Adjacent Cycle-Chains are e-Positive

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Abstract

We describe a way to decompose the chromatic symmetric function as a positive sum of smaller pieces. We show that these pieces are e-positive for cycles. Then we prove that attaching a cycle to a graph preserves the e-positivity of these pieces. From this, we prove an e-positive formula for graphs of cycles connected at adjacent vertices. We extend these results to graphs formed by connecting a sequence of cycles and cliques.

Mathematics Subject Classifications: 05E05, 05C15

1 Introduction

For a graph G, the chromatic symmetric function $X_G(\boldsymbol{x})$ was first defined by Stanley in [14], as a function on $\boldsymbol{x} = x_1, x_2, \ldots$, an infinite sequence of variables. The chromatic symmetric function can be written in many bases, and one particular field of interest is the positivity of the coefficients in these bases. One specific basis of interest is the elementary basis, or the e-basis. The famous Stanley–Stembridge conjecture in [16, Conjecture 5.5] claims that incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free partially ordered sets are e-positive, meaning the coefficients of the chromatic symmetric functions of these graphs in the e-basis are all positive. This conjecture is closely related to the immanants of Jacobi-Trudi matrices [16], the cohomology of Hessenberg varieties [2], and the characters of Kazhdan–Lusztig elements of the Hecke algebra [1]. Gasharov in [9] proved that such graphs are Schur-positive, a weaker condition than e-positivity. Guay-Paquet in [11] reduced the Stanley–Stembridge conjecture to proving that all natural unit interval graphs are e-positive. The converse is not true, as there are many non-unit interval graphs (such as cycles) that are e-positive. It is generally unknown when a graph is e-positive or not, but many papers have proven the e-positivity of specific families of graphs and derived explicit formulas for certain families of graphs [3, 4, 5, 6, 7, 17, 18].

Many results related to e-positivity have been achieved working with certain generalizations or alternate versions of the chromatic symmetric function, such as the chromatic

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symmetric function in non-commuting variables [10] or the quasisymmetric refinement [13].

The first author in [17] found a new method of finding the chromatic symmetric function for certain unit interval graphs, using objects called forest triples. With these he proved that K-chains, which are cliques connected at single vertices, are e-positive. We will use a similar method to prove that adjacent cycle chains are also e-positive. More precisely, our paper is structured as follows.

Section 2 introduces the definitions to describe the chromatic symmetric function, the e-basis, unit interval graphs, and the Stanley–Stembridge conjecture. In Section 3, we present a way to calculate the e-expansion of the chromatic symmetric function of a graph using objects called forest triples. In Section 4 we find a way to decompose the chromatic symmetric function into multiple pieces, which we conjecture are all e-positive. Section 5 presents a new proof of the already-known e-positivity of cycles, using forest triples and involutions. In Section 6, we prove that given an involution on "cycle+tree" graphs, attaching a cycle to a certain graphs at a single vertex preserves e-positivity. From this, we get that adjacent cycle chains are e-positive. Additionally, we derive explicit formulas for certain graphs, such as the graph of two cycles connected at a single vertex, $C_a + C_b$, with

$$X_{C_a+C_b}(\boldsymbol{x}) = \sum_{\substack{\alpha\models a,\beta\models b+\alpha_1 \\ \mathrm{len}(\beta)\geqslant 2,\beta_1\leqslant\alpha_1,\beta_2\geqslant\alpha_1}} \begin{array}{r}(\alpha_1-1)\cdots(\alpha_l-1)\cdot(\beta_2-\beta_1+1)(\beta_1+\beta_2-\alpha_1-1)\cdot \\ (\beta_3-1)\cdots(\beta_l-1)\cdot e_{\mathrm{sort}(\alpha\backslash\alpha_1\cdot\beta\backslash\beta_1\cdot(\beta_1-1))}.\end{array} \quad (1)$$

Finally, Section 7 constructs the involution on "cycle+tree" graphs used in Section 6, through similar methods as Section 5.

2 Background

In this paper, G always references a loopless non-directed graph with n vertices labeled 1 through n, and with a fixed total ordering on the edges. The vertex set of a graph is denoted as $V(G)$ and edge set as $E(G)$. The number of vertices in a graph is denoted as $|G|$. The clique K_n is the graph with n vertices and an edge between every pair of vertices. Let $\mathbb{N}$ represent the set of natural numbers $\{1, 2, \ldots\}$.

Definition 1. A *proper coloring* of graph G is a function $\kappa : V(G) \to \mathbb{N}$ such that if $(i, j) \in E(G)$, then $\kappa(i) \neq \kappa(j)$.

Definition 2. Let $\boldsymbol{x} = (x_i \text{ for all } i \in \mathbb{N})$ be an infinite tuple of variables. Then, the *chromatic symmetric function* of G is

$$X_G(\boldsymbol{x}) = \sum_{\kappa \text{ is proper}} \left(\prod_{i=1}^{n} x_{\kappa(i)} \right).$$

Remark 3. The function is symmetric since $X_G(\boldsymbol{x}) = X_G(\sigma(\boldsymbol{x}))$ for any permutation σ. The function is also homogeneous, since the degree of every term is n. Note that $X_G(\boldsymbol{x})$ is independent of the labeling of G and the ordering of the edges of G.

We use different bases to write symmetric functions without needing to use an infinite number of variables. The basis of interest in this paper is the e-basis. First, we must define a partition.

Definition 4. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a tuple of positive integers. Then, λ is a *partition* of n (denoted $\lambda \vdash n$) if λ is a weakly decreasing sequence such that $\sum_{i=1}^{l} \lambda_i = n$. We let $\mathrm{len}(\lambda) = l$ be the length of partition λ, which is the number of positive integers in the tuple λ.

Now, we define an elementary symmetric function.

Definition 5. For some partition λ of n, the *elementary symmetric function* e_λ of degree n is

$$e_\lambda = \prod_{i=1}^{\mathrm{len}(\lambda)} e_{(\lambda_i)}, \text{ where } e_{(k)} = \sum_{\substack{i_1,\ldots,i_k \in \mathbb{N} \\ i_1 < i_2 < \cdots < i_k}} x_{i_1} \cdots x_{i_k}.$$

Because every chromatic symmetric function is both symmetric and homogeneous, they can be written uniquely as the sum of finitely many elementary symmetric functions [15, Theorem 7.4.4].

Example 6. If G is the path graph with two vertices, its chromatic symmetric function can be written as

$$X_G(\boldsymbol{x}) = \sum_{\substack{i,j \in \mathbb{N} \\ i<j}} 2x_i x_j = 2e_{(2)}. \tag{2}$$

Definition 7. A graph G is *e-positive* if the expansion $X_G(\boldsymbol{x}) = \sum_{\lambda \vdash n} c_\lambda e_\lambda$ has only non-negative coefficients.

Now, we define unit interval graphs.

Definition 8. Graph G with labeled vertices 1 through n is a *natural unit interval graph* if for all $i < j < k$ where $(i, k) \in E(G)$, both $(i, j) \in E(G)$ and $(j, k) \in E(G)$.

We can finally state the Stanley–Stembridge conjecture.

Conjecture 9 ([16, Conjecture 5.5])**.** If G is a natural unit interval graph, then it is e-positive.

3 Forest Triples

Forest triples provide a way to calculate the chromatic symmetric function for a graph. First, we introduce the concept of a composition.

Definition 10. Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a tuple of positive integers. Then, α is a *composition* of n (denoted as $\alpha \models n$) if $\sum_{i=1}^{l} \alpha_i = n$. The length of the composition is denoted as $\operatorname{len}(\alpha)$, which is the number of positive integers, or *parts*, in the composition. We now define a few properties on compositions. The notation α_l always denotes the last part of composition α.

For some α, we denote $\alpha \setminus \alpha_i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_l)$. For two compositions α and β, we define $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l)$.

We denote a tuple of compositions as $\left(\alpha^{(1)}, \ldots, \alpha^{(m)}\right)$, where m is the number of compositions. This allows us to index the tuple for a specific composition and a specific part of the composition. For example, the first part of the second composition can be denoted as $\alpha_1^{(2)}$.

We can link a composition to a partition by sorting the elements.

Definition 11. If α is a composition of n, then $\operatorname{sort}(\alpha)$ is the partition of n formed by sorting the elements of α in non-increasing order.

Now, we will introduce the concept of broken circuits. Recall that G is a graph with labeled vertices 1 through n and ordered edges.

Definition 12. For every cycle C that is a subgraph of G, the corresponding *broken circuit* $B \subset G$ is the subgraph of C without the largest edge of C.

Definition 13. A *non-broken* circuit F of graph G is a spanning subgraph of G such that no subgraph of F is a broken circuit.

Every non-broken circuit is a forest, since if there is a cycle in F, then there exists a subgraph of that cycle that is a broken circuit.

Definition 14. A *tree triple* of G is a tuple $\mathcal{T} = (T, \alpha, r)$ where T is a tree in some non-broken circuit F, α is a composition of $|T|$, and r is an integer with $1 \leqslant r \leqslant \alpha_1$.

We can finally define what a forest triple is.

Definition 15. A *forest triple* $\mathcal{F}$ of G is a set of tree triples $\{\mathcal{T}_1, \ldots, \mathcal{T}_m\}$ where $\mathcal{T}_i = (T_i, \alpha^{(i)}, r_i)$ and the trees T_1 through T_m are the trees in some non-broken circuit F of graph G.

The type of a forest triple, $\operatorname{type}(\mathcal{F})$, is the partition formed by combining and sorting all the parts in every composition in decreasing order. The sign of a forest triple can be defined as

$$\operatorname{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^{m}\left(\operatorname{len}(\alpha^{(i)})-1\right)}.$$

Definition 16. A tree triple is *unit* if $\operatorname{len}(\alpha) = 1$. A *unit* forest triple is one where every tree triple is unit.

Remark 17. Note if $\mathcal{F}$ is unit, then $\operatorname{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^{m}(1-1)} = 1$. Additionally, if $\mathcal{T} = (T, \alpha, r)$ is unit, then $\alpha_1 = |T|$.

Let $\mathrm{FT}(G)$ be the set of all forest triples of G. We can now state the following theorem regarding forest triples.

Theorem 18 ([17, Theorem 5.10])**.** *The chromatic symmetric function of any graph G is*

$$X_G(\boldsymbol{x}) = \sum_{\mathcal{F}\in \mathrm{FT}(G)} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}(\mathcal{F})}. \tag{3}$$

Remark 19. Note that given a graph G with labeled vertices, the set $\mathrm{FT}(G)$ depends on how the edges are ordered. However, the formula in Equation 3 provides the same result for any arbitrary ordering of the edges, since the chromatic symmetric function of a graph does not depend on any ordering of the edges.

Example 20. Figure 1 shows the four forest triples of the path graph on 2 vertices, P_2, and how the sum of $\mathrm{sign}(\mathcal{F})\cdot e_{\mathrm{type}(\mathcal{F})}$ results in $X_{P_2}(\boldsymbol{x}) = 2e_{(2)}$ as calculated in Equation 2.

Figure 1: All forest triples for P_2.

Forest Triples	$\mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}(\mathcal{F})}$	Forest Triples	$\mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}(\mathcal{F})}$
$\alpha = (1), r = 1$ 1 $\alpha = (1), r = 1$ 2	$e_{(1,1)}$	$\alpha = (2), r = 1$ 1 2	$e_{(2)}$
$\alpha = (1,1), r = 1$ 1 2	$-e_{(1,1)}$	$\alpha = (2), r = 2$ 1 2	$e_{(2)}$

4 Sign-Reversing Involutions

Now, we describe a way to prove a graph is e-positive using forest triples.

Definition 21. A *sign-reversing involution* is a function $\varphi : \mathrm{FT}(G) \to \mathrm{FT}(G)$ with the following properties:

1. It is an involution, meaning $\varphi(\varphi(\mathcal{F})) = \mathcal{F}$.
2. It preserves type, so $\mathrm{type}(\mathcal{F}) = \mathrm{type}(\varphi(\mathcal{F}))$.
3. If $\mathcal{F} \neq \varphi(\mathcal{F})$, then $\mathrm{sign}(\mathcal{F}) \neq \mathrm{sign}(\varphi(\mathcal{F}))$.
4. If $\mathcal{F} = \varphi(\mathcal{F})$, then $\mathrm{sign}(\mathcal{F}) = 1$, and we say that $\mathcal{F}$ is a fixed point.

If there exists a sign-reversing involution on $\mathrm{FT}(G)$, then we can pair every forest triple with a negative sign to a non-fixed forest triple with a positive sign, meaning

$$X_G(\boldsymbol{x}) = \sum_{\substack{\mathcal{F}\in \mathrm{FT}(G) \\ \mathcal{F} \text{ is fixed under } \varphi}} e_{\mathrm{type}(\mathcal{F})}. \tag{4}$$

We will look at sign-reversing involutions with an additional property.

Definition 22. For $\mathcal{F} \in \mathrm{FT}(G)$, let $\mathcal{T}_{\min} = (T_{\min}, \alpha^{(\min)}, r_{\min})$ reference the unique tree triple where the smallest vertex of G is in $V(T_{\min})$.

Definition 23. Let $\varphi : \mathrm{FT}(G) \to \mathrm{FT}(G)$ be a sign-reversing involution. Suppose all fixed points in φ are units. If for all $\mathcal{F} \in \mathrm{FT}(G)$, letting $\mathcal{F}' = \varphi(\mathcal{F})$ with $\mathcal{T}'_{\min} = (T'_{\min}, \alpha^{(\min)\prime}, r'_{\min}) \in \mathcal{F}'$, we have $\alpha_1^{(\min)} = \alpha_1^{(\min)\prime}$ and $r_{\min} = r'_{\min}$, then φ is a *first-preserving involution.*

Definition 24. Let $\mathrm{FT}^{(i)}(G)$ be the set of forest triples where $\alpha_1^{(\min)} = i$ and $r_{\min} = 1$.

Letting $\mathcal{F} = \{\mathcal{T}_{\min}, \mathcal{T}_2, \ldots, \mathcal{T}_m\}$, we define $\mathrm{type}'(\mathcal{F}) = \mathrm{sort}(\alpha^{(\min)} \setminus \alpha_1^{(\min)} \cdot \alpha^{(2)} \cdots \alpha^{(m)})$. Note that $e_{\mathrm{type}'(\mathcal{F})} = e_{\mathrm{type}(\mathcal{F})} \,/\, e_{\alpha_1^{(\min)}}$.

If $\mathrm{FT}(G)$ has a first-preserving involution, then there exists sign-reversing involutions on the subsets $\mathrm{FT}^{(i)}(G)$.

Definition 25. For a graph G and a positive integer i, we denote

$$X_G^{(i)}(\boldsymbol{x}) = \sum_{\mathcal{F} \in \mathrm{FT}^{(i)}(G)} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})}. \tag{5}$$

Then we can see that $X_G^{(i)}$ is a homogeneous symmetric function of degree $|G| - i$.

Note that $X_G^{(i)}$ has non-negative coefficients for all integers i if and only if G has a first-preserving involution φ, letting Equation 5 be rewritten as

$$X_G^{(i)}(\boldsymbol{x}) = \sum_{\substack{\mathcal{F} \in \mathrm{FT}^{(i)}(G) \\ \mathcal{F} \text{ fixed under } \varphi}} e_{\mathrm{type}'(\mathcal{F})}. \tag{6}$$

The chromatic symmetric function can be written as

$$X_G(\boldsymbol{x}) = \sum_{i=1}^{|G|} e_i \cdot i \cdot X_G^{(i)}(\boldsymbol{x}). \tag{7}$$

Example 26. Returning to the forest triples of $G = P_2$ shown in Figure 1, we have $X_G^{(1)}(\boldsymbol{x}) = 0$ and $X_G^{(2)}(\boldsymbol{x}) = 1$.

The first author proved in [17, Theorem 4.10] that all K-chains, which are cliques joined at single vertices, have a first-preserving involution (for a specific labeling of vertices and ordering of edges). It is conjectured in [17, Section 5] that all natural unit interval graphs with a specific ordering of edges have a first-preserving involution.

To prove that adjacent cycle chains are e-positive, we will try to find a first-preserving involution for them. We will first look at cycles.

5 Cycle Graphs

In this section, we find a first-preserving involution on the set of forest triples of cycle graphs.

Definition 27. For $a \in \mathbb{N}$, we define $C_a = ([a], E = \{(1,2) < \cdots < (a-1,a) < (a,1)\})$ as the cycle graph with a vertices.

Stanley [14, Proposition 5.4] found a generating function formula to prove that the cycle C_a is e-positive. Ellzey gave the following e-positive expansion, which we will also prove using forest triples.

Theorem 28 ([8, Corollary 6.2])**.** *The cycle graph with a vertices, C_a, has a chromatic symmetric function equal to*

$$X_{C_a}(\boldsymbol{x}) = \sum_{\alpha \models a} e_{\mathrm{sort}(\alpha)} \cdot \alpha_1 \cdot (\alpha_1 - 1) \cdots (\alpha_l - 1). \tag{8}$$

Example 29. The graph C_6 has

$$\begin{matrix} \alpha = & (6) & (4,2) & (3,3) & (2,4) & (2,2,2) \\ X_{C_6}(\boldsymbol{x}) = & \overbrace{30e_6} & + \overbrace{12e_{4,2}} & + \overbrace{12e_{3,3}} & + \overbrace{6e_{4,2}} & + \overbrace{2e_{2,2,2}}. \end{matrix}$$

Let $\mathcal{F} = \{\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1), \ldots, \mathcal{T}_m\} \in \mathrm{FT}(C_a)$ be a forest triple with m tree triples. We order the tree triples so that $1 \in V(T_m)$ and for $1 \leqslant i < j \leqslant m-1$, we have $\min(V(T_i)) < \min(V(T_j))$.

For each $\mathcal{T} = (T, \alpha, r) \in \mathcal{F}$, we can find the size of T based on α (since $\alpha \models |T|$). Thus we can identify $\mathcal{T}$ with a tuple (v, α, r), where v is the vertex such that $V(T) = \{(v, v+1), \ldots, (v + |T| - 2, v + |T| - 1)\}$, with vertices taken mod a. We then denote forest triples as

$$\mathcal{F} = \left\langle (v_1, \alpha^{(1)}, r_1), \ldots, (v_m, \alpha^{(m)}, r_m) \right\rangle,$$

where T_i has edges $\{(v_i, v_i+1), \ldots, (v_{i+1} - 2, v_{i+1} - 1)\}$ mod a and $\mathcal{T}_i = (T_i, \alpha^{(i)}, r_i)$. A first-preserving involution preserves $\alpha_1^{(m)}$ and r_m, since $1 \in V(T_m)$.

An example is given in Subfigure 2a.

To define a first-preserving involution on $\mathrm{FT}(C_a)$, we break it into disjoint subsets such that every forest triple is in exactly one subset. Then, if there exists a first-preserving involution on each subset, they can be combined together to form a first-preserving involution on $\mathrm{FT}(C_a)$.

Say $\gamma : S_1 \to S_2$ is a bijective function between two disjoint subsets of $\mathrm{FT}(C_a)$, where γ preserves $\mathrm{type}(\mathcal{F})$ and $\alpha_1^{(m)}$ and r_m, but reverses $\mathrm{sign}(\mathcal{F})$. Then, we can define a first-preserving involution φ on $S_1 \cup S_2$, where φ either applies γ or its inverse. This involution has no fixed points.

Proof of Theorem 28. We now define the involution in three steps, breaking $\mathrm{FT}(C_a)$ into five subsets.

Step 1: We define sets A and B with indexed subsets A_i and B_i for $i \in \mathbb{N}$ as

$$A_i = \left\{\mathcal{F} \in \mathrm{FT}(C_a) : m-1 \geqslant i;\ r_i = 1;\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(i)}) = 1;\ r_1, \ldots, r_{i-1} \geqslant 2\right\},$$
$$B_i = \{\mathcal{F} \in \mathrm{FT}(C_a) : m \geqslant i;\ \mathrm{len}(\alpha^{(i)}) \geqslant 2;\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(i-1)}) = 1;\ r_1, \ldots, r_{i-1} \geqslant 2;$$
$$\text{either } m-1 \geqslant i \text{ or } \alpha_l^{(m)} \leqslant a - v_m + 1\}.$$

Intuitively, if $\mathcal{F} \in A_i$ or $\mathcal{F} \in B_i$, then i is the smallest integer such that either $\mathrm{len}(\alpha^{(i)}) \geqslant 2$ or $r_i = 1$. We define function $\mathrm{join}_i : A_i \to B_i$, where $\mathrm{join}_i(\mathcal{F})$ replaces $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ in $\mathcal{F}$ with tree triples

$$S = \left(v_i, \alpha^{(i+1)} \cdot \alpha^{(i)}, r_{i+1}\right).$$

The inverse map $\mathrm{break}_i : B_i \to A_i$ replaces $\mathcal{T}_i$ in $\mathcal{F}$ with

$$S_1 = \left(v_i, (\alpha_l^{(i)}), 1\right), S_2 = \left(v_i + \alpha_l^{(i)}, \alpha^{(i)} \setminus \alpha_l^{(i)}, r_i\right).$$

Notice that if $\mathcal{F} \in A_i$, then tree triples $\mathcal{T}_1$ through $\mathcal{T}_{i-1}$ all have $\mathrm{len}(\alpha) = 1$ and $r \geqslant 2$. Those tree triples are untouched in $\mathrm{join}_i(\mathcal{F})$, while $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ are replaced by S. Since S has a composition with length at least 2, then $\mathrm{join}_i(\mathcal{F}) \in B_i$. Applying $\mathrm{break}_i(\mathrm{join}_i(\mathcal{F}))$ breaks S while preserving the other tree triples, resulting in $\mathcal{F}$ again. Thus, join_i and break_i are inverses, meaning join_i is a bijection.

If $\mathcal{F} \in A_{m-1}$, then $\mathrm{join}_i(\mathcal{F})$ replaces $\mathcal{T}_{m-1}$ and $\mathcal{T}_m$ with S. Since every $\mathcal{F}$ has $|T_{m-1}| \leqslant a - v_{m-1} + 1$, then after joining S has $\alpha_l^{(m)} \leqslant a - v_m + 1$, explaining that condition. Also note that break_i preserves $\mathrm{type}(\mathcal{F})$, $\alpha_1^{(m)}$, r_m, and reverses $\mathrm{sign}(\mathcal{F})$. We then use this function to form a first-preserving involution on sets $A \cup B$.

Figure 2 shows examples of how join_i and break_i work, where we either join the red and blue tree triples or break the purple tree triple. In Subfigure 2b, joining $\mathcal{T}_{m-1}$ to $\mathcal{T}_m$ in $\mathcal{F}_A$ with $m = 3$ results in $\mathcal{F}_B$ having $2 = \alpha_l^{(m)} \leqslant a - v_m + 1 = 2$, hence we can do $\mathrm{break}_2(\mathcal{F}_B)$ to recover $\mathcal{F}_A$. Subfigure 2c has $\mathcal{F}$ with $m = 2$ where $3 = \alpha_l^{(m)} > a - v_m + 1 = 2$, and trying $\mathrm{break}_2(\mathcal{F})$ changes the order of the tree triples and moreover results in $\mathrm{break}_2(\mathcal{F}) \in A_1$, which is a problem.

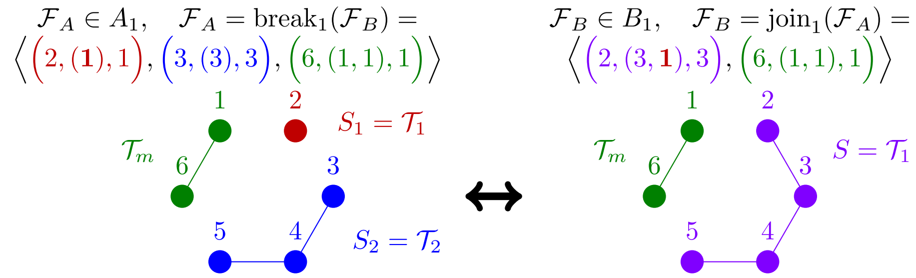
Step 2: To take care of the forest triples like the one on the left in Subfigure 2c, we will rotate the entire graph until we can break $\mathcal{T}_m$ while preserving $\alpha_1^{(m)}$ and r_m.

We define sets C, D with indexed subsets C_i and D_i for $i \in \mathbb{N}$ as

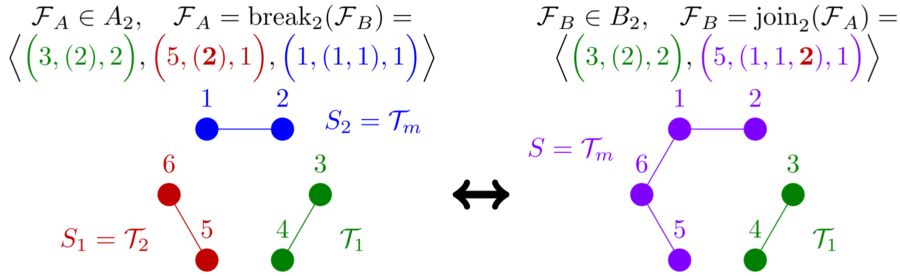
$$C_i = \{(a,1) \notin E(T_m);\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m-1)}) = 1;\ r_1, \ldots, r_{m-1} \geqslant 2;\ r_{m-1} - 1 = i\},$$
$$D_i = \{(a,1) \in E(T_m);\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m-1)}) = 1;\ r_1, \ldots, r_{m-1} \geqslant 2;$$
$$\mathrm{len}(\alpha^{(m)}) \geqslant 2;\ \alpha_l^{(m)} \geqslant a - v_m + 2;\ \alpha_l^{(m)} - a + v_m - 1 = i\}.$$

In other words, we now consider $\mathcal{F} \in \mathrm{FT}(C_a)$ where $\mathcal{F} \notin A \cup B$. If edge $(a, 1) \notin E(T_m)$, then we put $\mathcal{F} \in C_i$, where $i = r_{m-1} - 1$. If edge $(a, 1) \in E(T_m)$ and $\mathrm{len}(\alpha^{(m)}) \geqslant 2$, then we put $\mathcal{F} \in D_i$ where $i = \alpha_l^{(m)} - a + v_m - 1$.

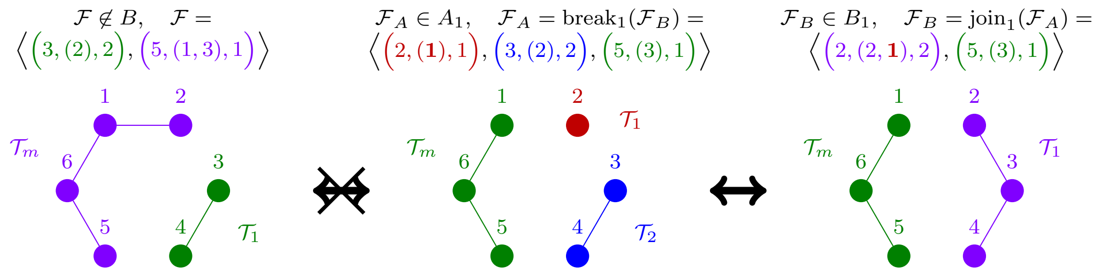
Figure 2: Examples of join and break for $\mathcal{F} \in \mathrm{FT}(C_6)$.



(a) Normal joining and breaking.



(b) Joining $\mathcal{T}_{m-1}$ to $\mathcal{T}_m$ and breaking $\mathcal{T}_m$.



(c) Trying $\mathrm{break}_2(\mathcal{F})$ when $\alpha_l^{(m)} > a - v_m + 1$ messes the order of tree triples.

We define function $\mathrm{rotatejoin}_i : C_i \to D_i$ as

$$\mathrm{rotatejoin}_i(\mathcal{F}) = \Big\langle (v_1 + i, \alpha^{(1)}, r_1), \ldots, (v_{m-2} + i, \alpha^{(m-2)}, r_{m-2}), \\ (v_{m-1} + i, \alpha^{(m)} \cdot \alpha^{(m-1)}, r_m) \Big\rangle.$$

Say $\mathcal{F}_C \in C_i$, and $\mathcal{F}_D = \mathrm{rotatejoin}_i(\mathcal{F}_C)$. Note that in $\mathcal{F}_C$, we have $\alpha_1^{(m-1)} = a - v_{m-1} + 1$. Then,

$$\begin{aligned} (v_m) \text{ in } \mathcal{F}_D &= (v_{m-1} + i) \text{ in } \mathcal{F}_C, \text{ so} \\ (\alpha_l^{(m)} - a + v_m - 1) \text{ in } \mathcal{F}_D &= (\alpha_1^{(m-1)} - a + v_{m-1} + i - 1) \text{ in } \mathcal{F}_C \\ &= i, \end{aligned}$$

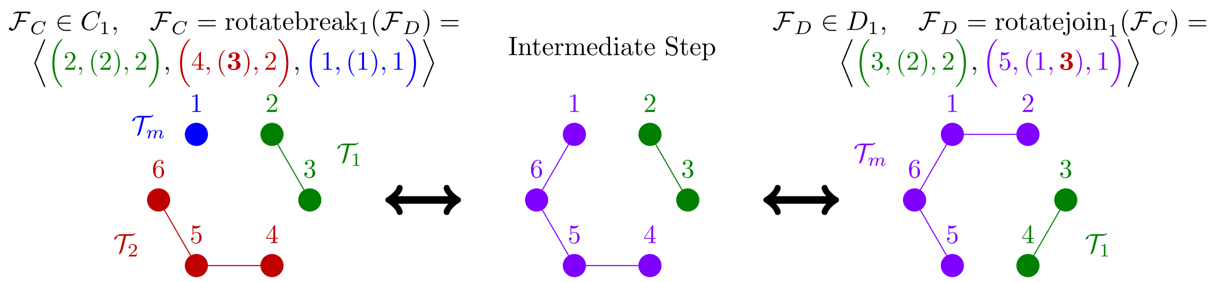
So $\mathcal{F}_D \in D_i$. The inverse map $\text{rotatebreak}_i : D_i \to C_i$ is defined as

$$\text{rotatebreak}_i(\mathcal{F}) = \Big\langle (v_1 - i, \alpha^{(1)}, r_1), \ldots, (v_{m-1} - i, \alpha^{(1)}, r_m), \\ (v_m - i, (\alpha_l^{(m)}), i+1), (1, \alpha^{(m)} \setminus \alpha_l^{(m)}, r_m) \Big\rangle.$$

We turn this bijection into a first-preserving involution on set $C \cup D$.

Figure 3 shows an example of rotatejoin_1 and rotatebreak_1, using the same forest triple in Subfigure 2c that could not be broken with normal break.

Figure 3: Example of rotatejoin and rotatebreak.



Step 3: The remaining $\mathcal{F} \in \text{FT}(C_a)$ have the property that $\mathcal{T}_1$ through $\mathcal{T}_m$ have $\text{len}(\alpha) = 1$, while $\mathcal{T}_1$ through $\mathcal{T}_{m-1}$ have $r \geqslant 2$. Additionally, edge $(a, 1) \in E(T_m)$. We place these $\mathcal{F}$ in set E.

Notice that all $\mathcal{F} \in E$ are unit, since every tree triple has $\text{len}(\alpha) = 1$. Thus, $\text{sign}(\mathcal{F})$ is always positive, meaning the identity function is a first-preserving involution on set E.

Counting Fixed Points: Combining the involutions on $A \cup B, C \cup D, E$ gives a first-preserving involution $\varphi : \text{FT}(C_a) \to \text{FT}(C_a)$, where all fixed points are in subset E.

We use Equation 4 to find the chromatic symmetric function of a cycle graph. For all $\mathcal{F} \in E$, we define $\text{comp}(\mathcal{F}) = \alpha^{(m)} \cdot \alpha^{(1)} \cdots \alpha^{(m-1)}$. Now, for some $\beta \models a$, we want to find the number of $\mathcal{F} \in E$ where $\text{comp}(\mathcal{F}) = \beta$. Since edge $(a, 1) \in T_m$, there are $|T_m| - 1 = \beta_1 - 1$ ways to place T_m, and one way to place the remaining T_1 through T_{m-1}. There are $|T_i| - 1$ possible r values for $\mathcal{T}_1$ through $\mathcal{T}_{m-1}$, and $|T_m|$ possible r values for $\mathcal{T}_m$. Thus, there are $\beta_1 \cdot (\beta_1 - 1) \cdots (\beta_l - 1)$ possible forest triples where $\text{comp}(\mathcal{F}) = \beta$.

If $\text{comp}(\mathcal{F}) = \beta$, then $\text{type}(\mathcal{F}) = \text{sort}(\beta)$. Thus, we get

$$X_{C_a}(\boldsymbol{x}) = \sum_{\beta \models a} e_{\text{sort}(\beta)} \cdot \beta_1 \cdot (\beta_1 - 1) \cdots (\beta_l - 1),$$

which is equivalent to Equation 8.

We can also calculate $X^{(i)}_{C_a}(\boldsymbol{x})$ using similar logic, counting the number of forest triples where $\text{comp}(\mathcal{F}) = \beta$ and $\beta_1 = i$ and $r_{\min} = 1$, getting that

$$X^{(i)}_{C_a}(\boldsymbol{x}) = \sum_{\alpha \models a-i} e_{\text{sort}(\alpha)} \cdot (i-1) \cdot (\alpha_1 - 1) \cdots (\alpha_l - 1). \tag{9}$$

This completes the proof. □

6 Adjacent Cycle Chains

In this section, we will prove that adding a cycle to a graph preserves the existence of a first-preserving involution. From this, we get that adjacent cycle chains are e-positive. First, we must define what adding a cycle to a graph means.

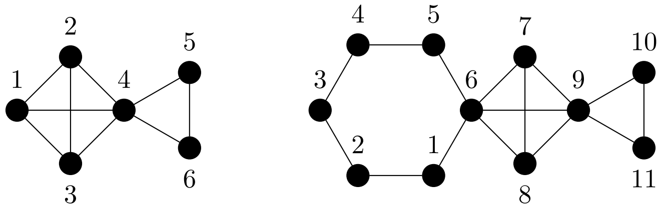
Definition 30. Let G_1 and G_2 be two graphs with k_1 and k_2 labeled vertices respectively. Both graphs have some fixed ordering of edges. Following the notation in [10], we define

$$G_1 + G_2 = ([k_1 + k_2 - 1], E = E(G_1) \cup \{(i + k_1 - 1, j + k_1 - 1) \mid (i,j) \in E(G_2)\}),$$

preserving the ordering of the edges in G_1 and G_2 and making the edges of G_1 smaller than the edges of G_2.

Example 31. Figure 4 shows a labeled graph G' as well as the new labeled graph $G = C_6 + G'$. Since G' is a K-chain, it has a first-preserving involution [17, Theorem 4.10], which we will prove in Theorem 35 also gives G a first-preserving involution.

Figure 4: The graphs G' and $C_6 + G'$.



Before jumping to cycle chains, let us first look at cycles connected to trees.

Definition 32. Let U_k be a tree with k vertices for some positive integer k. For every $\mathcal{F} \in \text{FT}(C_a + U_k)$, let $\mathcal{T} = (T, \alpha, r)$ be the tree triple where vertex $a \in V(T)$. We define $\text{FT}'(C_a + U_k) \subseteq \text{FT}(C_a + U_k)$ to be the set of all forest triples $\mathcal{F}$ where $E(U_k) \subseteq E(T)$ and $\alpha_l \geqslant k$.

Definition 33. We define $\text{FT}'^{(i)}(C_a + U_k) \subseteq \text{FT}'(C_a + U_k)$ as the subset of forest triples where $\alpha_1^{(\min)} = i$ and $r_{\min} = 1$.

In Section 7, we will prove the following lemma that the subset $\mathrm{FT}'(C_a + U_k)$ always has a first-preserving involution.

Lemma 34. *The subset* $\mathrm{FT}'(C_a + U_k)$ *has a first-preserving involution* φ. *Additionally,*

$$B_{a,k}(\boldsymbol{x}) := \sum_{\mathcal{F} \in \mathrm{FT}'(C_a+U_k)} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}(\mathcal{F})} = \sum_{\substack{\alpha \models a+k,\ \mathrm{len}(\alpha) \geqslant 2 \\ \alpha_1 \leqslant k,\ \alpha_2 \geqslant k}} (\alpha_2 - \alpha_1 + 1)(\alpha_1 + \alpha_2 - k - 1) \cdot (\alpha_3 - 1) \cdots (\alpha_l - 1) \cdot e_{\mathrm{sort}(\alpha_1 - 1, \alpha_2, \ldots, \alpha_l)}. \quad (10)$$

Moreover, we can break this function into e-positive pieces with

$$B^{(i)}_{a,k}(\boldsymbol{x}) := \sum_{\mathcal{F} \in \mathrm{FT}'^{(i)}(C_a+U_k)} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} = \begin{cases} \displaystyle\sum_{\substack{\alpha \models a+k-i-1 \\ \alpha_1 \geqslant k}} (k-i) \cdot (\alpha_2 - 1) \cdots (\alpha_l - 1) \cdot e_{\mathrm{sort}(\alpha)}, & \text{if } i \leqslant k-1 \\ \displaystyle\sum_{\substack{\alpha \models a+k-i \\ \alpha_1 \leqslant k}} (i-k) \cdot (\alpha_2 - 1) \cdots (\alpha_l - 1) \cdot e_{\mathrm{sort}(\alpha_1 - 1, \alpha \setminus \alpha_1)}, & \text{if } i \geqslant k \end{cases}, \quad (11)$$

where similar to Equation 7, we have

$$B_{a,k}(\boldsymbol{x}) = \sum_{i=1}^{a+k-1} e_i \cdot i \cdot B^{(i)}_{a,k}(\boldsymbol{x}). \quad (12)$$

Proving this lemma will involve constructing a first-preserving involution for the set $\mathrm{FT}'(C_a + U_k)$, in similar manner to Section 5. Then, we count the fixed points and derive the formulas above. It turns out that the shape of U_k does not matter, as $B_{a,k}(\boldsymbol{x})$ only depends on a and k.

Now, we turn to adding a cycle to a graph.

Theorem 35. *If* G' *is a graph where* $\mathrm{FT}(G')$ *has a first-preserving involution, then the graph* $G = C_a + G'$ *also has a first-preserving involution on* $\mathrm{FT}(G)$*, and the chromatic symmetric function for* G *can be written as*

$$X_G(\boldsymbol{x}) = \sum_{k=1}^{|G'|} X^{(k)}_{G'}(\boldsymbol{x}) \cdot B_{a,k}(\boldsymbol{x}). \quad (13)$$

The chromatic symmetric function can be broken into pieces

$$X^{(i)}_G(\boldsymbol{x}) = \sum_{k=1}^{|G'|} X^{(k)}_{G'}(\boldsymbol{x}) \cdot B^{(i)}_{a,k}(\boldsymbol{x}). \quad (14)$$

Before proving this theorem, we must introduce a few definitions.

Let $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_i, \ldots, \mathcal{T}_m\} \in \mathrm{FT}(G)$, with tree triples ordered based on their largest vertex, so if $1 \leqslant p < q \leqslant m$, then $\max(V(T_p)) < \max(V(T_q))$. Let $\mathcal{T}_i$ be the tree triple with vertex a. Note that tree triples $\mathcal{T}_1$ through $\mathcal{T}_{i-1}$ only contain vertices of the cycle (vertices 1 through a), while $\mathcal{T}_{i+1}$ through $\mathcal{T}_m$ only contain vertices of graph G'.

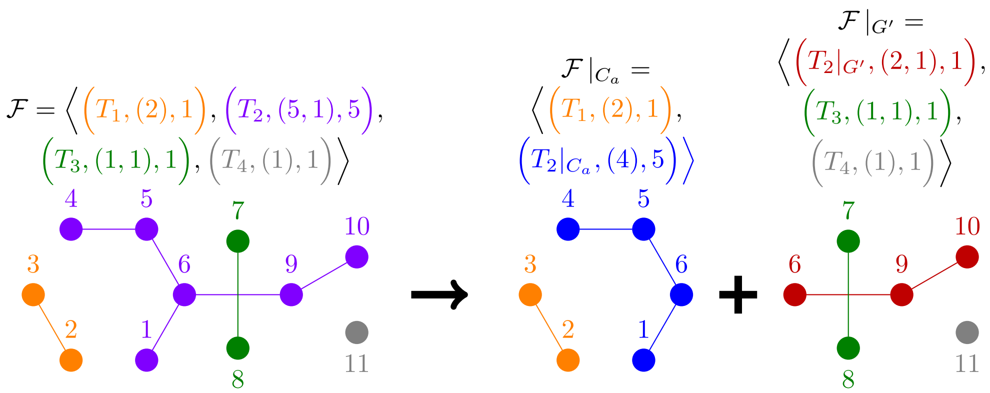
The idea behind the new involution will be to "restrict" a forest triple to G', apply the first-preserving involution on G' to that part, and then recombine the parts. Thus, we want to define how to restrict a forest triple $\mathcal{F}$ to a specific graph. To do this, we first define how to restrict a tree to a graph.

Definition 36. For subtree T of graph G, we define $T|_{G'}$, or T *restricted to* G', as the induced subgraph of T restricted to vertices in G'. Similarly, we define $T|_{C_a}$ as the induced subgraph of T restricted to vertices in C_a.

Example 37. The purple tree T_2 in Figure 5 on the left contains vertices in both the cycle and the main graph. Then, $T_2|_{G'}$ is denoted as the portion of the tree contained in G', which is the red tree in the right hand side of the diagram. Similarly, $T_2|_{C_a}$ is the blue tree in the right hand side of the diagram.

Definition 38. Let T_1 be a subtree of C_a and T_2 be a subtree of G', where vertex $a \in V(T_1)$ and $a \in V(T_2)$. Then, we define $T_1 + T_2 = T'$ to be a new tree with vertices $V(T_1) \cup V(T_2)$ and edges $E(T_1) \cup E(T_2)$. Note that $T|_{C_a} + T|_{G'} = T$.

Figure 5: A forest triple in FT(G) being broken into two parts.



We must also define how to break up a composition.

Definition 39. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \models n$ be a composition and let $1 \leqslant j \leqslant n$ be an integer. We define $\mathrm{last}(\alpha, j)$ as the unique composition of j of the form $(k, \alpha_{i+1}, \ldots, \alpha_m)$, such that $1 \leqslant k \leqslant \alpha_i$. We can also define $\mathrm{first}(\alpha, j)$ as the unique composition of j of the form $(\alpha_1, \ldots, \alpha_{i-1}, k)$, such that $1 \leqslant k \leqslant \alpha_i$.

Example 40. Let $\alpha = (3, 2, 8, 4) \models 17$, then $\mathrm{last}(\alpha, 4) = (4)$ and $\mathrm{last}(\alpha, 11) = (7, 4)$. Similarly, $\mathrm{first}(\alpha, 5) = (3, 2)$ and $\mathrm{first}(\alpha, 15) = (3, 2, 8, 2)$.

Definition 41. Let $\mathcal{F} = \{\mathcal{T}_1, \dots, \mathcal{T}_i, \dots, \mathcal{T}_m\} \in \mathrm{FT}(G)$, with the tree triples ordered as previously stated and with vertex $a \in V(T_i)$. We now define $\mathcal{F}\,|_{G'}$, or $\mathcal{F}$ *restricted to* G', as

$$\mathcal{F}\,|_{G'} = \left\{(T_i|_{G'}, \mathrm{last}\left(\alpha^{(i)}, \left| T_i|_{G'} \right|\right), 1)\right\} \cup \{\mathcal{T}_{i+1}, \dots, \mathcal{T}_m\}\,.$$

We also define $\mathcal{F}\,|_{C_a}$ as

$$\mathcal{F}\,|_{C_a} = \{\mathcal{T}_1 \dots, \mathcal{T}_{i-1}\} \cup \left\{(T_i|_{C_a}, \mathrm{first}\left(\alpha^{(i)}, \left| T_i|_{C_a} \right|\right), r_i)\right\}.$$

Remark 42. Note that $\mathcal{F}\,|_{G'}$ is a forest triple in $\mathrm{FT}(G')$. Similarly, $\mathcal{F}\,|_{C_a}$ is almost always a forest triple of C_a. However, it is possible for $r_i > \alpha_1^{(i)}$ in $\mathcal{F}\,|_{C_a}$. This does not matter, since when we reattach $\varphi(\mathcal{F}\,|_{G'})$ to $\mathcal{F}\,|_{C_a}$, we will still end up with a $\mathcal{F} \in \mathrm{FT}(G)$. This will be elaborated on in the following paragraphs.

Definition 43. We define an *almost forest triple* $\mathcal{F}$ of cycle C_a as an object $\{\mathcal{T}_1, \dots, \mathcal{T}_m\}$ (with the largest vertex in $V(T_m)$) with all the properties of a forest triple, except that if $\mathrm{len}(\alpha^{(m)}) = 1$ then it is possible for $r_m > \alpha_1^{(m)}$. All forest triples of C_a are also almost forest triples.

Example 44. Looking at Figure 5, we can break the forest triple $\mathcal{F}$ into two separate objects, $\mathcal{F}\,|_{C_a}$ and $\mathcal{F}\,|_{G'}$. Note that $\mathcal{F}\,|_{C_a}$ is not actually a forest triple because $r_2 > \alpha_1^{(2)}$, meaning it is an almost forest triple. Meanwhile, $\mathcal{F}\,|_{G'}$ is always a forest triple of graph G'.

We now define how to combine an almost forest triple of C_a and a forest triple of G'.

Definition 45. Let $\mathcal{F}_1 = \{\mathcal{T}_1, \dots, \mathcal{T}_i\}$ be an almost forest triple of C_a, where vertex $a \in T_i$. Let $\mathcal{F}_2 = \{\mathcal{T}_{i+1}, \dots, \mathcal{T}_m\}$ be a forest triple of G', with $\mathcal{T}_{\min} = \mathcal{T}_{i+1}$ and $r_{i+1} = 1$. Then, if $r_i \leqslant \alpha_1^{(i)}$, or both $\mathrm{len}(\alpha_1^{(i)}) = 1$ and $r_i \leqslant \alpha_1^{(i)} + \alpha_1^{(i+1)}$, we define $\mathcal{F}_1 + \mathcal{F}_2$ as

$$\{\mathcal{T}_1, \dots, \mathcal{T}_{i-1}\} \cup \left\{\left(T_i + T_{i+1}, (\alpha_1^{(i)}, \dots, \alpha_l^{(i)} + \alpha_1^{(i+1)}, \dots, \alpha_l^{(i+1)}), r_i\right)\right\} \cup \{\mathcal{T}_{i+2}, \dots, \mathcal{T}_m\}\,.$$

Remark 46. Note that for any forest triple $\mathcal{F}$ of G, then $\mathcal{F}\,|_{C_a} + \mathcal{F}\,|_{G'} = \mathcal{F}$.

We can now prove Theorem 35.

Proof of Theorem 35. Let $\varphi' : \mathrm{FT}(G') \to \mathrm{FT}(G')$ be a first-preserving involution on graph G'. We want to find a first-preserving involution $\varphi : \mathrm{FT}(G) \to \mathrm{FT}(G)$. For a forest triple $\mathcal{F}$, we denote tree triples $\mathcal{T}_1$ through $\mathcal{T}_m$ ordered by largest vertex, with vertex $a \in V(T_i)$.

We define subsets

$$A = \{\mathcal{F} \in \mathrm{FT}(G) : \mathcal{F}\,|_{G'} \text{ is not fixed under } \varphi'\},$$

and $B = \mathrm{FT}(G) \setminus A$. Now, say that $\mathcal{F} \in A$. Then $\varphi(\mathcal{F})$ is defined as

$$\varphi(\mathcal{F}) = (\mathcal{F}\,|_{C_a}) + \varphi'(\mathcal{F}\,|_{G'}).$$

Since φ' is a first-preserving involution, it preserves $\alpha_1^{(i+1)}$, meaning that $\mathcal{F}|_{C_a} + \varphi'(\mathcal{F}|_{G'})$ is defined. Additionally, $\varphi(\mathcal{F})|_{G'} = \varphi'(\mathcal{F}|_{G'})$ is not fixed under φ', so $\varphi(\mathcal{F}) \in A$. Notice that

$$\varphi(\varphi(\mathcal{F})) = (\mathcal{F}|_{C_a}) + \varphi'(\varphi'(\mathcal{F}|_{G'})) = (\mathcal{F}|_{C_a}) + (\mathcal{F}|_{G'}) = \mathcal{F},$$

meaning φ is an involution on A. It preserves type$(\mathcal{F})$ and reverses sign$(\mathcal{F})$, and thus is a sign-reversing involution. Additionally, it preserves α_1 and r for $\mathcal{T}_1$ through $\mathcal{T}_{i-1}$, and so it must be a first-preserving involution. Note $\varphi : A \to A$ has no fixed points.

The remaining forest triples $\mathcal{F} \in B$ have $\mathcal{F}|_{G'}$ fixed under φ'. We pick some $\mathcal{F}' \in \mathrm{FT}(G')$ where $\mathcal{F}'$ is fixed under φ' and $r'_{\min} = 1$. We look at forest triples $\mathcal{F}$ of G where $\mathcal{F}|_{G'} = \mathcal{F}'$, and denote this subset $B_{\mathcal{F}'}$.

Let $\mathrm{first}(\mathcal{F}) = \{\mathcal{T}_1, \dots, \mathcal{T}_i\}$ and $\mathrm{last}(\mathcal{F}) = \{\mathcal{T}_{i+1}, \dots, \mathcal{T}_l\}$. Note that $\mathrm{first}(\mathcal{F}) \in \mathrm{FT}'(C_a + T_i|_{G'}) = \mathrm{FT}'(C_a + T'_{\min})$, where $\mathcal{T}'_{\min} \in \mathcal{F}'$. By Lemma 34, $\mathrm{FT}'(C_a + T'_{\min})$ has a first-preserving involution, which we will call $\varphi_{\mathcal{F}'}$. We can define a first-preserving involution $\varphi : B_{\mathcal{F}'} \to B_{\mathcal{F}'}$, where

$$\varphi(\mathcal{F} \in B_{\mathcal{F}}) = \varphi_{\mathcal{F}'}(\mathrm{first}(\mathcal{F})) \cup \mathrm{last}(\mathcal{F}).$$

Then, we have

$$\varphi(\varphi(\mathcal{F})) = \varphi_{\mathcal{F}'}(\varphi_{\mathcal{F}'}(\mathrm{first}(\mathcal{F}))) \cup \mathrm{last}(\mathcal{F}) = \mathrm{first}(\mathcal{F}) \cup \mathrm{last}(\mathcal{F}) = \mathcal{F},$$

so φ is an involution. Since $\varphi_{\mathcal{F}'}$ is a first-preserving involution, we see φ preserves type$(\mathcal{F})$ and $\alpha_1^{(\min)}$ and $r_{\min}$. Additionally, φ reverses sign$(\mathcal{F})$, unless $\mathcal{F}$ is a fixed point. These fixed points in $B_{\mathcal{F}'}$ under φ have first$(\mathcal{F})$ fixed under $\varphi_{\mathcal{F}'}$, meaning fixed points are unit. Thus, $\varphi : B_{\mathcal{F}'} \to B_{\mathcal{F}'}$ is a first-preserving involution.

Note that for $\mathcal{F} \in B_{\mathcal{F}'}$, then

$$\mathrm{type}(\mathrm{last}(\mathcal{F})) = \mathrm{type}'(\mathcal{F}'),\ e_{\mathrm{type}'(\mathcal{F})} = e_{\mathrm{type}'(\mathrm{first}(\mathcal{F}))} \cdot e_{\mathrm{type}(\mathrm{last}(\mathcal{F}))},\ \mathrm{sign}(\mathcal{F}) = \mathrm{sign}(\mathrm{first}(\mathcal{F})).$$

We can now combine the first-preserving involutions on A and B, to make a new first-preserving involution on the set $\mathrm{FT}(G)$, and we can then calculate the value of $X_G^{(i)}(\boldsymbol{x})$ as

$$\begin{aligned} X_G^{(i)}(\boldsymbol{x}) &= \sum_{\substack{\mathcal{F} \in \mathrm{FT}^{(i)}(G) \\ \mathcal{F} \text{ fixed under } \varphi}} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} \\ &= \sum_{\substack{\mathcal{F}' \in \mathrm{FT}(G') \\ r_{\min}=1 \\ \mathcal{F}' \text{ fixed under } \varphi'}} \sum_{\mathcal{F} \in (B_{\mathcal{F}'}) \cap (\mathrm{FT}^{(i)}(G))} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} \\ &= \sum_{k=1}^{|G'|} \sum_{\substack{\mathcal{F}' \in \mathrm{FT}^{(k)}(G') \\ \mathcal{F}' \text{ fixed under } \varphi'}} \sum_{\substack{\mathcal{F} \in \mathrm{FT}^{(i)}(G) \\ \mathcal{F}|_{G'} = \mathcal{F}'}} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{|G'|} \sum_{\substack{\mathcal{F}' \in \mathrm{FT}^{(k)}(G') \\ \mathcal{F}' \text{ fixed under } \varphi}} \left(e_{\mathrm{type}'(\mathcal{F}')} \cdot \sum_{\substack{\mathcal{F} \in \mathrm{FT}^{(i)}(G) \\ \mathcal{F}|_{G'} = \mathcal{F}'}} \mathrm{sign}(\mathrm{first}(\mathcal{F})) \cdot e_{\mathrm{type}'(\mathrm{first}(\mathcal{F}))} \right) \\
&= \sum_{k=1}^{|G'|} \sum_{\substack{\mathcal{F}' \in \mathrm{FT}^{(k)}(G') \\ \mathcal{F}' \text{ fixed under } \varphi}} \left(e_{\mathrm{type}'(\mathcal{F}')} \cdot \sum_{\mathcal{F} \in \mathrm{FT}'^{(i)}(C_a + T'_{\min})} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} \right) \\
&= \sum_{k=1}^{|G'|} \sum_{\substack{\mathcal{F}' \in \mathrm{FT}^{(k)}(G') \\ \mathcal{F}' \text{ fixed under } \varphi}} \left(e_{\mathrm{type}'(\mathcal{F}')} \cdot B_{a,k}^{(i)}(\boldsymbol{x}) \right) = \sum_{k=1}^{|G'|} X_{G'}^{(k)}(\boldsymbol{x}) \cdot B_{a,k}^{(i)}(\boldsymbol{x}),
\end{aligned}
$$

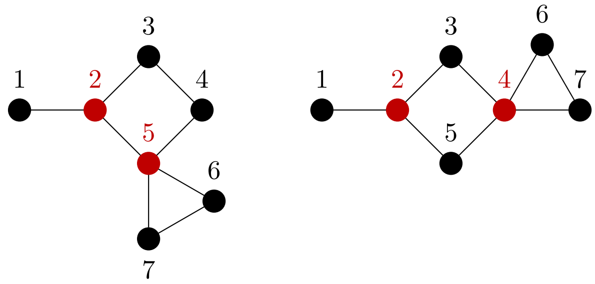
which is Equation 14. Then, using this equation with Equation 7 and Equation 12, we can easily derive Equation 13. □

We can now prove the e-positivity of adjacent cycle chains.

Definition 47. An *adjacent cycle chain* is a graph of the form $C_{a_1} + \cdots + C_{a_m}$, where $m \geqslant 1$ and $a_i \geqslant 2$ for $1 \leqslant i \leqslant m$. The labeling of the vertices and ordering of the edges of the cycles are defined in Section 5.

Example 48. Figure 6 shows the adjacent cycle chain $C_2 + C_4 + C_3$ on the left, next to a graph that is not an adjacent cycle chain on the right. Note that the cut vertices of adjacent cycle chains are adjacent to each other, while the graph on the right has non-adjacent cut vertices.

Figure 6: Two graphs with red cut vertices, on the left an adjacent cycle chain, on the right a non-adjacent cycle chain.



Corollary 49. *Adjacent cycle chains are e-positive.*

Proof. We use induction: if the adjacent cycle chain is just C_a, then it has a first-preserving involution by Theorem 28. Now, assume that all adjacent cycle chains with m cycles have a first-preserving involution. Then an adjacent cycle chain with $m + 1$ cycles can be written as $C_a + G'$, where G' is an adjacent cycle chain with m cycles and thus has a first-preserving involution. By Theorem 35, $C_a + G'$ also has a first-preserving involution, which means by induction all finite adjacent cycle chains have first-preserving involutions and are thus e-positive. □

This result provides an alternate proof for the e-positivity of tadpoles ($P_a + C_b$), hats ($P_a + C_b + P_c$), and dumbbells ($C_a + P_b + C_c$), which have been proven by previous papers [12, 19, 20]. We can extend this idea to adjacent cycle+clique chains.

Definition 50. An *adjacent cycle+clique chain* is a graph of the form $G_1 + \cdots + G_m$, where $m \geqslant 1$ and G_i is either a clique or a cycle.

Corollary 51. *Adjacent cycle+clique chains are e-positive.*

Proof. If a graph G' has a first-preserving involution, then the graph $K_a + G'$ also has a first-preserving involution [17, Theorem 4.10]. For the base case, both C_a and K_a have a first-preserving involution. Now, assume all adjacent cycle+clique chains with m cycles and cliques have a first-preserving involution. Then, an adjacent cycle+clique chain with $m + 1$ cycles, G, can be written as either $C_a + G'$ or $K_a + G'$, where G' is an adjacent cycle+clique chain with m cycles. Thus, G also has a first-preserving involution, and by induction, all finite adjacent cycle+clique chains have a first-preserving involution. □

Corollary 52. *The chromatic symmetric function of* $C_a + C_b$ *is*

$$X_{C_a+C_b}(\boldsymbol{x}) = \sum_{\substack{\alpha\models a,\beta\models b+\alpha_1 \\ \mathrm{len}(\beta)\geqslant 2,\beta_1\leqslant\alpha_1,\beta_2\geqslant\alpha_1}} (\alpha_1 - 1)\cdots(\alpha_l - 1)\cdot(\beta_2 - \beta_1 + 1)(\beta_1 + \beta_2 - \alpha_1 - 1)\cdot (\beta_3 - 1)\cdots(\beta_l - 1)\cdot e_{\mathrm{sort}(\alpha\backslash\alpha_1\cdot\beta\backslash\beta_1\cdot(\beta_1-1))}. \quad (15)$$

Proof. From Equation 9 and Equation 13, we get that

$$X_{C_a+C_b}(\boldsymbol{x}) = \sum_{k=1}^{a}\left(\sum_{\alpha\models a-k} e_{\mathrm{sort}(\alpha)}\cdot(k-1)\cdot(\alpha_1-1)\cdots(\alpha_l-1)\cdot B_{b,k}(\boldsymbol{x})\right).$$

We can combine the sum over k and the sum over $\alpha \models a-k$ to become a sum over $\alpha \models a$, where α_1 now represents the value of k. This gives

$$\sum_{\alpha\models a} e_{\mathrm{sort}(\alpha\backslash\alpha_1)}\cdot(\alpha_1-1)\cdots(\alpha_l-1)\cdot \sum_{\substack{\beta\models b+\alpha_1,\mathrm{len}(\beta)\geqslant 2 \\ \beta_1\leqslant\alpha_1,\beta_2\geqslant\alpha_1}} (\beta_2-\beta_1+1)(\beta_1+\beta_2-\alpha_1-1)\cdot(\beta_3-1)\cdots(\beta_l-1)\cdot e_{\mathrm{sort}(\beta_1-1,\beta_2,\ldots,\beta_l)},$$

which simplifies to Equation 15. □

Example 53. The chromatic symmetric function of $C_4 + C_3$ is

$$X_{C_4+C_3}\overset{\alpha,\beta\,=}{(\boldsymbol{x})} = \overbrace{4e_{3,2,1}}^{(2,2),(2,3)} + \overbrace{8e_{4,2}}^{(2,2),(1,4)} + \overbrace{12e_{4,2}}^{(4),(3,4)} + \overbrace{24e_{5,1}}^{(4),(2,5)} + \overbrace{36e_6}^{(4),(1,6)}.$$

7 Cycle Plus Tree Involution

In this section we will prove Lemma 34, which states the subset $\mathrm{FT}'(C_a + U_k)$ has a first-preserving involution and provides an explicit formula for $B_{a,k}(\boldsymbol{x})$. As a recap, we will redefine these terms below.

Definition 54. The graph $C_a + U_k$ is a cycle with a vertices connected to a tree with k vertices. For every $\mathcal{F} \in \mathrm{FT}(C_a + U_k)$, let $\mathcal{T} = (T, \alpha, r)$ where $a \in V(T)$. The set $\mathrm{FT}'(C_a + U_k)$ is the set of forest triples where $E(U_k) \subseteq E(T)$ and $\alpha_l \geqslant k$.

Then, $\mathrm{FT}'^{(i)}(C_a + U_k)$ is the subset of $\mathrm{FT}'(C_a + U_k)$ where $\alpha_1^{(\min)} = i$ and $r_{\min} = 1$.

We will define a way to denote forest triples similar to the manner described in Section 5. Say forest triple $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$ has edge $(a, 1)$. Then, we write $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_m, \mathcal{T}_{\min}\}$ as a forest triple with $m + 1$ tree triples where $m \geqslant 0$. We order the tree triples so vertex $1 \in V(T_{\min})$ and $\min(V(T_1)) < \cdots < \min(V(T_m))$.

If instead edge $(a, 1)$ is not in the forest triple, then we write $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_m, \mathcal{T}' = (T', \alpha', r'), \mathcal{T}_{\min}\}$ as a forest triple with $m + 2$ tree triples where $m \geqslant 0$. We order the tree triples so vertex $1 \in V(T_{\min})$ and $\min(V(T_1)) < \cdots < \min(V(T_m)) < \min(V(T'))$.

Similar to the cycles, we can identify the tree of a tree triple based on a single vertex. If edge $(a, 1) \in E(T_{\min})$, we write

$$\mathcal{F} = \left\langle (v_1, \alpha^{(1)}, r_1), \ldots, (v_m, \alpha^{(m)}, r_m), (v_{\min}, \alpha^{(\min)}, r_{\min}) \right\rangle,$$

where $2 \leqslant v_1 < \cdots < v_m < v_{\min} \leqslant a$ and

$$E(T_i) = \{(v_i, v_i + 1), \ldots, (v_i + |T_i| - 2, v_i + |T_i| - 1)\},$$
$$E(T_{\min}) = \{(v_{\min}, v_{\min} + 1), \ldots, (a - 1, a), (a, 1), (1, 2), \ldots, (v_1 - 2, v_1 - 1)\} \cup E(U_k).$$

If instead edge $(a, 1) \notin E(T_{\min})$, we write

$$\mathcal{F} = \left\langle (v_1, \alpha^{(1)}, r_1), \ldots, (v_m, \alpha^{(m)}, r_m), (v', \alpha', r'), (v_{\min}, \alpha^{(\min)}, r) \right\rangle,$$

where $1 = v_{\min} < v_1 < \cdots < v_m < v' \leqslant a$ and

$$E(T_i) = \{(v_i, v_i + 1), \ldots, (v_i + |T_i| - 2, v_i + |T_i| - 1)\},$$
$$E(T') = \{(v', v' + 1), \ldots, (a - 1, a)\} \cup E(U_k),$$
$$E(T_{\min}) = \{(1, 2), \ldots, (|T_{\min}| - 1, |T_{\min}|)\}.$$

Note that if $(a, 1) \in E(T_{\min})$, then for $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$ we need $\alpha_l^{(\min)} \geqslant k$. If $(a, 1) \notin E(T_{\min})$, then for $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$, we need $\alpha'_l \geqslant k$.

Figure 7 and Figure 9 provide examples of forest triples.

Proof of Lemma 34. A first-preserving involution always preserves $\alpha_1^{(\min)}$ and $r_{\min}$ for $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$. We will define this involution in 5 steps.

Step 1: This step is nearly identical to Step 1 in the cycle involution. We define sets A and B with indexed subsets A_i and B_i for $i \in \mathbb{N}$ as

$$A_i = \left\{\mathcal{F} : m-1 \geqslant i;\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(i)}) = 1;\ r_1, \ldots, r_{i-1} \geqslant 2;\ r_i = 1\right\},$$
$$B_i = \left\{\mathcal{F} : m \geqslant i;\ \mathrm{len}(\alpha^{(i)}) \geqslant 2;\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(i-1)}) = 1;\ r_1, \ldots, r_{i-1} \geqslant 2\right\}.$$

Intuitively, if $\mathcal{F} \in A_i$ or $\mathcal{F} \in B_i$, then i is the smallest integer such that either $\mathrm{len}(\alpha^{(i)}) \geqslant 2$ or $r_i = 1$. We define function $\mathrm{join}_i : A_i \to B_i$, where $\mathrm{join}_i(\mathcal{F})$ replaces $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ in $\mathcal{F}$ with tree triples

$$S = \left(v_i, \alpha^{(i+1)} \cdot \alpha^{(i)}, r_{i+1}\right).$$

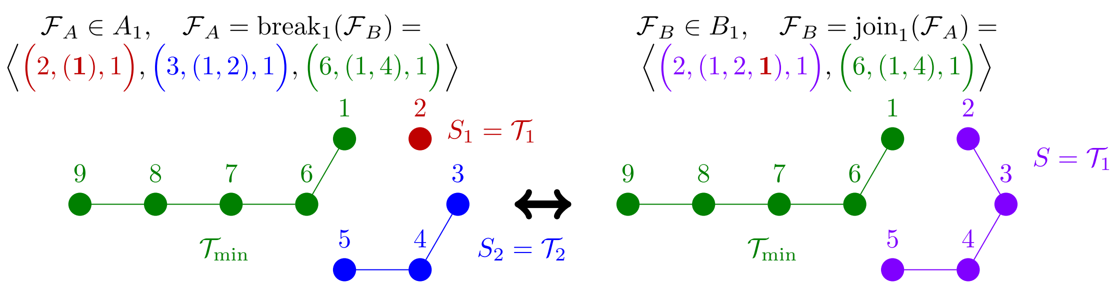
The inverse map $\mathrm{break}_i : B_i \to A_i$ replaces $\mathcal{T}_i$ in $\mathcal{F}$ with

$$S_1 = \left(v_i, (\alpha_l^{(i)}), 1\right), S_2 = \left(v_i + \alpha_l^{(i)}, \alpha^{(i)} \setminus \alpha_l^{(i)}, r_i\right).$$

If $\mathcal{F} \in A_i$, then tree triples $\mathcal{T}_1$ through $\mathcal{T}_{i-1}$ all have $\mathrm{len}(\alpha) = 1$ and $r \geqslant 2$. Those tree triples are untouched in $\mathrm{join}_i(\mathcal{F})$, while $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ are replaced by S. Since S has a composition with length at least 2, then $\mathrm{join}_i(\mathcal{F}) \in B_i$. Applying $\mathrm{break}_i(\mathrm{join}_i(\mathcal{F}))$ breaks S while preserving the other tree triples, resulting in $\mathcal{F}$ again. Thus, join_i and break_i are inverses, meaning join_i is a bijection.

Figure 7 shows an example, where we either join the red and blue tree triples or break the purple triple.

Figure 7: Example of join and break for $\mathcal{F} \in \mathrm{FT}'(C_6 + P_4)$



Step 2: This time, we focus on forest triples where edge $(a, 1) \in E(T_{\min})$, defining C and D as

$$C = \{\mathcal{F} : (a,1) \in E(T_{\min});\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m)}) = 1;\ r_1, \ldots, r_{m-1} \geqslant 2;\ r_m = 1;$$
$$\text{either } \mathrm{len}(\alpha^{(\min)}) \geqslant 2 \text{ or } |T_m| \geqslant k\},$$
$$D = \{\mathcal{F} : (a,1) \in E(T_{\min});\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m)}) = 1;\ r_1, \ldots, r_m \geqslant 2;$$
$$\mathrm{len}(\alpha^{(\min)}) \geqslant 2;\ \alpha_2^{(\min)} \leqslant a - v_{\min}\}.$$

The bijection from C to D will be similar to break_i and join_i, but instead of attaching the single part of $\alpha^{(m)}$ at the end of $\alpha^{(\min)}$, we insert it in the second position. We define $\text{secondjoin} : C \to D$, where $\text{secondjoin}(\mathcal{F})$ replaces $\mathcal{T}_m$ and $\mathcal{T}_{\min}$ with

$$S = \left(v_m, (\alpha_1^{(\min)}) \cdot \alpha^{(m)} \cdot (\alpha^{(\min)} \setminus \alpha_1^{(\min)}), r_m\right),$$

with inverse map $\text{secondbreak} : D \to C$ that replaces $\mathcal{T}_{\min}$ in $\mathcal{F}$ with tree triples

$$S_1 = \left(v_{\min}, \alpha_2^{(\min)}, 1\right), S_2 = \left(v_{\min} + \alpha_2^{(\min)}, \alpha^{(\min)} \setminus \alpha_2^{(\min)}, r_{\min}\right).$$

If $\mathcal{F}$ has $\text{len}(\alpha^{(\min)}) = 1$, then it must have $|T_m| \geqslant k$ in order for $\text{secondjoin}(\mathcal{F})$ to have $\alpha_l^{(\min)} \geqslant k$. The last condition is necessary for $\text{secondjoin}(\mathcal{F}) \in \text{FT}'(C_a + U_k)$. This explains one of the restrictions for C.

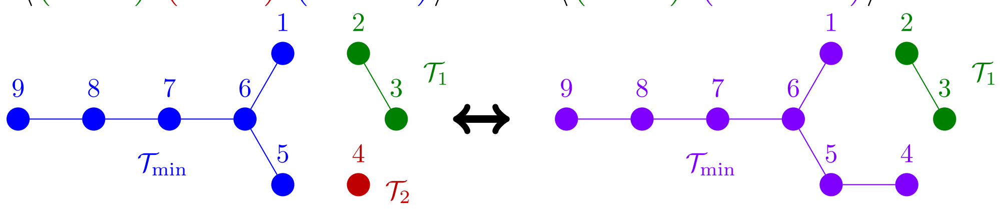
All $\mathcal{F} \in C$ have $\alpha_1^{(m)} = |T_m| \leqslant a - v_m$. Inspecting where $\alpha_1^{(m)}$ ends up after joining, we see that in $\text{secondjoin}(\mathcal{F})$, both $\text{len}(\alpha^{(\min)}) \geqslant 2$ and $\alpha_2^{(\min)} \leqslant a - v_{\min}$, which explains those restrictions in set D.

Figure 8 shows an example of secondjoin and secondbreak. Note that if $\mathcal{F}_C$ had $\alpha^{(\min)} = (6)$, then we would not be able to secondjoin.

Figure 8: Example of secondjoin and secondbreak for $\mathcal{F} \in \text{FT}'(C_6 + P_4)$.

$\mathcal{F}_C \in C, \quad \mathcal{F}_C = \text{secondbreak}(\mathcal{F}_D) = \left\langle \left(2, (2), 2\right), \left(4, (\mathbf{1}), 1\right), \left(5, (2,4), 2\right) \right\rangle$

$\mathcal{F}_D \in D, \quad \mathcal{F}_D = \text{secondjoin}(\mathcal{F}_C) = \left\langle \left(2, (2), 2\right), \left(4, (2, \mathbf{1}, 4), 2\right) \right\rangle$



Step 3: We now focus on forest triples where edge $(a, 1) \notin E(T_{\min})$, with

$$E = \{\mathcal{F} : (a,1) \notin E(T_{\min});\ \text{len}(\alpha^{(1)}) = \cdots = \text{len}(\alpha^{(m)}) = 1;\ r_1, \ldots, r_{m-1} \geqslant 2;\ r_m = 1\},$$
$$F = \{\mathcal{F} : (a,1) \notin E(T_{\min});\ \text{len}(\alpha^{(1)}) = \cdots = \text{len}(\alpha^{(m)}) = 1;\ r_1, \ldots, r_m \geqslant 2;\ \text{len}(\alpha^{(\min)}) \geqslant 2\}.$$

We define bijection $\text{shiftjoin} : E \to F$ where

$$\text{shiftjoin}(\mathcal{F}) = \left\langle \left(v_1 + \alpha_1^{(m)}, \alpha^{(1)}, r_1\right), \ldots, \left(v_{m-1} + \alpha_1^{(m)}, \alpha^{(m-1)}, r_{m-1}\right), \mathcal{T}', \left(1, \alpha^{(\min)} \cdot \alpha^{(m)}, r_{\min}\right) \right\rangle,$$

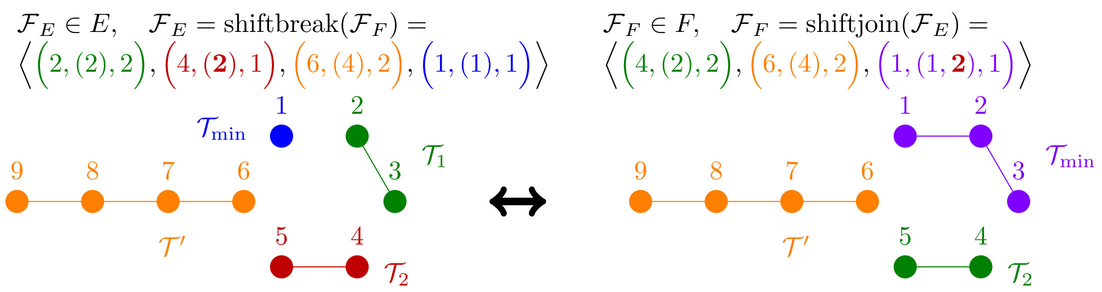
with inverse shiftbreak : $F \to E$ with

$$\text{shiftbreak}(\mathcal{F}) = \Big\langle \Big(v_1 - \alpha_l^{(\min)}, \alpha^{(1)}, r_1\Big), \dots, \Big(v_m - \alpha_l^{(\min)}, \alpha^{(m)}, r_m\Big), \\ \Big(v' - \alpha_l^{(\min)}, (\alpha_l^{(\min)}), 1\Big), \mathcal{T}', \Big(1, \alpha^{(\min)} \setminus \alpha_l^{(\min)}, r_{\min}\Big)\Big\rangle.$$

Essentially, we are either joining $\mathcal{T}_m$ to $\mathcal{T}_{\min}$ or breaking $\mathcal{T}_{\min}$, while ignoring $\mathcal{T}'$.

Figure 9 shows an example of shiftjoin and shiftbreak. Note that $\mathcal{T}'$ (which is the orange tree triple in the diagram) remains untouched.

Figure 9: Example of shiftjoin and shiftbreak for $\mathcal{F} \in \text{FT}'(C_6 + P_4)$.



Step 4: We define sets G and H with indexed subsets G_i, H_i for $i \in \mathbb{Z}$ where

$$G_i = \{\mathcal{F} : (a,1) \notin E(T_{\min});\ \text{len}(\alpha^{(1)}) = \dots = \text{len}(\alpha^{(m)}) = 1;\ r_1, \dots, r_m \geqslant 2; \\ \text{len}(\alpha^{(\min)}) = 1;\ r' \leqslant |T'| + |T_{\min}| - k;\ |T'| - k - r' + 1 = i\},$$
$$H_i = \{\mathcal{F} : (a,1) \in E(T_{\min});\ \text{len}(\alpha^{(1)}) = \dots = \text{len}(\alpha^{(m)}) = 1;\ r_1, \dots, r_m \geqslant 2; \\ \text{len}(\alpha^{(\min)}) \geqslant 2;\ \alpha_2^{(\min)} \geqslant a - v_{\min} + 1;\ v_1 - 1 - \alpha_1^{(\min)} = i\},$$

where for the final restriction on H_i, if $\mathcal{F} = \{\mathcal{T}_{\min}\}$ has $m = 0$, then $v_1 = v_{\min}$. We define a bijection, $\text{rotatejoin}_i : G_i \to H_i$, with

$$\text{rotatejoin}_i(\mathcal{F}) = \Big\langle \big(v_1 + i, \alpha^{(1)}, r_1\big), \dots, \big(v_m + i, \alpha^{(m)}, r_m\big), \\ \big(a - r' + 1, \alpha^{(\min)} \cdot \alpha', r_{\min}\big)\Big\rangle.$$

Say $\mathcal{F}_G \in G_i$ and $\mathcal{F}_H = \text{rotatejoin}_i(\mathcal{F}_G)$. Note that in $\mathcal{F}_G$, we have $v_1 = |T_{\min}| + 1 = \alpha_1^{(\min)} + 1$. After rotating, v_1 in $\mathcal{F}_G$ becomes $v_1 + |T'| - k - r' + 1$ in $\mathcal{F}_H$. In other words,

$$(v_1) \text{ in } \mathcal{F}_H = (v_1 + |T'| - k - r' + 1) \text{ in } \mathcal{F}_G, \text{ so}$$
$$(v_1 - 1 - \alpha_1^{(\min)}) \text{ in } \mathcal{F}_H = (v_1 + |T'| - k - r' + 1 - 1 - \alpha_1^{(\min)}) \text{ in } \mathcal{F}_G \\ = |T'| - k - r' + 1 = i,$$

meaning $\mathcal{F}_H \in H_i$. The inverse rotatebreak$_i : H_i \to G_i$ is defined as

$$\text{rotatebreak}_i(\mathcal{F}) = \Big\langle \big(v_1 - i, \alpha^{(1)}, r_1\big), \ldots, \big(v_m - i, \alpha^{(m)}, r_m\big), \big(v_{\min} - i, \alpha^{(\min)} \setminus \alpha_1^{(\min)}, a - v_{\min} + 1\big), \big(1, (\alpha_1^{(\min)}), r_{\min}\big)\Big\rangle.$$

Figure 10 shows examples of rotatejoin$_i$ and rotatebreak$_i$. Subfigure 10a and Subfigure 10b show how the value of r' in $\mathcal{F}_G$ determines the "rotation" of rotatejoin$_i(\mathcal{F}_G)$.

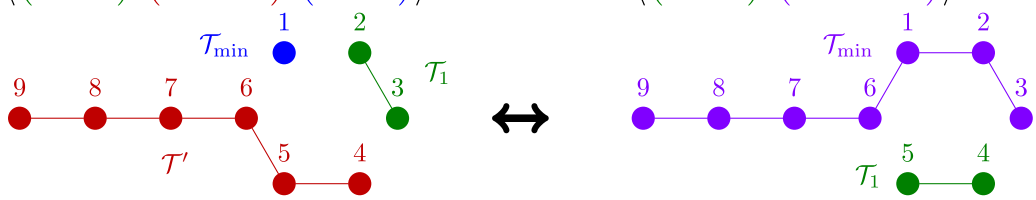
Subfigure 10c has $\mathcal{F}_G$ where $r' = 5 = |T'| + |T_{\min}| - k$. If $r' = 6$, then rotatejoin$_i(\mathcal{F}_G)$ would break edge $(1, 6)$, which would either mess the order of tree triples or in this example, result in a forest that is not a non-broken circuit. If $r' = 7$, then rotatejoin$_i(\mathcal{F}_G)$ would be the same graph as when $r' = 1$. Thus, if $r' \geqslant |T'| + |T_{\min}| - k + 1$, we cannot rotatejoin.

Figure 10: Examples of rotatejoin$_i$ and rotatebreak$_i$ for forest triples in $\mathrm{FT}'(C_6 + P_4)$.

(a) Simple example of rotate-joining where $r' = 1$ in F_G.

$\mathcal{F}_G \in G_2$, $\mathcal{F}_G = \text{rotatebreak}_2(\mathcal{F}_H) = \Big\langle \big(2, (2), 2\big), \big(4, (\mathbf{2{,}4}), 1\big), \big(1, (1), 1\big)\Big\rangle$

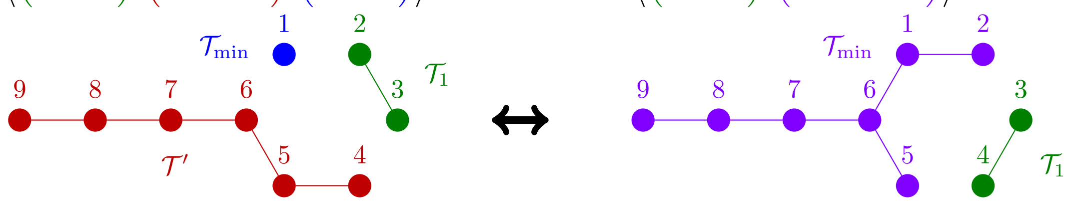
$\mathcal{F}_H \in H_2$, $\mathcal{F}_H = \text{rotatejoin}_2(\mathcal{F}_G) = \Big\langle \big(4, (2), 2\big), \big(6, (1, \mathbf{2{,}4}), 1\big)\Big\rangle$



(b) Same example as previous, but where $r' = 2$ in F_G.

$\mathcal{F}_G \in G_1$, $\mathcal{F}_G = \text{rotatebreak}_1(\mathcal{F}_H) = \Big\langle \big(2, (2), 2\big), \big(4, (\mathbf{2{,}4}), 2\big), \big(1, (1), 1\big)\Big\rangle$

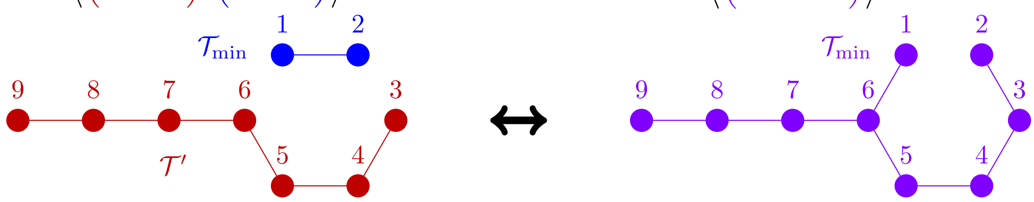
$\mathcal{F}_H \in H_1$, $\mathcal{F}_H = \text{rotatejoin}_1(\mathcal{F}_G) = \Big\langle \big(3, (2), 2\big), \big(5, (1, \mathbf{2{,}4}), 1\big)\Big\rangle$



(c) Rotate-joining where $r' = |T'| + |T_{\min}| - k$.

$\mathcal{F}_G \in G_{-1}$, $\mathcal{F}_G = \text{rotatebreak}_{-1}(\mathcal{F}_H) = \Big\langle \big(3, (\mathbf{7}), 5\big), \big(1, (2), 2\big)\Big\rangle$

$\mathcal{F}_H \in H_{-1}$, $\mathcal{F}_H = \text{rotatejoin}_{-1}(\mathcal{F}_G) = \Big\langle \big(2, (2, \mathbf{7}), 2\big)\Big\rangle$



Step 5: The remaining forest triples are in sets I_1 and I_2, where

$$I_1 = \{\mathcal{F} : (a,1) \in E(T);\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m)}) = \mathrm{len}(\alpha^{(\min)}) = 1;\ r_1, \ldots, r_{m-1} \geqslant 2; \\ \text{either } r_m \geqslant 2 \text{ or } |T_m| \leqslant k-1\},$$
$$I_2 = \{\mathcal{F} : (a,1) \notin E(T);\ \mathrm{len}(\alpha^{(1)}) = \cdots = \mathrm{len}(\alpha^{(m)}) = \mathrm{len}(\alpha^{(\min)}) = 1;\ r_1, \ldots, r_m \geqslant 2; \\ r' \geqslant |T'| + |T_{\min}| - k + 1\}.$$

First, note that if $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$ does not have edge $(a,1)$, then it must have $\alpha'_l \geqslant k$. If $\mathrm{len}(\alpha') \geqslant 2$, then

$$r' \leqslant \alpha'_1 \leqslant |T'| - \alpha'_l \leqslant |T'| - k \leqslant |T'| + |T| - k,$$

meaning $\mathcal{F} \notin I_2$. Thus, I_1 and I_2 contain only unit forest triples.

Combining the bijections on sets A through H as well as the identity function on I_1 and I_2 lets us construct a first-preserving involution on $\mathrm{FT}'(C_a + U_k)$, where the set of fixed points is $I_1 \cup I_2$.

Computing Sum of Fixed Points: The last step is to find an expression for the sum of fixed points, by counting the forest triples in I_1 and I_2. We recall

$$B^{(i)}_{a,k}(\boldsymbol{x}) = \sum_{\mathcal{F} \in \mathrm{FT}'^{(i)}(C_a+U_k)} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})} = \sum_{\substack{\mathcal{F} \in I_1 \cup I_2 \\ \alpha_1^{(\min)} = i, r_{\min} = 1}} \mathrm{sign}(\mathcal{F}) \cdot e_{\mathrm{type}'(\mathcal{F})}.$$

For $\mathcal{F} \in I_1$ (where edge $(a,1) \in E(T)$), we define $\mathrm{comp}'(\mathcal{F}) = \alpha^{(m)} \cdots \alpha^{(1)}$. For $\mathcal{F} \in I_2$ (where edge $(a,1) \notin E(T)$), we define $\mathrm{comp}'(\mathcal{F}) = \alpha' \cdot \alpha^{(m)} \cdots \alpha^{(1)}$. Note that $\mathrm{type}'(\mathcal{F}) = \mathrm{sort}(\mathrm{comp}'(\mathcal{F}))$, and $\mathrm{comp}'(\mathcal{F}) \models a + k - \alpha_1^{(\min)} - 1$.

Case 1: We will first find $B^{(i)}_{a,k}(\boldsymbol{x})$ where $i \leqslant k-1$. Let $\beta \models a + k - \alpha_1^{(\min)} - 1$ be some composition, we want to count the number of forest triples where $\mathrm{comp}'(\mathcal{F}) \models \beta$.

For $\mathcal{F} \in I_1$, note that $\alpha_1^{(\min)} = \alpha_l^{(\min)} \geqslant k$ for $\mathcal{F} \in \mathrm{FT}'(C_a + U_k)$, so $\alpha_1^{(\min)} \neq i$, and we have zero forest triples.

For $\mathcal{F} \in I_2$, since edge $(a,1) \notin E(T_{\min})$, there is only one way to place the tree triples so that $\mathrm{comp}'(\mathcal{F}) = \beta$. Note that $\alpha'_1 = \beta_1 \geqslant k$. There are $|T_m| - 1 = \beta_2 - 1$ possible values for r_m, and $|T_{m-1}| - 1 = \beta_3 - 1$ possible values for r_{m-1}, and we use similar logic up until r_1. There are $k - |T_{\min}| = k - i$ possible values for r'. Thus, there are $(k-i) \cdot (\beta_2 - 1) \cdots (\beta_l - 1)$ forest triples in I_2 where $\mathrm{comp}'(\mathcal{F}) = \beta$. This means

$$B^{(i)}_{a,k}(\boldsymbol{x}) = \sum_{\substack{\beta \models a+k-i-1 \\ \beta_1 \geqslant k}} (k-i) \cdot (\beta_2 - 1) \cdots (\beta_l - 1) \cdot e_{\mathrm{sort}(\beta)}, \qquad \text{if } i \leqslant k-1.$$

Case 2: We will now find $B^{(i)}_{a,k}(\boldsymbol{x})$ where $i \geqslant k$. Let $\beta \models a + k - \alpha_1^{(\min)} - 1$, we proceed like before.

First, for $\mathcal{F} \in I_1$, there are $|T_{\min}| - k = i - k$ ways to place the first tree triple, and then one way to place the remaining tree triples. For r_{m-1}, there are $|T_{m-1}| - 1 = \beta_2 - 1$ possible values. For r_{m-2}, there are $\beta_3 - 1$ possible values, and this is true till r_1. If $|T_m| = \beta_1 \leqslant k - 1$, then there are β_1 possible values for r_m (since r_m can equal 1), otherwise there are $\beta_1 - 1$ possible values (since $r_m \neq 1$).

If $\mathcal{F} \in I_2$, note that $r' \geqslant |T'| + |T_{\min}| - k + 1$, but since this case $|T_{\min}| = i \geqslant k$, we get $r' \geqslant |T'| + 1$. However, $\mathcal{F}$ must have $r' \leqslant |T'|$, so there are no forest triples to count in this case. Thus, summing up the forest triples in I_1 gets

$$\begin{aligned} B_{a,k}^{(i)}(\boldsymbol{x}) = & \sum_{\beta \models a+k-i-1} (i-k) \cdot (\beta_1 - 1) \cdots (\beta_l - 1) \cdot e_{\mathrm{sort}(\beta)} \\ & + \sum_{\substack{\beta \models a+k-i-1 \\ \beta_1 \leqslant k-1}} (i-k) \cdot (\beta_2 - 1) \cdots (\beta_l - 1) \cdot e_{\mathrm{sort}(\beta)} \\ & = \sum_{\substack{\beta \models a+k-i \\ \beta_1 \leqslant k}} (i-k) \cdot (\beta_2 - 1) \cdots (\beta_l - 1) \cdot e_{\mathrm{sort}(\beta_1 - 1, \beta \setminus \beta_1)}, \qquad \text{if } i \geqslant k. \end{aligned}$$

The final step involves combining the two sums into a single sum over compositions $\beta \models a + k - i$, where if $\beta_1 = 1$, the sum gives the value of the top summation, while if $2 \leqslant \beta_1 \leqslant k$, the sum gives the value of the second summation. Note that if $i = k$, the sum becomes 0.

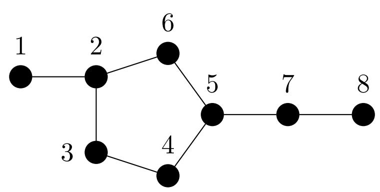
Thus, we have derived Equation 11, and using Equation 12 and some composition manipulation, we can derive Equation 10. □

8 Further Directions

Non-Adjacent Cycle Chains: This paper proved that cycle chains connected at adjacent vertices are e-positive. However, this property does not hold in general for cycles connected at non-adjacent vertices. For example, in Figure 6, the cycle chain on the left (which contains cycles connected at adjacent vertices) is e-positive, while the graph on the right containing cycles connected at non-adjacent vertices is not (with negative coefficient $-8e_{3,2,2}$).

All non-adjacent cycle chains with 6 and 7 vertices are not e-positive, but there are some non-adjacent cycle chains that are e-positive, with the smallest example being the

Figure 11: An e-positive non-adjacent cycle graph.



graph in Figure 11. One possible direction is finding out the exact conditions for when cycle chains are e-positive.

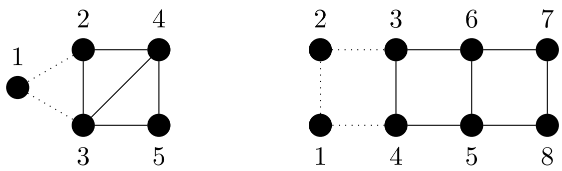
Attaching at Cut Vertices: We can generalize the method used for adjacent cycle chains to both non-adjacent cycle chains and more generally graphs attached at cut vertices. Say G' is an arbitrary graph and A is a graph with a vertices. Then, we define $B_{A,k}(\boldsymbol{x})$ similarly to how we defined it in Section 6, but instead looking at subsets of $\mathrm{FT}(A + U_k)$. The chromatic symmetric function of $A + G'$ can be written similar to Equation 13 as

$$X_{A+G'}(\boldsymbol{x}) = \sum_{k=1}^{|G'|} X_{G'}^{(k)}(\boldsymbol{x}) \cdot B_{A,k}(\boldsymbol{x}),$$

using the results from Section 6. Note that G' may not have a first-preserving involution, in which case the value of $X_{G'}^{(k)}$ would be negative for some integer k.

If G' does have a first-preserving involution and $B_{A,k}(\boldsymbol{x})$ is non-negative for all integers k, then the graph $A + G'$ is e-positive. Moreover, if $B_{A,k}^{(i)}$ is always e-positive for integer pairs k, i, then $A + G'$ also has a first-preserving involution. This paper proved this true for $A = C_a$, shown in Section 7, and a similar method could be used to prove that chains formed by connecting other graphs at single vertices are e-positive.

Figure 12: Attaching a vertex or a cycle to a graph at two vertices.



Attaching at Multiple Vertices: Another further direction could be attaching a vertex or a cycle at multiple edges. The graph on the left of Figure 12 shows how to attach a vertex to the triangle ladder with 4 vertices, using dotted edges. The right graph of Figure 12 shows attaching C_4 to a chain of cycles connected at two edges. Both graphs are e-positive; finding when it is possible to attach a vertex or cycle to two edges of a graph with a first-preserving involution is another further direction to explore.

Acknowledgements

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