Spectral Turán Problems for Nondegenerate Hypergraphs

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Abstract

Keevash, Lenz and Mubayi developed a general criterion for hypergraph spectral extremal problems in their seminal work (SIAM J. Discrete Math., 2014). Their framework shows that extremal results on the α-spectral radius (for $\alpha > 1$) may be deduced from a corresponding hypergraph Turán problem exhibiting stability properties, provided its extremal construction satisfies certain continuity assumptions. In this paper, we establish a spectral stability result for nondegenerate hypergraphs, extending the Keevash–Lenz–Mubayi criterion. Applying this result, we derive two general spectral Turán theorems for hypergraphs with bipartite or multipartite pattern, thereby transforming spectral Turán problems into the corresponding purely combinatorial problems related to degree-stability in nondegenerate k-graph families. As applications, we determine the maximum α-spectral radius for several classes of hypergraph and characterize the corresponding extremal hypergraphs, such as the expansion of complete graphs, the generalized fans, the cancellative hypergraphs, the generalized triangles, and a special book hypergraph.

Mathematics Subject Classifications: 15A18, 05C65, 05C31

1 Introduction

A *hypergraph* $H = (V(H), E(H))$ consists of a vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$ and an edge set $E(H) = \{e_1, e_2, \cdots, e_m\}$, where $e_i \subseteq V$ for $i \in [m] := \{1, 2, \ldots, m\}$. If $|e_i| = k$ for each $i \in [m]$ and $k \geqslant 2$, then H is called a *k-uniform* hypergraph (or simply *k-graph*). A simple graph is exactly a 2-uniform hypergraph. We denote by $e(H)$ the number of edges of H, that is, $e(H) = |E(H)|$. A k-graph $H' = (V(H'), E(H'))$ is called a *sub-hypergraph* of H if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. For $S \subseteq V(H)$, the k-graph with S as its vertex set and $E(H) \cap \binom{S}{k}$ as its edge set is called an *induced sub-hypergraph* of H, denoted by $H[S]$.

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Let $\mathcal{F}$ be a family of k-graphs. We say that a hypergraph H is *$\mathcal{F}$-free* if H does not contain any member of $\mathcal{F}$ as a sub-hypergraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is defined as the maximum number of edges of an $\mathcal{F}$-free k-graph on n vertices. Denote by $\mathrm{EX}(n, \mathcal{F})$ the set of all $\mathcal{F}$-free k-graphs with $ex(n, \mathcal{F})$ edges and n vertices. Determining the exact Turán number for a general k-graph is a classic and intractable problem in extremal combinatorics, but if we are satisfied with the asymptotic results, the simple graph is completely solved (see [6]). The *Turán density* of $\mathcal{F}$ is defined as

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}},$$

and $\mathcal{F}$ is called *nondegenerate* if $\pi(\mathcal{F}) > 0$. So, finding an asymptotic result for $\mathrm{ex}(n, \mathcal{F})$ is equivalent to determining the Turán density if $\mathcal{F}$ is nondegenerate.

The Turán problems are closely related to the phenomenon of stability, and many Turán problems can be solved by the stability results of the corresponding graphs or hypergraphs. The first stability theorem was proved independently by Erdős and Simonovits [33]. In addition, Simonovits [33] determined $\mathrm{ex}(n, F)$ exactly according to the stability theorem for a color-critical graph F. At present, there is much research on the stability of hypergraphs; for details see [3, 10, 20, 23, 29, 32]. In [21], Liu, Mubayi, and Reiher provided a unified framework for the stability of certain hypergraph families, which simplifies the proofs of many known results.

The spectral Turán problem of graphs or hypergraphs is a spectral version of the Turán problem. Nikiforov made important contributions to the spectral Turán problems of simple graphs. For example, Nikiforov [26] determined the maximum spectral radius for the K_{l+1}-free graph on n vertices, and showed that the Turán graph $T_l(n)$ is the unique spectral extremal graph, which is a generalization of Turán theorem. To date there are very few results on spectral Turán problems of hypergraph. In [18], Keevash, Lenz, and Mubayi gave two general criteria that formalize a generalized form of the strong stability of Turán problems. They also determined the maximum α-spectral radius of any 3-graph on n vertices not containing the Fano plane when n is sufficiently large. In [5], Ellingham, Lu, and Wang characterized the extremal hypergraph with maximum spectral radius among all outerplanar 3-graphs of n vertices by its shadow. In [27], Ni, Liu and Kang obtained the maximum α-spectral radius of cancellative 3-graphs, and characterized the extremal hypergraph. Hou, Liu, and Zhao [14] gave a result on spectral Turán problems for some hypergraphs which has degree-stability. Recently, the Turán and spectral Turán problems of linear hypergraphs have also been extensively studied; see [8, 11, 12, 13, 30, 31].

In this paper, we investigate the spectral Turán problems for nondegenerate hypergraphs. We show that for any family $\mathcal{F}$ of nondegenerate k-graphs, under a certain growth condition on α-spectral radius, the spectral Turán problems can be effectively reduced to the spectral extremal problems restricted to the class of $\mathcal{F}$-free hypergraphs with large minimum degree. Furthermore, we give two general results for hypergraphs with bipartite or multipartite pattern, which transform the corresponding spectral Turán problems into purely combinatorial problems involving the degree-stability of nondegenerate k-graph families; see Section 3. As an application, we determine the maximum α-spectral radius for some classes of hypergraphs and characterize the corresponding extremal hypergraphs, such as the expansion of complete graphs, the generalized fans, the cancellative hypergraphs, the generalized triangles, and a special book hypergraph; see Section 4.

2 Preliminaries

2.1 Stability

Let l, k be positive integers such that $l \geqslant k \geqslant 2$. A k-graph is called *l-partite* if its vertex set can be divided into l parts, so that each edge contains at most one vertex from each part. An edge maximal l-partite k-graph is called a *complete l-partite k-graph.* Let $T_l^k(n)$ be the balanced complete l-partite k-graph on n vertices, namely, any two parts have sizes differing by at most 1. Therefore, the number of edges in $T_l^k(n)$ is

$$t_l^k(n) := \sum_{S \in \binom{[l]}{k}} \prod_{i \in S} n_i,$$

where $n_i = \lfloor (n + i - 1)/l \rfloor$ for $i \in [l]$.

Although $t_l^k(n)$ has an explicit expression, the following asymptotic result is more useful later in our estimation.

Lemma 1. *Let $l \geqslant k \geqslant 2$. Then*

$$t_l^k(n) = \frac{(l)_k}{k! l^k} n^k + O(n^{k-2}),$$

where $(l)_k := l(l-1)\cdots(l-k+1)$.

Proof. Let $n = lq + s$, where $0 \leqslant s < l$. Then

$$\begin{aligned} t_l^k(n) &= \sum_{i=0}^{k} \binom{s}{i} \binom{l-s}{k-i} (q+1)^i q^{k-i} \\ &= \sum_{i=0}^{k} \binom{s}{i} \binom{l-s}{k-i} (q^k + iq^{k-1} + O(n^{k-2})) \\ &= \sum_{i=0}^{k} \binom{s}{i} \binom{l-s}{k-i} (\frac{n}{l} - \frac{s}{l})^k + \sum_{i=0}^{k} i \binom{s}{i} \binom{l-s}{k-i} (\frac{n}{l} - \frac{s}{l})^{k-1} + O(n^{k-2}). \end{aligned}$$

So we have

$$\begin{aligned}
t_l^k(n) =& \binom{l}{k}(\frac{n}{l}-\frac{s}{l})^k + \sum_{i=1}^{k} s\binom{s-1}{i-1}\binom{l-s}{k-i}(\frac{n}{l}-\frac{s}{l})^{k-1} + O(n^{k-2}) \\
=& \frac{(l)_k}{k!l^k}n^k - \frac{ks}{l^k}\binom{l}{k}n^{k-1} + s\sum_{i=0}^{k-1}\binom{s-1}{i}\binom{l-s}{k-1-i}(\frac{n}{l})^{k-1} + O(n^{k-2}) \\
=& \frac{(l)_k}{k!l^k}n^k - \frac{ks}{l^k}\binom{l}{k}n^{k-1} + \frac{s}{l^{k-1}}\binom{l-1}{k-1}n^{k-1} + O(n^{k-2}) \\
=& \frac{(l)_k}{k!l^k}n^k + O(n^{k-2}).
\end{aligned}$$

□

A k*-multiset* is a collection of k elements with repetitions allowed. A k*-pattern* is a pair $P = ([l], E)$ where l is a positive integer and E is a collection of k-multisets with elements from $[l]$. Clearly, k-patterns are generalizations of k-graphs. Given a k-graph H and a k-pattern $P = ([l], E)$, a map ϕ: $V(H) \to [l]$ is a *homomorphism* from H to P if the k-multiset $\{\phi(v_1), \ldots, \phi(v_k)\}$ belongs to E for every edge $\{v_1, \ldots, v_k\} \in E(H)$. We say H is P*-colorable* if there is a homomorphism from H to P. For example, any l-partite k-graph is K_l^k-colorable, where K_l^k is the complete k-graph on l vertices; conversely, every K_l^k-colorable k-graph is l-partite. Let $\mathcal{F}$ be a family of k-graphs and P be a k-pattern. We say $(\mathcal{F}, P)$ is a *Turán pair* if every P-colorable k-graph is $\mathcal{F}$-free and every edge maximum $\mathcal{F}$-free k-graph is P-colorable.

For a k-graph H and a vertex $v \in V(H)$, we denote by $E_H(v)$ the set of edges in H containing the vertex v. The degree $d_H(v)$ of v is defined as the cardinality $|E_H(v)|$. Let $\delta(H)$ denote the minimum degree of H. For simplicity, we write $H - v$ for the induced sub-hypergraph $H[V(H)\backslash\{v\}]$.

Definition 2 ([14])**.** Let $\mathcal{F}$ be a family of nondegenerate k-graphs, where $k \geqslant 2$, and let $\mathfrak{H}$ be a family of $\mathcal{F}$-free k-graphs.

(1) $\mathcal{F}$ is *edge-stable* with respect to $\mathfrak{H}$ if for every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that every $\mathcal{F}$-free k-graph $\mathcal{H}$ on $n \geqslant n_0$ vertices with $e(\mathcal{H}) \geqslant (\pi(\mathcal{F})/k! - \varepsilon)n^k$ becomes a member of $\mathfrak{H}$ after removing at most δn^k edges.

(2) $\mathcal{F}$ is *degree-stable* with respect to $\mathfrak{H}$ if there exist $\varepsilon > 0$ and n_0 such that every $\mathcal{F}$-free k -graph $\mathcal{H}$ on the $n \geqslant n_0$ vertices with $\delta(\mathcal{H}) \geqslant (\pi(\mathcal{F})/(k-1)! - \varepsilon)n^{k-1}$ is a member of $\mathfrak{H}$.

(3) $\mathcal{F}$ is *vertex-extendable* with respect to $\mathfrak{H}$ if there exist $\varepsilon > 0$ and n_0 such that every $\mathcal{F}$-free k-graph $\mathcal{H}$ on $n \geqslant n_0$ vertices with $\delta(\mathcal{H}) \geqslant (\pi(\mathcal{F})/(k-1)! - \varepsilon)n^{k-1}$ satisfies: if $\mathcal{H} - v$ is a member of $\mathfrak{H}$ for some vertex v, then $\mathcal{H}$ is a member of $\mathfrak{H}$ as well.

A class $\mathfrak{H}$ of k-graphs is called *hereditary* if it is closed under taking induced sub-hypergraphs, that is, for every $G \in \mathfrak{H}$ and $S \subseteq V(G)$, we have $G[S] \in \mathfrak{H}$. Note that the collection of all P-colorable hypergraphs is hereditary. In many cases, the extremal hypergraphs of Turán problems are P-colorable for some pattern P, so we usually choose $\mathfrak{H}$ as the collection of all P-colorable hypergraphs. For further developments on the hereditary property of hypergraphs, see [24, 25].

Theorem 3 ([14]). *Let $\mathcal{F}$ be a family of nondegenerate k-graphs and $\mathfrak{H}$ be a hereditary class of $\mathcal{F}$-free k-graphs. If $\mathcal{F}$ is both edge-stable and vertex-extendable with respect to $\mathfrak{H}$, then $\mathcal{F}$ is degree-stable with respect to $\mathfrak{H}$.*

2.2 Spectral radius

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $\alpha \geqslant 1$, the ℓ_α-norm of $\mathbf{x}$ is denoted and defined by $\|\mathbf{x}\|_\alpha := (|x_1|^\alpha + \cdots + |x_n|^\alpha)^{1/\alpha}$. Let G be a k-graph on n vertices. For any $\alpha > 1$, the *Lagrangian polynomial* $P_G(\mathbf{x})$ of G is defined as

$$P_G(\mathbf{x}) = k! \sum_{\{i_1,\dots,i_k\} \in E(G)} x_{i_1} \cdots x_{i_k},$$

and the *α-spectral radius* of G is defined as

$$\lambda_\alpha(G) = \max_{\|\mathbf{x}\|_\alpha = 1} P_G(\mathbf{x}).$$

Taking the vector $\mathbf{x} = (n^{-1/\alpha}, \dots, n^{-1/\alpha})$, we obtain

$$\lambda_\alpha(G) \geqslant k! n^{-k/\alpha} e(G). \tag{1}$$

If $\mathbf{x} \in \mathbb{R}^n$ is a unit vector with respect to ℓ_α-norm such that $\lambda_\alpha(G) = P_G(\mathbf{x})$, then $\mathbf{x}$ is called an *eigenvector* of G corresponding to $\lambda_\alpha(G)$. Obviously, any k-graph G always has a nonnegative eigenvector corresponding to $\lambda_\alpha(G)$. When $\alpha > 1$, the nonnegative eigenvector $\mathbf{x} = (x_1, \dots, x_n)$ of a k-graph G satisfies the following equations derived from Lagrange's method:

$$\lambda_\alpha(G) x_i^{\alpha-1} = (k-1)! \sum_{\{i,i_2,\dots,i_k\} \in E(G)} x_{i_2} \cdots x_{i_k}, \text{ for } 1 \leqslant i \leqslant n.$$

The *spectral Turán number*, denoted by $\mathrm{spex}(n, \mathcal{F})$, is defined as the maximum α-spectral radius over all $\mathcal{F}$-free k-graphs on n vertices. The *spectral Turán density* can be analogously defined by

$$\lambda^{(\alpha)}(\mathcal{F}) := \lim_{n \to \infty} \frac{\mathrm{spex}(n, \mathcal{F})}{n^{k-k/\alpha}}.$$

Note that any family of $\mathcal{F}$-free k-graphs is hereditary. The result of Nikiforov [24, Theorem 9.3] (see also [25, Theorem 12]) implies the following:

Lemma 4 ([24, 25]). *Let $\mathcal{F}$ be a family of k-graphs. Then for every $\alpha > 1$,*

$$\lambda^{(\alpha)}(\mathcal{F}) = \pi(\mathcal{F}).$$

Lemma 5 ([17])**.** *Let G be a k-graph of order n with at least one edge, and let u and v be vertices of G such that the transposition of u and v is an automorphism of G. If $\alpha > 1$, and $\mathbf{x}$ is an eigenvector corresponding to $\lambda_\alpha(G)$, then $x_u = x_v$.*

Lemma 6 ([17])**.** *Let $\alpha \geqslant 1$, and let G be a k-graph such that every nonnegative eigenvector corresponding to $\lambda_\alpha(G)$ is positive. If H is a sub-hypergraph of G, then $\lambda_\alpha(H) < \lambda_\alpha(G)$, unless $H = G$.*

Theorem 7 ([17])**.** *Let $l \geqslant k \geqslant 2$, and let G be an l-partite k-graph of order n. For every $\alpha > 1$,*

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(T_l^k(n)),$$

with equality if and only if $G = T_l^k(n)$.

Theorem 8 ([17])**.** *Let $l \geqslant k \geqslant 2$, and let G be an l-partite k-graph of order n. If $\alpha > 1$, then*

$$\lambda_\alpha(G) \leqslant \frac{(l)_k}{l^k} n^{k-k/\alpha},$$

with equality if and only if $l \mid n$ and $G = T_l^k(n)$.

3 Main Results

In this section, we establish a spectral stability result for nondegenerate hypergraphs. Based on this result, we present two general theorems for determining the maximum α-spectral radius among all n-vertex $\mathcal{F}$-free k-graphs, where $\mathcal{F}$ is a certain family of nondegenerate k-graphs.

Theorem 9 (Spectral stability)**.** *Let $k \geqslant 2$, $\alpha > 1$, $0 < \varepsilon < 1$, and $\mathcal{F}$ be a family of k-graphs with $\pi(\mathcal{F}) > 0$. Let $\mathcal{G}_n$ be the collection of all n-vertex $\mathcal{F}$-free k-graphs with minimum degree at least $(1-\varepsilon)\pi(\mathcal{F})\binom{n}{k-1}$, and let $\lambda_\alpha(\mathcal{G}_n) = \max\{\lambda_\alpha(G) : G \in \mathcal{G}_n\}$. Suppose that there exists N such that for all $n \geqslant N$,*

$$\lambda_\alpha(\mathcal{G}_n) \geqslant \lambda_\alpha(\mathcal{G}_{n-1}) + (k - k/\alpha)(1-\varepsilon')\pi(\mathcal{F})n^{k-k/\alpha-1}, \tag{2}$$

where $\varepsilon' = \varepsilon\pi(\mathcal{F})(\alpha-1)/(2k\alpha)$. Then there exists n_0 such that if H is an $\mathcal{F}$-free k-graph on $n \geqslant n_0$ vertices, then

$$\lambda_\alpha(H) \leqslant \lambda_\alpha(\mathcal{G}_n).$$

In addition, if the equality holds, then $H \in \mathcal{G}_n$.

Remark 10. The proof of Theorem 9 is lengthy, so we defer it to Section 5. Furthermore, for all sufficiently large n, if

$$\left|\mathrm{ex}(n,\mathcal{F}) - \mathrm{ex}(n-1,\mathcal{F}) - \pi(\mathcal{F})\binom{n}{k-1}\right| < cn^{k-1} \tag{3}$$

and

$$\left|\lambda_\alpha(\mathcal{G}_n) - k!\mathrm{ex}(n,\mathcal{F})n^{-k/\alpha}\right| \leqslant cn^{k-k/\alpha-1}, \tag{4}$$

then there exists a constant r such that

$$\lambda_\alpha(\mathcal{G}_n) \geqslant \lambda_\alpha(\mathcal{G}_{n-1}) + (k-k/\alpha)(1-rc)\pi(\mathcal{F})n^{k-k/\alpha-1}.$$

Therefore, under the conditions (3) and (4), for sufficiently small c, by Theorem 9, we also get the corresponding result, which generalizes the criterion by Keevash, Lenz and Mubayi [18, Theorem 1.4].

For a k-pattern P, we use $\mathrm{Col}(P)$ to denote the set of all P-colorable k-graphs.

Theorem 11. *Let $(\mathcal{F}, K_l^k)$ be a Turán pair, where $\mathcal{F}$ is a family of k-graphs that is degree-stable with respect to $\mathrm{Col}(K_l^k)$, and $l \geqslant k \geqslant 2$. For any $\alpha > 1$, there exists n_0 such that if G is an $\mathcal{F}$-free k-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_l^k(n))$, with equality if and only if $G = T_l^k(n)$.*

Proof. Since $(\mathcal{F}, K_l^k)$ is a Turán pair, by Lemma 1 we have

$$\mathrm{ex}(n,\mathcal{F}) = e(T_l^k(n)) = t_l^k(n),\ \pi(\mathcal{F}) = \lim_{n\to\infty} \frac{t_l^k(n)}{\binom{n}{k}} = \frac{(l)_k}{l^k}.$$

By the degree-stability of $\mathcal{F}$ with respect to $\mathrm{Col}(K_l^k)$, there exist n_0 and $\varepsilon' \in (0,1)$ such that every $\mathcal{F}$-free k-graph H on $n \geqslant n_0$ vertices with $\delta(H) \geqslant (\pi(\mathcal{F})/(k-1)! - \varepsilon')n^{k-1}$ is contained in $\mathrm{Col}(K_l^k)$. Let $\varepsilon := \varepsilon'/2$ and define $\mathcal{G}_n$ as the family of all n-vertex $\mathcal{F}$-free k-graphs G with $\delta(G) \geqslant (1-\varepsilon)\pi(\mathcal{F})\binom{n}{k-1}$. For sufficiently large n and any $G \in \mathcal{G}_n$, we have

$$\delta(G) \geqslant (1-\varepsilon)\pi(\mathcal{F})\binom{n}{k-1} \geqslant (\pi(\mathcal{F})/(k-1)! - \varepsilon')n^{k-1},$$

which implies that G is a member of $\mathrm{Col}(K_l^k)$ and hence G is l-partite.

By the definition of Turán pair, $T_l^k(n)$ is $\mathcal{F}$-free as it is K_l^k-colorable, and it satisfies

$$\delta(T_l^k(n)) \geqslant e(T_l^k(n)) - e(T_l^k(n-1)) = \frac{\pi(\mathcal{F})}{(k-1)!}n^{k-1} + O(n^{k-2}),$$

implying that $T_l^k(n) \in \mathcal{G}_n$. By Lemma 1 and Theorem 8, for sufficiently large n,

$$\begin{aligned}\lambda_\alpha(\mathcal{G}_n) - \lambda_\alpha(\mathcal{G}_{n-1}) &\geqslant \lambda_\alpha(T_l^k(n)) - \frac{(l)_k}{l^k}(n-1)^{k-k/\alpha} \\ &\geqslant k!e(T_l^k(n))n^{-k/\alpha} - \frac{(l)_k}{l^k}(n-1)^{k-k/\alpha} \\ &= \frac{(l)_k}{l^k}n^{k-k/\alpha} + O(n^{k-k/\alpha-2}) - \frac{(l)_k}{l^k}(n-1)^{k-k/\alpha} \\ &= (k-k/\alpha)\pi(\mathcal{F})n^{k-k/\alpha-1} + o(n^{k-k/\alpha-1}),\end{aligned}$$

where the 2nd inequality follows from (1). So, we get the result by Theorems 9 and 7. □

A $2k$-graph G is called *bipartite-like* if its vertex set has a bipartition such that each edge contains exactly k vertices from each part. An edge maximal bipartite-like $2k$-graph is called *complete bipartite-like.* Let $B_{2k}(n)$ be the complete balanced bipartite-like $2k$-graph on n vertices.

Lemma 12. *Let $\alpha > 1$, and let G be a complete bipartite-like 4-graph on n vertices. If H is a sub-hypergraph of G, then $\lambda_\alpha(H) < \lambda_\alpha(G)$, unless $H = G$.*

Proof. Let V_1 and V_2 be the two parts of G, and let $\mathbf{x}$ be a nonnegative eigenvector of G corresponding to $\lambda_\alpha(G)$. From Lemma 5, all entries of $\mathbf{x}$ indexed by vertices in the same part are equal. If there exists a vertex $u \in V_j$ ($j \in \{1, 2\}$) such that $x_u = 0$, then $x_v = 0$ for all $v \in V_j$. This implies $\lambda_\alpha(G) = 0$, which is impossible since G is nonempty. Thus, every nonnegative eigenvector corresponding to $\lambda_\alpha(G)$ is positive. The result follows from Lemma 6. $\square$

Lemma 13. *Let G be a bipartite-like 4-graph on n vertices. If $\alpha > 1$, then*

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(B_4(n)) \leqslant \frac{3}{8}(n-2)^2 n^{2-4/\alpha},$$

with left equality if and only if $G = B_4(n)$ and right equality if and only if $2 \mid n$.

Proof. Suppose that G is a bipartite-like 4-graph on n vertices with maximum α-spectral radius. By Lemma 12, G is a complete bipartite-like 4-graph with a positive eigenvector $\mathbf{x}$ corresponding to $\lambda_\alpha(G)$. Let V_1 and V_2 be two parts of G, where $|V_1| := t$. By Lemma 5, we can assume that $x_v := (\frac{\gamma}{t})^{1/\alpha}$ if $v \in V_1$ and $x_v := (\frac{1-\gamma}{n-t})^{1/\alpha}$ if $v \in V_2$, for some $\gamma \in (0, 1)$. Then

$$\begin{aligned}\lambda_\alpha(G) &= 4! \max_{0<\gamma<1} \binom{t}{2}\binom{n-t}{2}\left(\frac{\gamma}{t}\right)^{2/\alpha}\left(\frac{1-\gamma}{n-t}\right)^{2/\alpha} \\ &= 4! \times 2^{-4/\alpha}\binom{t}{2}\binom{n-t}{2} t^{-2/\alpha}(n-t)^{-2/\alpha}.\end{aligned}$$

Consider the following function of t on $[2, n-2]$:

$$\begin{aligned}f(t) &= \binom{t}{2}\binom{n-t}{2} t^{-2/\alpha}(n-t)^{-2/\alpha} \\ &= \frac{1}{4} t^{1-2/\alpha}(n-t)^{1-2/\alpha}(t-1)(n-t-1).\end{aligned}$$

Write $g_1(t) = \frac{1}{4} t^{1-2/\alpha}(n-t)^{1-2/\alpha}$ and $h_1(t) = (t-1)(n-t-1)$. If $\alpha \geqslant 2$, noting that $g_1(t)$ and $h_1(t)$ are nonnegative, symmetric with respect to $t = n/2$, increasing in $[2, n/2]$ and decreasing in $[n/2, n-2]$, we have $f(n/2) = \max\{f(t) : t \in [2, n-2]\}$. Observe that

$$f(t) = \frac{1}{4} t^{2-2/\alpha}(n-t)^{2-2/\alpha} + \frac{1}{4}(1-n) t^{1-2/\alpha}(n-t)^{1-2/\alpha}.$$

Write $g_2(t) = \frac{1}{4}t^{2-2/\alpha}(n-t)^{2-2/\alpha}$ and $h_2(t) = \frac{1}{4}(1-n)t^{1-2/\alpha}(n-t)^{1-2/\alpha}$. If $1 < \alpha < 2$, noting that $g_2(t)$ and $h_2(t)$ are symmetric with respect to $t = n/2$, increasing on $[2, n/2]$ and decreasing on $[n/2, n-2]$, we still have $f(n/2) = \max\{f(t) : t \in [2, n-2]\}$.

In summary, for $\alpha > 1$,

$$\lambda_\alpha(G) \leqslant 4! \times 2^{-4/\alpha} f(\lfloor n/2 \rfloor) \leqslant 4! \times 2^{-4/\alpha} f(n/2),$$

or equivalently,

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(B_4(n)) \leqslant \frac{3}{8}(n-2)^2 n^{2-4/\alpha}.$$

The result follows. □

Now let us focus on the pattern $P = (\{1,2\}, \{\{1,1,2,2\}\})$. Note that a 4-graph is P-colorable if and only if it is bipartite-like. Moreover, it is straightforward to verify that $B_4(n)$ attains the maximum number of edges among all bipartite-like 4-graphs on n vertices. By simple calculation, we have

$$e(B_4(n)) = \binom{\lfloor n/2 \rfloor}{2}\binom{\lceil n/2 \rceil}{2} = \frac{n^4 - 4n^3}{64} + O(n^2). \tag{5}$$

Theorem 14. *Let $(\mathcal{F}, P)$ be a Turán pair with $P = (\{1,2\}, \{\{1,1,2,2\}\})$, where $\mathcal{F}$ is a family of 4-graphs that is degree-stable with respect to $\mathrm{Col}(P)$. For any $\alpha > 1$, there exists n_0 such that if G is an $\mathcal{F}$-free 4-graph G on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(B_4(n))$, with equality if and only if $G = B_4(n)$.*

Proof. Since $(\mathcal{F}, P)$ is a Turán pair, by (5) we have

$$\mathrm{ex}(n, \mathcal{F}) = e(B_4(n)) = \frac{n^4 - 4n^3}{64} + O(n^2),\ \pi(\mathcal{F}) = \lim_{n\to\infty} \frac{e(B_4(n))}{\binom{n}{4}} = \frac{3}{8}.$$

By the degree-stability of $\mathcal{F}$ with respect to $\mathrm{Col}(P)$, there exist n_0 and $\varepsilon \in (0,1)$ such that every $\mathcal{F}$-free 4-graph $\mathcal{H}$ on $n \geqslant n_0$ vertices with $\delta(\mathcal{H}) \geqslant (1/16 - \varepsilon)n^3$ is contained in $\mathrm{Col}(P)$. Define $\mathcal{G}_n$ as the family of all n-vertex $\mathcal{F}$-free 4-graphs G with $\delta(G) \geqslant \frac{3(1-\varepsilon)}{8}\binom{n}{3}$. For sufficiently large n and any $G \in \mathcal{G}_n$, we have

$$\delta(G) \geqslant \frac{3(1-\varepsilon)}{8}\binom{n}{3} \geqslant \Big(\frac{1}{16} - \varepsilon\Big)n^3,$$

which implies that G is a member of $\mathrm{Col}(P)$ and hence G is a bipartite-like 4-graph.

By definition $B_4(n)$ is $\mathcal{F}$-free as it is P-colorable, and it holds

$$\delta(B_4(n)) \geqslant e(B_4(n)) - e(B_4(n-1)) = \frac{1}{16}n^3 + O(n^2).$$

This implies $B_4(n) \in \mathcal{G}_n$. By Lemma 13, for sufficiently large n, we have

$$\begin{aligned}
\lambda_\alpha(\mathcal{G}_n) - \lambda_\alpha(\mathcal{G}_{n-1}) \geqslant & \lambda_\alpha(B_4(n)) - \frac{3}{8}(n-3)^2(n-1)^{2-4/\alpha} \\
\geqslant & 4! e(B_4(n)) n^{-4/\alpha} - \frac{3}{8}\big(n^{4-4\alpha} - (8-4/\alpha)n^{3-4/\alpha}\big) + O(n^{2-4/\alpha}) \\
\geqslant & \frac{3}{8}\big(n^{4-4/\alpha} - 4n^{3-4/\alpha}\big) - \frac{3}{8}\big(n^{4-4\alpha} - (8-4/\alpha)n^{3-4/\alpha}\big) + O(n^{2-4/\alpha}) \\
= & (4-4/\alpha)\pi(\mathcal{F})n^{3-4/\alpha} + o(n^{3-4/\alpha}).
\end{aligned}$$

The result now follows from Theorem 9 and Lemma 13. □

4 Applications

In this section, we apply the results in Section 3 to the spectral Turán problems of $\mathcal{F}$-free hypergraphs for some special families $\mathcal{F}$.

4.1 The expansion of complete graphs

For a simple graph G and an integer $k \geqslant 3$, the *k-expansion of G*, denoted by $G^{(k)}$, is the k-graph obtained from G by enlarging each edge of G with $k-2$ new vertices disjoint from $V(G)$ such that distinct edges of G are enlarged by distinct vertices. In [29], Pikhurko proved that $\mathrm{EX}(n, K_{l+1}^{(k)}) = \{T_l^k(n)\}$ for any $l \geqslant k \geqslant 3$ when n is sufficiently large. So $(K_{l+1}^{(k)}, K_l^k)$ is a Turán pair. Pikhurko also proved that $K_{l+1}^{(k)}$ is edge-stable with respect to $\mathrm{Col}(K_l^k)$ (see [29, Lemma 3]), and is also vertex-extendable (see Page 12 in [15]). By Theorem 3, we know that $K_{l+1}^{(k)}$ is degree-stable with respect to $\mathrm{Col}(K_l^k)$. Hence, by Theorem 11, we obtain the following corollary.

Corollary 15. *For any $l \geqslant k \geqslant 3$ and $\alpha > 1$, there exists n_0 such that if G is a $K_{l+1}^{(k)}$-free k-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_l^k(n))$, with equality if and only if $G = T_l^k(n)$.*

4.2 The extension of hypergraphs

Let $k \geqslant 3$ and F be a k-graph. The *extension* of F, denoted by H^F, is the k-graph obtained from F by adding a $(k-2)$-set B_{ij} of new vertices and an edge $\{v_i, v_j\} \cup B_{ij}$ for each pair of vertices v_i, v_j of F that is not contained in any edge of F, and moreover, these $(k-2)$-sets of new vertices are pairwise disjoint.

The *generalized k-fan*, denoted by Fan^k, is the extension of the k-graph on $k+1$ vertices with only one edge. In [23], Mubayi and Pikhurko proved that $\mathrm{EX}(n, \mathrm{Fan}^k) = \{T_k^k(n)\}$ for $k \geqslant 3$ and sufficiently large n, and proved that Fan^k is edge-stable with respect to $\mathrm{Col}(K_k^k)$. So (Fan^k, K_k^k) is a Turán pair. It follows from Page 13 in [15] that Fan^k is vertex-extendable. So, by Theorem 3, we know that Fan^k is degree-stable with respect to $\mathrm{Col}(K_k^k)$. Therefore, by Theorem 11, we obtain the following result.

Corollary 16. *For any $l \geqslant k \geqslant 3$ and $\alpha > 1$, there exists n_0 such that if G is a* Fan^k*-free k-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_k^k(n))$, with equality if and only if $G = T_k^k(n)$.*

Let M_t^k denote the k-graph consisting of t vertex-disjoint edges, also called a *t-matching*. Let $S_t^{(k)}$ denote the k-expansion of a star S_t with t edges, which is also called a *t-hyperstar*. By Corollary 1.13 and Concluding Remarks in [21], we see that the extension $H^{M_t^3}$ (respectively, $H^{S_t^{(3)}}$, $H^{S_t^{(4)}}$) is degree-stable with respect to $\mathrm{Col}(K_{3t-1}^3)$(respectively, $\mathrm{Col}(K_{2t}^3)$, $\mathrm{Col}(K_{3t}^4)$) for $t \geqslant 2$. By Theorems 5.1-5.3 in [16], for $t \geqslant 2$ and sufficiently large n,

$$\mathrm{EX}(n, H^{M_t^3}) = \{T_{3t-1}^3(n)\},\ \mathrm{EX}(n, H^{S_t^{(3)}}) = \{T_{2t}^3(n)\},\ \mathrm{EX}(n, H^{S_t^{(4)}}) = \{T_{3t}^4(n)\}.$$

So $(H^{M_t^3}, K_{3t-1}^3)$, $(H^{S_t^{(3)}}, K_{2t}^3)$ and $(H^{S_t^{(4)}}, K_{3t}^4)$ are all Turán pairs. By Theorem 11, we get the following spectral extremal results immediately.

Corollary 17. *For any $\alpha > 1$ and $t \geqslant 2$, there exists n_0 such that if G is an $H^{M_t^3}$-free 3-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{3t-1}^3(n))$, with equality if and only if $G = T_{3t-1}^3(n)$.*

Corollary 18. *For any $\alpha > 1$ and $t \geqslant 2$, there exists n_0 such that if G is an $H^{S_t^{(3)}}$-free 3-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{2t}^3(n))$, with equality if and only if $G = T_{2t}^3(n)$.*

Corollary 19. *For any $\alpha > 1$ and $t \geqslant 2$, there exists n_0 such that if G is an $H^{S_t^{(4)}}$-free 4-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{3t}^4(n))$, with equality if and only if $G = T_{3t}^4(n)$.*

4.3 Cancellative hypergraphs and generalized triangles

A k-graph G is called *cancellative* if whenever A, B, C are edges of G with $A \cup B = A \cup C$ (or equivalently, $B \Delta C \subseteq A$, where Δ denotes the symmetric difference), we have $B = C$. In particular, a graph G is cancellative if and only if it is triangle-free.

The Turán problems of cancellative hypergraphs are closely related to the generalized triangles. The *generalized triangle* T_k is the k-graph with vertex set $[2k-1]$ and edge set

$$\{\{1, \ldots, k-1, k\}, \{1, \ldots, k-1, k+1\}, \{k, k+1, \ldots, 2k-1\}\}.$$

Note that a 3-graph is cancellative if and only if it is $\{F_4, T_3\}$-free, where F_4 is a hypergraph with edge set $\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$. Bollobás [2] showed that $\mathrm{EX}(n, \{F_4, T_3\}) = \{T_3^3(n)\}$. Subsequently, Frankl and Füredi [7] proved that $\mathrm{EX}(n, T_3) = \{T_3^3(n)\}$ for all $n \geqslant 3000$, and this was improved to $n \geqslant 33$ by Keevash and Mubayi [19]. In [28], Pikhurko proved that $\mathrm{EX}(n, T_4) = \{T_4^4(n)\}$ for sufficiently large n. So (T_3, K_3^3) and (T_4, K_4^4) are both Turán pairs. From Theorem 1.10 in [21], we know that T_3 (respectively, T_4) is degree-stable with respect to $\mathrm{Col}(K_3^3)$ (respectively, $\mathrm{Col}(K_4^4)$). Therefore, by Theorem 11, we have the following result.

Corollary 20. *For any $\alpha > 1$ and $k \in \{3, 4\}$, there exists n_0 such that if G is a T_k-free k-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_k^k(n))$, with equality if and only if $G = T_k^k(n)$.*

Since cancellative k-graphs must be T_k-free, we immediately obtain the following corollary, which implies the result in [27, Theorem 3.2].

Corollary 21. *For any $\alpha > 1$ and $k \in \{3, 4\}$, there exists n_0 such that if G is a cancellative k-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(T_k^k(n))$, with equality if and only if $G = T_k^k(n)$.*

4.4 4-book with three pages

Let F_7 denote the 4-graph with vertex set [7] and edge set

$$\{\{1234\}, \{1235\}, \{1236\}, \{4567\}\},$$

also called *a 4-book with three pages*. Füredi, Pikhurko, and Simonovits [9] proved that $\mathrm{EX}(n, F_7) = \{B_4(n)\}$ for sufficiently large n. So (F_7, P) is a Turán pair, where $P = (\{1, 2\}, \{\{1, 1, 2, 2\}\})$. They also proved that F_7 is degree-stable with respect to $\mathrm{Col}(P)$. Hence, by Theorem 14, we obtain the following corollary.

Corollary 22. *For any $\alpha > 1$, there exists n_0 such that if G is an F_7-free 4-graph on $n \geqslant n_0$ vertices, then $\lambda_\alpha(G) \leqslant \lambda_\alpha(B_4(n))$, with equality if and only if $G = B_4(n)$.*

5 Proof of Theorem 9

In this section, we establish the proof of Theorem 9. We begin by presenting several preliminary lemmas derived under the assumptions and notations of Theorem 9.

Fact 23. *For $p > -1$ and sufficiently large n,*

$$\sum_{i=1}^{n} i^p = (1 + o(1))\frac{n^{p+1}}{p+1}.$$

Proof. The result holds trivially for $p = 0$. Consider the function $f(x) = x^p$, which is increasing on the interval $[0, n+1]$ for $p > 0$, and is decreasing on the interval $[1, n]$ for $-1 < p < 0$.

Case 1: $p > 0$. In this case, we have

$$\int_0^n x^p dx \leqslant \sum_{i=1}^{n} i^p \leqslant \int_1^{n+1} x^p dx.$$

By a direct calculation,

$$\frac{n^{p+1}}{p+1} \leqslant \sum_{i=1}^{n} i^p \leqslant \frac{(n+1)^{p+1} - 1}{p+1},$$

which implies the desired result.

Case 2: $-1 < p < 0$. We have

$$\int_1^n x^p dx + f(n) \leqslant \sum_{i=1}^n i^p \leqslant \int_1^n x^p dx + f(1),$$

where $f(1) = 1$, $f(n) = n^p$. Therefore,

$$\frac{n^{p+1} - 1}{p+1} + n^p \leqslant \sum_{i=1}^n i^p \leqslant \frac{n^{p+1} - 1}{p+1} + 1.$$

The result follows. □

Fact 24. *If* $0 \leqslant x < 1$ *and* $\beta > 0$*, then* $(1-x)^{-\beta} \geqslant 1 + \beta x$.

Fact 25. *If* $0 < x < \frac{1}{2}$*, then* $1 - x \geqslant e^{-x-x^2}$.

Fact 26. *If* $x > 1$*, then* $\frac{1}{x} < \ln x - \ln(x-1)$ *and* $\frac{1}{x^2} < \frac{1}{x-1} - \frac{1}{x}$.

Lemma 27. *If* n *is sufficiently large, then* $\lambda_\alpha(\mathcal{G}_n) \geqslant \pi(\mathcal{F})(1 - 2\varepsilon')n^{k-k/\alpha}$.

Proof. By (2) and Fact 23, for sufficiently large $n \geqslant N$ we have

$$\begin{aligned}
\lambda_\alpha(\mathcal{G}_n) &= \sum_{i=N}^n (\lambda_\alpha(\mathcal{G}_i) - \lambda_\alpha(\mathcal{G}_{i-1})) + \lambda_\alpha(\mathcal{G}_{N-1}) \\
&\geqslant \sum_{i=N}^n (k - k/\alpha)\pi(\mathcal{F})(1-\varepsilon')i^{k-k/\alpha-1} + \lambda_\alpha(\mathcal{G}_{N-1}) \\
&= \sum_{i=1}^n (k - k/\alpha)\pi(\mathcal{F})(1-\varepsilon')i^{k-k/\alpha-1} + O(1) \\
&= (1 + o(1))\pi(\mathcal{F})(1-\varepsilon')n^{k-k/\alpha} \\
&\geqslant \pi(\mathcal{F})(1-2\varepsilon')n^{k-k/\alpha}.
\end{aligned}$$

□

Let H be an n-vertex $\mathcal{F}$-free k-graph, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a nonnegative eigenvector corresponding to $\lambda_\alpha(H)$. Note that by definition $\mathbf{x}$ has unit length with respect to ℓ_α-norm. Define $x_{\min} = \min\{x_v : v \in V(H)\}$. For a subset $U \subseteq V(H)$, denote $x_U := \Pi_{v\in U} x_v$. The following two lemmas mainly discuss the spectral property of H.

Lemma 28. *Let* $\varepsilon'' = \varepsilon\pi(\mathcal{F})/(2(k-1))$*. If* $\lambda_\alpha(H) \geqslant \lambda_\alpha(\mathcal{G}_n)$ *and* $\delta(H) < (1-\varepsilon)\pi(F)\binom{n}{k-1}$*, then for sufficiently large* n*,*

$$x_{\min}^\alpha < \frac{1 - \varepsilon''}{n}.$$

Proof. Let $V := V(H)$, $\lambda := \lambda_\alpha(H)$, $\delta := \delta(H)$, and let $u \in V$ be a vertex with degree δ. By eigenequation at vertex u,

$$\lambda x_{\min}^{\alpha-1} \leqslant \lambda x_u^{\alpha-1} = (k-1)! \sum_{e\in E_H(u)} x_{e\setminus\{u\}}.$$

Applying Hölder's inequality, we have

$$\left(\frac{\lambda x_{\min}^{\alpha-1}}{(k-1)!}\right)^{\alpha} \leqslant \delta^{\alpha-1} \sum_{e\in E_H(u)} x_{e\setminus\{u\}}^{\alpha}. \tag{6}$$

We estimate the right-hand sum above as follows:

$$\begin{aligned}
\sum_{e\in E_H(u)} x_{e\setminus\{u\}}^{\alpha} &= \sum_{S\in\binom{V}{k-1}} x_S^{\alpha} - \sum_{T\in\binom{V}{k-1} \text{ and } T\cup\{u\}\notin E_H(u)} x_T^{\alpha} \\
&\leqslant \sum_{S\in\binom{V}{k-1}} x_S^{\alpha} - \sum_{T\in\binom{V}{k-1} \text{ and } T\cup\{u\}\notin E_H(u)} x_{\min}^{\alpha(k-1)} \\
&= \sum_{S\in\binom{V}{k-1}} x_S^{\alpha} - \left(\tbinom{n}{k-1} - \delta\right) x_{\min}^{\alpha(k-1)}.
\end{aligned} \tag{7}$$

By Maclaurin's inequality, we have

$$\sum_{S\in\binom{V}{k-1}} x_S^{\alpha} \leqslant \binom{n}{k-1}\left(\frac{1}{n}\sum_{i\in V} x_i^{\alpha}\right)^{k-1} = \frac{\binom{n}{k-1}}{n^{k-1}}. \tag{8}$$

Assume to the contrary that $x_{\min}^{\alpha} \geqslant \frac{1-\varepsilon''}{n}$. Combining (7), (8) and the assumption $\delta(H) < (1-\varepsilon)\pi(F)\binom{n}{k-1}$, we have

$$\begin{aligned}
\sum_{e\in E_H(u)} x_{e\setminus\{u\}}^{\alpha} &\leqslant \frac{\binom{n}{k-1}}{n^{k-1}} - \big(1-(1-\varepsilon)\pi(\mathcal{F})\big)\tbinom{n}{k-1} x_{\min}^{\alpha(k-1)} \\
&\leqslant \frac{\binom{n}{k-1}}{n^{k-1}} - \big(1-(1-\varepsilon)\pi(\mathcal{F})\big)\tbinom{n}{k-1} \frac{(1-\varepsilon'')^{k-1}}{n^{k-1}} \\
&= \frac{\binom{n}{k-1}}{n^{k-1}}\Big(1 - \big(1-(1-\varepsilon)\pi(\mathcal{F})\big)\big(1-\varepsilon''\big)^{k-1}\Big).
\end{aligned}$$

By Bernoulli's inequality and the definition of ε'', we have

$$(1-\varepsilon'')^{k-1} \geqslant 1-(k-1)\varepsilon'' = 1 - \varepsilon\pi(\mathcal{F})/2 \geqslant 1-\varepsilon\pi(\mathcal{F}).$$

Therefore,

$$\begin{aligned}\sum_{e\in E_H(u)} x^{\alpha}_{e\setminus\{u\}} &\leqslant \frac{\binom{n}{k-1}}{n^{k-1}}\Big(1-\big(1-(1-\varepsilon)\pi(\mathcal{F})\big)\big(1-\varepsilon\pi(\mathcal{F})\big)\Big)\\ &= \frac{\binom{n}{k-1}}{n^{k-1}}\Big(\pi(\mathcal{F})-\varepsilon(1-\varepsilon)\pi(\mathcal{F})^2\Big)\\ &\leqslant \frac{\pi(\mathcal{F})\binom{n}{k-1}}{n^{k-1}}.\end{aligned} \tag{9}$$

Combining this inequality with (6), we have

$$\begin{aligned}\left(\frac{\lambda x_{\min}^{\alpha-1}}{(k-1)!}\right)^{\alpha} &\leqslant \frac{\pi(\mathcal{F})\binom{n}{k-1}}{n^{k-1}}\left((1-\varepsilon)\pi(\mathcal{F})\binom{n}{k-1}\right)^{\alpha-1}\\ &\leqslant \frac{(1-\varepsilon)^{\alpha-1}\pi(\mathcal{F})^{\alpha}n^{(k-1)(\alpha-1)}}{((k-1)!)^{\alpha}}.\end{aligned} \tag{10}$$

However, by Lemma 27, for sufficiently large n we have

$$\lambda \geqslant \lambda_{\alpha}(\mathcal{G}_n) \geqslant \pi(\mathcal{F})(1-2\varepsilon')n^{k-k/\alpha}.$$

This gives the lower bound:

$$\left(\frac{\lambda x_{\min}^{\alpha-1}}{(k-1)!}\right)^{\alpha} \geqslant \frac{(1-\varepsilon'')^{\alpha-1}(1-2\varepsilon')^{\alpha}\pi(\mathcal{F})^{\alpha}n^{(k-1)(\alpha-1)}}{((k-1)!)^{\alpha}}. \tag{11}$$

Combining (10) and (11) and noting that $\varepsilon'' \leqslant \varepsilon/2$, we have

$$1-2\varepsilon' \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon''}\right)^{\frac{\alpha-1}{\alpha}} \leqslant \left(\frac{1-\varepsilon}{1-\varepsilon/2}\right)^{\frac{\alpha-1}{\alpha}} < \left(1-\frac{\varepsilon}{2}\right)^{\frac{\alpha-1}{\alpha}}.$$

Thus, by Bernoulli's inequality,

$$2\varepsilon' > 1-\left(1-\frac{\varepsilon}{2}\right)^{\frac{\alpha-1}{\alpha}} \geqslant 1-\left(1-\frac{\varepsilon(\alpha-1)}{2\alpha}\right) = \frac{\varepsilon(\alpha-1)}{2\alpha}.$$

This contradicts the fact that

$$\varepsilon' = \frac{\varepsilon\pi(\mathcal{F})(\alpha-1)}{2k\alpha} \leqslant \frac{\varepsilon(\alpha-1)}{4\alpha}.$$

This completes the proof of Lemma 28. □

Lemma 29. *Let $\varepsilon'' = \varepsilon\pi(\mathcal{F})/(2(k-1))$. If $\lambda_{\alpha}(H) \geqslant \lambda_{\alpha}(\mathcal{G}_n)$ and v is a vertex of H such that $x_v^{\alpha} < (1-\varepsilon'')/n$, then for sufficiently large n,*

$$\lambda_{\alpha}(H-v) \geqslant (1-(k-k/\alpha)(1-\varepsilon''/2)n^{-1})\lambda_{\alpha}(H)$$

and

$$\lambda_{\alpha}(H-v) > \lambda_{\alpha}(\mathcal{G}_{n-1}).$$

Proof. Let $\mathbf{x}'$ be a sub-vector of $\mathbf{x}$ only by removing the component x_v. For the k-graph $H-v$, we have

$$P_{H-v}(\mathbf{x}') = \lambda_\alpha(H) - k! x_v \sum_{e\in E_H(v)} x_{e\setminus\{v\}} = \lambda_\alpha(H) - k\lambda_\alpha(H)x_v^\alpha.$$

Note that $P_{H-v}(\mathbf{x}') \leqslant \lambda_\alpha(H-v)(\|\mathbf{x}'\|_\alpha)^k$. Since $x_v^\alpha < (1-\varepsilon'')/n$, by Fact 24 we have

$$\begin{aligned} \frac{\lambda_\alpha(H-v)}{\lambda_\alpha(H)} &\geqslant \frac{1-kx_v^\alpha}{(1-x_v^\alpha)^{k/\alpha}} \\ &\geqslant (1-kx_v^\alpha)(1+kx_v^\alpha/\alpha) \\ &= 1-(k-k/\alpha)x_v^\alpha - k^2x_v^{2\alpha}/\alpha \\ &\geqslant 1-\frac{(k-k/\alpha)(1-\varepsilon'')}{n} - \frac{k^2(1-\varepsilon'')^2}{\alpha n^2}, \end{aligned} \tag{12}$$

which implies that for sufficiently large n,

$$\lambda_\alpha(H-v) \geqslant (1-(k-k/\alpha)(1-\varepsilon''/2)n^{-1})\lambda_\alpha(H).$$

Note that H is an n-vertex $\mathcal{F}$-free k-graph with $\lambda_\alpha(H) \geqslant \lambda_\alpha(\mathcal{G}_n)$. For sufficiently large n, by (2) we have

$$\lambda_\alpha(H) \geqslant \lambda_\alpha(\mathcal{G}_n) \geqslant \lambda_\alpha(\mathcal{G}_{n-1}) + (k-k/\alpha)\pi(\mathcal{F})(1-\varepsilon')n^{k-k/\alpha-1} \tag{13}$$

and by Lemma 4,

$$\lambda_\alpha(\mathcal{G}_{n-1}) \leqslant (1+o(1))\pi(\mathcal{F})(n-1)^{k-k/\alpha} \leqslant (1+o(1))\pi(\mathcal{F})n^{k-k/\alpha}. \tag{14}$$

Substituting (13) and (14) into (12), we derive

$$\begin{aligned} \lambda_\alpha(H-v) \geqslant & \lambda_\alpha(\mathcal{G}_{n-1}) + (k-k/\alpha)\pi(\mathcal{F})(1-\varepsilon')n^{k-k/\alpha-1} \\ & -(k-k/\alpha)\pi(\mathcal{F})(1-\varepsilon'')(1+o(1))n^{k-k/\alpha-1} + O(n^{k-k/\alpha-2}) \\ \geqslant & \lambda_\alpha(\mathcal{G}_{n-1}) + \varepsilon(1-1/\alpha)\pi(\mathcal{F})^2(2(k-1))^{-1}n^{k-k/\alpha-1} + o(n^{k-k/\alpha-1}) \\ > & \lambda_\alpha(\mathcal{G}_{n-1}), \end{aligned}$$

where the second equality follows from

$$\begin{aligned} (1-\varepsilon')-(1-\varepsilon'')(1+o(1)) &= \varepsilon''-\varepsilon'+o(1) \\ &= \varepsilon\pi(\mathcal{F})\left(\frac{1}{2(k-1)} - \frac{1}{2k}\left(1-\frac{1}{\alpha}\right)\right) + o(1) \\ &\geqslant \varepsilon\pi(\mathcal{F})\left(\frac{1}{2(k-1)} - \frac{1}{2k}\right) \\ &= \varepsilon\pi(\mathcal{F})\frac{1}{2k(k-1)}. \end{aligned}$$

This completes the proof of Lemma 29. □

Finally, we will finish the proof of Theorem 9.

Proof of Theorem 9. Let H be an n-vertex $\mathcal{F}$-free k-graph such that $\lambda_\alpha(H) \geqslant \lambda_\alpha(\mathcal{G}_n)$. We aim to show that for sufficiently large n, $H \in \mathcal{G}_n$, or equivalently, $\delta(H) \geqslant (1-\varepsilon)\pi(F)\binom{n}{k-1}$. We may assume that N_0 is sufficiently large to apply Lemmas 27, 28 and 29 for $n \geqslant N_0$. Define

$$n_0 = \left(\frac{N_0^{k-k/\alpha}\mathrm{e}^{k^2}}{(1-2\varepsilon')\pi(\mathcal{F})}\right)^{\frac{2}{(k-k/\alpha)\varepsilon''}}.$$

Clearly, we have $n_0 > N_0$.

We assert that when $n \geqslant n_0$, H has a sub-hypergraph G on m vertices with $\delta(G) \geqslant (1-\varepsilon)\pi(F)\binom{m}{k-1}$ and $m > N_0$. The idea is that we can keep removing the vertex of minimum value given by the nonnegative eigenvector associated with the α-spectral radius, and then we must eventually get the sub-hypergraph G as wanted. Suppose this does not give us a suitable sub-hypergraph even after we have got N_0 vertices left. This means we can find a sequence of k-graphs:

$$H = H_n \supseteq H_{n-1} \supseteq \cdots \supseteq H_{N_0},$$

where, for each $i > N_0$, H_i has i vertices with $\delta(H_i) < (1-\varepsilon)\pi(F)\binom{i}{k-1}$, and $H_{i-1} = H_i - u_i$, where the vertex $u_i \in V(H_i)$ satisfies $x_{u_i} = \min\{x_v : v \in V(H_i)\}$, and $\mathbf{x}$ is a nonnegative eigenvector for $\lambda_\alpha(H_i)$.

By Lemma 28, if $\lambda_\alpha(H_i) \geqslant \lambda_\alpha(\mathcal{G}_i)$ and $\delta(H_i) < (1-\varepsilon)\pi(F)\binom{i}{k-1}$, then $x_{u_i}^\alpha < (1-\varepsilon'')/i$; and by Lemma 29,

$$\lambda_\alpha(H_{i-1}) \geqslant \lambda_\alpha(H_i)(1-(k-k/\alpha)(1-\varepsilon''/2)i^{-1}), \tag{15}$$

and

$$\lambda_\alpha(H_{i-1}) > \lambda_\alpha(\mathcal{G}_{i-1}). \tag{16}$$

We note that equation (16) guarantees the repeated application of Lemmas 28 and 29 such that both equations (15) and (16) hold for all $i > N_0$.

By (15), we have

$$\begin{aligned}
\lambda_\alpha(H_{N_0}) \geqslant& \lambda_\alpha(H_{N_0+1})\left(1-\frac{(k-k/\alpha)(1-\varepsilon''/2)}{N_0+1}\right)\\
\geqslant& \lambda_\alpha(H_n)\prod_{i=N_0+1}^{n}\left(1-\frac{(k-k/\alpha)(1-\varepsilon''/2)}{i}\right)\\
\geqslant& \lambda_\alpha(H_n)\exp\left(-\sum_{i=N_0+1}^{n}\left(\frac{(k-k/\alpha)(1-\varepsilon''/2)}{i}+\frac{k^2}{i^2}\right)\right)\\
\geqslant& \lambda_\alpha(H_n)\exp\left(-(k-k/\alpha)(1-\varepsilon''/2)\ln\frac{n}{N_0}-k^2\right)\\
\geqslant& (1-2\varepsilon')\pi(\mathcal{F})n^{k-k/\alpha}\left(\frac{n}{N_0}\right)^{-(k-k/\alpha)(1-\varepsilon''/2)}\mathrm{e}^{-k^2}
\end{aligned}$$

$$\geqslant (1-2\varepsilon')\pi(\mathcal{F})n^{(k-k/\alpha)\varepsilon''/2}\mathrm{e}^{-k^2}$$
$$\geqslant (1-2\varepsilon')\pi(\mathcal{F})n_0^{(k-k/\alpha)\varepsilon''/2}\mathrm{e}^{-k^2}$$
$$\geqslant N_0^{k-k/\alpha},$$

where the 3rd and 4th inequalities follow from Facts 25 and 26, respectively. This yields a contradiction, since the α-spectral radius of any k-graph on N_0 vertices is less than $N_0^{k-k/\alpha}$.

Hence, the removal process must terminate at H_t for some $t > N_0$. By the stopping condition, we have $\delta(H_t) \geqslant (1-\varepsilon)\pi(\mathcal{F})\binom{t}{k-1}$, which implies $H_t \in \mathcal{G}_t$, and hence $\lambda_\alpha(H_t) \leqslant \lambda_\alpha(\mathcal{G}_t)$. If $t < n$, by (16) we have $\lambda_\alpha(H_t) > \lambda_\alpha(\mathcal{G}_t)$, yielding a contradiction. Therefore $t = n$, and hence $\delta(H) = \delta(H_n) \geqslant (1-\varepsilon)\pi(\mathcal{F})\binom{n}{k-1}$, which implies that $H \in \mathcal{G}_n$. □

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