worry about a locally dense host Γ as in the proof of Theorem 18 (see the discussion at the end of this section). This leads to some simplifications in the proof. As in Definition 19, for the hypergraph H in question, an ordering σ of $V(H)$, an integer $r \geqslant 1$ and a list assignment $\mathcal{L} : E(K_n^{(k)}) \to \mathbb{N}^r$, we define an auxiliary hypergraph $\mathcal{H} = \mathcal{H}_H^\sigma(\mathcal{L})$ on vertex set $V(\mathcal{H}) = E(K_n^{(k)}) \times [r]$ whose edges encode canonical copies (Definition 10) of H with respect to σ, when choosing colours from the lists. As in Lemma 4.2, it follows from the definition (1) of $m_k(H)$ that

$$\Delta_j(\mathcal{H}) \leqslant r^{e(H)}(n^{-1/m_k(H)})^{j-1} n^{v(H)-k}, \tag{5}$$

for all $1 \leqslant j \leqslant e(H)$. Also as in the proof for graphs, for a subset $W \subseteq V(\mathcal{H}) = E(K_n^{(k)}) \times [r]$, we define the *hypergraph shadow* G_W to be the subhypergraph of $K_n^{(k)}$ obtained by taking the hyperedges featured in W. We further define $\mathcal{F} = \mathcal{F}(\gamma) := \{W \subseteq V(\mathcal{H}) : e(G_W) \geqslant (1-\gamma)\binom{n}{k}\}$ to be the vertex subsets of $V(\mathcal{H})$ with large hypergraph shadow.

We then fix some small $\gamma, \varepsilon > 0$ and need to show that $\mathcal{F} = \mathcal{F}(\gamma)$ is $(\mathcal{H}, \varepsilon)$-abundant (Definition 12). Conditions (1) and (2) follow directly from the definition of $\mathcal{F}$ and for (3), we need to prove supersaturation of canonical copies. As in Lemma 21, this can be done by averaging. Indeed, by the canonical Ramsey theorem of Erdős and Rado [8], there is some $R \in \mathbb{N}$ such that any colouring of $K_R^{(k)}$ results in a canonical copy of $K_{v(H)}^{(k)}$ and hence a canonical copy of H (with respect to σ). On the other hand, every copy of H in $K_n^{(k)}$ is contained in at most $n^{R-v(H)}$ copies of $K_R^{(k)}$. Therefore, as in Lemma 21, any colouring of $K_n^{(k)}$ results in at least $c_0 n^{v(H)}$ canonical copies of H with respect to σ, for some $c_0 > 0$. This in turn implies that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense (recall (3)), as given some $A \in \mathcal{F}$, one can colour the edges e of G_A with a colour in $\mathcal{L}(e)$ corresponding to a vertex $(e,s) \in A$ and colour edges of $K_n^{(k)} \setminus G_A$ arbitrarily. This gives rise to at least $c_0 n^{v(H)}$ canonical copies of H and by deleting any such copy of H containing an edge of $K_n^{(k)} \setminus G_A$, we destroy at most $\gamma n^{v(H)}$ of these copies. Thus, as γ is taken to be much smaller than c_0, we get at least $c_0 n^{v(H)}/2 \geqslant \varepsilon e(\mathcal{H})$ canonical copies of H, each of which corresponds to an edge in $\mathcal{H}[A]$.

We are therefore in a position to apply Theorem 13 to $\mathcal{H}$ and $\mathcal{F}$ with $q = n^{-1/m_k(H)}$, also considering (5). The rest of the proof follows analogously to the proof of Theorem 18, by performing a union bound over the containers $C(S) := f(S) \cup S \subseteq V(\mathcal{H})$ output by Theorem 13 (as well as a union bound over orderings σ of $V(H)$). As in (2), we crucially use that if $\mathbf{G} \sim \mathbf{G}^{(k)}(n,p)$ can be coloured in a canonical H-free way that corresponds to an independent set lying in some fixed container $C(S)$, then as $f(S) \notin \mathcal{F}$, there is some set of $\Omega(n^k)$ edges of $K_n^{(k)}$ that are not hit by $\mathbf{G}$, an event that occurs with probability less than $e^{-\Omega(n^k p)}$.

We close by remarking that a generalisation of Theorem 18 to hypergraphs would be more complicated than the version (Theorem 11) we give here. Indeed, the straightforward generalisation of a locally dense k-graph Γ, namely that sets $S \subseteq V(\Gamma)$ of size $o(n)$ have positive density, is not enough to guarantee the supersaturation of copies of cliques, or even a single copy of a $K_{k+1}^{(k)}$ in Γ (as noted by Rödl using a construction from [7]). A much stronger pseudorandom condition would be needed for Γ so that an analogue of Theorem 18 holds for hypergraphs of larger uniformity. Whilst this should certainly be possible, as we do not yet have applications for such a result, we do not pursue this here.

Acknowledgements

We are grateful to the anonymous referees for their careful reading of the previous version of the manuscript and their helpful suggestions.

J. D. Alvarado was partially supported by FAPESP (2020/10796-0). Y. Kohayakawa was partially supported by FAPESP (2023/03167-5) and CNPq (407970/2023-1, 420838/2025-2, 315258/2023-3). P. Morris was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Walter Benjamin program (project number 504502205). G. O. Mota was partially supported by CNPq (306620/2020-0, 406248/2021-4) and FAPESP (2018/04876-1, 2019/13364-7, 2023/03167-5). This study was financed in part by CAPES, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brazil, Finance Code 001. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

References

[1] J. D. Alvarado, Y. Kohayakawa, P. Morris, and G. O. Mota, *A canonical Ramsey theorem with list constraints in random graphs*, Procedia Comput. Sci. **223** (2023), 13–19. MR 4742584

[2] ———, *A canonical Ramsey theorem for even cycles in random graphs*, 2024. Available as arXiv:2411.14566.

[3] M. Axenovich and R. E. Jamison, *Canonical pattern Ramsey numbers*, Graphs Combin. **21** (2005), no. 2, 145–160. MR 2191012

[4] J. Balogh, R. Morris, and W. Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. **28** (2015), no. 3, 669–709. MR 3327533

[5] C. Bowtell, R. Hancock, and J. Hyde, *Proof of the Kohayakawa–Kreuter conjecture for the majority of cases*, 2023. Available as arXiv:2307.16760.

[6] D. Conlon and W. T. Gowers, *Combinatorial theorems in sparse random sets*, Ann. of Math. (2) **184** (2016), no. 2, 367–454. MR 3548529

[7] P. Erdős and A. Hajnal, *On Ramsey like theorems. Problems and results*, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, pp. 123–140. MR 337636

[8] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255. MR 37886

[9] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without K_4*, Graphs Combin. **2** (1986), no. 2, 135–144. MR 932121

[10] E. Friedgut, V. Rödl, and M. Schacht, *Ramsey properties of random discrete structures*, Random Structures & Algorithms **37** (2010), no. 4, 407–436.

[11] L. Gugelmann, R. Nenadov, Y. Person, N. Škorić, A. Steger, and H. Thomas, *Symmetric and asymmetric Ramsey properties in random hypergraphs*, Forum Math. Sigma **5** (2017), Paper No. e28, 47. MR 3725732

[12] R. E. Jamison and D. B. West, *On pattern Ramsey numbers of graphs*, Graphs Combin. **20** (2004), no. 3, 333–339. MR 2093488

[13] N. Kamčev and M. Schacht, *Canonical colourings in random graphs*, Procedia Comput. Sci. **223** (2023), 5–12. MR 4742583

[14] ———, *Canonical colourings in random graphs*, J. Lond. Math. Soc. (2) **112** (2025), no. 2, Paper No. e70239, 29. MR 4938284

[15] T. Łuczak, A. Ruciński, and B. Voigt, *Ramsey properties of random graphs*, J. Combin. Theory Ser. B **56** (1992), no. 1, 55–68. MR 1182457

[16] R. Nenadov, Y. Person, N. Škorić, and A. Steger, *An algorithmic framework for obtaining lower bounds for random Ramsey problems*, J. Combin. Theory Ser. B **124** (2017), 1–38. MR 3623165

[17] R. Nenadov and A. Steger, *A short proof of the random Ramsey theorem*, Combin. Probab. Comput. **25** (2016), no. 1, 130–144. MR 3438289

[18] J. Nešetřil and V. Rödl, *Sparse Ramsey graphs*, Combinatorica **4** (1984), no. 1, 71–78. MR 739415

[19] F. P. Ramsey, *On a problem of formal logic*, Proc. Lond. Math. Soc. (2) **s2-30** (1930), 264–286.

[20] V. Rödl and A. Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, pp. 317–346. MR 1249720

[21] ———, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures Algorithms **5** (1994), no. 2, 253–270. MR 1262978

[22] ———, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942. MR 1276825

[23] D. Saxton and A. Thomason, *Hypergraph containers*, Invent. Math. **201** (2015), no. 3, 925–992. MR 3385638

[24] H. Thomas, *Aspects of games on random graphs*, Ph.D. thesis, ETH Zurich, 2013.