Hypertree Shrinking Avoiding Low Degree Vertices

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Abstract

The shrinking operation converts a hypergraph into a graph by choosing, from each hyperedge, two endvertices of a corresponding graph edge. A hypertree is a hypergraph which can be shrunk to a tree on the same vertex set. Klimošová and Thomassé [J. Combin. Theory Ser. B 156 (2022), 250–293] proved (as a tool to obtain their main result on edge-decompositions of graphs into paths of equal length) that any rank 3 hypertree T can be shrunk to a tree where the degree of each vertex is at least $1/100$ times its degree in T. We prove a stronger and a more general bound, replacing the constant $1/100$ with $1/2k$ when the rank is k. In place of entropy compression (used by Klimošová and Thomassé), we use a hypergraph orientation lemma combined with a characterisation of edge-coloured graphs admitting rainbow spanning trees.

Mathematics Subject Classifications: 05C65, 05C05, 05C07

1 Introduction

Hypertrees are a generalisation of trees to the context of hypergraphs. Similarly to [3], we define a *hypertree* as a hypergraph H satisfying the following two conditions: for each set X of vertices, at most $|X| - 1$ hyperedges are subsets of X, and the equality holds if X is the full vertex set of H.[1] Clearly, a graph is a hypertree if and only if it is a tree.

In further support for viewing hypertrees as hypergraphic analogues of trees, it is known [5] (cf. also [3, Theorem 2.3]) that spanning hypertrees of a hypergraph H are the bases of a matroid associated with H, just like spanning trees are the bases of the cycle matroids of graphs.

It was proved by Lovász [6] that hypertrees may equivalently be characterised as hypergraphs in which it is possible to choose two vertices from each hyperedge in such

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[1] This structure is called 'spanning hypertree' in [3]. We define a *spanning hypertree* of a hypergraph H' as a subhypergraph of H' which is a hypertree and contains all the vertices of H'.

a way that the chosen pairs, viewed as edges of a graph on the vertex set of H, form a tree. Using this equivalence, Lovász further proved that hypertrees are 2-colourable as conjectured by Erdős.

The above operation that produces a graph from a hypergraph by choosing a pair of vertices in each hyperedge will be called *shrinking*. Thus, a hypertree is a hypergraph which can be shrunk to a tree.

Klimošová and Thomassé [4] derived a result on hypertree shrinking that preserves vertex degree up to a constant factor, and used it as one of the tools needed to obtain their main result about decompositions of 3-edge-connected graphs into paths of equal length. Their lemma on shrinking [4, Lemma 22] is as follows:

Lemma 1. *Let H be a hypertree with hyperedges of size at most three. It is possible to shrink H to a tree T such that for every vertex v of H,*

$$d_T(v) \geqslant \frac{d_H(v)}{100}.$$

Lemma 1 is proved using entropy compression, the method devised to prove an algorithmic version of the Lovász Local Lemma [7].

In this paper, we use a different method to strengthen Lemma 1 in two ways: first, our version ensures a stronger degree bound, and second, it applies to hypergraphs with an arbitrary size of the hyperedges. In addition, the proof is conceptually simpler. We prove:

Theorem 2. *Let H be a hypertree with hyperedges of size at most k. It is possible to shrink H to a tree T such that for every vertex v of H,*

$$d_T(v) \geqslant \max\left\{1, \left\lfloor \frac{d_H(v)}{k} \right\rfloor\right\}. \tag{1}$$

In particular, for every vertex v of H, $d_T(v)$ is at least $d_H(v)/2k$.

2 Preliminaries

We review some of the basic definitions on hypergraphs. A *hypergraph* is a pair $H = (V, E)$, where V is a finite set and E is a set of subsets of V. The elements of V are the *vertices* of H and the elements of E are the *hyperedges* of H.

The *rank* of a hypergraph H is the maximum size of a hyperedge of H. The number of hyperedges of H containing a vertex $v \in V$ is the *degree* of v and is denoted by $d_H(v)$. A hyperedge is *incident* with a set X of vertices of H if it contains at least one vertex of X.

A hypergraph is *simple* if it does not contain *loops* (i.e., hyperedges of size 1) nor repeated hyperedges (distinct hyperedges with identical vertex sets). All the hypergraphs discussed in this paper will be simple (and we will henceforth drop this adjective).

It will be useful in our argument to consider directed hypergraphs, consisting of a set of hyperarcs on a finite vertex set. A *hyperarc* is a hyperedge e together with a designated *head*. The other vertices of e are called the *tails* of e.

The *indegree* of vertex v in a directed hypergraph $\vec{H}$, denoted by $d^{IN}_{\vec{H}}(v)$, is the number of hyperarcs whose head is v. Similarly, the *outdegree* of v (denoted by $d^{OUT}_{\vec{H}}(v)$) is the number of hyperarcs in which v is a tail. Observe that a hyperarc of size k contributes to the indegree of exactly one vertex and to the outdegree of $k-1$ vertices.

3 Tools

In the proof of Theorem 2, we will need two main tools. The first one is an orientation lemma for hypergraphs with lower bounds on the indegrees. The second one is a characterisation of edge-coloured graphs admitting rainbow spanning trees, described in the last few paragraphs of this section.

We begin with the topic of hypergraph orientation. A result of Frank, Királyi and Királyi [2, Lemma 3.3] gives a necessary and sufficient condition for the existence of an orientation of a hypergraph with prescribed indegrees. In an early version of this paper, we derived our orientation lemma (Lemma 3 below) from this characterisation, using an approach inspired by the proof of [1, Lemma 3]. However, as pointed out to us by a reviewer, it is easier to derive it directly from Hall's Marriage Theorem as we do below. Given an integer-valued function f on the vertex set of a hypergraph and a set X of vertices, we write $f(X)$ for the sum of the values of f on the vertices in X.

Lemma 3. *Let $H = (V, E)$ be a hypergraph and let $f : V \to \mathbb{Z}^+$ be a mapping of the vertex set V of H to the set of non-negative integers. Assume that for every $F \subseteq V$,*

$$f(F) \leqslant e^*(F), \tag{2}$$

where $e^*(F)$ denotes the number of hyperedges incident with F. Then there is an orientation $\vec{H}$ of H such that

$$d^{IN}_{\vec{H}}(v) \geqslant f(v) \tag{3}$$

for every $v \in V$.

Proof. Let H' be a bipartite graph with parts E and W, where the vertices in E are the hyperedges of H, and W is comprised of $f(v)$ copies of each vertex $v \in V$. A vertex $e \in E$ is adjacent to a vertex $w \in W$ if w is (a copy of) some vertex belonging to the hyperedge e of H.

We claim that H' contains a matching covering all of W. To prove this, we need to verify the condition in Hall's Marriage Theorem. That is, for each set $A \subseteq W$, we need to show that $|A| \leqslant |N(A)|$, where $N(A)$ is the set of vertices of H' with at least one neighbour in A. Let A be given. We may assume that for each vertex $v \in V$, A contains either all $f(v)$ copies of v or none, because adding another copy of v to A (if there is one already) increases $|A|$ but not $|N(A)|$. Under this assumption, $|A| = f(F)$, where F is the set of vertices of H whose copies are included in A. At the same time, $|N(A)|$ equals the number $e^*(F)$ of hyperedges of H incident with F. Thus, $|A| \leqslant |N(A)|$ by (2). Hall's Marriage Theorem implies that H' contains the required matching M.

We use M to determine the orientation $\vec{H}$ of H. If there is an edge of M joining vertices $e \in E$ and $w \in W$, then the vertex of H corresponding to w is designated to be the head of the hyperedge e of H. The orientation of the hyperedges unmatched by M is chosen arbitrarily. Since M covers all $f(v)$ copies of any vertex v of H, we conclude that $d^{IN}_{\vec{H}}(v) \geqslant f(v)$ as desired. □

The second main tool which we will use to prove Theorem 2 is a necessary and sufficient condition for an edge-coloured graph to contain a rainbow spanning tree.

Let G be a graph with a (not necessarily proper) edge colouring. A subgraph of G is *rainbow* if it does not contain two edges with the same color. The following necessary and sufficient condition for the existence of a rainbow spanning tree has been derived in [9] and, independently, in [8, Section 41.1a].

Theorem 4 ([9])**.** *Let G be a (possibly improperly) edge-coloured graph of order n. There exists a rainbow spanning tree of G if and only if*

> *for any set of r colours ($0 \leqslant r \leqslant n-2$), the removal of all edges coloured with these r colours from G results in a graph with at most $r+1$ components.* (4)

Observe that the case $r = 0$ of condition (4) corresponds to G being connected.

4 Shrinking hypertrees

In this section, we prove Theorem 2. We begin with an application of Lemma 3 concerning a specific lower bound for the indegrees in an orientation of a hypergraph.

Lemma 5. *Let $H = (V, E)$ be a hypergraph of rank at most k. There exists an orientation $\vec{H}$ of H in which*

$$d^{IN}_{\vec{H}}(v) \geqslant \left\lfloor \frac{d_H(v)}{k} \right\rfloor$$

for every vertex $v \in V$.

Proof. We apply Lemma 3 with $f(v) = \lfloor d_H(v)/k \rfloor$. For a set $F \subseteq V$ of vertices of H, we verify condition (2). Clearly,

$$f(F) \leqslant \sum_{v \in F} \frac{d_H(v)}{k}. \tag{5}$$

Each hyperedge e of H contributes at most 1 to the degree of a specific vertex v, and since $|e| \leqslant k$, its total contribution to the right hand side of (5) is at most $k \cdot \frac{1}{k} = 1$. It follows that the right hand side is at most $e^*(F)$, i.e., (2) holds. □

Let H be a hypertree of rank k, and let $\vec{H}$ be the orientation from Lemma 5. We construct an edge-coloured graph $G(\vec{H})$ on the vertex set of $\vec{H}$ by the following rule: for each hyperarc $\vec{e}$ of $\vec{H}$, add to $G(\vec{H})$ a star whose center is the head of e and whose leaves are the tails of e. Furthermore, all the edges of this star have the same colour, which differs from the colours used for the other stars.

We claim that $G(\vec{H})$ has a rainbow spanning tree. To prove this, we need to verify condition (4) of Theorem 4. Consider first a simple variant of the above construction: let the graph G_H be obtained by adding, for each hyperedge e of H, a complete graph on the vertex set of e. Observe that since H is a hypertree, G_H has a rainbow spanning tree. Consequently, condition (4) holds for G_H.

Now in the construction of $G(\vec{H})$, we used stars in place of the complete subgraphs. It is, however, easy to see that this replacement has no effect on the validity of condition (4): indeed, if instead of each complete subgraph we take any connected subgraph on the same vertex set, then their union has the same number of components in each case, and so the validity of condition (4) remains without change.

Thus, $G(\vec{H})$ has a rainbow spanning tree as well. Let T be the tree obtained from it by disregarding the edge colours. It is possible to shrink H to T, since to each hyperedge e of H, there corresponds an edge e' of T whose endvertices are contained in e (namely the edge selected from the star corresponding to an orientation $\vec{e}$ of e) and the correspondence is bijective. Moreover, the head of $\vec{e}$ is an endvertex of e', which implies that if v is a vertex of H, then its degree in T is at least the indegree of v in $\vec{H}$ — that is, at least $\lfloor d_H(v)/k \rfloor$. At the same time, $d_T(v) \geqslant 1$ since T is a tree. Inequality (1) of Theorem 2 follows.

The second inequality, namely $d_T(v) \geqslant d_H(v)/2k$, clearly holds if $d_H(v)$ is at most $2k$. Otherwise it follows from (1) and the inequality $\lfloor x \rfloor \geqslant x/2$ which is valid for all real $x \geqslant 1$. This concludes the proof of Theorem 2.

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