Generalized Turán problem with bounded matching number

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Abstract

For a graph T and a set of graphs $\mathcal{H}$, let $\mathrm{ex}(n,T,\mathcal{H})$ denote the maximum number of copies of T in an n-vertex $\mathcal{H}$-free graph. Recently, Alon and Frankl (Journal of Combinatorial Theory, Series B, 2024) determined the exact value of $\mathrm{ex}(n,K_2,\{K_{k+1},M_{s+1}\})$, where K_{k+1} and M_{s+1} are complete graph on $k+1$ vertices and matching of size $s+1$, respectively. In this paper, we continue the study of the function $\mathrm{ex}(n,T,\{K_{k+1},M_{s+1}\})$. We determine the exact value of $\mathrm{ex}(n,K_r,\{K_{k+1},M_{s+1}\})$ for $r\geqslant 3$ and the exact value of $\mathrm{ex}(n,S_r,\{K_{k+1},M_{s+1}\})$ for $n\geqslant 2(s+1)(r+1)$ and $r\geqslant 2$.

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1 Introduction

Let $G=(V,E)$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)\subset\binom{V}{2}$. We may write G instead of $E(G)$.

Let T be a fixed graph and $\mathcal{H}$ be a set of given graphs. A graph G is called *$\mathcal{H}$-free* if G contains no copy of any member in $\mathcal{H}$ as its subgraph. Write $N(G,T)$ for the number of copies of T in a graph G. Define the generalized Turán number as

$$\mathrm{ex}(n,T,\mathcal{H})=\max\{N(G,T): G \text{ is an } n\text{-vertex } \mathcal{H}\text{-free graph}\}.$$

We call an n-vertex graph G with $N(G,T)$ attaining the maximum an *extremal graph* of $\mathcal{H}$. This function has been systematically studied by Alon and Shikhelman [2] and has received much attention, for example, in [7, 8, 9, 10, 12, 13, 14, 15, 16, 20]. When $T=K_2$, it is the classical Turán number $\mathrm{ex}(n,\mathcal{H})$.

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Let K_r denote a complete graph on r vertices for some integer r. For a set U, write $K[U]$ for a complete graph on vertex set U. Let $U_1, U_2, \ldots, U_r$ be disjoint sets and $\mathcal{U} = \{U_1, \ldots, U_r\}$, write $K[\mathcal{U}] = K[U_1, U_2, \ldots, U_r]$ for a complete r-partite graph with partition sets $U_1, \ldots, U_r$. Let $G = (V, E)$ be a graph. For a set $U \subseteq V$, write $G[U]$ for the subgraph induced by U. For disjoint sets $U_1, U_2, \cdots, U_r \subseteq V$, write $G[U_1, \cdots, U_r]$ for the induced r-partite subgraph of G, i.e. $G[U_1, \cdots, U_r] = K[U_1, \cdots, U_r] \cap G$. Let $K_{a_1, a_2, \cdots, a_r}$ denote the complete r-partite graph with partition sets of size $a_1, a_2, \cdots, a_r$. In particular, $S_r = K_{1,r}$ is the star with r edges. We call the only vertex incident to all the edges in a star be the center of the star and call every other vertex a leaf of the star. For graphs $G_1, \cdots, G_r$, let $\sum_{i=1}^{r} G_i$ be the union of vertex-disjoint copies of $G_1, \cdots, G_r$.

A Turán graph $T_k(n)$ is a complete k-partite graph on n vertices whose partition sets have sizes as equal as possible. Let $t_k(n) = |T_k(n)| = N(T_k(n), K_2)$.

The famous Turán Theorem [4] states that $\mathrm{ex}(n, K_2, K_{k+1}) = t_k(n)$. Zykov [21] gave the generalized version of Turán Theorem as follows.

Theorem 1 ([21], see also [5])**.** *For all $n \geqslant k \geqslant r \geqslant 2$,*

$$ex(n, K_r, K_{k+1}) = N(T_k(n), K_r),$$

and $T_k(n)$ is the unique extremal graph.

Write $\chi(G)$ for the chromatic number of graph G. We say a graph is *edge-critical* if there exists some edge whose deletion reduces its chromatic number. Simonovits [19] proved that for any edge-critical graph H with $\chi(H) = k + 1 \geqslant 3$, $\mathrm{ex}(n, K_2, H) = t_k(n)$ for sufficiently large n, and $T_k(n)$ is the unique extremal graph. This result was extended by Ma and Qiu [18] as follows: For sufficiently large n, $\mathrm{ex}(n, K_r, H) = N(T_k(n), K_r)$, and $T_k(n)$ is the unique extremal graph, where H is an edge-critical graph with $\chi(H) = k + 1 > r \geqslant 2$.

Let $G_k(n, s) = \overline{K_{n-s}} \vee T_{k-1}(s)$, the join of an empty graph $\overline{K_{n-s}}$ and Turán graph $T_{k-1}(s)$, i.e. a complete k-partite graph on n vertices with one partition set of size $n - s$ and the others having sizes as equal as possible. Write M_k for a matching consisting of k edges. Another fundamental result in graph theory is the Erdős-Gallai Theorem, showing that

$$\mathrm{ex}(n, K_2, M_{s+1}) = \max\left\{|E(G_{s+1}(n, s))|, \binom{2s+1}{2}\right\}.$$

Recently, Alon and Frankl [1] studied the function $\mathrm{ex}(n, K_2, \mathcal{H})$ when $\mathcal{H} = \{K_{k+1}, M_{s+1}\}$.

Theorem 2 ([1])**.** *For $n \geqslant 2s + 1$ and $k \geqslant 2$,*

$$ex(n, K_2, \{K_{k+1}, M_{s+1}\}) = \max\{|T_k(2s + 1)|, |G_k(n, s)|\}.$$

In this article, we first extend the result of Alon and Frankl [1] as shown in the following.

Theorem 3. *For $n \geqslant 2s + 1$ and $k \geqslant r \geqslant 3$,*

$$ex(n, K_r, \{K_{k+1}, M_{s+1}\}) = \max\{N(T_k(2s + 1), K_r), N(G_k(n, s), K_r)\}.$$

We also give a similar result counting the number of stars.

Theorem 4. *For $n \geqslant 2(s+1)(r+1)$ and $r \geqslant 2$,*

$$ex(n, S_r, \{K_{k+1}, M_{s+1}\}) = N(G_k(n,s), S_r).$$

We should mention the following result given by Gerbner [11], which is a weakening of Theorem 3 and Theorem 4 for any complete multipartite graph with at most r parts instead of K_r or S_r.

Theorem 5 ([11])**.** *If H is a complete multipartite graph with at most r parts, then $ex(n, H, \{K_{r+1}, M_{s+1}\}) = N(H, K)$, where K is either a complete multipartite n-vertex graph with at most r parts and a part of order at least $n - s$, or consists of a complete multipartite graph on at most $2s+1$ vertices with at most r parts and possibly some isolated vertices.*

2 Preliminaries

We need the following fundamental theorem in graph theory.

Theorem 6 (Tutte-Berge Theorem [3], see also [17])**.** *A graph G is M_{s+1}-free if and only if there is a set $B \subset V(G)$ such that all the components $G_1, \ldots, G_m$ of $G - B$ are odd (i.e. $|V(G_i)| \equiv 1 \pmod 2$ for $i \in [m]$), and*

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} \leqslant s.$$

Let G be a graph and r be a positive integer. For a vertex $v \in V(G)$, define $N_G^{(r)}(v) = \{U \in \binom{V}{r} : G[U \cup \{v\}] \cong K_{r+1}\}$ be the *r-clique neighborhood* of v and $d_G^{(r)}(v) = |N_G^{(r)}(v)|$ be the *r-clique-degree* of v. For a set $U \subset V(G)$, write $d_G^{(r)}(U) = \sum_{u \in U} d_G^{(r)}(u)$. As usual, write neighborhood $N_G(v)$ and degree $d_G(v)$ instead of 1-clique neighborhood $N_G^{(1)}(v)$ and 1-clique-degree $d_G^{(1)}(v)$ for short.

For two non-adjacent vertices u, v in a graph G, we define the *switching operation* $u \to v$ as deleting the edges joining u to its neighbors and adding new edges connecting u to vertices in $N_G(v)$. Let $G_{u \to v}$ be the graph obtained from G by the switching operation $u \to v$, that is $V(G_{u\to v}) = V(G)$ and

$$E(G_{u \to v}) = (E(G) \setminus E(G[\{u\}, N_G(u)])) \cup E(G[\{u\}, N_G(v)]).$$

Note that the edges between u and the common neighbors of u and v remain unchanged by the definition of $G_{u\to v}$. For two disjoint independent sets S and T in a graph G, if all of the vertices in S (resp. T) have the same neighborhood $N_G(S)$ (resp. $N_G(T)$) and $E(G[S,T]) = \emptyset$, we then call $\{S, T\}$ an *independent pair*. For any independent pair $\{S, T\}$, we similarly define $G_{S \to T}$ to be the graph obtained from G by deleting the edges between S and $N_G(S)$ and adding new edges connecting S and $N_G(T)$.

For any fixed graph H, we say H is *switchable* if for any graph G and any independent pair $\{S,T\}$ in G, either $G' = G_{S\to T}$ or $G' = G_{T\to S}$ has the property that $N(G',H) \geqslant N(G,H)$.

Proposition 7. *For $r \geqslant 2$, K_r and S_r are both switchable.*

Proof. Let $\{S,T\}$ be an independent pair of G.

We firstly prove for K_r. Without loss of generality, suppose $d_G^{(r-1)}(T) \geqslant d_G^{(r-1)}(S)$. Let $G' = G_{S\to T}$. Then

$$N(G',K_r) = N(G,K_r) - d_G^{(r-1)}(S) + d_G^{(r-1)}(T) \geqslant N(G,K_r),$$

the equality holds if and only if $d_G^{(r-1)}(T) = d_G^{(r-1)}(S)$.

For S_r, note that $N(G,S_r) = \sum_{v\in G}\binom{d_G(v)}{r}$. Let $X = N(G_{S\to T},S_r) - N(G,S_r)$ and $Y = N(G_{T\to S},S_r) - N(G,S_r)$, then

$$\begin{aligned} X &= |S|\left(\binom{|N_G(T)|}{r} - \binom{|N_G(S)|}{r}\right) \\ &+ \sum_{u\in N_G(S)\setminus N_G(T)}\left(\binom{d_G(u)-|S|}{r} - \binom{d_G(u)}{r}\right) \\ &+ \sum_{w\in N_G(T)\setminus N_G(S)}\left(\binom{d_G(w)+|S|}{r} - \binom{d_G(w)}{r}\right), \end{aligned}$$

and

$$\begin{aligned} Y &= |T|\left(\binom{|N_G(S)|}{r} - \binom{|N_G(T)|}{r}\right) \\ &+ \sum_{u\in N_G(S)\setminus N_G(T)}\left(\binom{d_G(u)+|T|}{r} - \binom{d_G(u)}{r}\right) \\ &+ \sum_{w\in N_G(T)\setminus N_G(S)}\left(\binom{d_G(w)-|T|}{r} - \binom{d_G(w)}{r}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} |T|X + |S|Y &= \sum_{u\in N_G(S)\setminus N_G(T)}\left(|S|\binom{d_G(u)+|T|}{r} + |T|\binom{d_G(u)-|S|}{r}\right) \\ &- \sum_{u\in N_G(S)\setminus N_G(T)}(|S|+|T|)\binom{d_G(u)}{r} \\ &+ \sum_{w\in N_G(T)\setminus N_G(S)}\left(|T|\binom{d_G(w)+|S|}{r} + |S|\binom{d_G(w)-|T|}{r}\right) \\ &- \sum_{w\in N_G(T)\setminus N_G(S)}(|S|+|T|)\binom{d_G(w)}{r}. \end{aligned}$$

Now since $\binom{x}{r}$ is a convex function of x, by Jensen's inequality, $|T|X + |S|Y \geqslant 0$. This means either $X \geqslant 0$ or $Y \geqslant 0$, which completes the proof. □

Let G be a graph and u_1, u_2, v_1, v_2 be distinct vertices in G. Define $G_{(u_1,v_1)\to(u_2,v_2)} = (G_{u_1\to u_2})_{v_1\to v_2} = (G_{v_1\to v_2})_{u_1\to u_2}$.

Proposition 8. *Let G be a graph and u_1, u_2, v_1, v_2 be distinct vertices in G. Suppose $u_1u_2, v_1v_2, u_1v_2, v_1u_2 \notin E(G)$. Then either $G' = G_{(u_1,v_1)\to(u_2,v_2)}$ or $G' = G_{(u_2,v_2)\to(u_1,v_1)}$ has $N(G', S_r) \geqslant N(G, S_r)$ for $r \geqslant 2$.*

Proof. For any two vertices x and y in G, let $\varepsilon_{xy} = 1$ if $xy \in E(G)$ and $\varepsilon_{xy} = 0$ otherwise. For $i = 1, 2$, Let $\varepsilon_i = \varepsilon_{u_iv_i}$. Also, for any vertex $v \in V(G)$, let $\varepsilon_v = \varepsilon_{vu_2} + \varepsilon_{vv_2} - \varepsilon_{vu_1} - \varepsilon_{vv_1}$. Let $X = N(G_{(u_1,v_1)\to(u_2,v_2)}, S_r) - N(G, S_r)$ and $Y = N(G_{(u_2,v_2)\to(u_1,v_1)}, S_r) - N(G, S_r)$. Similarly as the proof in Proposition 7, we see that

$$\begin{aligned} X &= 2\left(\binom{d_G(u_2)+\varepsilon_2}{r} + \binom{d_G(v_2)+\varepsilon_2}{r}\right) \\ &- \left(\binom{d_G(u_1)}{r} + \binom{d_G(v_1)}{r} + \binom{d_G(u_2)}{r} + \binom{d_G(v_2)}{r}\right) \\ &+ \sum_{v\neq u_1,u_2,v_1,v_2} \left(\binom{d_G(v)+\varepsilon_v}{r} - \binom{d_G(v)}{r}\right), \end{aligned}$$

and

$$\begin{aligned} Y &= 2\left(\binom{d_G(u_1)+\varepsilon_1}{r} + \binom{d_G(v_1)+\varepsilon_1}{r}\right) \\ &- \left(\binom{d_G(u_1)}{r} + \binom{d_G(v_1)}{r} + \binom{d_G(u_2)}{r} + \binom{d_G(v_2)}{r}\right) \\ &+ \sum_{v\neq u_1,u_2,v_1,v_2} \left(\binom{d_G(v)-\varepsilon_v}{r} - \binom{d_G(v)}{r}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} X+Y &= 2\left(\binom{d_G(u_1)+\varepsilon_1}{r} + \binom{d_G(v_1)+\varepsilon_1}{r} + \binom{d_G(u_2)+\varepsilon_2}{r} + \binom{d_G(v_2)+\varepsilon_2}{r}\right) \\ &- 2\left(\binom{d_G(u_1)}{r} + \binom{d_G(v_1)}{r} + \binom{d_G(u_2)}{r} + \binom{d_G(v_2)}{r}\right) \\ &+ \sum_{v\neq u_1,u_2,v_1,v_2} \left(\binom{d_G(v)-\varepsilon_v}{r} + \binom{d_G(v)+\varepsilon_v}{r} - 2\binom{d_G(v)}{r}\right). \end{aligned}$$

Note that $\varepsilon_1, \varepsilon_2 \geqslant 0$. Now since $\binom{x}{r}$ is a convex function of x, by Jensen's inequality, $X + Y \geqslant 0$. This means either $X \geqslant 0$ or $Y \geqslant 0$, which completes the proof. □

We give the following observation about combination numbers without proof.

Observation 9. *For integers $n > m \geqslant r \geqslant 2$,*

$$1-\frac{r(n-m)}{n-r+1} < \left(1-\frac{n-m}{n-r+1}\right)^r = \left(\frac{m-r+1}{n-r+1}\right)^r < \frac{\binom{m}{r}}{\binom{n}{r}} < \left(\frac{m}{n}\right)^r.$$

Proposition 10. *Let $r \geqslant 2$, $s \geqslant 1$, $k \geqslant 2$, $0 \leqslant b \leqslant s$ and $n \geqslant 2(s+1)(r+1)$ be integers. Let $h_{n,k}(x_1,\cdots,x_{k-1}) = \sum_{i=1}^{k-1} x_i\binom{n-x_i}{r}$.*
(1) Let

$$g_{n,k,r,b,y}(x_1,\cdots,x_k) := h_{n,k}(x_1,\cdots,x_{k-1}) + x_k\binom{y-x_k}{r}$$

be a function on $\{(x_1,\cdots,x_k)\in\mathbb{Z}^k : x_1+\cdots+x_k = b,\ x_i \geqslant 0 \text{ for any } i\in[k]\}$, where $s \leqslant y \leqslant 2s$ is an integer. Then $g_{n,k,r,b,y}(x_1,\cdots,x_k)$ reaches its maximum when $x_k = 0$ and $|x_i - x_j| \leqslant 1$ for any $i,j\in[k-1]$.
(2) Let $x = \sum_{i=1}^{k-1} x_i$ and

$$g_{n,k,r,s}(x_1,\cdots,x_{k-1}) := h_{n,k}(x_1,\cdots,x_{k-1}) + (2s+1-2x)\binom{2s-x}{r} + (n-1-2s+x)\binom{x}{r}$$

be a function on $\{(x_1,\cdots,x_{k-1})\in\mathbb{Z}^{k-1} : x\leqslant s,\ x_i\geqslant 0 \text{ for any } i\in[k-1]\}$. Then the function $g_{n,k,r,s}(x_1,\cdots,x_{k-1})$ reaches its maximum when $x = s$ and $|x_i - x_j|\leqslant 1$ for any $i,j\in[k-1]$.
(3) $N(G_k(n,s),S_r)$ is an increasing function of s.

Proof. Before the proof of (1), (2) and (3), we firstly prove the following claim.

Claim 11. *On $\{(x_1,\cdots,x_{k-1})\in\mathbb{Z}^{k-1} : x_1+\cdots+x_{k-1} = b,\ x_i\geqslant 0 \text{ for any } i\in[k-1]\}$, $h_{n,k}(x_1,\cdots,x_{k-1})$ reaches its maximum when $|x_i-x_j|\leqslant 1$ for any $i,j\in[k-1]$.*

Proof. If $k=2$, then we are done. For $k\geqslant 3$, without loss of generality, suppose on the contrary that $h_{n,k}$ reaches its maximum on $(b_1,\cdots,b_{k-1})$ while $b_1-b_2\geqslant 2$. Let $X = h_{n,k}(b_1-1,b_2+1,b_3,\cdots,b_{k-1}) - h_{n,k}(b_1,b_2,b_3,\cdots,b_{k-1})$, then

$$\begin{aligned}
X &= (b_1-1)\binom{n-b_1+1}{r} + (b_2+1)\binom{n-b_2-1}{r} - b_1\binom{n-b_1}{r} - b_2\binom{n-b_2}{r}\\
&= \left(\binom{n-b_2-1}{r} - \binom{n-b_1+1}{r}\right) + \left(b_1\binom{n-b_1}{r-1} - b_2\binom{n-b_2-1}{r-1}\right)\\
&= \left(\frac{n-b_2-r}{r}\binom{n-b_2-1}{r-1} - \frac{n-b_1+1}{r}\binom{n-b_1}{r-1}\right)\\
&+ \left(b_1\binom{n-b_1}{r-1} - b_2\binom{n-b_2-1}{r-1}\right)\\
&= \frac{n-(r+1)(b_2+1)+1}{r}\binom{n-b_2-1}{r-1} - \frac{n-(r+1)b_1+1}{r}\binom{n-b_1}{r-1}\\
&> 0,
\end{aligned}$$

where the last inequality holds because $n-(r+1)(b_2+1)+1 > n-(r+1)b_1+1 > 0$ and $n-b_2-1 > n-b_1 \geqslant r-1$. This is a contradiction by the maximality of $h_{n,k}(b_1\cdots,b_{k-1})$. $\square$

By the proof of the Claim 11, it is easy to see that (3) is correct since $N(G_k(n,s),S_r) = h_{n,k+1}(x_1,\cdots,x_{k-1},n-s)$ with $\sum_{i=1}^{k-1} x_i = s$, $|x_i - x_j| \leqslant 1$ for any $i,j \in [k-1]$ and so $|(n-s)-x_i| > 1$ for $i \in [k-1]$. It remains to prove (1) and (2).

(1) By Claim 11, it remains to prove $x_k = 0$ when $g_{n,k,r,b,y}$ reaches its maximum. Suppose on the contrary that $g_{n,k,r,b,y}$ reaches its maximum on $(b_1,\cdots,b_k)$ while $b_k \geqslant 1$. Let $X = g_{n,k,r,b,y}(b_1+b_k,b_2,\cdots,b_{k-1},0) - g_{n,k,r,b,y}(b_1,b_2,\cdots,b_{k-1},b_k)$, then

$$\begin{aligned}
X &= (b_1+b_k)\binom{n-b_1-b_k}{r} - \left(b_1\binom{n-b_1}{r} + b_k\binom{y-b_k}{r}\right) \\
&> (b_1+b_k)\left(1-\frac{rb_k}{n-b_1-r+1}\right)\binom{n-b_1}{r} - \left(b_1\binom{n-b_1}{r} + b_k\binom{y-b_k}{r}\right) \\
&= b_k\left(\left(1-\frac{r(b_1+b_k)}{n-b_1-r+1}\right)\binom{n-b_1}{r} - \binom{y-b_k}{r}\right) \\
&> b_k\binom{y-b_k}{r}\left(\left(1-\frac{r(b_1+b_k)}{n-b_1-r+1}\right)\left(\frac{n-b_1}{y-b_k}\right)^r - 1\right) \\
&\geqslant b_k\binom{y-b_k}{r}\left(\left(1-\frac{sr}{2sr}\right)\left(\frac{2sr}{2s}\right)^r - 1\right) \\
&\geqslant 0,
\end{aligned}$$

where the first two inequality holds by Observation 9 and the last holds since $b_1+b_k \leqslant s$, $n-b_1-r+1 > 2sr$, $n-b_1 > 2sr$ and $y-b_k < 2s$. This is a contradiction by the maximality of $g_{n,k,r,b,y}(b_1\cdots,b_k)$.

(2) Similarly, we only need to prove $x = s$ when $g_{n,k,r,s}$ reaches its maximum. Suppose on the contrary that $g_{n,k,r,s}$ reaches its maximum on $(b_1,\cdots,b_{k-1})$ while $x = b_1+\cdots+b_{k-1} \leqslant s-1$. Note that $p(x) = (n-1-2s+x)\binom{x}{r}$ is an increasing function on $x \in [s]$.

Let $X = g_{n,k,r,s}(b_1 + (s-x), b_2, \cdots, b_{k-1}) - g_{n,k,r,s}(b_1, b_2, \cdots, b_{k-1})$, then

$$\begin{aligned}
X &= (b_1 + (s-x))\binom{n-b_1-(s-x)}{r} - b_1\binom{n-b_1}{r} - (2s+1-2x)\binom{2s-x}{r} \\
&+ \binom{s}{r} + p(s) - p(x) \\
&> (b_1 + (s-x))\binom{n-b_1-(s-x)}{r} - b_1\binom{n-b_1}{r} - (2s+1-2x)\binom{2s-x}{r} \\
&> (b_1 + (s-x))\left(1 - \frac{r(s-x)}{n-b_1-r+1}\right) - b_1\binom{n-b_1}{r} - (2s+1-2x)\binom{2s-x}{r} \\
&= (s-x)\left(1 - \frac{r(b_1+(s-x))}{n-b_1-r+1}\right)\binom{n-b_1}{r} - (2s+1-2x)\binom{2s-x}{r} \\
&> \left((s-x)\left(1 - \frac{r(b_1+(s-x))}{n-b_1-r+1}\right)\left(\frac{n-b_1}{2s-x}\right)^r - (2s+1-2x)\right)\binom{2s-x}{r} \\
&> \left((s-x)\left(\left(1-\frac{sr}{2sr}\right)\left(\frac{2s(r+\frac{1}{2})}{2s}\right)^r - 2\right) - 1\right)\binom{2s-x}{r} \\
&> 0,
\end{aligned}$$

where the second inequality and the third inequality hold by Observation 9; the forth inequality holds since $r(b_1 + (s-x)) \leqslant sr$, $n - b_1 - r + 1 > 2sr$, $n - b_1 > 2s(r + \frac{1}{2})$ and $2s - x \leqslant 2s$. This is a contradiction by the maximality of $g_{n,k,r,s}(b_1 \cdots, b_{k-1})$. □

Let $\Delta^r_{t,k} = N(T_k(t), K_r)$ for some positive integers t, k, r.

Observation 12. *(1) For positive integers $t, k, r \geqslant 2$,*

$$\Delta^r_{t+1,k} - \Delta^r_{t,k} = \Delta^{r-1}_{t-\lfloor \frac{t}{k} \rfloor,k-1} \text{ and } \Delta^r_{t,k} = \Delta^r_{t-\lfloor \frac{t}{k} \rfloor,k-1} + \left\lfloor \frac{t}{k} \right\rfloor \Delta^{r-1}_{t-\lfloor \frac{t}{k} \rfloor,k-1}.$$

(2) For $n \geqslant 2s+1$ and $s \geqslant t, k, r \geqslant 3$, define

$$g_{n,k,r}(t) := (n-t)\Delta^{r-1}_{t,k-1} + \Delta^r_{t,k-1}.$$

Then $g_{n,k,r}(t)$ is a strictly increasing function of t. In particular, $g_{n,k,r}(s) = N(G_k(n,s), K_r)$.

Proof. (1) can be checked directly by the definitions of $\Delta^r_{t,k}$ and the Turán graph.

(2) By (1), for $t \leqslant s-1$,

$$\begin{aligned}
g_{n,k,r}(t+1) - g_{n,k,r}(t) &= (n-t-1)\Delta^{r-2}_{t-\lfloor \frac{t}{k-1} \rfloor,k-2} - \Delta^{r-1}_{t,k-1} + \Delta^{r-1}_{t-\lfloor \frac{t}{k-1} \rfloor,k-2} \\
&= (n-t-1)\Delta^{r-2}_{t-\lfloor \frac{t}{k-1} \rfloor,k-2} - \left\lfloor \frac{t}{k-1} \right\rfloor \Delta^{r-2}_{t-\lfloor \frac{t}{k-1} \rfloor,k-2} \\
&= \left(n-1-t-\left\lfloor \frac{t}{k-1} \right\rfloor\right)\Delta^{r-2}_{t-\lfloor \frac{t}{k-1} \rfloor,k-2} > 0.
\end{aligned}$$

This completes the proof. □

3 Proof of Theorem 3 and Theorem 4

Now we are ready to give the proofs of Theorem 3 and Theorem 4.

Let G be an extremal graph of $\{K_{k+1}, M_{s+1}\}$ (with the maximum number of K_r (resp. S_r) in it) on $n \geqslant 2s+1$ vertices. By Theorem 6, there is a vertex set $B \subset V(G)$ such that $G - B$ consists of odd components $G_1, \ldots, G_m$, and

$$|B| + \sum_{i=1}^{m} \frac{|V(G_i)| - 1}{2} \leqslant s.$$

Let $A_i = V(G_i)$ and $|A_i| = a_i$ for $i \in [m]$. Denote $A = \cup_{i=1}^{m} A_i$. Let $I_G(A) = \{i \in [m] : a_i = 1\}$. We may choose G maximizing $|I_G(A)|$ (assumption (*)). Let $|B| = b$.

Define two vertices u and v in B are equivalent if and only if $N_G(u) = N_G(v)$. Clearly, it is an equivalent relation. Therefore, the vertices of B can be partitioned into equivalent classes according to the equivalent relation defined above. We may choose G (among graphs G satisfying assumption (*)) with the minimum number of equivalent classes of B (assumption (**)). Note that each equivalent class of B is an independent set of G by the definition of the equivalent relation. We firstly claim that every two non-adjacent vertices of B have the same neighborhood (a clique version of Lemma 2.1 of [1]), which is also a simple consequence of the Zykov symmetrization method introduced in [21], for completeness we include the proof.

Lemma 13. *Every two non-adjacent vertices of B have the same neighborhood.*

Proof. Suppose there are two non-adjacent vertices $u, w \in B$ with $N_G(u) \neq N_G(w)$. Then u and w must be in distinct equivalent classes U and W by the definition of the equivalence. Since $uw \notin E(G)$, we have $E(G[U, W]) = \emptyset$. Thus, $\{U, W\}$ is an independent pair. Without loss of generality, let $G' = G_{U \to W}$. By Proposition 7, we can suppose $N(G', K_r) \geqslant N(G, K_r)$ (resp. $N(G', S_r) \geqslant N(G, S_r)$). Now we show that G' is $\{K_{k+1}, M_{s+1}\}$-free too. Clearly, $G' - B$ still consists of odd components $G_1, \ldots, G_m$. Hence G' is M_{s+1}-free by Theorem 6. If G' contains a copy T of K_{k+1}, we must have a vertex $u \in V(T) \cap U$. Choose a vertex $w \in W$. Since $N_{G'}(u) = N_{G'}(w) = N_G(w)$, $(V(T) \setminus \{u\}) \cup \{w\}$ induces a copy of K_{k+1} in G, a contradiction. Hence, $G_{U \to W}$ is $\{K_{k+1}, M_{s+1}\}$-free. By the extremality of G, we have $N(G', K_r) = N(G, K_r)$ (resp. $N(G', S_r) = N(G, S_r)$). But the number of equivalent classes of G' (U and W merge into one class in G') is less than the one in G, a contradiction to assumption (**). □

In fact, with similar proofs, we also have the following two lemmas.

Lemma 14. *For any fixed $i \in [m]$, every two non-adjacent vertices of A_i have the same neighborhood.*

Lemma 15. *For $i, j \in [m]$ with $a_i = a_j = 1$, the only vertex $v_i \in A_i$ and the only vertex $v_j \in A_j$ have the same neighborhood.*

Here we firstly give the proof of Theorem 3.

Proof of Theorem 3: Let G be the extremal graph mentioned above with the maximum number of K_r. By Lemma 13 and G being K_{k+1}-free, $G[B]$ is a complete ℓ-partite graph with $\ell \leqslant k$. Let its partition sets be $B_1, \ldots, B_\ell$ and let $B_{\ell+1} = \cdots = B_k = \emptyset$ if $\ell < k$. Let $b_i = |B_i|$ for $i \in [k]$. Without loss of generality, assume $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_k \geqslant 0$. Write $\mathcal{B} = \{B_1, \ldots, B_{k-1}\}$. Let $\Delta_{\mathcal{B}}^{r-1} = N(K[\mathcal{B}], K_{r-1})$. Since $\sum_{i=1}^{k-1} b_i = b - b_k$, by Theorem 1, $\Delta_{\mathcal{B}}^{r-1} \leqslant \Delta_{b-b_k,k-1}^{r-1}$.

Recall that $A = \cup_{i=1}^m A_i$. For those isolated vertices in $G[A]$, we have the following claim.

Claim 16. *For $v \in A$ with $d_{G[A]}(v) = 0$, we have $d_G^{(r-1)}(v) \leqslant \Delta_{\mathcal{B}}^{r-1} \leqslant \Delta_{b-b_k,k-1}^{r-1}$.*

Proof. Let T be the vertex set of a copy of K_r covering v. Then $T \cap A = \{v\}$ and $|T \cap B| = r - 1$ because $d_{G[A]}(v) = 0$. Therefore, $G[T \cap B] \cong K_{r-1}$. Note that $N_G(v) \subset B$. We have $d_G^{(r-1)}(v) \leqslant N(G[N_G(v)], K_{r-1})$.

Let $I(v) = \{i \in [k] : N_G(v) \cap B_i \neq \emptyset\}$. Apparently, $|I(v)| \leqslant k - 1$. Otherwise, $I(v) = [k]$. Note that $G[B] = K[B_1, \ldots, B_k]$ in this case. Thus $G[B \cup \{v\}] = K[\{v\}, B_1, \ldots, B_k]$ contains a copy of K_{k+1}, a contradiction. Now let $\mathcal{B}' = \cup_{i \in I(v)} \{B_i\} \subseteq \mathcal{B}$. Then

$$d_G^{(r-1)}(v) \leqslant N(G[N_G(v)], K_{r-1}) \leqslant N(K[\mathcal{B}'], K_{r-1}) \leqslant N(K[\mathcal{B}], K_{r-1}) = \Delta_{\mathcal{B}}^{r-1}.$$

□

When A is an independent set of G, we have the following claim.

Claim 17. *If A is an independent set of G, then $G \cong G_k(n, s)$.*

Proof. Let $\mathcal{B}_0 = \{B_1, \cdots, B_k\}$. Recall that $\mathcal{B} = \{B_1, \ldots, B_{k-1}\}$. Then

$$N(K[\mathcal{B}_0], K_r) = \Delta_{\mathcal{B}_0}^r = \Delta_{\mathcal{B}}^{r-1}|B_k| + \Delta_{\mathcal{B}}^r.$$

Note that $b + \sum_{i=1}^m \frac{a_i - 1}{2} \leqslant s$. Hence when $A = \cup_{i=1}^m A_i$ is an independent set, we have $a_1 = \ldots = a_m = 1$ and $b \leqslant s$. We can also suppose $b - b_k \geqslant 3$ since the small cases are easy to check. By Claim 16,

$$\begin{aligned} N(G, K_r) &\leqslant \Delta_{\mathcal{B}_0}^r + \sum_{v \in A} d_G^{r-1}(v) \\ &\leqslant \Delta_{\mathcal{B}_0}^r + (n - b)\Delta_{\mathcal{B}}^{r-1} \\ &= (n - b + |B_k|)\Delta_{\mathcal{B}}^{r-1} + \Delta_{\mathcal{B}}^r \\ &\leqslant [n - (b - b_k)]\Delta_{b-b_k,k-1}^{r-1} + \Delta_{b-b_k,k-1}^r \\ &= g_{n,k,r}(b - b_k). \end{aligned}$$

By Observation 12 (2), we have

$$N(G, K_r) \leqslant g_{n,k,r}(b - b_k) \leqslant g_{n,k,r}(s) = N(G_k(n, s), K_r).$$

When the equality holds, we must have $b_k = 0$, $b = s$, $G[B] \cong T_{k-1}(s)$ by Theorem 1, and $G[B, A] = K[B, A]$. This implies that $G \cong G_k(n, s)$. □

Claim 18. $a_2 = a_3 = \cdots = a_m = 1$.

Proof. If A is an independent set of G, then we are done. Now suppose $|G[A]| > 0$. Then $b < s$. Let v_0 be a vertex in A with $d_G^{(r-1)}(v_0) = \max_{v \in A} d_G^{(r-1)}(v)$. Without loss of generality, suppose $v_0 \in A_1$.

If $d_{G[A]}(v_0) = 0$, let G' be the resulting graph by applying the switching operations $u \to v_0$ for all vertices $u \in A \backslash \{v_0\}$ one by one. Then we have $|G'[A]| = 0$. By Proposition 7 (and its proof), $N(G', K_r) \geqslant N(G, K_r)$. With the same discussion as in the proof of Lemma 13, we have that G' is still $\{K_{k+1}, M_{s+1}\}$-free. But $|I_{G'}(A)| = m > |I_G(A)|$, a contradiction to the assumption (*).

If $d_{G[A]}(v_0) > 0$, then $a_1 \geqslant 3$. If $a_2 = \ldots = a_m = 1$, we are done. Now, without loss of generality, assume $a_2 \geqslant 3$. Since $G[A_2]$ is connected, we can pick two vertices, say u_1, u_2 in A such that $G[A_2 \backslash \{u_1, u_2\}]$ is still connected (u_1, u_2 exist, for example, we can take two leaves of a spanning tree of $G[A_2]$). Let G_1 be the resulting graph by applying the switching operations $u_1 \to v_0$ and $u_2 \to v_0$ one by one. With similar discussion as in the above case, we have $N(G_1, K_r) \geqslant N(G, K_r)$ and G_1 is $\{K_{k+1}, M_{s+1}\}$-free. Continue the process after $t = \frac{a_2 - 1}{2}$ steps, we obtain a graph G_t with $N(G_t, K_r) \geqslant N(G, K_r)$ and G_t is $\{K_{k+1}, M_{s+1}\}$-free. But $|I_{G_t}(A)| = |I_G(A)| + 1$, a contradiction to the assumption (*). $\square$

Now by Claim 18 and Theorem 6, we have $s' := b + \frac{a_1 - 1}{2} \leqslant s$. Thus, $a_1 = 2s' - 2b + 1$ and then $|B \cup A_1| = a_1 + b = 2s' - b + 1$. Note that $0 \leqslant b \leqslant s'$. By Claim 16, for vertex $v \notin B \cup A_1$, $d_G^{(r-1)}(v) \leqslant \Delta^{r-1}_{b-b_k, k-1} \leqslant \Delta^{r-1}_{b,k-1}$. Also, since $G[B \cup A_1]$ is K_{k+1}-free, by Theorem 1, $N(G[B \cup A_1], K_r) \leqslant \Delta^r_{2s'-b+1,k}$. Therefore,

$$N(G, K_r) \leqslant \Delta^r_{2s'-b+1,k} + (n - 2s' + b - 1)\Delta^{r-1}_{b,k-1}.$$

Define $f_{n,k,r,s'}(b) := \Delta^r_{2s'-b+1,k} + (n - 2s' + b - 1)\Delta^{r-1}_{b,k-1}$. If $b = 0$, then $f_{n,k,r,s'}(0) = N(T_k(2s'+1), K_r) \leqslant N(T_k(2s+1), K_r)$, and the proof is done. If $b = s'$, then $a_1 = 1$ and thus A is an independent set of G, the proof is done by Claim 17. In particular, $f_{n,k,r,s'}(s') \leqslant N(G_k(n, s'), K_r) \leqslant N(G_k(n, s), K_r)$.

By Observation 12 (1), for $0 \leqslant b \leqslant s' - 1$,

$$f_{n,k,r,s'}(b+1) - f_{n,k,r,s'}(b) = -\Delta^{r-1}_{(2s'-b)-\lfloor \frac{2s'-b}{k} \rfloor, k-1} + (n - 2s' + b)\Delta^{r-2}_{b - \lfloor \frac{b}{k-1} \rfloor, k-2} + \Delta^{r-1}_{b,k-1}.$$

For fixed k and r, $\Delta^r_{t,k}$ is an increasing function of t, and $t - \lfloor \frac{t}{k} \rfloor$ is a non-decreasing function of t. It is easy to check that $g(b) = f_{n,k,r,s'}(b+1) - f_{n,k,r,s'}(b)$ is an increasing function on $0 \leqslant b \leqslant s' - 1$. This implies that $f_{n,k,r,s'}(b)$ is convex on $[0, s'-1]$. Therefore,

$$f_{n,k,r,s'}(b) \leqslant \max\{f_{n,k,r,s'}(0), f_{n,k,r,s'}(s')\} \leqslant \max\{N(T_k(2s+1), K_r), N(G_k(n, s), K_r)\}.$$

$\square$

Now we prove Theorem 4.

Proof of Theorem 4: Let G be the extremal graph mentioned above with the maximum number of S_r.

Claim 19. $a_2 = a_3 = \cdots = a_m = 1$.

Proof. For any $i \in [m]$, a pair of vertices $\{u, v\} \subset A_i$ is said to be 2-switchable if $G[A_i \backslash \{u, v\}]$ is still connected. For any distinct $i, j \in [m]$ and any two ordered pairs $\{u_1, v_1\} \subset A_i$ and $\{u_2, v_2\} \subset A_j$, we call $(u_1, v_1) \to (u_2, v_2)$ a 2-switching if $\{u_1, v_1\}$ is 2-switchable. we call it a reversible 2-switching if both $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are 2-switchable. A 2-switching $(u_1, v_1) \to (u_2, v_2)$ is said to be good if $N(G_{(u_1,v_1)\to(u_2,v_2)}, S_r) > N(G, S_r)$; it is said to be bad if $N(G_{(u_1,v_1)\to(u_2,v_2)}, S_r) < N(G, S_r)$; otherwise, $N(G_{(u_1,v_1)\to(u_2,v_2)}, S_r) = N(G, S_r)$ and we say it is a fair 2-switching. Clearly, there is no good 2-switching in G.

Since there is no good 2-switching, any 2-switching $(u_1, v_1) \to (u_2, v_2)$ is bad or fair. If it is bad and reversible, then by proposition 8, $(u_2, v_2) \to (u_1, v_1)$ must be a good 2-switching, a contradiction. In other words, a reversible 2-switching cannot be bad, so it must be fair.

By Lemma 14, for any $i \in [m]$, $G[A_i]$ is a complete multipartite graph. One can check that there is at most one pair within A_i that is not 2-switchable (only if $a_i \geqslant 5$ and all the vertices in A_i other than this pair form a partition set of $G[A_i]$). Hence we can always pick some 2-switchable pair in A_i if $a_i \geqslant 3$.

Now suppose on the contrary that $a_1 \geqslant a_2 \geqslant 3$. Suppose $\{u_0, v_0\} \subset A_1$ is 2-switchable. For any 2-switchable pair $\{u, v\} \in A_i$ for some $i \geqslant 2$, $\{u, v\} \to \{u_0, v_0\}$ is reversible, so it must be fair. Thus, $G_{\{u,v\}\to\{u_0,v_0\}}$ is also an extremal graph. By repeatedly doing 2-switching $\{u, v\} \to \{u_0, v_0\}$ for 2-switchable pair $\{u, v\} \in A_i$ with $i \geqslant 2$ (note that during these operations, $\{u_0, v_0\}$ keeps being 2-switchable), we end up with a graph G' which is also an extremal graph. Moreover, there is no 2-switchable pair in A_i for any $i \geqslant 2$, which implies that $a_i = 1$ for $i \geqslant 2$. So $I_{G'}(A) = m - 1 > I_G(A)$, a contradiction. □

By Lemma 13 and G is K_{k+1}-free, $G[B]$ is a complete k-partite graph (some of its partition set may be empty). Suppose these partition set be B_1, B_2, $\cdots$, B_k and let $b_i = |B_i|$ for $i \in [k]$. If there exists some $i \in [k]$ with $b_i = 0$, then let $B_k = \emptyset$ without loss of generality. Otherwise, by Lemma 15, Claim 19 and since $n \geqslant 2(s + 1)(r + 1)$, it holds $A \backslash A_1$ is not empty and all its vertices share the same neighborhood. Clearly, at least one of $B_1, \cdots, B_k$ is not adjacent to $A \backslash A_1$, or we would get a copy of K_{k+1} in G. In this case, we let B_k be a partition set not adjacent to $A \backslash A_1$. Now define

$$f'_{n,k,r,s}(b_1, \cdots, b_k, a_1) = g_{n,k,r,b,a_1+b}(b_1, \cdots, b_k) + a_1 \binom{a_1 + b - 1}{r} + (n - a_1 - b) \binom{b}{r}.$$

Note that: the leaves of a star whose center is in B_i ($i \neq k$) must be in $V(G) \backslash B_i$; the leaves of a star whose center is in B_k must be in $A_1 \cup (B \backslash B_k)$; the leaves of a star whose center is in A_1 must be in $A_1 \cup B$; the leaves of a star whose center is in $A \backslash A_1$ must be in B. Therefore, we have $N(G, S_r) \leqslant f'_{n,k,r,s}(b_1, \cdots, b_k, a_1)$. Now by Proposition 10 (1),

$$f'_{n,k,r,s}(b_1, \cdots, b_{k-1}, b_k, a_1) < f'_{n,k,r,s}(b'_1, \cdots, b'_{k-1}, 0, a_1),$$

where $\sum_{i=1}^{k-1} b'_i = b = \sum_{i=1}^{k} b_i$ and $|b'_i - b'_j| \leqslant 1$ for any $i, j \in [k - 1]$. By Proposition 10 (2) and the fact that $a_1 = 2s' + 1 - 2b$ for some $s' \leqslant s$,

$$f'_{n,k,r,s'}(b'_1, \cdots, b'_{k-1}, 0, a_1) < f'_{n,k,r,s'}(b''_1, \cdots, b''_{k-1}, 0, 1),$$

where $\sum_{i=1}^{k-1} b''_i = s'$ and $|b''_i - b''_j| \leqslant 1$ for any $i, j \in [k-1]$. Therefore, $N(G, S_r) \leqslant f'_{n,k,r,s'}(b''_1, \cdots, b''_{k-1}, 0, 1)$. One can check that $f'_{n,k,r,s'}(b''_1, \cdots, b''_{k-1}, 0, 1) = N(G_k(n, s'), S_r) \leqslant N(G_k(n, s), S_r)$, which means we are done. □

4 Concluding Remarks

In this paper, we study on the function $\mathrm{ex}(n, T, \{K_{k+1}, M_{s+1}\})$ and determine the exact values when $T = K_r$ and $T = S_r$. However, for $T = S_r$, our result holds only when $n \geqslant 2(s+1)(r+1)$. One may conjecture that $\mathrm{ex}(n, S_r, \{K_{k+1}, M_{s+1}\}) = \max\{N(T_k(2s+1), S_r), N(G_k(n, s), S_r)\}$ for general $n \geqslant 2s+1$. Unluckily, this is not true. In fact, when r is somehow larger than k (for example $r > 20k$), one may find some conterexamples. Hence, we give the following conjecture instead.

Conjecture 20. There exists some fixed positive number α, for $n \geqslant 2s+1$ and $2 \leqslant r \leqslant \alpha k$,

$$\mathrm{ex}(n, S_r, \{K_{k+1}, M_{s+1}\}) = \max\{N(T_k(2s+1), S_r), N(G_k(n, s), S_r)\}.$$

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