

Pattern-avoiding shallow permutations

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Abstract

Shallow permutations are those permutations for which the lower bound of the Diaconis-Graham inequalities is an equality, i.e., those for which the total displacement is equal to the sum of the length and reflection length. Recently, there has been renewed interest in these permutations. In particular, Berman and Tenner showed they satisfy certain pattern avoidance conditions in their cycle form and Woo showed they are exactly those whose cycle diagrams are unlinked. Shallow permutations that avoid 321 have appeared in many contexts; they are those permutations for which depth equals the reflection length, they have unimodal cycles, and they have been called Boolean permutations. Motivated by this interest in 321-avoiding shallow permutations, we investigate σ -avoiding shallow permutations for all $\sigma \in \mathcal{S}_3$. To do this, we develop more general structural results about shallow permutations, and apply them to enumerate shallow permutations avoiding any pattern of length 3.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction and background

Let \mathcal{S}_n denote the set of permutations on $[n] = \{1, 2, \dots, n\}$ and we write these permutations in their one-line notation as $\pi = \pi_1\pi_2 \dots \pi_n$ where $\pi_i := \pi(i)$. There are multiple measures of disorder or disarray of a permutation $\pi \in \mathcal{S}_n$. Three of these, namely the total displacement $D(\pi)$, the length $I(\pi)$, and the reflection length $T(\pi)$, are connected by the Diaconis-Graham inequalities [5]:

$$I(\pi) + T(\pi) \leq D(\pi) \leq 2I(\pi).$$

Here, the total displacement $D(\pi)$, also called Spearman's measure of disarray, is given by

$$D(\pi) = \sum_{i=1}^n |\pi_i - i|.$$

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The length $I(\pi)$ is equal to the minimal number of simple transpositions required to produce π . It is also called the inversion number and is given by

$$I(\pi) = \sum_{i=1}^n |\{i < j \mid \pi_i > \pi_j\}|.$$

The reflection length $T(\pi)$ is the minimal number of transpositions required to produce π from the identity permutation, which is well-known (see for example, [3]) to be

$$T(\pi) = n - \text{cyc}(\pi),$$

where $\text{cyc}(\pi)$ denotes the number of cycles in the disjoint cycle decomposition of π .

It is well-known that the upper Diaconis-Graham inequality is achieved, i.e., $D(\pi) = 2I(\pi)$, when π avoids the pattern 321, meaning there is no set of indices $i < j < k$ with $\pi_i > \pi_j > \pi_k$. A permutation is called *shallow* when it satisfies the lower inequality, i.e., when $I(\pi) + T(\pi) = D(\pi)$. We note that shallow permutations have recently been investigated from various perspectives: In [2], the authors use pattern functions to characterize the cycle form of these permutations in terms of pattern-avoidance, and in [17], the author proves that shallow permutations are exactly those permutations whose cycle diagrams are equivalent to the unknot when viewed as a knot diagram.

Permutations which satisfy both the upper and lower bounds of the Diaconis-Graham inequalities have been well-studied in their own right. These permutations are exactly those that are shallow 321-avoiding permutations; these have been called Boolean permutations [12, 13, 16], unimodal permutations [6] (because of their unimodal cycle form), and are characterized as avoiding both 321 and 3412 [11]. It was stated, without proof, in [5], that these permutations are enumerated by F_{2n-1} , the $(2n - 1)$ -st Fibonacci number. A proof of this fact does appear in other places, including [8] and [11], and we provide an independent proof of this fact in this paper, directly using shallowness.

Motivated by this interesting answer regarding 321-avoiding shallow permutations, in this paper we investigate shallow permutations which avoid σ for $\sigma \in \mathcal{S}_3$. In Section 3, we describe certain properties of general shallow permutations which we use in follow-on sections. In particular, we show how to build shallow permutations from smaller ones, and we prove that all shallow permutations necessarily avoid certain mesh patterns. In Sections 4, 5, 6, and 7 we enumerate σ -avoiding shallow permutations for $\sigma \in \mathcal{S}_3$. Additionally, we enumerate σ -avoiding shallow permutations by number of descents and by three symmetry properties. In particular, we enumerate those shallow σ -avoiding permutations that are fixed under inverse, reverse-complement, and reverse-complement-inverse. The sections are ordered by the complexity of the proofs involved, with the exception of $\sigma = 321$, which we do last since these have been investigated in previous papers. We conclude the paper with open questions and directions for future study.

2 Preliminaries

For a given permutation $\pi = \pi_1\pi_2 \dots \pi_n \in \mathcal{S}_n$, we say that π *contains* the pattern $\sigma = \sigma_1\sigma_2 \dots \sigma_k \in \mathcal{S}_k$ if there is some $i_1 < i_2 < \dots < i_k$ so that $\sigma_r < \sigma_s$ if and only if

σ	number of shallow σ -avoiding permutations	Theorem	OEIS
132	$t_n(\sigma) = F_{2n-1}$	Theorem 9	A001519
213			
231	g.f. $T_\sigma(x) = \frac{1 - 3x + 2x^2 - x^3 - x^4 - x^5}{1 - 4x + 4x^2 - 2x^3 - x^4 - x^5}$	Theorem 17	A382570
312			
123	g.f. $T_\sigma(x) = \frac{1 - 3x + 11x^3 - 13x^4 + 7x^5 + 6x^6 + 3x^7}{(1-x)^4(1-4x^2+x^4)}$	Theorem 26	A382574
321	$t_n(\sigma) = F_{2n-1}$	Theorem 32	A001519

Table 1: In this table, $t_n(\sigma)$ denotes the number of shallow permutations avoiding a given pattern σ , and $T_\sigma(x) = \sum_{n \geq 0} t_n(\sigma)x^n$ is the corresponding generating function.

$\pi_{i_r} < \pi_{i_s}$ (i.e., if there is a subsequence of π that appears in the same relative order as the elements of σ). We say π *avoids* σ if π does not contain σ . For example, the permutation $\pi = 65748231$ contains the pattern 231 since $\pi_1\pi_5\pi_7 = 683$ is an example of a 231 pattern. The same permutation π avoids the pattern 132 since there is no subsequence of π in the same relative order as 132.

Pattern avoidance was first defined in 1968 by Knuth [9] in the context of sorting algorithms. The study of pattern-avoiding permutations from an enumerative perspective began in earnest in 1985 [15] and has continued since then. Variations on pattern avoidance have also been considered including consecutive, vincular, bivincular, and mesh patterns, each of which have extra requirements on the position and/or values that appear in a given pattern. We make use of so-called mesh patterns in Section 3 where we show shallow permutations necessarily avoid certain mesh patterns. These specific patterns are defined in that section. (See [1] for a more general discussion of mesh patterns.)

Throughout this paper, it will be convenient to refer to certain permutations that have been built from smaller ones. For that reason, we define the direct sum and skew sum of two permutations. Given two permutations $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$ and $\tau = \tau_1 \dots \tau_m \in \mathcal{S}_m$, we can form their *direct sum* $\sigma \oplus \tau \in \mathcal{S}_{n+m}$, and their *skew sum* $\sigma \ominus \tau \in \mathcal{S}_{n+m}$ as follows,

$$\sigma \oplus \tau := \sigma_1 \dots \sigma_n (\tau_1 + n) \dots (\tau_m + n) \quad \text{and} \quad \sigma \ominus \tau := (\sigma_1 + m) \dots (\sigma_n + m) \tau_1 \dots \tau_m.$$

For example, $132 \oplus 4123 = 1327456$, and $132 \ominus 4123 = 5764123$. We will often take direct and skew sums involving the decreasing permutation, so we let $\delta_n := n(n-1) \dots 21$.

Next, let us define certain symmetries on permutations.

- We denote by π^{-1} the *algebraic inverse* of π . That is, $\pi_j^{-1} = i$ if and only if $\pi_i = j$. This corresponds to a reflection of the diagram of the permutation (given by points (j, π_j) for each $j \in [n]$) about the main diagonal. Permutations which are their own inverse are called *involutions*.
- We define π^{rc} to be the *reverse-complement* of π , so that $\pi_{n+1-i}^{rc} = n + 1 - \pi_i$. This corresponds to a 180° rotation of the diagram of the permutation. Permutations satisfying $\pi = \pi^{rc}$ are called *centrosymmetric*; see, for example, [7, 10].
- We let $\pi^{rci} := (\pi^{rc})^{-1}$ be the *reverse-complement-inverse* of the permutation, corresponding to the reflection of the diagram of the permutation about the anti-diagonal. We will refer to permutations satisfying $\pi = \pi^{rci}$ as *persymmetric*.

Let us note that each of these symmetries preserves the cycle type of a permutation, which will make them quite useful for our purposes. In other words, π has a given cycle type if and only if π^{-1} , π^{rc} , and π^{rci} have the same cycle type. This is a well-known fact about these symmetries and can be seen quickly from the cycle diagram of the permutation.

We define a *descent* in a permutation π to be an index i with $1 \leq i \leq n - 1$ such that $\pi_i > \pi_{i+1}$, and we denote by $\text{des}(\pi)$ the number of descents the permutation π has. For example, the permutation $\pi = 65748231$ has a descent in positions 1, 3, 5, and 7, so $\text{des}(\pi) = 4$. We say that π_i is a *left-to-right maximum* if $i = 1$ or if

$$\pi_i > \max\{\pi_1, \pi_2, \dots, \pi_{i-1}\} \quad \text{for } i > 1.$$

Similarly, we say π_i is a *right-to-left minimum* if $i = n$ or if

$$\pi_i < \min\{\pi_{i+1}, \pi_{i+2}, \dots, \pi_n\} \quad \text{for } i < n.$$

For example, in the permutation $\pi = 24175683$, the elements $\pi_1 = 2, \pi_2 = 4, \pi_4 = 7$, and $\pi_7 = 8$ are left-to-right maxima and $\pi_8 = 3$ and $\pi_3 = 1$ are right-to-left minima.

Finally, we let \mathcal{T}_n denote the permutations $\pi \in \mathcal{S}_n$ that are shallow (i.e., those permutations π for which $I(\pi) + T(\pi) = D(\pi)$), and let $t_n = |\mathcal{T}_n|$. Additionally, we let $\mathcal{T}_n(\sigma)$ denote the permutations $\pi \in \mathcal{S}_n$ that are shallow and avoid σ . We set $t_n(\sigma) = |\mathcal{T}_n(\sigma)|$ and $T_\sigma(x) = \sum_{n \geq 0} t_n(\sigma)x^n$.

3 Structure of shallow permutations

We will often make use of the following recursive formulation of shallow permutations that is due to Hadjicostas and Monico [8].

Theorem 1. [8, Theorem 4.1] *Suppose $\pi \in \mathcal{S}_n$ and for $n \geq 2$, define*

$$\pi^R = \begin{cases} \pi_1\pi_2 \dots \pi_{n-1} & \text{if } \pi_n = n \\ \pi_1\pi_2 \dots \pi_{j-1}\pi_n\pi_{j+1} \dots \pi_{n-1} & \text{if } \pi_j = n \text{ with } j < n \end{cases}$$

Then $\pi = 1 \in \mathcal{S}_1$ is shallow, and when $n \geq 2$,

- if $\pi_n = n$, then π is shallow exactly when π^R is, and
- if $\pi_j = n$ with $j < n$, then π is shallow exactly when π^R is shallow and $\pi_j^R = \pi_n$ is a left-to-right maximum or right-to-left minimum in π^R .

Let us see an example of Theorem 1. Suppose $\pi = 421635 \in \mathcal{T}_6$, a shallow permutation of 6 elements. Notice $\pi_4 = 6$, and $\pi_6 = 5$; applying the π^R map we see

$$421\mathbf{6}3\mathbf{5} \xrightarrow{\pi^R} 421\mathbf{5}3,$$

and $42153 \in \mathcal{T}_5$. Notice that we can use the inverse of this map to construct new shallow permutations from old ones. Given any $\tau \in \mathcal{T}_{n-1}$ and any position i for which τ_i is either a left-to-right maximum or a right-to-left minimum, we can construct a permutation π for which $\pi^R = \tau$ by taking $\pi_j = \tau_j$ for $j \neq i$, $\pi_i = n$, and $\pi_n = \tau_i$. Notice that we can get every shallow permutation on $[n]$ from the shallow permutations on $[n-1]$ in this way since every shallow permutation $\pi \in \mathcal{T}_n$ has an image $\tau = \pi^R$ in \mathcal{T}_{n-1} .

We will define a similar operation π^L which acts on the left of π . To do this, we will make use of the three symmetries defined in Section 2. In the following proposition, we show that each of these three symmetries preserves shallowness.

Proposition 2. *If π is shallow, then so are the permutations π^{-1} , π^{rc} , and π^{rci} .*

Proof. To see that $\pi^{-1} \in \mathcal{T}_n$ notice first that $D(\pi^{-1}) = \sum_{i=1}^n |\pi_{\pi_i}^{-1} - \pi_i| = \sum_{i=1}^n |i - \pi_i| = D(\pi)$. Next, $I(\pi^{-1}) = I(\pi)$ since (i, j) is an inversion of π (i.e., $i < j$ and $\pi_i > \pi_j$) if and only if (π_j, π_i) is an inversion of π^{-1} . Similarly, $T(\pi^{-1}) = T(\pi)$ since the cycle type of π and π^{-1} are the same. So, since $I(\pi) + T(\pi) = D(\pi)$, the same is true for π^{-1} , which means that π^{-1} is shallow.

We similarly check π^{rc} . First,

$$D(\pi^{rc}) = \sum_{i=1}^n |\pi_i^{rc} - i| = \sum_{i=1}^n |(n - \pi_{n-i+1} + 1) - i| = \sum_{i=1}^n |(n - i + 1) - \pi_{n-i+1}| = D(\pi).$$

Next, $I(\pi^{rc}) = I(\pi)$ since π has an inversion in position (i, j) if and only if π^{rc} has one in position $(n - i + 1, n - j + 1)$. Indeed $\pi_i > \pi_j$ with $i < j$ if and only if $\pi_{n-i+1}^{rc} = n - \pi_i + 1 < n - \pi_j + 1 = \pi_{n-j+1}^{rc}$ with $n - i + 1 > n - j + 1$. Finally π^{rc} and π have the same cycle type because $\pi^{rc} = \sigma^{-1}\pi\sigma$ where $\sigma = n(n-1)(n-2)\dots 21$. Two permutations have the same cycle type if and only if they are conjugate. Finally, π^{rci} preserves shallowness because the reverse-complement and inverse do. \square

We can now use Proposition 2 to define a similar operation to π^R , which we denote by π^L , which also preserves shallowness and is defined as follows. Here, the reduction operator, red , takes the elements of its input of length ℓ and returns a permutation in \mathcal{S}_ℓ in the same relative order. For example, $\text{red}(48291) = 34251$ and $\text{red}(9482) = 4231$.

Theorem 3. Suppose $\pi \in \mathcal{S}_n$ and for $n \geq 2$, define

$$\pi^L = \begin{cases} \text{red}(\pi_2 \dots \pi_n) & \text{if } \pi_1 = 1 \\ \text{red}(\pi_2 \dots \pi_{j-1} \pi_1 \pi_{j+1} \dots \pi_n) & \text{if } \pi_j = 1 \text{ with } j > 1 \end{cases}$$

Then $\pi = 1 \in \mathcal{S}_1$ is shallow, and when $n \geq 2$,

- if $\pi_1 = 1$, then π is shallow exactly when π^L is, and
- if $\pi_j = 1$ with $j > 1$, then π is shallow exactly when π^L is shallow and $\pi_j^L = \pi_1 - 1$ is a left-to-right maximum or right-to-left minimum in π^L .

Proof. This follows immediately from Theorem 1 and Proposition 2 since $\pi^L = [(\pi^{rc})^R]^{rc}$. \square

Let us see an example. If $\pi = 421635 \in \mathcal{T}_6$, we can apply π^L to get

$$\underline{4}2\underline{1}635 \xrightarrow{\pi^L} \text{red}(\underline{2}\underline{4}635) = \underline{1}3\underline{5}24,$$

and note that $13524 \in \mathcal{T}_5$. Similar to our observation about the right operator above, this left operator can also be “inverted” to produce all shallow permutations on $[n]$ from those on $[n - 1]$.

We denote by π^{R^m} and π^{L^m} the application of the right and left operators from Theorems 1 and 3, respectively, applied m times. For example, $\pi^{L^3} = ((\pi^L)^L)^L$. On occasion, after applying the left operator to a permutation, we will work with the entries of the resulting permutation without reducing, for ease of notation. When we do this, we mark the entries. For example, we may write $(421635)^L$ as $2'4'6'3'5'$ with $i' = i - 1$, instead of writing 13524 . More generally, if $\pi = \pi_1 \pi_2 \dots \pi_{j-1} \pi_j \pi_{j+1} \dots \pi_n$ and $\pi_j = 1$, we may refer to π^L as $\pi^L = \pi'_2 \dots \pi'_{j-1} \pi'_1 \pi'_{j+1} \dots \pi'_n$ with $\pi'_i = \pi_i - 1$ for each $i \neq j$ instead of writing $\pi^L = (\pi_2 - 1)(\pi_3 - 1) \dots (\pi_{j-1} - 1)(\pi_1 - 1)(\pi_{j+1} - 1) \dots (\pi_n - 1)$.

Next, let us make some general observations about shallowness. In the following lemma, we will see that shallowness is preserved under direct sums.

Lemma 4. If $\pi \in \mathcal{T}_n$ and $\sigma \in \mathcal{T}_m$, then $\pi \oplus \sigma \in \mathcal{T}_{n+m}$.

Proof. First, notice that $D(\pi \oplus \sigma) = \sum_{i=1}^n |\pi_i - i| + \sum_{i=1}^m |(\sigma_i + n) - (i + n)| = D(\pi) + D(\sigma)$. Next, $I(\pi \oplus \sigma) = I(\pi) + I(\sigma)$ since there can be no additional inversions between the elements of π and σ . Finally, $T(\pi \oplus \sigma) = T(\pi) + T(\sigma)$ since the number of cycles in $\pi \oplus \sigma$ is the sum of the number of cycles in π plus the number of those in σ . It then follows from the original definition of a shallow permutation that if π and σ are shallow, so is $\pi \oplus \sigma$. \square

In the next lemma, we see that we can always add n to the beginning and 1 to the end of a shallow permutation of length $n - 2$ and the result would be a shallow permutation of length n , and we can similarly delete those elements from a shallow permutation of length n to get a shallow permutation of length $n - 2$.

Lemma 5. Let $\pi \in \mathcal{S}_{n-2}$ and $\tau = (n)(\pi_1 + 1)(\pi_2 + 1) \dots (\pi_{n-2} + 1)1$. Then $\pi \in \mathcal{T}_{n-2}$ if and only if $\tau \in \mathcal{T}_n$.

Proof. Let $\pi \in \mathcal{T}_{n-2}$. Then by Lemma 4, $\sigma = 1 \oplus \pi \in \mathcal{T}_{n-1}$. By Theorem 1 and with $\sigma_1 = 1$ a left-to-right maximum, we apply the inverse recursion from Theorem 1 which replaces the 1 with n and moves 1 to the end. Thus we arrive at a shallow permutation in the form of τ as defined in the statement of the lemma. \square

By repeatedly applying this lemma, we can obtain the following corollary, which we will use frequently in this paper.

Corollary 6. The decreasing permutation $\delta_n = n(n-1) \dots 21$ is shallow.

Remark 7. It is possible to prove a stronger version of the above results. If $\pi \in \mathcal{T}_n$, $\tau \in \mathcal{T}_m$ and $\pi_i = i$, then the inflation of i by τ (that is, the larger permutation created by replacing π_i with a contiguous block of elements that is order isomorphic to τ) remains shallow. Indeed, it is straightforward to check that the sum of the inversion number and reflection length of the inflated permutation still equals its total displacement. Lemma 4 is the inflation of 12 by two shallow permutations and Lemma 5 is the inflation of 321 at 2 by a shallow permutation. We do not use this stronger result, so we omit its full proof.

We end this section by noting that shallow permutations necessarily avoid certain so-called mesh patterns. Recall that these are not classical patterns, but those that require an extra condition on the position and/or values in the pattern. We will denote by **3412** the permutation pattern 3412 where the “4” is equal to n and the “1” is equal to 1. For example, $\pi = 642981537$ contains the subsequence 4913 which is a **3412**. It also contains 6815 which is a 3412 pattern, but not a **3412** pattern since the “4” in this pattern is not equal to $n = 9$.

We denote by 3412 the permutation pattern 3412 where the “3” occurs in the first position and the “2” occurs in the last position. For example, the permutation $\pi = 672198435$ contains the subsequence 6835 which is a 3412 pattern since 6, which is the “3” in this pattern, appears in the first position and 5, which is the “2” in this pattern, appears in the last position.

Theorem 8. If $\pi \in \mathcal{T}_n$, then π avoids the patterns **3412** and 3412.

Proof. Let us proceed by contradiction. Suppose π contains a **3412** pattern. We will show that upon repeatedly applying the right operator R , we will eventually move an element that will be neither a left-to-right maximum nor a right-to-left minimum in the new permutation, contradicting that π is shallow.

To this end, suppose $\pi_r \pi_i \pi_j \pi_s$ (with $r < i < j < s$) is a **3412** pattern, so $\pi_i = n$, $\pi_j = 1$, and $\pi_r > \pi_s$. Notice that when we apply the right operator once, we get

$$\pi^R = \pi_1 \dots \pi_{i-1} \pi_n \pi_{i+1} \dots \pi_{j-1} 1 \pi_{j+1} \dots \pi_{n-1}.$$

If $s = n$, then we have a contradiction since $1 < \pi_s < \pi_r$ and so $\pi_s = \pi_n$ is neither a left-to-right maximum nor a right-to-left minimum in π^R . If $s < n$, then we must have

that π_n is a left-to-right maximum, or else π would not be shallow, so the element in position i is still larger than all elements in positions 1 through $i - 1$.

Now, let us continue to apply the right operator, R . Each time, the last element is either deleted (if it is the largest element), moved to a position to the right of 1 (if the largest element is also to the right of 1), or it is moved to the left of 1, in which case it must be a left-to-right maximum. Note that each time an element is moved to the left of 1, it must be in a position greater than or equal to i since each element moved over is itself larger than all elements in positions 1 through $i - 1$. Eventually, π_s will be moved to the left of 1, and it will be moved to a position greater than or equal to i . However, $\pi_s < \pi_r$ with $r < i$. Thus π_s cannot be a left-to-right maximum in this permutation. It also cannot be a right-to-left minimum since 1 is to its right. Thus the original permutation is not shallow.

The other avoidance follows from Proposition 2 and the fact that π avoids $\underline{3412}$ if and only if π^{-1} avoids $\underline{3412}$. \square

4 Shallow permutations that avoid 132 or 213

In this section, we enumerate shallow permutations that avoid the pattern 132. We also consider the number of such permutations with a given number of descents, as well as those that exhibit a certain symmetry. Note that by Proposition 2 and the fact $132^{rc} = 213$, $\mathcal{T}_n(132) = \mathcal{T}_n(213)$, so proving Theorem 9 for shallow permutations avoiding 132 holds for 213 as well.

4.1 Enumeration of 132-avoiding shallow permutations

In this subsection, we will prove the following theorem.

Theorem 9. *For $n \geq 1$ and $\sigma \in \{132, 213\}$, $t_n(\sigma) = F_{2n-1}$, the $(2n - 1)$ st Fibonacci number.*

We will first establish a few lemmas. This first lemma guarantees that for any shallow 132-avoiding permutation π , we must have that, if π does not start or end with n , it must end with 1, and in the case that n is not in the second-to-last position, π must start with $(n - 1)$.

Lemma 10. *For $n \geq 3$, suppose $\pi \in \mathcal{T}_n(132)$.*

- *If $\pi_j = n$ with $2 \leq j \leq n - 1$ then $\pi_n = 1$, and*
- *If $\pi_j = n$ with $2 \leq j \leq n - 2$ then $\pi_1 = n - 1$.*

Proof. Let us consider the first bullet point. Suppose $\pi \in \mathcal{T}_n(132)$ with $\pi_j = n$ for $2 \leq j \leq n - 1$. Note that $\pi_i > \pi_k$ for any $i < j < k$ since π avoids 132. By Theorem 1, $\pi_j^R = \pi_n$ must be either a left-to-right maximum or right-to-left minimum. It cannot be a left-to-right maximum because $\pi_{j-1}^R = \pi_{j-1} > \pi_n = \pi_j^R$. So π_j^R must be a right-to-left

minimum. However, since π is 132-avoiding we know that 1 appears to the right of n in π , so the only way for π_j^R to be a right-to-left minimum is if $\pi_j^R = 1$, and thus $\pi_n = 1$.

Now let us prove the second bullet point. Since π is 132 avoiding and $j > 1$, $n - 1$ must occur to the left of n in π . This means that $n - 1$ occurs to the left of 1 in π^R . Suppose $\pi_k^R = n - 1$ with $1 \leq k < j$, we will show that $k = 1$. Again by Theorem 1, $\pi_k^{R^2}$ must be a left-to-right maximum or right-to-left minimum. But now it cannot possibly be a right-to-left minimum because $\pi_j^{R^2} = 1$ by the first part of the lemma and $k < j \leq n - 2$. So $\pi_k^{R^2}$ must be a left-to-right maximum. Since π was 132 avoiding, every entry to the left of 1 in π^R will be larger than every entry to the right of 1. So the only way $\pi_k^{R^2}$ is a left-to-right maximum is if $k = 1$. \square

Proof of Theorem 9. Let $t_n = |\mathcal{T}_n(132)|$ and b_n be the number of $\pi \in \mathcal{T}_n(132)$ ending with 1. Notice that b_n is also the number of $\pi \in \mathcal{T}_n(132)$ beginning with n since π is a 132-avoiding shallow permutation if and only if π^{-1} is. By Lemma 10, we know that each $\pi \in \mathcal{T}_n(132)$ either begins with n , ends with n or ends with 1. There are clearly t_{n-1} such permutations that end in n (by removing that fixed point) and by Lemma 5, there are t_{n-2} such permutations that start with n and end with 1. Thus it follows that

$$t_n = t_{n-1} + 2b_n - t_{n-2}.$$

Next, let us find a recurrence for b_n ; let $\pi \in \mathcal{T}_n(132)$ with $\pi_n = 1$ and consider the position of n . If $\pi_{n-1} = n$, then $\pi^R \in \mathcal{T}_n(132)$ ending in 1 and so there are b_{n-1} such permutations. If $\pi_j = n$ for $2 \leq j \leq n - 2$, then $\pi_1 = n - 1$ and so $\pi^{RL} = \pi'_2 \pi'_3 \dots \pi'_{j-1} (n - 2) \pi'_{j+1} \dots \pi'_{n-1}$, which is still 132 avoiding because π is and we do not change the relative order of any elements. Furthermore, since in π^R , $\pi_j^R = \pi_n = 1$ is clearly a right-to-left minimum and in π^{RL} , $\pi_{j-1}^{RL} = (\pi_1^R)' = (n - 1)' = n - 2$ will clearly be a left-to-right maximum, π^{RL} is also shallow. So, the resulting permutation is any shallow 132-avoiding permutation that does not end in $n - 2$ since $\pi_{j-1}^{RL} = n - 2$ and $1 \leq j - 1 \leq n - 3$. There are $t_{n-2} - t_{n-3}$ such permutations. Finally, if $\pi_1 = n$, there are clearly t_{n-2} such permutations by Lemma 5.

Altogether, we find

$$b_n = b_{n-1} + 2t_{n-2} - t_{n-3}.$$

Substituting this back into our recursion for t_n ,

$$\begin{aligned} t_n &= t_{n-1} + 2(b_{n-1} + 2t_{n-2} - t_{n-3}) - t_{n-2} \\ &= t_{n-1} + (t_{n-2} + 2b_{n-1} - t_{n-3}) + 2t_{n-2} - t_{n-3} \\ &= 2t_{n-1} + 2t_{n-2} - t_{n-3} \end{aligned}$$

which is precisely the recursion satisfied by F_{2n-1} . \square

4.2 132-avoiding shallow permutation by number of descents

In this subsection, we will refine the enumeration of 132-avoiding shallow permutations by their number of descents. We first present the following lemma.

Lemma 11. *If $\pi \in \mathcal{S}_n(132)$ has k descents, then $\pi^{-1} \in \mathcal{S}_n(132)$ also has k descents.*

Proof. We will proceed by strong induction on n . The result is clear for $n \leq 3$. So assume $n \geq 4$ and $\pi \in \mathcal{S}_n(132)$. This means $\pi = (\tau \oplus 1) \ominus \sigma$ for some $\tau \in \mathcal{S}_m(132)$ and $\sigma \in \mathcal{S}_{n-m-1}(132)$. In this case, $\pi^{-1} = \sigma^{-1} \ominus (\tau^{-1} \oplus 1)$. By induction σ^{-1} and τ^{-1} have the same number of descents as σ and τ ; we lose one descent in position $m+1$ of π , but gain an additional descent in position $n-m-1$ of π^{-1} , when transitioning from σ^{-1} to τ^{-1} , that does not come from any of the descents in σ^{-1} or τ^{-1} . We therefore preserve the number of descents, and the result follows by induction. \square

Example 12. Consider $\pi = 5346721 \in \mathcal{S}_7(132)$ which has 3 descents. Then we have $\pi = (3124 \oplus 1) \ominus (21)$ and $\pi^{-1} = 7623145$ which is $(21)^{-1} \ominus ((3124)^{-1} \oplus 1)$. The number of descents in $(21)^{-1}$ and $(3124)^{-1}$ are the same as in 21 and 3124, respectively; we lose the descent in position $m=5$ of π , but gain a descent in position $n-m-1=6-5-1=2$ of π^{-1} when we transition from σ^{-1} to τ^{-1} .

We now adapt the proof of Theorem 9 to keep track of descents to arrive at the following result.

Theorem 13. For $n \geq 2$, the number of shallow, 132-avoiding permutations with k descents is equal to

$$\binom{2n-2-k}{k}.$$

Proof. Let $t_{n,k}$ be the number of shallow, 132-avoiding permutations of length n with k descents and let $b_{n,k}$ be the number of such permutations π that end with $\pi_n = 1$. Note that by Lemma 11, $b_{n,k}$ is also the number of shallow, 132-avoiding permutations π with k descents that start with $\pi_1 = n$.

As in the proof of Theorem 9, we know that for any permutation $\pi \in \mathcal{T}_n(132)$, by Lemma 10, π has either $\pi_n = 1, \pi_n = n$ or $\pi_1 = n$. It is clear there are $t_{n-1,k}$ shallow, 132-avoiding permutations with k descents ending in n since adding an n to the end preserves shallowness and does not change the number of descents. For $n \geq 3$, it is also clear that there are $t_{n-2,k-2}$ permutations that both begin with n and end with 1, which is seen by deleting both n and 1 to obtain a shallow permutation that still avoids 132 and has two fewer descents. This means

$$t_{n,k} = t_{n-1,k} + 2b_{n,k} - t_{n-2,k-2}.$$

Now, let us consider those permutations with $\pi_n = 1$. As before, if $\pi_{n-1} = n$, then there are still k descents in π^R , which still ends in 1, and so $b_{n-1,k}$ permutations. If $\pi_j = n$ for $2 \leq j \leq n-2$, then $\pi_1 = n-1$ by Lemma 10. If $j = 2$, then π^{RL} has one fewer descent and begins with its largest element $n-2$, meaning there are $b_{n-2,k-1}$ such permutations. If $3 \leq j \leq n-2$, then π^{RL} has two fewer descents, and it must not end or begin with its largest element, $n-2$. There are $t_{n-2,k-2} - b_{n-2,k-2} - t_{n-3,k-2}$ such permutations. Indeed, subtracting $b_{n-2,k-2}$ accounts for the permutations beginning with the largest element and subtracting $t_{n-3,k-2}$ accounts for the permutations that end with the largest element. Finally, if $\pi_1 = n$, as stated above there are $t_{n-2,k-2}$ such permutations. In total, we have

$$b_{n,k} = 2t_{n-2,k-2} + b_{n-1,k} + b_{n-2,k-1} - b_{n-2,k-2} - t_{n-3,k-2}.$$

We now claim that $b_{n,k} = t_{n-1,k-1}$. We will prove this claim by strong induction on n , showing that for fixed $n \geq 2$, $b_{n,k} = t_{n-1,k-1}$ for all $k \geq 0$. It is straightforward to check this claim for values of $n \leq 3$ so let us assume $n \geq 4$. We note that if k is negative, $b_{n,k} = t_{n,k} = 0$, and for $n \geq 4$, $b_{n,0} = 0 = t_{n-1,-1}$. Now, by the previous paragraph,

$$\begin{aligned} b_{n,k} &= 2t_{n-2,k-2} + b_{n-1,k} + b_{n-2,k-1} - b_{n-2,k-2} - t_{n-3,k-2} \\ &= 2b_{n-1,k-1} + t_{n-2,k-1} + b_{n-2,k-1} - t_{n-3,k-3} - b_{n-2,k-1} \\ &= t_{n-2,k-1} + 2b_{n-1,k-1} - t_{n-3,k-3} \\ &= t_{n-1,k-1}, \end{aligned}$$

where the last equality follows from the recurrence for $t_{n,k}$ above. Notice that by taking $b_{n,k} = t_{n-1,k-1}$, we now obtain a recurrence for $t_{n,k}$ as follows:

$$t_{n,k} = t_{n-1,k} + 2t_{n-1,k-1} - t_{n-2,k-2},$$

which together with the initial conditions is exactly the recurrence satisfied by $t_{n,k} = \binom{2n-2-k}{k}$. \square

4.3 132-avoiding shallow permutations with symmetry

In this subsection, we consider 132-avoiding shallow permutations that are involutions (so that $\pi = \pi^{-1}$), that are centrosymmetric (so that $\pi = \pi^{rc}$), and that are persymmetric (so that $\pi = \pi^{rci}$).

Theorem 14. *For $n \geq 1$, the number of shallow, 132-avoiding involutions of length n is F_{n+1} , where F_{n+1} is the $(n+1)$ -st Fibonacci number.*

Proof. Let i_n be the number of shallow, 132-avoiding permutations of length n that are involutions. We will show that $i_n = i_{n-1} + i_{n-2}$, and with initial conditions $i_1 = 1$ and $i_2 = 2$, we have the Fibonacci sequence shifted by 1.

There are clearly i_{n-1} shallow, 132-avoiding involutions π of length n with $\pi_n = n$ since adding the fixed point n to the end of an involution in $\mathcal{T}_{n-1}(132)$ gives us a permutation that is still an involution, still avoids 132, and is still shallow by Theorem 1.

If $\pi \in \mathcal{T}_{n-1}(132)$ does not end in n , then by Lemma 10, $\pi_1 = n$ or $\pi_n = 1$. However, if π is an involution, then one of these will imply the other. Note that by Lemma 5, we can add an n to the beginning and 1 to the end of an involution in $\mathcal{T}_{n-1}(132)$, and the resulting permutation is still shallow. Additionally the permutation still avoids 132 and is still an involution since we have only added the 2-cycle $(1, n)$. Thus there are i_{n-2} involutions in $\mathcal{T}_n(132)$ beginning with n and ending with 1. The recurrence, and thus the result follows. \square

Theorem 15. *For $n \geq 2$, the number of 132-avoiding shallow centrosymmetric permutations is $\lceil (n+1)/2 \rceil$.*

Proof. Notice that if π avoids 132 and $\pi = \pi^{rc}$, π must also avoid $132^{rc} = 213$. By Lemma 10, we know that either $\pi_n = n$, $\pi_n = 1$, or $\pi_1 = n$. However, if $\pi = \pi^{rc}$, then $\pi_1 = n$ implies that $\pi_n = 1$. Therefore, either $\pi_n = n$, or $\pi_1 = n$ and $\pi_n = 1$. In the first case, since $\pi_n = n$ and π avoids 213, π is the increasing permutation. In the second case, by Lemma 5, by deleting n and 1, we obtain a shallow 132-avoiding centrosymmetric permutation of length $n - 2$. Letting c_n be the number of centrosymmetric permutations in $\mathcal{T}_n(132)$, we thus have $c_n = c_{n-2} + 1$, which together with the initial conditions that $c_1 = 1$ and $c_2 = 2$ implies the result. \square

Theorem 16. *Let $p_n(132)$ be the number of 132-avoiding shallow persymmetric permutations and let $P_{132}(x)$ be the generating function for $p_n(132)$. Then*

$$P_{132}(x) = \frac{1 - x^2 + 2x^3}{(1 - x)(1 - 2x^2 - x^4)}.$$

Proof. For the cases $1 \leq n \leq 4$, it is straightforward to check that the 132-avoiding shallow persymmetric permutations are

$$\{1, 12, 21, 123, 231, 312, 321, 1234, 2341, 3421, 4123, 4231, 4312, 4321\}$$

and so $p_1(132) = 1, p_2(132) = 2, p_3(132) = 4$, and $p_4(132) = 7$. Additionally, $p_0(132) = 1$ by considering the empty permutation.

Now, let $n \geq 5$ and let $\pi \in \mathcal{T}_n(132)$ with $\pi = \pi^{rci}$. We use Lemma 10 to determine a few possible cases. First, if $\pi_n = n$, then since $\pi = \pi^{rci}$ we must have $\pi_1 = 1$, which implies that π is the increasing permutation. If $\pi_{n-1} = n$, then by Lemma 10, we must have $\pi_n = 1$. Since $\pi = \pi^{rci}$, we have $\pi_1 = 2$, which implies that $\pi = 2345 \dots n1$ since π is 132-avoiding. Note that this permutation is clearly shallow.

Next, consider a permutation where $\pi_j = n$ for some $2 \leq j \leq n - 2$. By Lemma 10, this permutation must end with 1 and start with $n - 1$. But this implies that $\pi_2 = n$ and so $\pi = (n - 1)n\pi_3 \dots \pi_{n-1}1$. Note that π^{RL} can be obtained by deleting n and 1 from π . This permutation is still shallow, still avoids 132, and still is persymmetric, and furthermore begins with n . Let $q_n(132)$ be the number of persymmetric permutations in $\mathcal{T}_n(132)$ that begin with n (which is also equal to the number of persymmetric permutations in $\mathcal{T}_n(132)$ that end with 1). Note that we must have $q_0(132) = 0$ since the empty permutation does not start with n . We thus have

$$p_n(132) = 2 + q_n(132) + q_{n-2}(132).$$

Let us now enumerate those permutations in $\mathcal{T}_n(132)$ that end with 1 (or equivalently start with n , since π is persymmetric if and only if π^{-1} is). We clearly have $p_{n-2}(132)$ permutations that start with n and end with 1 since removing these will leave a persymmetric shallow permutation avoiding 132. Now let $\pi \in \mathcal{T}_n(132)$ with $\pi_n = 1$ and $\pi_1 \neq n$. If $\pi_{n-1} = n$, then since π is persymmetric, we have $\pi_1 = 2$, and so $\pi = 23 \dots n1$. If $\pi_j = n$ for $2 \leq j \leq n - 2$, then by Lemma 10, $\pi_1 = n - 1$. But since π is persymmetric, we must have $j = 2$. Thus $\pi = (n - 1)n\pi_3 \dots \pi_{n-1}1$. Note that π^{RL} is also shallow of length $n - 2$

and starts with $\pi_1 = n - 2$, avoids 132, and is persymmetric. Thus there are $q_{n-2}(132)$ such permutations. Thus we have

$$q_n(132) = 1 + p_{n-2}(132) + q_{n-2}(132).$$

Letting $Q_{132}(x)$ be the generating function for $q_n(132)$, and taking into account the initial conditions, we get

$$Q_{132}(x) = x + \frac{x^4}{1-x} + x^2 P_{132}(x) + x^2 Q_{132}(x)$$

and

$$P_{132}(x) = 1 + x^2 + x^3 + \frac{2x^4}{1-x} + (1+x^2)Q_{132}(x).$$

Solving for $Q_{132}(x)$, plugging the result into the equation for $P_{132}(x)$, solving for $P_{132}(x)$, and then simplifying gives the result in the statement of the theorem. \square

5 Shallow permutations that avoid 231 or 312

In this section, we enumerate shallow permutations that avoid the pattern 231. We also consider the number of such permutations with a given number of descents, as well as those that exhibit certain symmetry. Note that by Proposition 2 and the fact that $231^{-1} = 312$, $\mathcal{T}_n(231) = \mathcal{T}_n(312)$, and so Theorem 17 holds for shallow permutations avoiding 312 as well.

5.1 Enumeration of 231-avoiding shallow permutations

Theorem 17. *Let $T_{231}(x) = \sum_{n \geq 0} t_n(231)x^n$ be the generating function for $t_n(231)$. Then,*

$$T_{231}(x) = \frac{1 - 3x + 2x^2 - x^3 - x^4 - x^5}{1 - 4x + 4x^2 - 2x^3 - x^4 - x^5}.$$

We will prove this theorem via a series of lemmas. First, we prove that permutations of a particular form built from decreasing permutations are shallow.

Lemma 18. *Let $\delta_m = m(m-1)\dots 21$ be the decreasing permutation of length m . If $\pi \in \mathcal{S}_n$ is one of the permutations below:*

- $12 \oplus \delta_k$
- $21 \oplus (\delta_j \oplus \delta_k)$,
- $\delta_i \oplus (1 \oplus \delta_k)$, or
- $\delta_i \oplus (\delta_j \oplus 1)$,

where $i, j, k \geq 0$, then π is a shallow permutation.

Proof. For the first bullet point, notice that $\pi^{RR} = (12 \ominus \delta_k)^{RR} = \delta_2 \oplus \delta_{k-2}$, which is a direct sum of shallow permutations and is therefore shallow. Furthermore, $\pi_2^R = 1$ is a right-to-left minimum and $\pi_1^{RR} = 2$ is a left-to-right maximum. Therefore, Theorem 1 implies the original permutation is shallow.

For the second bullet point, notice $\pi^{LL} = (21 \ominus (\delta_j \oplus \delta_k))^{LL} = \delta_{j-2} \oplus (12 \ominus \delta_k)$ which, using the previous paragraph, is a direct sum of shallow permutations and is therefore shallow. Furthermore, $n' = n - 1$ is a left-to-right maximum in π^L and $(n - 1)'' = n - 3$ is a left-to-right maximum in π^{LL} . Therefore, Theorem 3 implies the original permutation is shallow.

We prove the third and fourth bullet points by induction on the length of the permutation, $n = k + i + 1$. Let us first consider the third bullet point, when $\pi = \delta_i \ominus (1 \oplus \delta_k) \in \mathcal{S}_n$. If $k = 0$, then $\pi = \delta_{i+1}$ which is shallow, and if $i = 0$, then π is a direct sum of shallow permutations and thus is shallow. Therefore, let us consider the cases when $i, k \geq 1$. It is straightforward to check the base cases when $n \leq 3$, so let us assume $n \geq 4$. Notice that $\pi^L = (\delta_{i-1} \oplus 1) \ominus \delta_k$ and $\pi^{LR} = (\delta_{i-1} \ominus (1 \oplus \delta_{k-1}))$. Since $n' = n - 1$ is a left-to-right maximum of π^L , $2' = 1$ is a right-to-left minimum of π^{LR} , and π^{LR} is shallow by induction, we conclude by Theorems 1 and 3 that π is also shallow. The result follows by induction.

An identical argument works for the fourth bullet point since $(\delta_i \ominus (\delta_j \oplus 1))^{LR} = \delta_{i-1} \ominus (\delta_{j-1} \oplus 1)$. \square

In order to enumerate $\mathcal{T}_n(231)$, we will decompose these permutations into a direct sum of two shallow permutations that avoid 231, one of which begins with $\pi_1 = n$. In order to enumerate those permutations in $\mathcal{T}_n(231)$ that begin with n we will decompose them further, enumerating them in terms of those that begin with $\pi_1\pi_2 = n(n - 1)$.

Lemma 19. *Let $n \geq 2$ and suppose b_n is the number of permutations in $\mathcal{T}_n(231)$ with $\pi_1 = n$ and let c_n be the number of permutations in $\mathcal{T}_n(231)$ with $\pi_1 = n$ and $\pi_2 = n - 1$. Then we have*

$$b_n = \sum_{i=1}^{n-1} b_i c_{n-i+1}.$$

Proof. We will show that if $\alpha \in \mathcal{T}_m(231)$ satisfies $\alpha_1 = m$, $\beta \in \mathcal{T}_\ell(231)$ with $\beta_1 = \ell$ and $\beta_2 = \ell - 1$, and $\pi \in \mathcal{S}_{m+\ell-1}$ with $\pi_1 = m + \ell - 1$, $\pi_i = \alpha_i$ for $2 \leq i \leq m$ and $\pi_{m+j-1} = \beta_j + m - 1$ for $2 \leq j \leq \ell$, then $\pi \in \mathcal{T}_{m+\ell-1}(231)$. In other words, taking $n = m + \ell - 1$, we have

$$\pi = n\alpha_2\alpha_3 \dots \alpha_m(n-1)\beta'_3\beta'_4 \dots \beta'_\ell$$

where $\beta'_i = \beta_i + m - 1$ for $3 \leq i \leq \ell$. Let us first see that this permutation is also shallow.

Note that since α and β are shallow, we have that $I(\alpha) + m - \text{cyc}(\alpha) = D(\alpha)$ and $I(\beta) + \ell - \text{cyc}(\beta) = D(\beta)$. It will be enough for us to show that $I(\pi) + n - \text{cyc}(\pi) = D(\pi)$.

First, notice that $I(\pi) = I(\alpha) + I(\beta)$. Indeed, if (i, j) is an inversion of π (so that $i < j$ and $\pi_i > \pi_j$), then we have a few cases to consider. If $1 \leq i, j \leq m$, then (i, j) is also an inversion of α , and in fact, all inversions of α are counted this way. If $(1, j)$ is an inversion of π with $m + 1 \leq j \leq n$, then $(1, j - m + 1)$ is an inversion of β (since $\pi_1 = n$). If (i, j)

is an inversion of π with $m + 1 \leq i, j \leq n$, then $(i - m + 1, j - m + 1)$ is an inversion of β . Furthermore, the previous two cases count all inversions of β . Finally, since $\pi_r < \pi_s$ for all $2 \leq r \leq m$ and $m + 1 \leq s \leq n$, there are no other inversions of π .

Next, let us show that $\text{cyc}(\pi) = \text{cyc}(\alpha) + \text{cyc}(\beta) - 1$. Notice that any cycles of α that do not contain 1 are still cycles of π since their values and positions are unchanged. Similarly, all cycles of β that do not contain 1 correspond to cycles of π with values scaled up by $m - 1$. Let $(1, m, a_3, \dots, a_r)$ and $(1, \ell, b_3, \dots, b_s)$ be the cycles in α and β , respectively, that contain 1. Then in π , we have the corresponding cycle $(1, n, b_3 + m - 1, \dots, b_s + m - 1, m + 1, a_3, \dots, a_r)$.

Finally, let us consider displacement; we will see that $D(\pi) = D(\alpha) + D(\beta)$. Indeed we have

$$\begin{aligned} D(\pi) &= \sum_{i=1}^n |\pi_i - i| \\ &= (n - 1) + \sum_{i=2}^m |\pi_i - i| + \sum_{i=m+1}^n |\pi_i - i| \\ &= (n - 1) + \sum_{i=2}^m |\alpha_i - i| + \sum_{j=2}^{\ell} |\beta_j + m - 1 - (m + j - 1)| \\ &= (n - 1) + D(\alpha) - (m - 1) + D(\beta) - (\ell - 1) \\ &= D(\alpha) + D(\beta), \end{aligned}$$

where the last equality holds since $n = m + \ell - 1$.

Taken altogether, we can see that

$$\begin{aligned} I(\pi) + n - \text{cyc}(\pi) &= I(\alpha) + I(\beta) + (m + \ell - 1) - (\text{cyc}(\alpha) + \text{cyc}(\beta) - 1) \\ &= I(\alpha) + m - \text{cyc}(\alpha) + I(\beta) + \ell - \text{cyc}(\beta) \\ &= D(\alpha) + D(\beta) \\ &= D(\pi). \end{aligned}$$

□

Remark 20. One could also use Berman and Tenner's characterization of shallow permutations in [2] to prove Lemma 19 by considering the cycle form of π . We opted for a different proof to avoid introducing additional terminology.

Lemma 21. *Let c_n be the number of permutations in $\mathcal{T}_n(231)$ with $\pi_1 = n$ and $\pi_2 = n - 1$. Then for $n \geq 5$, $c_n = 3n - 11$.*

Proof. Let $n \geq 5$ and π be a shallow permutation in \mathcal{S}_n that avoids 231. Let us first consider permutations $\pi = n(n - 1)\pi_3 \dots \pi_n$ so that $\pi_{k+3} = n - 3$ for some $1 \leq k \leq n - 4$. Thus we have

$$\pi = n(n - 1)\pi_3 \dots \pi_{k+2}(n - 2)\pi_{k+4} \dots \pi_n,$$

where $\{\pi_3, \dots, \pi_{k+2}\} = \{1, 2, \dots, k\}$ and $\{\pi_{k+4}, \dots, \pi_n\} = \{k+1, \dots, n-3\}$ since π avoids 231. Furthermore, suppose $\pi_s = 1$ for some $3 \leq s \leq k+2$.

Notice that π^L deletes n from the beginning and replaces $\pi_s = 1$ with the first element n and re-sizes the elements, so that

$$\pi^L = (n-2)(\pi_3-1) \dots (\pi_{s-1}-1)(n-1)(\pi_{s+1}-1) \dots (\pi_{k+2}-1)(n-3)(\pi_{k+4}-1) \dots (\pi_n-1).$$

If the original permutation π is shallow, then π^L is as well since $n-1$ is necessarily a left-to-right maximum in a permutation in \mathcal{S}_{n-1} . Next, we find $\pi^{LR} = (\pi^L)^R$ by replacing $n-1$ in π^L with the last element (π_n-1) and deleting (π_n-1) from the end. This cannot be a left-to-right maximum in π^{LR} since π^{LR} necessarily starts with its largest element. Notice it can only be a right-to-left minimum if π_n is the smallest element among $\{\pi_{k+4}, \dots, \pi_n\}$ and if the largest element in π^L appeared after all elements smaller than the last element of π^L . In other words, $\pi_n = k+1$ and $\pi_{k+2} = 1$. Since π avoids 231, this implies that

$$\pi = n(n-1)k(k-1) \dots 21(n-2)(n-3) \dots (k+1).$$

A similar argument proves that if π ends with $n-2$, it must be of the form

$$\pi = n(n-1)(n-3)(n-4) \dots 21(n-2).$$

Since by Lemma 18, these permutations are shallow, this gives us $n-3$ shallow permutations π that avoid 231 and begin with $\pi_1\pi_2 = n(n-1)$ with the property that $\pi_3 \neq n-2$. Next we need to show that there are $2n-8$ such permutations with $\pi_3 = n-2$.

Suppose that $\pi = n(n-1)(n-2)(n-3) \dots (n-m)\pi_{m+2} \dots \pi_{s-1}(n-m-1)\pi_{s+1} \dots \pi_n$ for some $m \geq 3$ and $m+3 \leq s \leq n$. We will show that there is one shallow permutation avoiding 231 with $s = m+3$ and one with $s = n$. First suppose $\pi_n = n-m$. Then by the same argument above (i.e. by considering the shallowness of π^{LR}), we must have that $\pi = n(n-1) \dots (n-m)(n-m-2) \dots 21(n-m)$. If $s = m+3$, then by the same argument as above, $\pi = n(n-1) \dots (n-m)1(n-m-1)(n-m-2) \dots 32$. Note that by Lemma 18, these are both shallow permutations.

Now for the sake of contradiction, suppose $m+3 < s < n$. Then, by the argument above, $\pi = n(n-1) \dots (n-m)k(k-1) \dots 21(n-m-1)(n-m-2) \dots (k+1)$ for some $k \geq 2$. By considering π^{LL} , we get a permutation in \mathcal{S}_{n-2} equal to

$$\pi^{LL} = (n-2)' \dots (n-m)'k'(k-1)' \dots 3'(n-1)'n'(n-m-1)'(n-m-2)' \dots (k+1)'$$

where $j' = j-2$ for each element $3 \leq j \leq n$. Now taking $\pi^{LLR} = (\pi^{LL})^R$, we get

$$\pi^{LLR} = (n-2)' \dots (n-m)'k'(k-1)' \dots 3'(n-1)'(k+1)'(n-m-1)'(n-m-2)' \dots (k+2)'.$$

Finally, we consider π^{LLRR} . First suppose $k < n-m-2$. Since π^{LLRR} must start with its largest element $(n-2)' = n-4$, the element $(k+2)' = k$ must not be a left-to-right maximum. However, since it is to the left of $(k+1)' = k-1$ it is also not a right-to-left minimum and thus the permutation π is not shallow. If $k = n-m-2$, then π^{LLR} ends

with $(n - m - 1)'$, which is also smaller than $(n - 2)'$ and larger than $(k + 1)'$, and so will not be a left-to-right maximum or right-to-left minimum in π^{LLRR} . Thus there are $2(n - 4)$ shallow permutations avoiding 231 starting with $\pi_1\pi_2\pi_3 = n(n - 1)(n - 2)$.

Since we have a total of $n - 3 + 2(n - 4)$ shallow permutations that begin with $\pi_1\pi_2 = n(n - 1)$, the proof is complete. \square

We now have the tools necessary to prove the main theorem.

Proof of Theorem 17. As above, suppose b_n is the number of permutations in $\mathcal{T}_n(231)$ with $\pi_1 = n$ and let c_n be the number of permutations in $\mathcal{T}_n(231)$ with $\pi_1 = n$ and $\pi_2 = n - 1$. Let $B(x) = \sum_{n \geq 1} b_n x^n$ and $C(x) = \sum_{n \geq 2} c_n x^n$.

Since any 231-avoiding permutation is the direct sum of a 231-avoiding permutation and a 231-avoiding permutation starting with n , we can use Lemma 4 to write that $t_n(231) = \sum_{i=0}^{n-1} t_i(231)b_{n-i}$, where we let $t_0(231) = 1$. Therefore, we have $T_{231}(x) = T_{231}(x)B(x) + 1$.

By Lemma 19, we also have that $B(x) = \frac{1}{x}B(x)C(x) + x$. Finally, by Lemma 21, we know that for $n \geq 5$, $c_n = 3n - 11$. Together with the fact that $c_2 = 1, c_3 = 1$, and $c_4 = 2$, we have that

$$C(x) = x^4 + \frac{x^2}{1-x} + \frac{3x^5}{(1-x)^2}.$$

Since $T_{231}(x) = \frac{1}{1-B(x)}$ and $B(x) = \frac{x}{1-\frac{1}{x}C(x)}$, the result follows. \square

5.2 231-avoiding shallow permutations by number of descents

We can refine the generating function in the previous section with respect to descents. Notice that since $312 = 231^{rc}$ and the reverse-complement preserves the number of descents, this result holds for 312-avoiding shallow permutations as well.

For the purposes of this subsection, let $t_{n,k}(231)$ be the number of permutations in $\mathcal{T}_n(231)$ with k descents, let $b_{n,k}$ be the number of such permutations that begin with $\pi_1 = n$, and let $c_{n,k}$ be the number of such permutations that begin with $\pi_1\pi_2 = n(n - 1)$. Furthermore, let $T_{231}(x, t) = \sum t_{n,k}(231)x^n t^k$, $B(x, t) = \sum b_{n,k}x^n t^k$, and $C(x, t) = \sum c_{n,k}x^n t^k$.

Theorem 22.

$$C(x, t) = t^2 x^4 + \frac{tx^2}{1-xt} + \frac{3t^3 x^5}{(1-xt)^2}$$

and

$$B(x, t) = \frac{x + C(x, t) - \frac{1}{t}C(x, t)}{1 - \frac{1}{xt}C(x, t)}$$

and finally,

$$T_{231}(x, t) = \frac{1}{1-B(x, t)}.$$

Proof. We first note that by the proof of Lemma 21, shallow permutations that avoid 231 and begin with $\pi_1\pi_2 = n(n-1)$ must either be the decreasing permutation or have at most one ascent. It follows that for each n , the coefficient of x^n in $C(x, t)$ must be the polynomial $(3n-10)t^{n-2} + t^{n-1}$ for $n \geq 5$. It follows that

$$C(x, t) = t^2x^4 + \frac{tx^2}{1-xt} + \frac{3t^3x^5}{(1-xt)^2}.$$

Next, by the proof of Lemma 19, permutations in $\mathcal{T}_n(231)$ that start with n are built from smaller permutations: α that starts with n and β that starts with $n(n-1)$. When the α is at least size 2, we have that $\text{des}(\pi) = \text{des}(\alpha) + \text{des}(\beta) - 1$ since the first descent in β is lost in this process. Therefore, we get that

$$B(x, t) = x + C(x, t) + \frac{1}{xt}C(x, t)(B(x, t) - x).$$

Finally, the number of descents in the direct sum of two permutations is the sum of the number of descents in each summand. Therefore $T_{231}(x, t) = T_{231}(x, t)B(x, t) + 1$. \square

5.3 231-avoiding shallow permutations with symmetry

In this subsection, we consider those 231-avoiding shallow permutations that exhibit certain symmetries. In particular, we enumerate 231-avoiding shallow involutions, in which $\pi = \pi^{-1}$, 231-avoiding shallow centrosymmetric permutations, in which $\pi = \pi^{rc}$, and 231-avoiding shallow persymmetric permutations, in which $\pi = \pi^{rci}$. We show that in fact all 231-avoiding involutions and centrosymmetric permutations are shallow, but this same result does not hold for persymmetric permutations.

Theorem 23. *For $n \geq 1$, the number of shallow, 231-avoiding involutions of length n is 2^{n-1} .*

Proof. In [15], Simion and Schmidt show there are 2^{n-1} involutions of length n that avoid 231. In their proof, it is shown that each of these permutations is a direct sum of decreasing permutations, i.e., $\pi = \bigoplus_{i=1}^k \delta_{m_i}$ for some composition $\sum_{i=1}^k m_i = n$. Since the decreasing permutation is always shallow, as is the direct sum of shallow permutations by Lemma 4, all 231-avoiding involutions are shallow. \square

Theorem 24. *For $n \geq 1$, the number of shallow, 231-avoiding centrosymmetric permutations of length n is $2^{\lfloor n/2 \rfloor}$.*

Proof. In [7], Egge shows there are $2^{\lfloor n/2 \rfloor}$ centrosymmetric permutations of length n that avoid 231. In his proof, it is shown that each of these permutations is a direct sum of decreasing permutations, i.e., $\pi = \bigoplus_{i=1}^k \delta_{m_i}$ for a palindromic composition $\sum_{i=1}^k m_i = n$. Since the decreasing permutation is always shallow, as is the direct sum of shallow permutations by Lemma 4, all 231-avoiding centrosymmetric permutations are shallow. \square

Theorem 25. For $n \geq 1$, if the number of shallow, 231-avoiding persymmetric permutations of length n is $p_n(231)$ and the corresponding generating function is $P_{231}(x)$, then

$$P_{231}(x) = \frac{x^{10} + 2x^8 + x^7 + x^6 - x^5 - 2x^4 + x^3 + 2x^2 - x - 1}{x^{10} + x^8 + 2x^6 - 4x^4 + 4x^2 - 1}.$$

Proof. Let $P^B(x)$ be the generating function for shallow 231-avoiding persymmetric permutations that begin with n and $P^C(x)$ be the generating function for those beginning with $n(n-1)$. Then, since the only 231-avoiding shallow permutations that begin with $n(n-1)$ (of the form described in Lemma 21) are the decreasing permutation $\pi = n(n-1)\dots 21$, the permutations $\pi = n(n-1)\dots 4312$, and when n is even, the permutation $21 \ominus (\delta_{n/2-1} \oplus \delta_{n/2-1})$. Therefore for $n \geq 6$, there are 2 such permutations when n is odd and 3 such permutations when n is even, giving us

$$P^C(x) = x^2 + x^3 + \frac{2x^4}{1-x} + \frac{x^6}{1-x^2}.$$

For those permutations beginning with n , if $\pi_i = n-1$ for $i > 2$, then we must have that $\pi_i \pi_{i+1} \dots \pi_n$ are composed of the numbers $\{i-1, i, \dots, n-1\}$ and is order-isomorphic to the reverse-complement-inverse of $\pi_2 \pi_3 \dots \pi_{n-i+2}$ which is composed of the elements in $\{1, 2, \dots, m-1\}$. The remaining permutation is itself a shallow 231-avoiding persymmetric permutation beginning with n . Thus, we have that

$$P^B(x) = x + P^C(x) + \frac{1}{x^2} C(x^2) P^B(x)$$

where $C(x)$ is the generating function given in the proof of Theorem 17. Finally, if a given persymmetric permutation in $\mathcal{T}_n(231)$ does not begin with n , it is the direct sum $\gamma \oplus \nu \oplus \gamma^{rci}$ where ν is a shallow 231-avoiding persymmetric permutation and γ is any shallow 231-avoiding permutation beginning with n . Thus,

$$P_{231}(x) = 1 + P^B(x) + B(x^2) P_{231}(x)$$

where $B(x)$ is the generating function given in the proof of Theorem 17. The result follows. \square

6 Shallow permutations that avoid 123

In this section, we consider those shallow permutations that avoid the pattern 123, as well as those that exhibit the three symmetries of inverse, reverse-complement, and reverse-complement-inverse. We omit the enumeration of 123-avoiding shallow permutations with a given number of descents, though this is likely tractable (but tedious) by following the proof of Theorem 26 below.

6.1 Enumeration of 123-avoiding shallow permutations

Let us start by stating the main theorem in this section.

Theorem 26. *Let $T_{123}(x)$ be the generating function for the number of shallow permutations that avoid 123. Then,*

$$T_{123}(x) = \frac{1 - 3x + 11x^3 - 13x^4 + 7x^5 + 6x^6 + 3x^7}{(1-x)^4(1-4x^2+x^4)}.$$

We first establish a few lemmas based on the position of n and 1 in the permutation. In Lemma 27, we consider those permutations that do not start with n or end with 1, and in Lemma 28, we consider those that do start with n and have a 1 in any other position.

Lemma 27. *For $n \geq 3$, the number of 123-avoiding shallow permutations π with $\pi_1 \neq n$ and $\pi_n \neq 1$ is equal to $2 \binom{n-1}{3} + (n-1)$.*

Proof. Let us first consider the case when $\pi_i = n$ and $\pi_j = 1$ for some $1 < i < j < n$. We will see that there are $j-i$ such 123-avoiding shallow permutations. In particular, these $j-i$ permutations are of the form

$$\pi = \underline{(t-1) \dots (t-i+1)} \underline{n} \underline{(t-i) \dots 2} \underline{(n-1) \dots (t+n-j)} \underline{1} \underline{(t+n-j-1) \dots t}$$

for any $i+1 \leq t \leq j$, where the underlined regions are decreasing.

We will first show that π is shallow. Let us consider the permutation $\pi^{R^{n-t}}$. Since upon each iteration of the right operator, the last element replaces the largest element, all elements that appear before $n-1$, except for n , will remain unchanged. Each time, a term will be deleted, leaving us with

$$\pi^{R^{n-t}} = \underline{(t-1) \dots (t-i+1)} \underline{t} \underline{(t-i) \dots 2} \underline{1} \in \mathcal{S}_t.$$

For example, if $\pi = 493287165$, we have $n = 9$ and $t = 5$, so $\pi^{R^4} = 45321$. In the first step, t is a left-to-right maximum in π^R , and in all the subsequent steps the element we move is a right-to-left minimum in its new position. Furthermore, $\pi^{R^{n-t}} = (\delta_{i-1} \oplus 1) \ominus \delta_{t-i}$ is shallow by an identical argument to Lemma 18. These two facts in combination with Theorem 1 imply that π is shallow.

Now let us see that these are indeed the only shallow 123-avoiding permutations with $\pi_i = n$ and $\pi_j = 1$ for some $1 < i < j < n$. Indeed, since π avoids 123, we must have $\pi_1 \dots \pi_{i-1}$ and $\pi_{j+1} \dots \pi_n$ are decreasing. Furthermore, by considering π^R , we would have that π_n is to the left of 1 and thus must be a left-to-right maximum, implying that $\pi_n > \pi_1$, which in turn implies that each element of $\{\pi_1, \dots, \pi_{i-1}\}$ is less than each element of $\{\pi_{j+1}, \dots, \pi_n\}$. This implies that if $\pi_r = 2$ then either $r = i-1$ or $i < r < j$. Clearly if $\pi_{i-1} = 2$, then the subsequence $\pi_{i+1} \dots \pi_{j-1} \pi_{j+1} \dots \pi_n$ is decreasing and thus is of the above form with $t = i+1$. Similarly, if $\pi_s = n-1$, then either $s = j+1$ or $i < s < j$. If $\pi_{j+1} = n-1$, then π must be of the form above with $t = j$. We can thus

assume $i < r, s < j$. If $r < s$, then it is of the form above, so for the sake of contradiction, suppose $r > s$ (so, suppose 2 appears after $n - 1$). However, in this case, π^{RL} contains the subsequence $\pi'_n(n-1)2'\pi'_1$ which is a **3412** pattern, contradicting Theorem 8.

Next, let us consider those permutations with 1 appearing before n in π . Since π avoids 123, it must be that $\pi = \pi_1 \dots \pi_{i-1} 1 n \pi_{i+2} \dots \pi_n$ for $1 \leq i \leq n - 1$. Furthermore, we must have $\pi_1 > \pi_2 > \dots > \pi_{i-1}$ and $\pi_{i+2} > \pi_{i+3} > \dots > \pi_n$.

We claim that if $\pi_i = 1$ and $\pi_{i+1} = n$, then either $\pi_1 < \pi_n$ in which case

$$\pi = i(i-1) \dots 1 n(n-1)(n-2) \dots (i+1),$$

or $\pi_1 > \pi_n$. Since the elements preceding 1 are decreasing and those after n are decreasing, we must have that $\pi_1 \in [i+1, n-1]$, $\pi_n \in [2, i]$. Furthermore, we can show that $\pi_{n-1} > \pi_2$. For the sake of contradiction, suppose not. Then $\pi_1 > \pi_2 > \pi_{n-1} > \pi_n$. But then π^{RL} contains the sequence $\pi'_2 \pi'_1 \pi'_n \pi'_{n-1}$ which is a **3412** pattern, contradicting Theorem 8. Thus once i, π_1 , and π_n are selected, the rest of the permutation is determined.

So in total for each $1 \leq i \leq n - 1$ there are $(n - i - 1)(i - 1)$ permutations with $\pi_1 > \pi_n$ and 1 with $\pi_1 < \pi_n$. Summing over all possible values of i , we obtain

$$\sum_{i=1}^{n-1} (1 + (n - i - 1)(i - 1)) = \binom{n-1}{3} + (n-1)$$

total permutations with 1 appearing before n .

Altogether, there are $\sum_{j=3}^{n-1} \sum_{i=2}^{j-1} (j-i) = \binom{n-1}{3}$ permutations with n appearing before 1 and $\binom{n-1}{3} + (n-1)$ permutations where 1 appears before n . Adding these gives us the result. \square

Let b_n be the number of permutations in $\mathcal{T}_n(123)$ that start with $\pi_1 = n$ and let $b_n(j)$ be the number of such permutations that also have $\pi_j = 1$. Note that by considering the reverse-complement, we have that b_n is also the number that end with $\pi_n = 1$ and $b_n(j)$ is also the number with $\pi_{n-j+1} = n$ and $\pi_n = 1$.

Lemma 28. *For $n \geq 5$, we have $b_n(2) = 1$, $b_n(n-1) = b_n(n) = t_{n-2}(123)$, and for $3 \leq j \leq n-2$ we have*

$$b_n(j) = 2 + (n-j-2)(2j-5) + 4 \binom{j-2}{2} + b_{n-2}(j-1).$$

Proof. Let us first consider those permutations with $\pi_1 = n$, $\pi_j = 1$ and $\pi_n = 2$ with $j \leq n - 2$. Notice that π^{RL} is still shallow of length $n - 2$ and has the property that 1 appears in the $j - 1$ position where $j - 1 \leq n - 3$, so π^{RL} does not end with 1. It avoids 123 since it was essentially obtained by “deleting” 1 and n . By considering the position of $n - 2$ in $\pi^{RL} \in \mathcal{T}_{n-2}(123)$, by the proof of Lemma 27, there are $1 + \binom{j-2}{2} + (j-2)(n-2-j) + b_{n-2}(j-1)$ such permutations.

Next, let us consider those with $\pi_i = 2$ with $1 < i < j$. First, let us consider those permutations with $\pi_{j+1} = n - 1$. In this case, we must have $i = j - 1$, so we have

$$\pi = n \pi_2 \dots \pi_{j-2} 2 1 (n-1) \pi_{j+2} \dots \pi_n$$

where $\pi_2 \dots \pi_{j-2}$ and $\pi_{j+2} \dots \pi_n$ are both decreasing segments since π is 123-avoiding. We claim that the only such permutations are either

$$n(j-1) \dots 21(n-1) \dots j$$

or those that have $\pi_2 \in \{j, \dots, n-2\}$ and $\pi_n \in \{3, \dots, j-1\}$, with all the remaining elements before 2 being smaller than all the remaining elements after 1. If π is not of one of these forms, then we have $\pi_2 > \pi_3 > \pi_{n-1} > \pi_n$, in which case π^{LLR} would contain a $\underline{3412}$ pattern, contradicting Theorem 8. These are clearly shallow since π^{LLR} is the direct sum of two shallow permutations, and it is clear there are $(j-3)(n-j-1) + 1$ such permutations based on the choices of π_2 and π_n .

Next, consider those with $\pi_2 = n-1$, so

$$\pi = n(n-1)\pi_3 \dots \pi_{i-1}2\pi_{i+1} \dots \pi_{j-1}1\pi_{j+1} \dots \pi_n$$

where $\pi_{i+1} \dots \pi_{j-1}\pi_{j+1} \dots \pi_n$ is decreasing since π avoids 123. Notice that by first considering π^{RR} , we get a permutation

$$\pi^{RR} = \pi_n\pi_{n-1}\pi_3 \dots \pi_{i-1}2\pi_{i+1} \dots \pi_{j-1}1\pi_{j+1} \dots \pi_{n-2}$$

with $\pi_{n-1} > \pi_n$ since $j \leq n-2$. This is clearly still shallow if the original π was. Now, taking π^{RRL} , we see that our original permutation is only shallow if π_{n-1} is a left-to-right maximum in π^{RRL} since $\pi_n < \pi_{n-1}$ will appear to its right. Thus we must have that the elements of the segment $\pi_3 \dots \pi_{i-1}$ are all less than π_{n-1} , as is π_n . Thus $\pi_{n-1} = i+1$ and there are $i-2$ choices of π , all of which are shallow. Summing over all possible choices of i , we see there are $\binom{j-2}{2}$ permutations.

Now let us consider the final case, when $\pi_j = 1$, $\pi_i = n-1$ with $3 \leq i \leq j-1$, and $\pi_n \neq 2$. We claim that $\pi_n \in \{3, \dots, j-1\}$ for each possible value of i and that the other terms are determined, for a total of $(j-3)^2$ permutations.

Indeed, in this case, we have $\pi = n\pi_2 \dots (n-1) \dots 1 \dots \pi_{n-1}\pi_n$, and so $\pi^{RL} = \pi'_2 \dots (n-1)' \dots \pi'_n \dots \pi'_{n-1}$. Note that if we show that both $\pi_2 < \pi_{n-1}$ and 2 appears before $n-2$, the rest of the permutation π must be determined since π must avoid 123. Notice that in π^{RL} , if $\pi_2 > \pi_{n-1}$, then $\pi'_2(n-1)'\pi'_n\pi'_{n-1}$ is a $\underline{3412}$ pattern, contradicting Theorem 8. Note also that $\pi_2 \neq n-2$ since otherwise $\pi'_2(n-1)'\pi'_n\pi'_{n-1}$ would be a $\underline{3412}$ pattern in π^{RL} . If $n-2$ does occur before 2, then we would have

$$\pi = n\pi_2 \dots (n-1) \dots (n-2) \dots 2 \dots 1 \dots \pi_{n-1}\pi_n,$$

but then π^{RLR} contains $\pi'_{n-1}(n-2)'\pi'_n$ which is a $\underline{3412}$ pattern, contradicting Theorem 8.

Thus we have $(j-3)(n-j-1) + 1 + \binom{j-2}{2} + (j-3)^2$ permutations that do not end in 2. Adding all these possible cases together gives us the result in the statement of the theorem. \square

We are now ready to prove the main theorem of this section.

Proof of Theorem 26. Let $t_n = |\mathcal{T}_n(123)|$ and let b_n be the number of permutations in $\mathcal{T}_n(123)$ that start with n . Since there are clearly also b_n permutations that end with 1 and t_{n-2} permutations that both start with n and end with 1, using the results of Lemma 27, we have

$$t_n(123) = 2b_n - t_{n-2}(123) + 2\binom{n-1}{3} + n - 1.$$

Using Lemma 28, we obtain

$$\begin{aligned} b_n &= \sum_{j=2}^n b_n(j) \\ &= 1 + 2t_{n-2} + \sum_{j=3}^{n-2} \left(2 + (n-j-2)(2j-5) + 4\binom{j-2}{2} + b_{n-2}(j-1) \right) \\ &= 1 + 2t_{n-2} + 2(n-3) + 5\binom{n-3}{3} + \binom{n-4}{3} + b_{n-2} - b_{n-2}(n-2) \\ &= 1 + 2t_{n-2} + 2(n-3) + 5\binom{n-3}{3} + \binom{n-4}{3} + b_{n-2} - t_{n-4}. \end{aligned}$$

Thus if $B(x)$ is the generating function for the sequence $\{b_n\}$, we have

$$T_{123}(x) = 2B(x) - x^2T_{123}(x) + \frac{x^2}{(1-x)^2} + \frac{2x^4}{(1-x)^4} + 1 - x$$

and

$$B(x) = (2x^2 - x^4)T_{123}(x) + x^2B(x) + \frac{x}{1-x} + \frac{5x^6 + x^7}{(1-x)^4} + \frac{2x^5}{(1-x)^2} - 2(x^2 + x^3).$$

Solving for $T_{123}(x)$, we obtain the result in the statement of the theorem. □

6.2 123-avoiding shallow permutations with symmetry

In this subsection, we consider 123-avoiding permutations that exhibit certain symmetry.

Theorem 29. *For $n \geq 1$, the number of shallow, 123-avoiding involutions of length n is $\lfloor \frac{n^2}{4} \rfloor + 1$.*

Proof. Let a_n be the number of shallow, 123-avoiding permutations that are involutions. We will show that $a_n = a_{n-2} + n - 1$. This together with the initial conditions $a_1 = 1$ and $a_2 = 2$ implies the formula as given in the statement of the theorem.

Note that by Lemma 5, there are a_{n-2} shallow 123-avoiding involutions that start with n and end with 1 since these comprise a 2-cycle and thus removing them leaves us with

an involution. Also note that all involutions that have $\pi_n = 1$ must also have $\pi_1 = n$ and thus all involutions starting with $\pi_1 = n$ are counted in this way.

Next suppose $\pi_i = n$ for $i > 1$. Then, since π is an involution, $\pi_n = i$. We claim that $\pi_1 \leq \pi_n$. For the sake of contradiction, suppose not. If $\pi_1 > \pi_n = i$, then since π is an involution $\pi_{\pi_1} = 1$. Since $\pi_1 > i$, this 1 appears after n and before π_n . Thus, in π^L , π_1 replaces this 1, but cannot be a left-to-right maximum since n is to its left and cannot be a right-to-left minimum since it is larger than π_n . Thus $\pi_1 \leq \pi_n$.

Finally, since π avoids 123 and $\pi_1 \leq \pi_n$, the only permutations that satisfy this are of the form

$$\pi = m(m-1) \dots 1n(n-1) \dots (m+1)$$

for $m \in [n-1]$. There are clearly $n-1$ such permutations, and so the result follows. \square

Theorem 30. *For $n \geq 1$, the number of shallow, 123-avoiding centrosymmetric permutations of length n is $\frac{n^2}{4} + 1$ when n is even and 1 when n is odd.*

Proof. Let c_n be the number of centrosymmetric 123-avoiding permutations. First, let us consider the case when n is odd. Since $\pi = \pi^{rc}$, we must have that $\pi_{(n+1)/2} = (n+1)/2$. Since π avoids 123, it must be the case that the elements in $\pi_1\pi_2 \dots \pi_{(n+1)/2-1}$ are greater than $(n+1)/2$ and the elements in $\pi_{(n+1)/2+1} \dots \pi_n$ are less than $(n+1)/2$. In particular, n occurs in the first half and 1 occurs in the second half. If 1 occurs at the end of π , then since $\pi = \pi^{rc}$, $\pi_1 = n$. Thus by Lemma 5, there are c_{n-2} such permutations. If 1 does not occur at the end, then n necessarily does not occur at the beginning. But then, in π^R , π_n is neither a left-to-right maximum nor a right-to-left minimum. Thus, when n is odd, we have $c_n = c_{n-2}$. Since $c_1 = 1$, the result for odd n follows.

Now, suppose n is even. We will show that π either starts with n , in which case there are c_{n-2} for the same reasons as above, or is either of the form

$$\pi = (n-k)(n/2)(n/2-1) \dots (k+1)(k-1) \dots 21n(n-1) \dots (n-k+1)(n-k-1) \dots (n/2+1)k$$

for $2 \leq k \leq n/2 + 1$, or of the form

$$\pi = (n/2)(n/2-1) \dots (k+1)nk \dots 2(n-1)(n-2) \dots (n-k+1)1(n-k) \dots (n/2+1)$$

for $1 \leq k < n/2$.

Let us first consider the case when n appears after 1 in π . Since π avoids 123 and is centrosymmetric, it must be that $\pi_{n/2} = 1$ and $\pi_{n/2+1} = n$. Note that if $\pi_1 < \pi_n$, then we must have the first case above with $k = n/2 + 1$, so let us assume $\pi_1 > \pi_n$. In that case, in π^{RL} , we will have a $\underline{3412}$ pattern unless $\pi_2 < \pi_{n-1}$, contradicting Theorem 8. Since π is centrosymmetric, the only possibility is the first one listed above.

Next consider when n appears before 1 in π . In this case, we must have $\pi_1 > \pi_n$ or else we will have a $\underline{3412}$ pattern, contradicting Theorem 8. Therefore, since π avoids 123 and is centrosymmetric, we must have $\pi_1 = n/2$ and $\pi_n = n/2 + 1$. Furthermore, the elements that appear before n are decreasing and consecutive and those after 1 are decreasing and consecutive, since otherwise we would have a 123 pattern. This implies that either 1 appears immediately after n , in which case we have the second case above with $k = 1$,

or the 2 and $n - 1$ appear between the n and 1 in π . In fact, we must have 2 appearing before $n - 1$, or else π^{RL} will have a **3412** pattern, contradicting Theorem 8. It is a straightforward exercise to check that these permutations listed above are indeed shallow, and we now have shown they are the only possible shallow 123-avoiding permutations of length n .

Thus when n is even, $c_n = c_{n-2} + n - 1$, which together with the fact that $c_2 = 2$, implies that $c_n = \frac{n^2}{4} + 1$. \square

Theorem 31. *For $n \geq 3$, the number of shallow 123-avoiding persymmetric permutations of length n has the associated generating function*

$$P_{123}(x) = \frac{x^6 + x^5 + x^3 - 2x^2 + 1}{(x - 1)^2(x + 1)(1 - 2x^2 - x^4)}.$$

Proof. Let p_n denote the number of persymmetric 123-avoiding permutations and let q_n denote those that start with $\pi_1 = n$.

First note that if $\pi_2 = n$, then we must have $\pi_1 = n - 1$ since π is persymmetric. Also, we must have $\pi_n = 1$ since if 1 appeared anywhere else in π , then in π^L , the element $n - 1$ would not be a left-to-right maximum nor a right-to-left minimum, and so π would not be shallow. Thus since $\pi_1\pi_2 = (n - 1)n$ and $\pi_n = 1$, then $\pi^{RL} \in \mathcal{S}_{n-2}$ will be a shallow 123-avoiding persymmetric permutation that starts with $n - 2$. Since any such permutation can be obtained this way, there are q_{n-2} persymmetric permutations $\pi \in \mathcal{T}_n(123)$ with $\pi_2 = n$.

Now, we will show there is exactly one shallow persymmetric 123-avoiding permutation with $\pi_i = n$ for $3 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$ and none with $i > \lfloor \frac{n}{2} \rfloor + 1$. First note that if $i > \lfloor \frac{n}{2} \rfloor + 1$, then $\pi_1 = n + 1 - i$. But since π avoids 123, the elements before n must be decreasing, which is impossible in this case since π_1 is too small. Now assume $i \leq \lfloor \frac{n}{2} \rfloor + 1$. Since π is persymmetric, this means $\pi_1 = n + 1 - i$ and since π avoids 123, we have $\pi_1 \dots \pi_{i-1}$ is decreasing. If the 1 appears before n , then we must have that $\pi_{i-1} = 1$ and $\pi_n = n + 2 - i$, and that every element after n is decreasing in order to avoid 123. The only way this is possible is if $i = \lfloor \frac{n}{2} \rfloor + 1$ and $\pi = (n/2) \dots 21n(n - 1) \dots (n/2 + 1)$. In fact, this is the only possibility for $i = \lfloor \frac{n}{2} \rfloor + 1$, so assume $i \leq \lfloor \frac{n}{2} \rfloor$ and that the 1 appears after n . Note that if $\pi_j = i$ for $i + 1 \leq j \leq n - 1$, then $\pi_n = n + 1 - j$ which implies π contains **3412**. In order to avoid this, we must have $\pi_n = 1$. Since π^{RL} must avoid 3412, we must have that either $\pi_{n-1} = 2$ or $\pi_{n-1} > \pi_2$. In the second case, since π avoids 123 and is persymmetric, the only possibility is that n is odd and taking $r = (n + 1)/2$ we get

$$\pi = (n + 1 - j)(r - 1)(r - 2) \dots (r - i + 2)n(r - i + 1) \dots 32(n - 1)(n - 2) \dots r1.$$

If $\pi_{n-1} = 2$, then we must not have $\pi_{n-2} = 3$ since π^{RRLL} would send π_2 to where π_{n-1} is in π and it would not be a left-to-right maximum since $\pi_1 > \pi_2$ would appear before it and would not be a right-to-left minimum since $\pi_{n-2} = 3$ would appear to its right. Thus for similar reasons to above, we would have to have $\pi_{n-2} > \pi_2$ and there would only be one case: that n is even and taking $r = n/2 + 1$, we have

$$\pi = (n + 1 - j)(r - 1)(r - 2) \dots (r - i + 2)n(r - i + 1) \dots 32(n - 1)(n - 2) \dots r21.$$

Again it is straightforward to check these permutations are indeed shallow. Taken altogether, this implies that

$$p_n = q_n + q_{n-2} + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Next, let us consider those that have $\pi_1 = n$. If $\pi_n = 1$, then by Lemma 5, there are p_{n-2} such permutations. If $\pi_{n-1} = 1$, then since π is persymmetric, we must have $\pi_n = 2$. Then π^{RL} is a persymmetric permutation that ends with 1, and such permutations are enumerated by q_{n-2} . Finally, by a proof similar to the one above, there is exactly one 123-avoiding shallow permutation that starts with n and has $\pi_i = 1$ for $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 2$. Now, this implies that

$$q_n = p_{n-2} + q_{n-2} + \left\lfloor \frac{n-1}{2} \right\rfloor - 1.$$

Taking $P_{123}(x)$ and $Q_{123}(x)$ to be the respective generating functions. These recurrences together with the initial conditions imply that

$$P_{123}(x) = (1 + x^2)Q_{123}(x) + \frac{x^5 + x^4}{(1 - x^2)^2} + 1 + x^2$$

and

$$Q_{123}(x) = x^2P_{123}(x) + x^2Q_{123}(x) + \frac{x^6 + x^5}{(1 - x^2)^2} + x.$$

Solving for $P_{123}(x)$ gives us the generating function in the statement of the theorem. \square

7 Shallow permutations that avoid 321

7.1 Enumeration of 321-avoiding shallow permutations

Diaconis and Graham [5] pointed out that permutations which satisfy the upper and lower bound of their inequality are enumerated by the bisection of the Fibonacci numbers, F_{2n-1} . These permutations were further discussed and characterized in [8]. We start this section by providing an independent proof of this enumeration. We then enumerate these permutations by their descent count as well as those that exhibit certain symmetry.

Theorem 32. *For $n \geq 1$, $t_n(321) = F_{2n-1}$, where F_{2n-1} is the $(2n - 1)$ -st Fibonacci number.*

Before proving this theorem, we will prove the following lemma, which determines what these permutations must look like when n occurs before position $n - 1$.

Lemma 33. *Let $n \geq 3$. If $\pi \in \mathcal{T}_n(321)$ has $\pi_j = n$ with $1 \leq j < n - 1$ then:*

- $\pi_n = n - 1$
- $\pi^R \in \mathcal{T}_{n-1}(321)$
- $\pi_k = k - 1$ for $j + 2 \leq k \leq n$,

Proof. Let $\pi \in \mathcal{T}_n(321)$ with $n \geq 3$. Since π avoids 321 we must have $\pi_{j+1} < \pi_{j+2} < \dots < \pi_n$. By Theorem 1, since π is shallow, π_n must be either a left-to-right maximum or right-to-left minimum in $\pi^R = \pi_1 \dots \pi_{j-1} \pi_n \pi_{j+1} \dots \pi_{n-1}$. It cannot be a right-to-left minimum because $j < n - 1$ and $\pi_{j+1}^R = \pi_{j+1} < \pi_n = \pi_j^R$. So π_n must be a left-to-right maximum in π^R . If $\pi_n \neq n - 1$, since it is a left-to-right maximum in π^R , $n - 1$ must occur after position j in π^R , and thus in π . However, this means π contains $n(n - 1)\pi_n$ as a subsequence, which is a 321 pattern. Thus $\pi_n = n - 1$. This completes the proof of the first bullet point.

Note that the previous paragraph also implies that if $\pi \in \mathcal{T}_n(321)$ with $\pi_j = n$ where $1 \leq j < n - 1$ then $\pi^R \in \mathcal{T}_{n-1}(321)$. Indeed, by Theorem 1 π^R is still shallow and we form π^R by replacing n with $n - 1$, so π^R is still 321 avoiding since π was. This establishes the second bullet point.

We can combine the first two bullet points to prove the third. If $\pi \in \mathcal{T}_n(321)$ has $\pi_j = n$ with $1 \leq j < n - 1$ then the first and second bullet point imply that $\pi^{R^m} \in \mathcal{T}_{n-m}(321)$ with $\pi_j = n - m$ for $1 \leq m \leq n - j - 1$. When $1 \leq m \leq n - j - 2$ we have $j \leq n - m - 2$, in this case the first bullet point shows that $\pi_{n-m} = \pi_{n-m}^{R^m} = n - m - 1$. This is equivalent to $\pi_k = k - 1$ for $j + 2 \leq k \leq n - 1$ which in combination with the first bullet point proves the third. \square

As an example, if we have a permutation $\pi \in \mathcal{T}_{13}(321)$ with the element 13 in position 8, then we must have that the permutation π ends with $\pi_{10}\pi_{11}\pi_{12}\pi_{13} = 9(10)(11)(12)$. Note that π_9 is not determined by this lemma.

We are now able to prove Theorem 32.

Proof of Theorem 32. Let $\pi \in \mathcal{T}_n(321)$. If $\pi_n = n$, by Theorem 1, π^R obtained by removing n will be shallow and still 321 avoiding. Similarly, we can append n to the end of any $\tau \in \mathcal{T}_{n-1}(321)$ to obtain a permutation in $\mathcal{T}_n(321)$. Therefore, there are $t_{n-1}(321)$ permutations $\pi \in \mathcal{T}_n(321)$ with $\pi_n = n$. Similarly, if $\pi_{n-1} = n$, then π^R is obtained by replacing n with π_n , which is equivalent to deleting n from π . One can clearly add n into the $(n - 1)$ st position of any $\pi \in \mathcal{T}_{n-1}(321)$ and obtain a permutation that is still shallow and 321-avoiding. This shows that there are $t_{n-1}(321)$ permutations $\pi \in \mathcal{T}_n(321)$ with $\pi_{n-1} = n$.

Now let us see that there are $t_j(321)$ permutations $\pi \in \mathcal{T}_n(321)$ with $\pi_j = n$ for $1 \leq j \leq n - 2$. Suppose $\pi \in \mathcal{T}_n(321)$ with $\pi_j = n$ and $1 \leq j \leq n - 2$. A direct consequence of Lemma 33 is that $\pi^{R^{n-j}} \in \mathcal{T}_j(321)$. This is actually a bijection. Indeed, given any $\tau \in \mathcal{T}_j(321)$, we can form a new permutation $\hat{\tau} \in \mathcal{S}_n$ with $\hat{\tau}_m = \tau_m$ for $1 \leq m < j$, $\hat{\tau}_j = n$, $\hat{\tau}_{j+1} = \tau_j$, and $\hat{\tau}_k = k - 1$ for $j + 2 \leq k \leq n$. For example, given the permutation $\tau = 41263857 \in \mathcal{T}_8(321)$, we can obtain the permutation $\pi = 4126385(13)79(10)(11)(12) \in \mathcal{T}_{13}(321)$. It is clear that the permutation $\hat{\tau}$ formed is 321-avoiding, and it is shallow since $\hat{\tau}^{R^{n-j}} = \tau$ is.

As this exhausts all the possible positions of n , we conclude that

$$t_n(321) = 2t_{n-1}(321) + \sum_{i=1}^{n-2} t_i(321)$$

which, together with the initial conditions, is satisfied by F_{2n-1} . \square

7.2 321-avoiding shallow permutations by number of descents

In this subsection, we consider those shallow 321-avoiding permutations with k descents.

Theorem 34. *Let $t_{n,k}$ be the number of permutations in $\mathcal{T}_n(321)$ with k descents and let $T(x, z) = \sum_{n,k} t_{n,k} x^k z^n$. Then,*

$$T(x, z) = \frac{z - 2z^2 + xz^2 + z^3 - xz^3}{1 - 3z + 3z^2 - 2xz^2 - z^3 + xz^3}.$$

Proof. Let $t_{n,k}$ denote the number of permutations $\pi \in \mathcal{T}_n(321)$ with k descents and $b_{n,k}$ denote the number of such permutations with $\pi_{n-1} = n$.

Let $\pi \in \mathcal{T}_n(321)$ have $\pi_{n-1} = n$ and k descents and consider the value of π_{n-2} . If $\pi_{n-2} = n - 1$ then $\pi^R \in \mathcal{S}_{n-1}$ is still a shallow 321-avoiding permutation and has $\pi_{n-2}^R = n - 1$. Since π has k descents, π^R will also have k descents. These are precisely the permutations enumerated by $b_{n-1,k}$. This construction is clearly reversible so there are $b_{n-1,k}$ permutations $\pi \in \mathcal{T}_n(321)$ with k descents, $\pi_{n-1} = n$ and $\pi_{n-2} = n - 1$.

If $\pi_{n-2} \neq n - 1$ this forces $\pi_{n-2} < \pi_n$, otherwise we have a 321 consisting of $(n - 1)\pi_{n-2}\pi_n$. This means π^R will have one fewer descent, since we are removing the descent in position $n - 1$. In other words, π^R can be any permutation $\pi' \in \mathcal{T}_{n-1}(321)$ with $k - 1$ descents and $\pi'_{n-2} \neq n - 1$. These are precisely enumerated by $t_{n-1,k-1} - b_{n-1,k-1}$. Again, this construction is reversible, so there are $t_{n-1,k-1} - b_{n-1,k-1}$ shallow 321-avoiding permutations of size n with k descents, $\pi_{n-1} = n$ and $\pi_{n-2} \neq n - 1$.

This implies the following recursion for $b_{n,k}$:

$$b_{n,k} = b_{n-1,k} + t_{n-1,k-1} - b_{n-1,k-1}.$$

Now, if $\pi \in \mathcal{T}_n(321)$ with k descents and $\pi_n = n$, then $\pi^R \in \mathcal{T}_{n-1}(321)$ with k descents. This is reversible, so there are $t_{n-1,k}$ such permutations. If $\pi_j = n$ with $1 \leq j \leq n - 1$, then since π is 321 avoiding we must have $\pi_{j+1} < \pi_{j+2} < \dots < \pi_n$. In order to have k descents we therefore must have $k + 1 \leq j \leq n - 1$. We claim there are $b_{j+1,k}$ such permutations with $\pi_j = n$. This is clearly true when $j = n - 1$ by construction. Now, if $k + 1 \leq j \leq n - 2$, by Lemma 33 since $\pi \in \mathcal{T}_n(321)$ has $\pi_j = n$ with $1 \leq j \leq n - 2$, we have $\pi_k = k - 1$ for $j + 2 \leq k \leq n$.

As a result, $\pi^{R^{n-j-1}} \in \mathcal{S}_{j+1}$ is a flat permutation with k descents and $\pi_j^{R^{n-j-1}} = j + 1$; these are precisely enumerated by $b_{j+1,k}$. Even stronger, thanks to Lemma 33, reversing this operation produces all the $\pi \in \mathcal{T}_n(321)$ with k descents and $\pi_j = n$. This proves the claim that such permutations are enumerated by $b_{j+1,k}$.

Summing over all the possible positions of n we find that

$$t_{n,k} = t_{n-1,k} + \sum_{j=k+1}^{n-1} b_{j+1,k}.$$

Now, let $B(x, z) = \sum_{n,k} b_{n,k} x^k z^n$. These recursions together with the initial conditions imply that

$$B(x, z) = zB(x, z) + xzA(x, z) - xzB(x, z) + z + z^2x.$$

and

$$A(x, z) = zA(x, z) + (B(x, z) - z - z^2x)(1 + z + z^2 + \cdots + z^{n-k-2}) + z.$$

We need to remove z and z^2x because in the recursion we always have n at least two greater than k , the number of descents. These are the only two terms in $B(x, z)$ where this does not occur. Furthermore, we can replace $1 + z + z^2 + \cdots + z^{n-k-2}$ by $\frac{1}{1-z}$ because the coefficients of $z^j x^k$ in $(B(x, z) - z - z^2x)$ for $0 \leq j \leq k + 1$ are all zero. We can therefore conclude,

$$A(x, z) = zA(x, z) + \frac{xzA(x, z)}{(1-z)(1-z+xz)} + z.$$

This gives us

$$A(x, z) = \frac{z}{1-z - \frac{xz}{(1-z)(1-z+xz)}}$$

which simplifies to the desired generating function. \square

Recall that Grassmannian permutations are those permutations with at most one descent. These permutations necessarily avoid 321 and thus we can obtain the following corollary.

Corollary 35. *For $n \geq 2$, the number of shallow permutations of length n that are Grassmannian is equal to $\binom{n+1}{3} + 1$.*

Proof. It follows from the generating function in Theorem 34 that the generating function for shallow 321-avoiding permutations with exactly one descent is $\frac{\partial}{\partial x} \Big|_{x=0} A(x, z) = \frac{z^2}{(1-z)^4}$, which tells us there are $\binom{n+1}{3}$ permutations in $\mathcal{T}_n(321)$ with 1 descent. Since there is 1 permutation of any size with zero descents and that permutation is shallow, the result follows. \square

7.3 321-avoiding shallow permutations with symmetry

In this last subsection, let us consider those shallow 321-avoiding permutations that exhibit certain symmetry.

Theorem 36. *For $n \geq 1$, the number of involutions in $\mathcal{T}_n(321)$ is F_{n+1} , where F_n is the n -th Fibonacci number.*

Proof. Let $i_n(321)$ be the number of 321-avoiding shallow involutions. First we note that if $\pi \in \mathcal{T}_n(321)$ is an involution with $\pi_j = n$, then $j = n$ or $j = n - 1$. To see this, consider $j < n - 1$. Then by Lemma 33 we have $\pi_n = n - 1$. But then since π is an involution, we must have $\pi_{n-1} = n$, a contradiction. Therefore n is in position n or $n - 1$. It is clear that there are $i_{n-1}(321)$ such permutations that have $\pi_n = n$. Since any involution with $\pi_{n-1} = n$ must also have $\pi_n = n - 1$, there are $i_{n-2}(321)$ permutations in $\mathcal{T}_n(321)$ with $\pi_{n-1} = n$. With the initial conditions $i_1(321) = 1$ and $i_2(321) = 2$, the result follows. \square

Theorem 37. *The number of centrosymmetric 321-avoiding shallow permutations is F_{n+1} when n is even and F_{n-2} when n is odd, where F_n is the n -th Fibonacci number.*

Proof. Let $c_n(321)$ be the number of shallow 321-avoiding centrosymmetric permutations and let us consider the position of n . If $\pi_n = n$, then since π is centrosymmetric, $\pi_1 = 1$. By removing both n and 1 , we are left with a centrosymmetric shallow 321-avoiding permutation of size $n - 2$. Since this is reversible for any centrosymmetric permutation in $\mathcal{T}_n(321)$, there are $c_{n-2}(321)$ such permutations. If $\pi_{n-1} = n$, then we must have $\pi_2 = 1$. In this case π^{RL} is the same as the permutation obtained by deleting both 1 and n (scaling as appropriate). The remaining permutation is a centrosymmetric shallow 321-avoiding permutation of size $n - 2$. Again, this is reversible for any centrosymmetric permutation in $\mathcal{T}_n(321)$, so there are $c_{n-2}(321)$ such permutations.

Now consider the case where $\pi_{n-j} = n$ for $n - j \leq n - 2$. Then since π is centrosymmetric and satisfies Lemma 33, we must have

$$\pi = 23 \dots j \pi_j 1 \dots n \pi_{n-j+1} (n - j + 1) (n - j + 2) \dots (n - 1).$$

Note that $\pi^{(RL)^j}$ leaves us with a centrosymmetric 321-avoiding shallow permutation of length $n - 2j$. Thus, we get

$$c_n = c_{n-2} + c_{n-2} + c_{n-4} + c_{n-6} + \dots$$

which is equivalent to $c_n = 3c_{n-2} - c_{n-4}$ which together with the initial conditions is satisfied by F_{n+1} when n is even and F_{n-2} when n is odd. \square

Theorem 38. *The number of persymmetric 321-avoiding shallow permutations is F_{n+1} , where F_n is the n -th Fibonacci number.*

Proof. Let $p_n(321)$ be the number of shallow 321-avoiding persymmetric permutations and let us consider the position of n . If $\pi_n = n$, then since π is persymmetric, $\pi_1 = 1$. By removing both n and 1 , we are left with a persymmetric shallow 321-avoiding permutation of size $n - 2$. Since this is reversible for any persymmetric permutation in $\mathcal{T}_n(321)$, there are $p_{n-2}(321)$ such permutations. If $\pi_{n-1} = n$, then we must have $\pi_1 = 2$. In this case π^{RL} is the same as the permutation obtained by deleting both 2 and n (scaling as appropriate). The remaining permutation is a persymmetric shallow 321-avoiding permutation of size $n - 2$. Again, this is reversible for any persymmetric permutation in $\mathcal{T}_n(321)$, so there are $p_{n-2}(321)$ such permutations.

Now consider the case where $\pi_{n-j} = n$ for $n - j \leq n - 2$. Then since π is persymmetric, we must have

$$\pi = (j + 1)12 \dots (j - 1) \dots n \pi_{n-j+1} (n - j + 1) (n - j + 2) \dots (n - 1).$$

Note that $\pi^{(RL)^j}$ leaves us with a persymmetric 321-avoiding shallow permutation of length $n - 2j$. Thus, we get

$$p_n = p_{n-2} + p_{n-2} + p_{n-4} + p_{n-6} + \dots$$

which is equivalent to $p_n = 3p_{n-2} - p_{n-4}$ which together with the initial conditions is satisfied by the Fibonacci numbers. \square

8 Future directions and concluding remarks

Theorems 9 and 32 imply that $t_n(132) = t_n(321)$, since both are equal to F_{2n-1} . In this paper, we prove these separately and directly, but it does raise the following question.

Question 39. Is there a bijective proof that $t_n(132) = t_n(321)$?

Based on the numerical data, we can conjecture something stronger. We conjecture that there is a bijection $f : \mathcal{T}_n(132) \rightarrow \mathcal{T}_n(321)$ with the property that $\text{cyc}(\pi) = \text{cyc}(f(\pi))$ and $\text{des}(\pi) + 1 = \text{lrmax}(f(\pi))$, where $\text{lrmax}(\sigma)$ is the number of left-to-right maxima in a permutation σ . It seems likely that there are more statistics that could be preserved in bijections between shallow 132-avoiding and shallow 321-avoiding permutations.

It may even be the case that this relationship between $\mathcal{T}_n(132)$ and $\mathcal{T}_n(321)$ goes deeper and could imply more interesting things about 132-avoiding shallow permutations. For example, it is known that the 321-avoiding shallow permutations have many nice properties (see [7, 11], among others): they have unimodal cycles, they avoid the patterns 321 and 3412, they satisfy both the upper and lower bound of the Diaconis-Graham inequality, etc.

Question 40. Are there any interesting characterizations of 132-avoiding shallow permutations that are in the same vein as those listed above for 321-avoiding shallow permutations?

Another possibility for future work is related to Theorem 8. In that theorem we show that shallow permutations avoid certain mesh patterns. However, this is not a complete characterization of shallow permutations. This leads us to ask the following question.

Question 41. Can shallow permutations be characterized completely in terms of mesh pattern avoidance? That is, is there a set of mesh patterns S so that π is shallow if and only if it avoids all patterns in S ?

One other question to consider comes from an observation by Sawin [14]. He claims that shallow permutations are in bijection with sequences q_1, \dots, q_n of nonnegative integers with the property that $q_1 = 0$ and $q_{i+1} \leq 1 + q_i + \epsilon$ where ϵ is $q_i \bmod 3$. For example, there are six shallow permutations of length 3 and six sequences that satisfy this requirement, namely: $\{000, 001, 010, 011, 012, 013\}$. If one could establish a bijection between these two combinatorial objects, there is a clear follow-up question in the context of the results of this paper.

Question 42. Does pattern avoidance in shallow permutations correspond to some notion of avoidance in these sequences of nonnegative integers?

Finally, there are many other questions about pattern-avoiding shallow permutations that we did not consider in this paper. In many cases, it seems reasonable to count these permutations by various statistics, like number of cycles. One might also consider shallow permutations that avoid longer patterns or sets of patterns. As a more general

question, one could attempt to count pattern-avoiding permutations that are not shallow, but perhaps satisfy $I(\pi)+T(\pi)+k = D(\pi)$ for a fixed value k , or perhaps pattern-avoiding permutations whose cycle diagrams correspond to different knots/links (see, for example, [4, 17]).

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