A signed graph analogue of acyclic orientation polynomials and the sink theorem

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Abstract

We introduce the acyclic orientation polynomial of a signed graph, defined as the generating function that counts sinks of its acyclic orientations, thereby refining the number of acyclic orientations. We prove that our acyclic orientation polynomial satisfies the deletion-contraction recurrence with the recurrence rule depending on the sign of the edge involved. Using this recurrence, we expand the polynomial in terms of its subgraphs as an analogue of the expansion of the chromatic polynomial. The main application is an alternative proof of the signed graph analogue of Stanley's sink theorem for chromatic symmetric functions, which does not rely on a signed version of quasi-symmetric functions and P-partitions.

Mathematics Subject Classifications: 05C31, 05C22, 05C30, 05E05, 05C15

1 Introduction

This paper aims to introduce acyclic orientation polynomials for signed graphs and to present their expansions in terms of subgraphs, providing a new proof of Stanley's sink theorem for signed graphs.

A *signed* graph Σ with vertex set V and edge set E is a graph in which each edge is assigned either a positive or a negative sign [3]. An orientation $\mathfrak{o}$ of a signed graph is an assignment of an arrow to each half-edge. Each arrow may point either toward or away from its incident vertex, subject to the condition that, for a positive edge, one arrow must point toward a vertex and the other must point away from it, while for a negative edge, both arrows must point either toward their incident vertices or away from them. An orientation $\mathfrak{o}$ is an *acyclic orientation* if $\mathfrak{o}$ contains no directed cycles [13]. Let $\mathcal{A}(\Sigma)$ denote the set of acyclic orientations of Σ. A vertex v is called a sink in an orientation $\mathfrak{o}$ if every half-edge incident to v is directed toward it.

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Define the acyclic orientation polynomial $A_\Sigma(V)$ of the signed graph Σ to be the generating function that counts sinks of its acyclic orientations (Definition 1), i.e.,

$$A_\Sigma(V) = \sum_{\mathfrak{o}\in\mathcal{A}(\Sigma)} \prod_{v\in\mathrm{Sink}(\Sigma,\mathfrak{o})} v,$$

where V also denotes the set of variables corresponding to the vertices. Let $a_\Sigma(t)$ be a polynomial obtained by substituting $v = t$ for each vertex $v \in V$, so that the coefficient of t^j equals the number of acyclic orientations of Σ with j sinks.

We develop a signed graph version of the deletion-contraction recurrence (Theorem 4) for $A_\Sigma(V)$. Take a non-loop edge $e = u_1u_2$ of Σ. Let $\Sigma \setminus e$ be the signed graph obtained by deleting e from Σ. Let Σ/e be the signed graph obtained from Σ by first switching the signs of all non-loop edges incident to one endpoint of e and then contracting e into a new vertex v_e. The vertex set V/e is $(V \setminus \{u_1, u_2\}) \cup \{v_e\}$. A change of variables in the contraction Σ/e is required for our deletion-contraction recurrence to hold.

We define the variable v_e according to the sign of the edge e. If e is a positive edge, we set $v_e = 1 - (1 - u_1)(1 - u_2)$, which coincides with the change of variables used in the recursive contraction formula in [4, Theorem 2.3]. If e is a negative edge, we set $v_e = 1$, a choice that does *not* arise in the usual (unsigned) case and is specific to the signed graph setting. Then the following recurrence holds:

$$A_\Sigma(V) = A_{\Sigma\setminus e}(V) + A_{\Sigma/e}(V/e).$$

Using the new recurrence, we expand our acyclic orientation polynomial $A_\Sigma(V)$ in terms of subgraphs (Theorem 8) as an "acyclic orientation" analogue of the chromatic polynomial for signed graphs in [13, Theorem 2.4]. The connection between the chromatic polynomial and acyclic orientations originates from a result of [7], which asserts that the number of acyclic orientations of a graph is given by evaluating its chromatic polynomial at -1, along with a signed analogue presented in [13, Corollary 4.1]. Motivated by this result and the subgraph expansion of the chromatic polynomial, an analogue in terms of acyclic orientations was developed in [4, Theorem 3.2]. In this paper, we introduce its signed version.

As a main application, we provide an alternative proof (Theorem 15) of *Stanley's sink theorem* for signed graphs [1, Theorem 4]. This theorem is a signed graph analogue of [8, Theorem 3.3], which states that for the chromatic symmetric function X_G of a graph G, the coefficients of elementary symmetric functions of length j in the expansion of X_G count acyclic orientations of G with j sinks. A signed graph analogue of the chromatic symmetric function was introduced in [2, 5, 1] as a symmetric function associated with the Coxeter group of type B, and is therefore referred to as the *chromatic B-symmetric function.* The proof of [1, Theorem 4] employs a signed analogue of the theory of quasi-symmetric functions and P-partitions, following the approach taken in [8, Theorem 3.3].

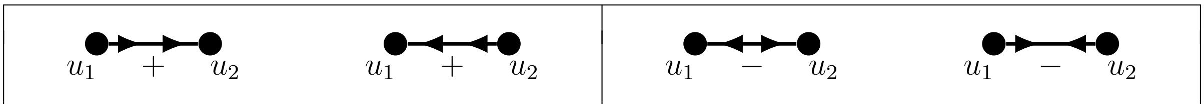
Our proof is established by comparing the expansion of the acyclic orientation polynomials of signed graphs (Theorem 8) with that of the chromatic B-symmetric function ([5, Theorem 3.8]). This approach is in the same spirit as [4, Theorem 4.5], which addressed a question posed by Stanley in [9, 10], seeking a simple and conceptual proof of the theorem.

This paper is organized as follows. Section 2 introduces the acyclic orientation polynomials of signed graphs, while Section 3 establishes a deletion-contraction recurrence for these polynomials. Section 4 builds on this recurrence to present a subgraph expansion. Section 5 then reviews the B-chromatic symmetric function along with its subgraph expansion. Finally, Section 6 offers an alternative proof of Stanley's sink theorem for signed graphs.

2 Acyclic orientation polynomials of signed graphs

A *signed graph* Σ is a graph with vertex set V and edge set E, where each edge is assigned either a positive sign, $+$, or a negative sign, $-$, referred to as a *positive edge* or a *negative edge*, respectively [3].

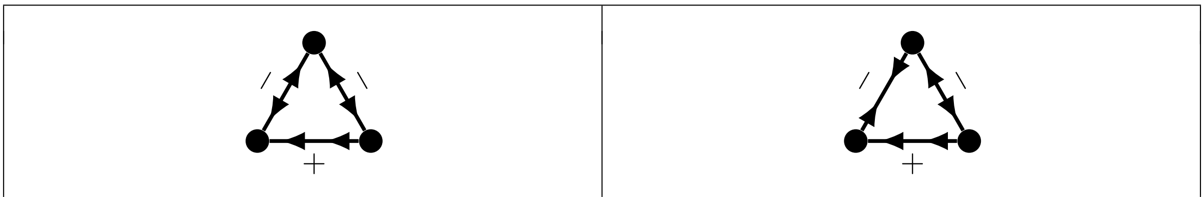
An orientation $\mathfrak{o}$ of a signed graph assigns an arrow to each *half-edge*, i.e., to each pair consisting of a vertex and an incident edge $e = u_1u_2$ [13]. Each such arrow may point either toward or away from its vertex according to the following rules: for a *positive edge*, one half-edge arrow must point toward its vertex and the other must point away from it, which will be denoted by $\overrightarrow{u_1}\overrightarrow{u_2}$ or $\overleftarrow{u_1}\overleftarrow{u_2}$, respectively. For a *negative edge*, both half-edge arrows must either point toward their respective vertices or away from them, which will be denoted by $\overleftarrow{u_1}\overrightarrow{u_2}$ and $\overrightarrow{u_1}\overleftarrow{u_2}$, respectively. See Figure 1 for an illustration of this arrow assignment.



(a) A positive edge. (b) A negative edge.

Figure 1: The assignment of orientations in signed graphs.

A *directed cycle* in a signed graph Σ is a closed path in which every vertex along the path has exactly one incident arrow pointing toward it and one arrow pointing away from it. An orientation $\mathfrak{o}$ is said to be *acyclic* if it contains no directed cycles. See Figure 2 for an example.



(a) A directed cycle. (b) A non-directed cycle.

Figure 2: A directed cycle and a non-directed cycle in a signed graph.

Our main focus in this paper is on acyclic orientations. We denote by $\mathcal{A}(\Sigma)$ the set of all acyclic orientations of Σ. A sink is a vertex v such that all incident half-edges are directed toward v. For an orientation $\mathfrak{o}$, let $\mathrm{Sink}(\Sigma, \mathfrak{o})$ denote the set of sinks, and define

$\mathrm{sink}(\Sigma, \mathfrak{o}) := |\operatorname{Sink}(\Sigma, \mathfrak{o})|$ as the number of sinks. To simplify notation, we identify each vertex $v \in V$ with a corresponding commuting variable v. Consequently, we also use V to denote the set of variables corresponding to the vertices. For each acyclic orientation $\mathfrak{o} \in \mathcal{A}(\Sigma)$, we associate the following monomial:

$$\prod_{v \in \operatorname{Sink}(\Sigma, \mathfrak{o})} v.$$

We are now ready to define the acyclic orientation polynomial of a signed graph.

Definition 1. For a signed graph Σ, define the *acyclic orientation polynomial* $A_\Sigma(V)$ of Σ to be the generating function that counts sinks of acyclic orientations of Σ, i.e.,

$$A_\Sigma(V) = \sum_{\mathfrak{o} \in \mathcal{A}(\Sigma)} \prod_{v \in \operatorname{Sink}(\Sigma, \mathfrak{o})} v.$$

Let $a_\Sigma(t)$ be the polynomial obtained from $A_\Sigma(V)$ by setting $v = t$ for every $v \in V$, i.e.,

$$a_\Sigma(t) = \sum_{\mathfrak{o} \in \mathcal{A}(\Sigma)} t^{\mathrm{sink}(\Sigma, \mathfrak{o})}.$$

We present examples illustrating the definition of the acyclic orientation polynomials.

Example 2. Let Σ be a triangle signed graph on the vertex set $V = \{v_1, v_2, v_3\}$ with a negative edge v_1v_2 and two positive edges v_1v_3, v_2v_3. Then Σ has 8 acyclic orientations with

$$A_\Sigma(V) = v_1v_2 + v_1 + v_2 + 2v_3 + 3 \text{ and } a_\Sigma(t) = t^2 + 4t + 3. \tag{1}$$

Let Σ' be a signed graph on the vertex set $V' = \{v_1, v_2, v_3\}$ with two positive edges v_1v_3, v_2v_3. Then Σ' has 4 acyclic orientations with

$$A_{\Sigma'}(V') = v_1v_2 + v_1 + v_2 + v_3. \tag{2}$$

Let Σ'' be a signed graph with vertex set $V'' = \{v_{12}, v_3\}$ consisting of two parallel edges, one positive and one negative. Then Σ'' has 4 acyclic orientations with

$$A_{\Sigma''}(V'') = v_3 + v_{12} + 2. \tag{3}$$

All the acyclic orientations are illustrated in Figure 3.

Example 3. Let Σ_1 be a triangle signed graph on the vertex set $V = \{v_1, v_2, v_3\}$ with two negative edges v_1v_2, v_1v_3 and one positive edge v_2v_3. Then Σ_1 has 6 acyclic orientations with

$$A_{\Sigma_1}(V_1) = v_1v_2 + v_1v_3 + v_2 + v_3 + 2 \text{ and } a_{\Sigma_1}(t) = 2t^2 + 2t + 2. \tag{4}$$

Let Σ_1' be a signed graph on the vertex set $V' = \{v_1, v_2, v_3\}$ with a negative edge v_1v_3 and a positive edge v_2v_3. Then Σ_1' has 4 acyclic orientations with

$$A_{\Sigma_1'}(V_1') = v_1v_2 + v_1v_3 + v_2 + 1. \tag{5}$$

Let Σ_1'' be a signed graph with the vertex set $V'' = \{v_{12}, v_3\}$ consisting of two parallel positive edges. Then Σ_1'' has 2 acyclic orientations with

$$A_{\Sigma_1''}(V_1'') = v_3 + v_{12}. \tag{6}$$

All the acyclic orientations are illustrated in Figure 4.

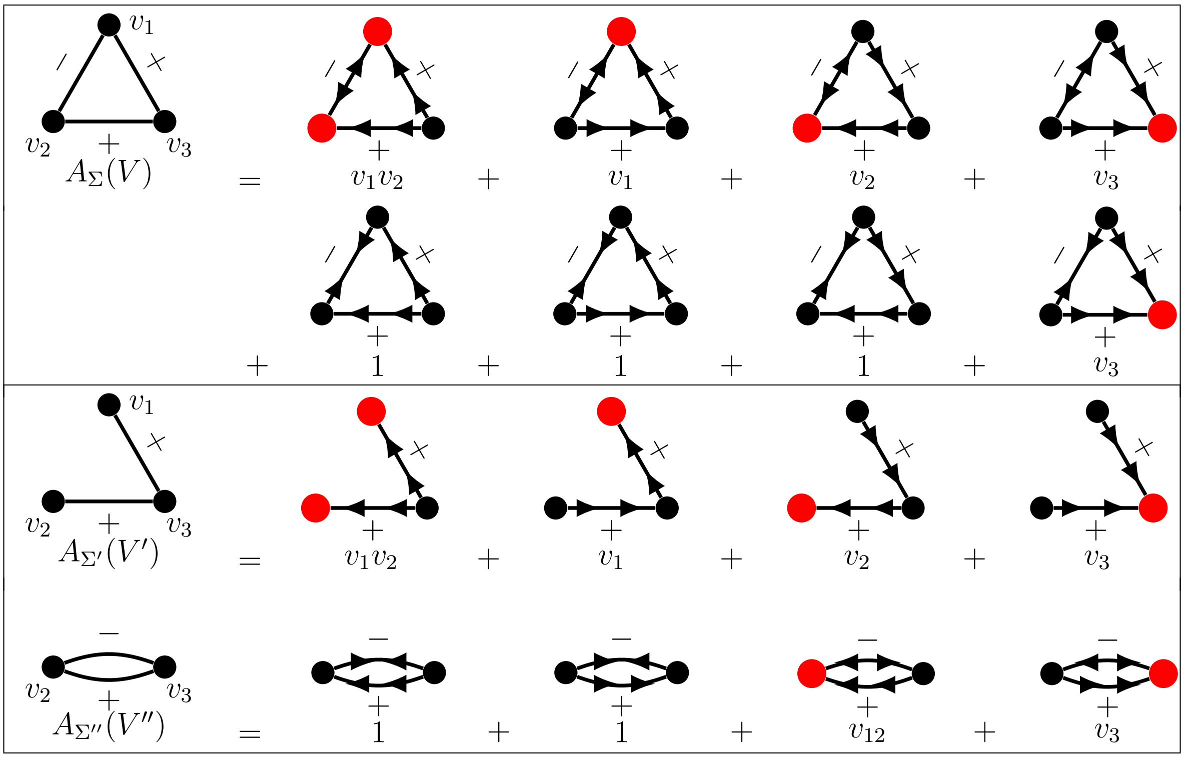


Figure 3: The acyclic orientations of $\Sigma, \Sigma', \Sigma''$ with sinks marked in red.

3 The deletion-contraction recurrence for $A_\Sigma(V)$

We establish a deletion-contraction recurrence for $A_\Sigma(V)$, as the signed graph analogue of [4, Theorem 2.3]. Let Σ be a signed graph and $e = u_1u_2$ be a non-loop edge. The *deletion* $\Sigma \setminus e$ is the signed graph obtained by removing e from Σ. The *contraction* Σ/e is obtained by identifying the endpoints of e into a new vertex v_e so that its vertex set V/e is $(V \setminus \{u_1, u_2\}) \cup \{v_e\}$. If e is a negative edge, we adjust the signs of all non-loop edges incident to an endpoint of e by switching their signs [12, Section 3] before performing the identification, so that e is treated as a positive edge. This contraction process is shown in Figure 5.

Define the weight of the contracted vertex v_e as follows. For a positive edge $e = u_1u_2$, let

$$v_e = u_1 + u_2 - u_1u_2, \quad \text{equivalently,} \quad (1 - v_e) = (1 - u_1)(1 - u_2)$$

as in the graph formulation of [4, Eq. (1)]. For a negative edge e, we set $v_e = 1$, ignoring whether v_e is a sink under this weighting. With this weighting, the deletion-contraction recurrence for $A_\Sigma(V)$ takes the following form.

Theorem 4 (Deletion-contraction recurrence)**.** *For a signed graph Σ with vertex set V, the acyclic orientation polynomial $A_\Sigma(V)$ satisfies the following deletion-contraction recurrence: for any non-loop edge e,*

$$A_\Sigma(V) = A_{\Sigma\setminus e}(V) + A_{\Sigma/e}(V/e).$$

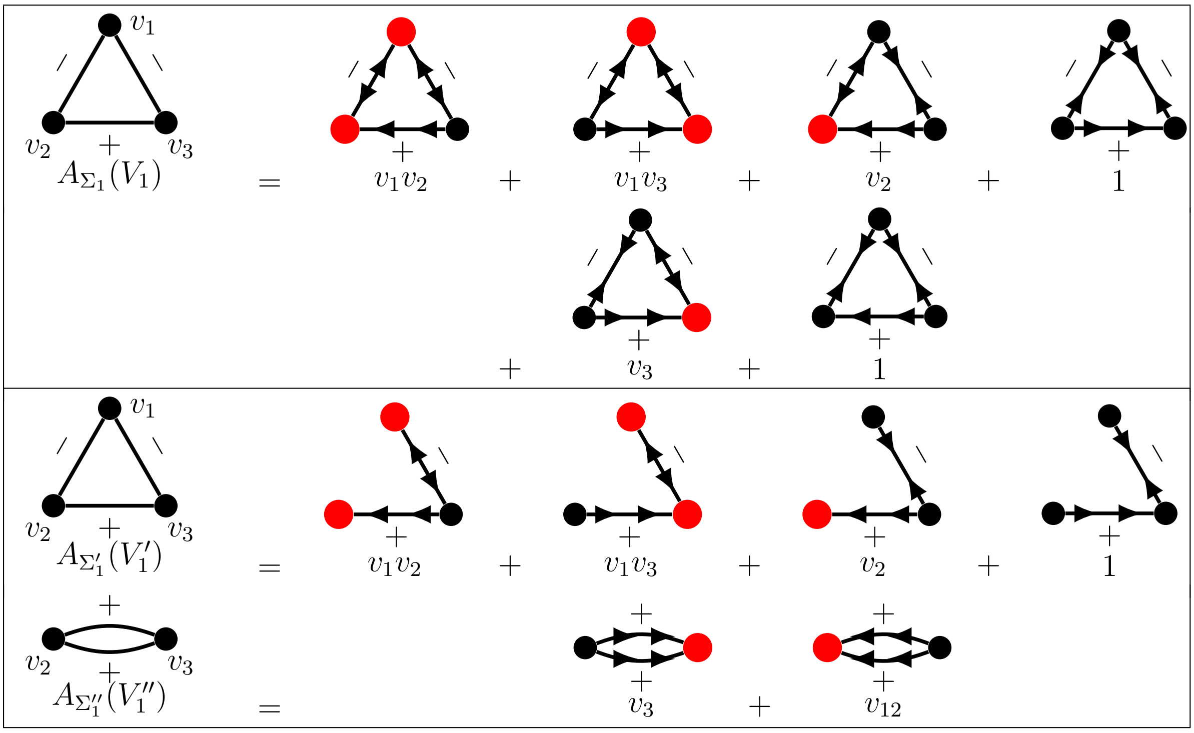
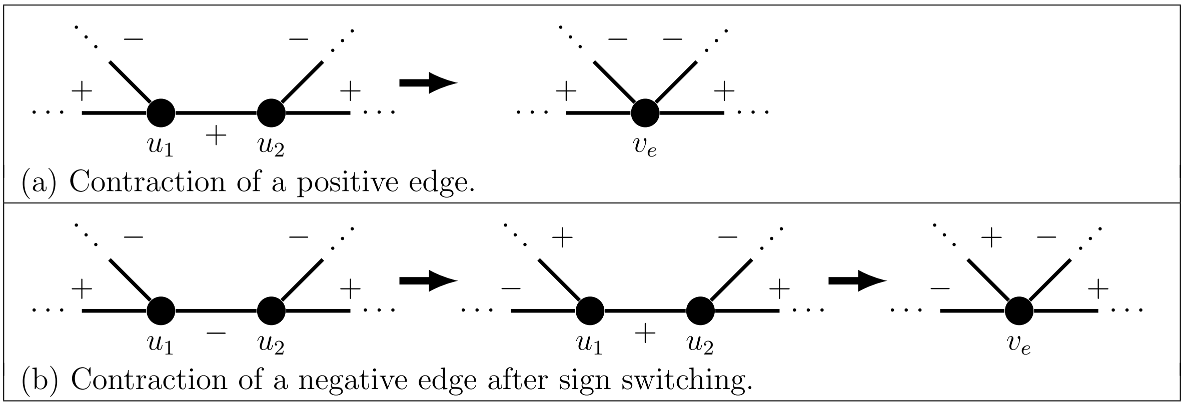


Figure 4: The acyclic orientations of $\Sigma_1, \Sigma_1', \Sigma_1''$ with sinks marked in red.



(a) Contraction of a positive edge.

(b) Contraction of a negative edge after sign switching.

Figure 5: Contraction of the edge $e = u_1u_2$ to a vertex v_e in signed graphs.

Before proving the theorem, we present examples that illustrate the recurrence.

Example 5. Let $\Sigma, \Sigma', \Sigma''$ be signed graphs defined in Example 2. By applying Theorem 4 to the negative edge $e = v_1v_2$ and subsequently using Eqs. (2) and (3), we compute

$$\begin{aligned} A_\Sigma(V) &= A_{\Sigma\setminus e}(V) + A_{\Sigma/e}(V/e) \\ &= A_{\Sigma'}(V') + A_{\Sigma''}(V'')\big|_{v_{12}=1} \\ &= \big(v_1v_2 + v_1 + v_2 + v_3\big) + \big(v_3 + v_{12} + 2\big)\big|_{v_{12}=1} \\ &= v_1v_2 + v_1 + v_2 + 2v_3 + 3, \end{aligned}$$

which coincides with Eq. (1).

Example 6. Let $\Sigma_1, \Sigma_1', \Sigma_1''$ be signed graphs given in Example 3. By applying Theorem 4 to the negative edge $e = v_1v_2$ and then using Eqs. (5) and (6), we derive

$$\begin{aligned} A_{\Sigma_1}(V) &= A_{\Sigma_1\setminus e}(V) + A_{\Sigma_1/e}(V/e) \\ &= A_{\Sigma_1'}(V') + A_{\Sigma_1''}(V'')\big|_{v_{12}=1} \\ &= \big(v_1v_2 + v_1v_3 + v_2 + 1\big) + \big(v_3 + v_{12}\big)\big|_{v_{12}=1} \\ &= v_1v_2 + v_1v_3 + v_2 + v_3 + 2, \end{aligned}$$

which aligns with Eq. (4).

Proof of Theorem 4. For a positive edge, the proof is the same as that of [4, Theorem 2.3], to which we refer the reader.

Let $e = u_1u_2$ be a negative edge in Σ. Fix a non-empty subset $U \subseteq V$, and let a, a_D, and a_C denote the coefficients of $\prod_{v\in U} v$ in $A_\Sigma(V)$, $A_{\Sigma\setminus e}(V)$, and $A_{\Sigma/e}(V/e)$, respectively. Thus, it suffices to show that $a = a_D + a_C$. Note that $a = a(\Sigma, U)$ and $a_D = a(\Sigma \setminus e, U)$.

We first consider the case where either u_1 or u_2 belongs to U. Since the variables u_1 and u_2 do not appear in V/e, it follows that $a_C = 0$. For each $\mathfrak{o} \in \mathcal{A}(\Sigma, U)$, the edge e must be oriented toward both u_1 and u_2, and deleting the edge defines a bijection between $\mathcal{A}(\Sigma, U)$ and $\mathcal{A}(\Sigma \setminus e, U)$, showing that $a = a_D$.

Now consider the case where $u_1, u_2 \notin U$. Since the contracted vertex v_e has weight 1 in V/e, the coefficient a_C is given by

$$a_C = |\mathcal{A}(\Sigma/e, U, v_e)|,$$

where $\mathcal{A}(\Sigma/e, U, v_e) := \mathcal{A}(\Sigma/e, U) \sqcup \mathcal{A}(\Sigma/e, U \cup \{v_e\})$.

For convenience, we say that an orientation $\mathfrak{o}$ contains a directed path from u_1 to u_2 (resp. from u_2 to u_1) if $\mathfrak{o} \cup \overset{\rightarrow\leftarrow}{u_1u_2}$ (resp. $\mathfrak{o} \cup \overleftrightarrow{u_1u_2}$) forms a directed cycle. Let $\mathcal{A}_{\nleftrightarrow}$ denote the set of orientations of $\Sigma \setminus e$ that contain neither a directed path from u_1 to u_2 nor from u_2 to u_1. We now decompose $\mathcal{A}(\Sigma, U)$ into the following subsets:

- $\mathcal{A}_1$: the set of acyclic orientations for which either u_1 or u_2 becomes a sink upon removing e:

$$\mathcal{A}_1 := \{\mathfrak{o} \in \mathcal{A}(\Sigma, U) \mid \mathfrak{o} \setminus e \notin \mathcal{A}(\Sigma \setminus e, U)\}.$$

- $\mathcal{A}_2$: the set of orientations whose restriction to $\Sigma \setminus e$ remains acyclic, but in which $\Sigma \setminus e$ contains a directed path between u_1 and u_2:

$$\mathcal{A}_2 := \{\mathfrak{o} \in \mathcal{A}(\Sigma, U) \mid \mathfrak{o} \setminus e \in \mathcal{A}(\Sigma \setminus e, U),\, \mathfrak{o} \setminus e \notin \mathcal{A}_{\not\leftrightarrow}\}\,.$$

- $\mathcal{A}_3$: the remaining orientations, i.e., those where removing e yields an acyclic orientation with no directed path between u_1 and u_2 in either direction:

$$\mathcal{A}_3 := \{\mathfrak{o} \in \mathcal{A}(\Sigma, U) \mid \mathfrak{o} \setminus e \in \mathcal{A}(\Sigma \setminus e, U),\, \mathfrak{o} \setminus e \in \mathcal{A}_{\not\leftrightarrow}\}\,.$$

  We further divide $\mathcal{A}_3$ according to the orientation of the edge e in $\mathfrak{o}$:

$$\mathcal{A}_{3,+} = \{\mathfrak{o} \in \mathcal{A}_3 \mid \overset{\rightarrow\leftarrow}{u_1u_2} \in \mathfrak{o}\}\,, \text{ and } \mathcal{A}_{3,-} = \{\mathfrak{o} \in \mathcal{A}_3 \mid \overleftrightarrow{u_1u_2} \in \mathfrak{o}\}\,.$$

Thus,

$$\mathcal{A}(\Sigma, U) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3, \quad \text{and} \quad \mathcal{A}_3 = \mathcal{A}_{3,+} \sqcup \mathcal{A}_{3,-}.$$

We now claim that $a_D = |\mathcal{A}_2 \sqcup \mathcal{A}_{3,-}|$ and$a_C = |\mathcal{A}_1 \sqcup \mathcal{A}_{3,+}|$, which will follow once we verify that the following two maps are bijections:

$$\begin{aligned} \Phi_D : \quad & \mathcal{A}_2 \sqcup \mathcal{A}_{3,-} \longrightarrow \mathcal{A}(\Sigma \setminus e, U), \quad \mathfrak{o} \mapsto \mathfrak{o} \setminus e, \\ \Phi_C : \quad & \mathcal{A}_1 \sqcup \mathcal{A}_{3,+} \longrightarrow \mathcal{A}(\Sigma / e, U), \quad \mathfrak{o} \mapsto \mathfrak{o}/e, \end{aligned}$$

where $\mathfrak{o}/e$ is obtained by reversing orientations of all half-edges which were incident to u_1 in $\mathfrak{o}$, being non-loops. This reversal is necessary due to the sign switching in the contraction. Observe that the sink status of all vertices other than u_1 and u_2 remains unchanged. To justify the claim, we now explicitly construct the inverses of these two maps.

We first define a map from $\mathcal{A}(\Sigma \setminus e, U)$ to $\mathcal{A}_2 \sqcup \mathcal{A}_{3,-}$ as follows. Given an orientation $\mathfrak{o}' \in \mathcal{A}(\Sigma \setminus e, U)$, define

$$\mathfrak{o}' \mapsto \begin{cases} \mathfrak{o}' \cup \overleftrightarrow{u_1u_2} \in \mathcal{A}_2, & \text{if there exists a directed path from } u_1 \text{ to } u_2 \text{ in } \mathfrak{o}', \\ \mathfrak{o}' \cup \overset{\rightarrow\leftarrow}{u_1u_2} \in \mathcal{A}_2, & \text{if there exists a directed path from } u_2 \text{ to } u_1 \text{ in } \mathfrak{o}', \\ \mathfrak{o}' \cup \overleftrightarrow{u_1u_2} \in \mathcal{A}_{3,-}, & \text{otherwise.} \end{cases}$$

Note that the first two cases are disjoint since $\mathfrak{o}'$ is acyclic and cannot contain both directed paths. It is straightforward that this map is the inverse of the deletion map Φ_D.

To define the inverse of the contraction map Φ_C, we proceed as follows. Given $\mathfrak{o}' \in \mathcal{A}(\Sigma/e, U)$, denote by $\mathfrak{o}''$ the orientation on $\Sigma \setminus e$ obtained by splitting the contracted vertex v_e back into u_1 and u_2, and by reversing orientations in all non-loop half-edges incident to u_1, which is required because of the sign switching in the contraction. Since $\mathfrak{o}'$ is acyclic, there cannot be a directed path from u_1 to u_2 or from u_2 to u_1 in $\mathfrak{o}''$. Then define the map

$$\mathfrak{o}' \mapsto \begin{cases} \mathfrak{o}'' \cup \overset{\rightarrow\leftarrow}{u_1u_2} \in \mathcal{A}_1, & \text{if } u_1 \text{ or } u_2 \text{ is a sink in } \mathfrak{o}'', \\ \mathfrak{o}'' \cup \overset{\rightarrow\leftarrow}{u_1u_2} \in \mathcal{A}_{3,+}, & \text{otherwise.} \end{cases}$$

This map is the inverse of the contraction map Φ_C. $\square$

We conclude this section with an illustration of the preceding proof. Figures 3 and 4 display the acyclic orientations, arranged to reflect the bijection described in the proof, with corresponding orientations shown in matching positions.

In Figure 3, the first three acyclic orientations in the first row have either u_1 or u_2 as a sink, while the last one belongs to $\mathcal{A}_{3,-}$. These orientations therefore correspond to those in the deletion $\Sigma \setminus e$. The first three orientations in the second row belong to $\mathcal{A}_1$, and the last one belongs to $\mathcal{A}_{3,+}$. These orientations hence correspond to the contraction Σ/e.

Similarly, in Figure 4, the former three acyclic orientations in the first row have either u_1 or u_2 as a sink, and the last one belongs to $\mathcal{A}_2$. Thus, these correspond to those in the deletion $\Sigma \setminus e$. All orientations in the second row belong to $\mathcal{A}_1$, and hence correspond to the contraction Σ/e.

4 A subgraph expansion of acyclic orientation polynomials

In this section, we express the acyclic orientation polynomial $A_\Sigma(V)$ in terms of spanning subgraphs of the signed graph Σ. This provides a signed graph analogue of [4, Theorem 3.2].

Let S be a subset of the edge set E. We also write S for the spanning subgraph of Σ with edge set S, and let $|S|$ denote the number of edges in this subgraph. Let $\mathcal{S}(\Sigma)$ denote the collection of all spanning subgraphs of Σ. For an edge $e = u_1u_2 \in E(\Sigma)$, define the following subsets of $\mathcal{S}(\Sigma)$:

$$\mathcal{S}(\Sigma)^e = \{\, S \in \mathcal{S}(\Sigma) \mid e \notin E(S) \,\}, \quad \text{and} \quad \mathcal{S}(\Sigma)_e = \mathcal{S}(\Sigma) \setminus \mathcal{S}(\Sigma)^e.$$

Note that every subgraph in $\mathcal{S}(\Sigma)_e$ must contain a (connected) component that includes both vertices, u_1 and u_2.

Proposition 7. *For an edge $e \in E$, deleting e results in $\mathcal{S}(\Sigma)^e = \mathcal{S}(\Sigma \setminus e)$ and contracting e establishes a bijection between $\mathcal{S}(\Sigma)_e$ and $\mathcal{S}(\Sigma/e)$.*

Let $\mathcal{C}(S)$ denote the set of components of a subgraph S. A component is said to be *positive* if all its edges are positive. For each component $C \in \mathcal{C}(S)$, define the variable v_C as

$$v_C = \begin{cases} 1 - \prod_{v \in V(C)}(1 - v), & \text{if } C \text{ is positive,} \\ 1, & \text{otherwise,} \end{cases}$$

and let t_C be defined as

$$t_C = \begin{cases} 1 - (1 - t)^{|V(C)|}, & \text{if } C \text{ is positive,} \\ 1, & \text{otherwise.} \end{cases}$$

We recall the notion of *balance*, a fundamental concept in signed graph theory. A cycle of length $k \geqslant 2$ is said to be *balanced* if it contains an even number of negative edges;

otherwise, it is called *unbalanced*. A signed graph Σ is called *balanced* if all of its cycles are balanced; otherwise, it is said to be *unbalanced*.

For a subgraph S, let $\mathcal{C}_b(S) \subseteq \mathcal{C}(S)$ denote the set of its balanced components, and define

$$s_b(S) := |S| - d + |\mathcal{C}_b(S)|. \tag{7}$$

Theorem 8. *For a signed graph $\Sigma = (V, E)$, its acyclic orientation polynomial $A_\Sigma(V)$ is given by*

$$A_\Sigma(V) = \sum_{S \subseteq E} (-1)^{s_b(S)} \prod_{C \in \mathcal{C}(S)} v_C.$$

Consequently,

$$a_\Sigma(t) = \sum_{S \subseteq E} (-1)^{s_b(S)} \prod_{C \in \mathcal{C}_b(S)} t_C. \tag{8}$$

Proof. To simplify notation, for any $S \subseteq E$, define

$$v(S) := (-1)^{s_b(S)} \prod_{C \in \mathcal{C}(S)} v_C \quad \text{and} \quad B_\Sigma(V) := \sum_{S \subseteq E} v(S).$$

Our goal is to show $A_\Sigma(V) = B_\Sigma(V)$.

The proof proceeds by induction on the number of non-loop edges. For the base case, consider a signed graph where all edges are loops. For disjoint signed graphs Σ_1 and Σ_2 on vertex sets V_1 and V_2, respectively, we have

$$A_{\Sigma_1 \sqcup \Sigma_2}(V_1 \sqcup V_2) = A_{\Sigma_1}(V_1) A_{\Sigma_2}(V_2),$$

and the same holds for $B_\Sigma(V)$. Thus, it suffices to consider the case where Σ consists of a single vertex v with m loops. For an illustration, see Figure 6.

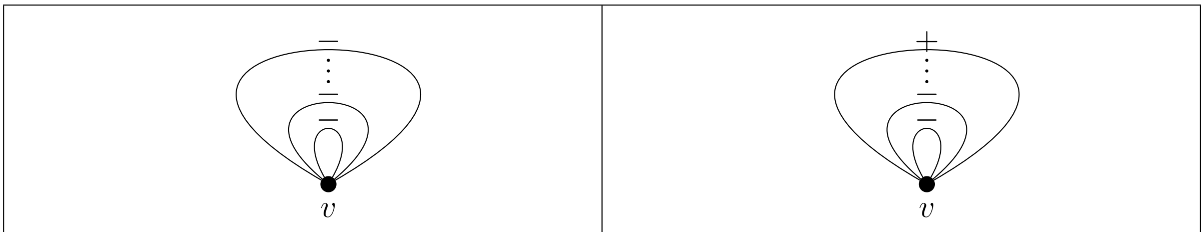
First, assume that all m loop edges are negative. In this case, $A_\Sigma(V) = v + 1$, and

$$B_\Sigma(V) = v + \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} = v + 1.$$

Now suppose that Σ contains at least one positive loop edge e. In this case, $A_\Sigma(V) = 0$. On the other hand, for any subset $S \subset E$ with $e \notin S$, we have $v(S) = v(S \cup \{e\})$, so such pairs cancel each other out in the sum, implying $B_\Sigma(V) = 0$ as well.

For the induction step, consider a non-loop edge $e = u_1u_2 \in E$. Let C' denote the component in $S \in \mathcal{S}(\Sigma)_e$ that contains e, so that C'/e denotes the corresponding component in S/e containing the contracted vertex v_e. We claim that for every $S \in \mathcal{S}(\Sigma)_e$, the following identity holds:

$$v(S) = (-1)^{s_b(S)} \left(\prod_{C \in \mathcal{C}(S) \setminus \{C'\}} v_C \right) v_{C'} = (-1)^{s_b(S/e)} \left(\prod_{C \in \mathcal{C}(S/e) \setminus \{C'/e\}} v_C \right) v_{C'/e}.$$



(a) All edges are negative. (b) At least one edge is positive.

Figure 6: A signed graph with a single vertex v and multiple loops.

Using this claim, we can compute:

$$\begin{aligned} A_\Sigma(V) &= A_{\Sigma\backslash e}(V) + A_{\Sigma/e}(V/e) \\ &= \sum_{S\in\mathcal{S}(\Sigma\backslash e)} v(S) + \sum_{S/e\in\mathcal{S}(\Sigma/e)} (-1)^{s_b(S/e)} \left(\prod_{C\in\mathcal{C}(S)\backslash\{C'/e\}} v_C \right) v_{C'/e} \\ &= \sum_{S\in\mathcal{S}(\Sigma)^e} v(S) + \sum_{S\in\mathcal{S}(\Sigma)_e} v(S) = \sum_{S\in\mathcal{S}(\Sigma)} v(S), \end{aligned}$$

where the first equality follows from the deletion-contraction recurrence (Theorem 4), the second is justified by the induction hypothesis, the third equality uses Proposition 7 together with the above claim, and the last follows from $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)^e \sqcup \mathcal{S}(\Sigma)_e$.

To prove the claim, it is sufficient to consider the component C' since contracting the edge e does not affect the other components. We first show that the variable $v_{C'/e}$ is equal to $v_{C'}$, i.e., $v_{C'/e} = v_{C'}$. If C' is positive, then

$$v_{C'/e} = 1 - (1 - v_e) \prod_{v\in V(C'/e)\backslash\{v_e\}} (1 - v) = 1 - (1 - u_1)(1 - u_2) \prod_{v\in V(C')\backslash\{u_1,u_2\}} (1 - v) = v_{C'}.$$

Let C' be negative so that $v_{C'} = 1$. If C'/e is also negative, then clearly $v_{C'/e} = 1$. If C'/e is positive, then e must be a negative edge, and we have

$$v_{C'/e} = \left(1 - (1 - v_e) \prod_{v\in V(C')\backslash\{u_1,u_2\}} (1 - v) \right) \Bigg|_{v_e=1} = 1 = v_{C'}.$$

Next, since contraction preserves the balance of a component, it follows that

$$s_b(S) = |S| - d + |\mathcal{C}_b(S)| = |S| - 1 - (d-1) + |\mathcal{C}_b(S/e)| = s_b(S/e).$$

This completes the proof of the claim, and thus the theorem follows. $\square$

Example 9. Using the previous theorem, we compute the acyclic orientation polynomial $A_\Sigma(V)$ of Σ from Example 2 as follows:

$$\begin{aligned} A_\Sigma(V) &= v_1v_2v_3 + v_2\big(1 - (1 - v_1)(1 - v_3)\big) + v_1\big(1 - (1 - v_2)(1 - v_3)\big) + \big(1 - (1 - v_1)(1 - v_2)(1 - v_3)\big) \\ &\quad + v_3 + 1 + 1 + 1 = v_1v_2 + v_1 + v_2 + 2v_3 + 3, \end{aligned}$$

agreeing with Example 2. The detailed computation is depicted in Figure 7.

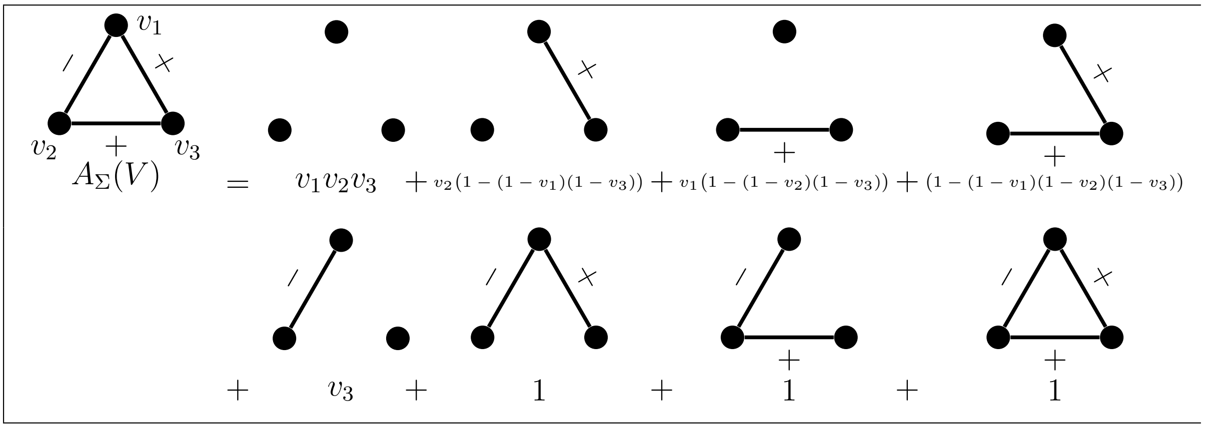


Figure 7: A subgraph expansion of the acyclic orientation polynomial of Σ.

Example 10. Similarly, the acyclic orientation polynomial $A_{\Sigma_1}(V_1)$ of Σ_1 from Example 3 is computed as

$$\begin{aligned}A_{\Sigma_1}(V_1) &= v_1v_2v_3 + v_2 + v_1\left(1-(1-v_2)(1-v_3)\right) + 1 + v_3 + 1 + 1 + (-1)\\ &= v_1v_2 + v_1v_3 + v_2 + v_3 + 2,\end{aligned}$$

which is consistent with Example 3. The detailed expansion is shown in Figure 8.

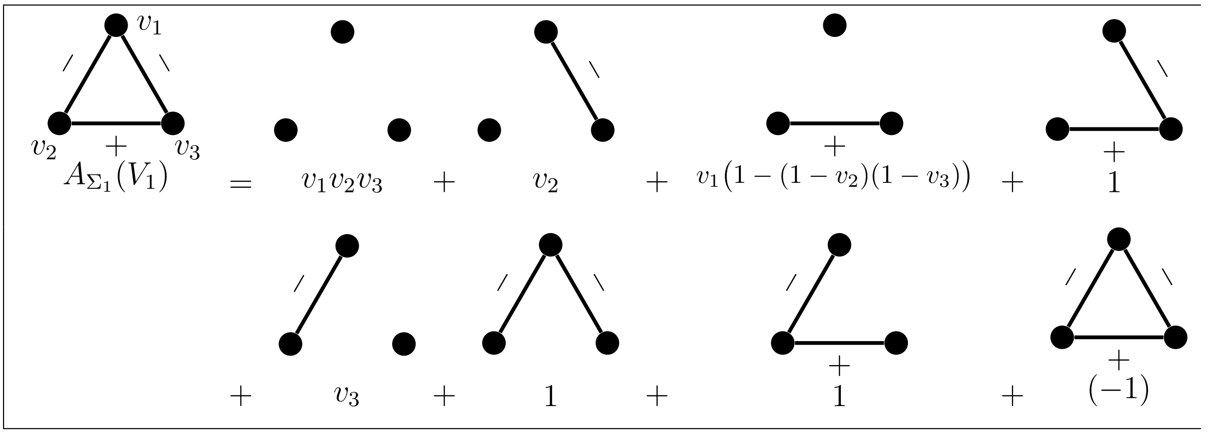


Figure 8: A subgraph expansion of the acyclic orientation polynomial of Σ_1.

5 Chromatic symmetric functions for signed graphs

The *chromatic B-symmetric function* of a signed graph, introduced in [2, 5, 1], generalizes the chromatic symmetric function of a graph [8, Definition 2.1]. In this section, we review its subgraph expansion as presented in [5, Theorem 3.8].

Stanley [8] introduced the *chromatic symmetric function* X_G for a graph G with vertex set V, defined as follows. A coloring κ is said to be a *proper coloring* of G if $\kappa : V \to \mathbb{N}$ with the natural numbers $\mathbb{N}$, and $\kappa(v) \neq \kappa(v')$ for every edge vv' in G. Let $\mathcal{P}(G)$ denote the set of all such proper colorings. Then the chromatic symmetric function X_G of G is defined as

$$X_G(x_1, x_2, \dots) := \sum_{\kappa \in \mathcal{P}(G)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where $v_1, v_2, \dots, v_d$ are the vertices of G. The function X_G is *symmetric* because it remains invariant under any permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the variables, i.e.,

$$X_G(x_{\pi(1)}, x_{\pi(2)}, \dots) = X_G(x_1, x_2, \dots).$$

The chromatic symmetric function X_G has been extended to a signed graph Σ with vertex set V. For the signed graph Σ, a coloring κ is called a *proper coloring* using colors from the integers $\mathbb{Z}$ if $\kappa : V \to \mathbb{Z}$ and $\kappa(v) \neq \operatorname{sgn}(e)\kappa(v')$ for every edge $e = vv'$ of Σ, where $\operatorname{sgn}(e)$ denotes the sign of the edge e. Let $\mathcal{P}(\Sigma)$ denote the set of all such proper colorings. The *chromatic B-symmetric function* X_Σ of Σ is defined as

$$X_\Sigma(\dots, x_{-1}, x_0, x_1, \dots) := \sum_{\kappa \in \mathcal{P}(\Sigma)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where $v_1, v_2, \dots, v_d$ are the vertices of Σ. The function X_Σ exhibits the following symmetry property:

$$X_\Sigma(\dots, x_{\pi(-1)}, x_{\pi(0)}, x_{\pi(1)}, \dots) = X_\Sigma(\dots, x_{-1}, x_0, x_1, \dots),$$

for any permutation $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying $\pi(k) = -\pi(-k)$ for all $k \in \mathbb{Z}$. Such a permutation is called a *signed permutation.* The function X_Σ is referred to as *B-symmetric* because the group of all such permutations is isomorphic to the Coxeter group of type B.

The function X_Σ admits a subgraph expansion [5, Theorem 3.8], analogous to that of the chromatic symmetric function of a graph [8, Theorem 2.5]. This expansion requires an extension of the classical power-sum basis $\{p_a \mid a \geqslant 1\}$, where $p_a := \sum_{i \in \mathbb{N}} x_i^a$, to a basis suitable for B-symmetric functions. The set

$$\{p_{\binom{a}{b}} \mid a \geqslant 1,\ b \geqslant 0\} \cup \{x_0\},$$

where $p_{\binom{a}{b}} := \sum_{i \in \mathbb{Z}} x_i^a x_{-i}^b$, forms a basis for an algebra of B-symmetric functions. This set is referred to as the *p-basis.*

For the subgraph expansion of X_Σ, we recall a fundamental result in the theory of signed graphs, *Harary's balance theorem*:

Theorem 11 ([3], Theorem 3)**.** *A signed graph Σ is balanced if and only if its vertex set V can be partitioned into two disjoint subsets $A \sqcup B$ such that every positive edge has both endpoints in the same subset (either A or B), and every negative edge has one endpoint in A and the other in B.*

Let S be a spanning subgraph of Σ with r balanced components, and let $U \subseteq V$ denote the set of vertices contained in the unbalanced components of S. By Harary's balance theorem, the vertex set of each balanced component can be partitioned into two disjoint subsets A_j and B_j for $1 \leqslant j \leqslant r$. Hence, the entire vertex set V can be decomposed as

$$V = U \sqcup A_1 \sqcup B_1 \sqcup \cdots \sqcup A_r \sqcup B_r.$$

Let $a_j = |A_j|$ and $b_j = |B_j|$ for $1 \leqslant j \leqslant r$, so that

$$|U| + \sum_{j=1}^{r}(a_j + b_j) = |V|. \tag{9}$$

We define the *type*, denoted $\lambda(S)$, of the subgraph S as the two-line array:

$$\lambda(S) = \begin{pmatrix} |U| & a_1 & \dots & a_r \\ 0 & b_1 & \dots & b_r \end{pmatrix}.$$

Now, define the function $p_{\lambda(S)}$ by

$$p_{\lambda(S)} = x_0^{|U|} p_{\binom{a_1}{b_1}} \cdots p_{\binom{a_r}{b_r}}.$$

Theorem 12 ([5], Theorem 3.8)**.** *For a signed graph Σ, the following identity holds:*

$$X_\Sigma = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}. \tag{10}$$

Proof. The proof can be found in [5, Theorem 3.8]. □

Example 13. Let Σ be the signed graph presented in Example 2. By Theorem 12, the chromatic B-symmetric function X_Σ is computed as

$$X_\Sigma = p^3_{\binom{1}{0}} - p_{\binom{1}{0}} p_{\binom{1}{1}} - 2p_{\binom{1}{0}} p_{\binom{2}{0}} + 2p_{\binom{2}{1}} + p_{\binom{3}{0}} - x_0^3.$$

Figure 9 offers a detailed illustration of this computation.

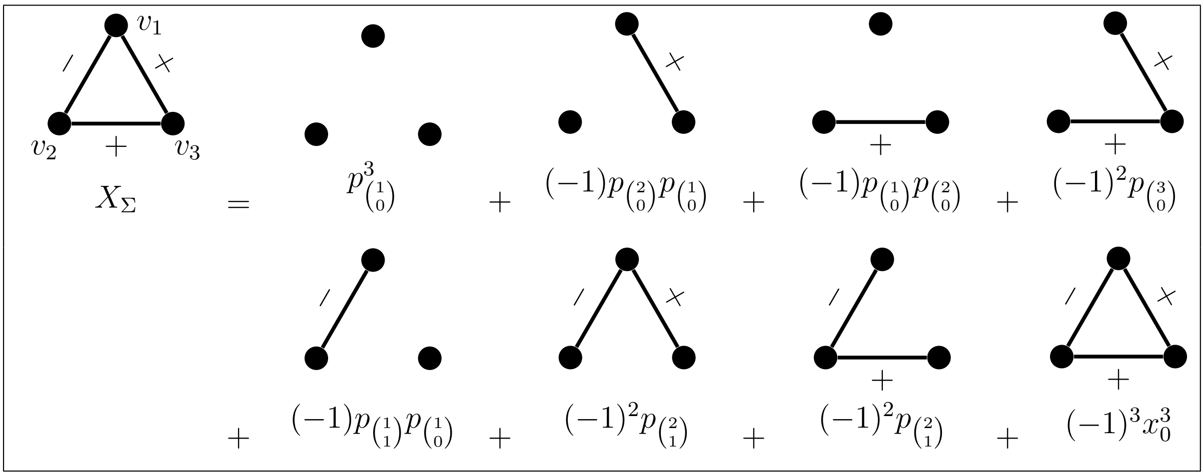


Figure 9: A subgraph expansion of the chromatic B-symmetric function X_Σ.

Example 14. The signed graph Σ_1 from Example 3 has the chromatic B-symmetric function

$$X_{\Sigma_1} = p^3_{\binom{1}{0}} - 2p_{\binom{1}{0}} p_{\binom{1}{1}} - p_{\binom{1}{0}} p_{\binom{2}{0}} + 2p_{\binom{2}{1}},$$

using similar computations as in the previous example. See Figure 10 for a detailed derivation.

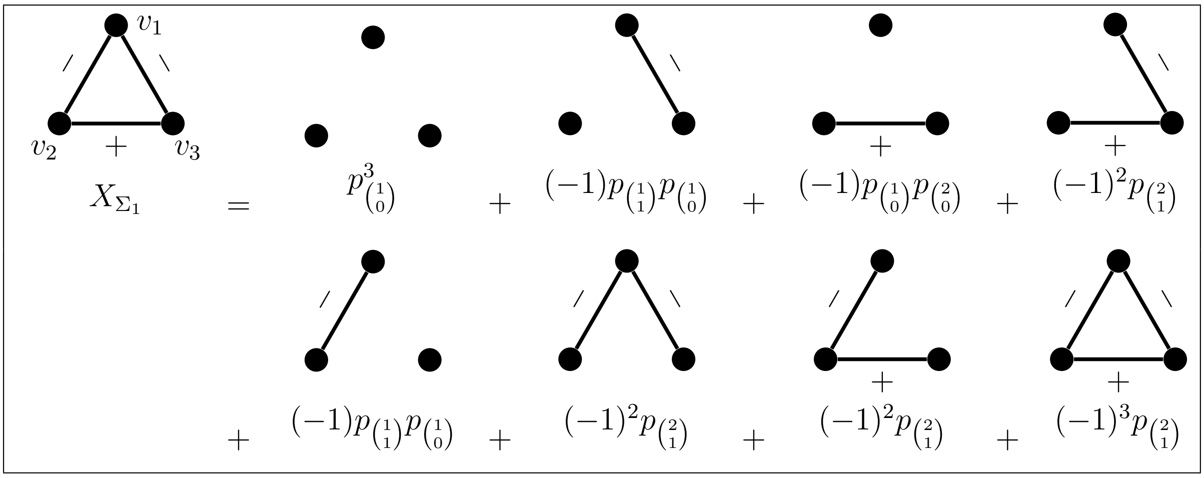


Figure 10: A subgraph expansion of the chromatic B-symmetric function X_{Σ_1}.

6 Stanley's sink theorem for signed graphs

We present an alternative proof of Stanley's sink theorem for signed graphs [1, Theorem 4] as a signed graph analogue of [8, Theorem 3.3] and [4, Theorem 4.5]. Our approach utilizes our acyclic orientation polynomials, thereby avoiding the use of a signed version of the theory of quasi-symmetric functions and P-partitions.

To state the theorem, we first introduce the concept of the *augmented elementary B-symmetric basis* [1, Section 3]. The elementary B-symmetric functions e_n are defined as

$$e_n := \sum_{i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

Additionally, define $q_{\binom{a}{b}} := (-1)^{a+b+1}p_{\binom{a}{b}}$ and let $z := -x_0$. With these definitions, the set

$$\{e_n \mid n \geqslant 1\} \cup \{q_{\binom{a}{b}} \mid a, b \geqslant 1\} \cup \{z\} \tag{11}$$

forms a basis for the algebra of B-symmetric functions, which is referred to as the *augmented elementary B-symmetric basis* [1, Definition 3].

Theorem 15. *[1, Theorem 4] Let X_Σ be the chromatic B-symmetric function of a signed graph Σ, expressed in the augmented elementary B-symmetric basis. Then the number of acyclic orientations of Σ with j sinks is given by the sum of the coefficients of terms involving j elementary symmetric function factors.*

Before presenting the proof, we first illustrate the theorem with an example. Recall that Newton's identities provide a recursive expression for the power-sum symmetric polynomial $p_{\binom{n}{0}}$ using the elementary symmetric polynomials e_i (see, e.g., [6, (2.11')]):

$$p_{\binom{n}{0}} = (-1)^{n-1}ne_n + \sum_{i=1}^{n-1}(-1)^{i-1}e_i \cdot p_{\binom{n-i}{0}}.$$

Example 16. Let Σ be the signed graph given in Example 2. Applying the equation above along with Newton's identities, the function X_Σ in terms of the augmented elementary B-symmetric basis can be written as

$$\begin{aligned} X_\Sigma &= p^3_{\binom{1}{0}} - p_{\binom{1}{0}}p_{\binom{1}{1}} - 2p_{\binom{1}{0}}p_{\binom{2}{0}} + 2p_{\binom{2}{1}} + p_{\binom{3}{0}} - x_0^3 \\ &= \big(e_1e_2\big) + \big(3e_3 + q_{\binom{1}{1}}e_1\big) + \big(2q_{\binom{2}{1}} + z^3\big). \end{aligned}$$

Based on this expansion, Theorem 15 tells us that Σ has:

- 1 acyclic orientation with 2 sinks (from e_1e_2),
- 4 acyclic orientations with 1 sink (combining $3e_3$ and $q_{\binom{1}{1}}e_1$),
- 3 acyclic orientations with 0 sinks (from $2q_{\binom{2}{1}} + z^3$),

yielding a total of $1 + 4 + 3 = 8$ acyclic orientations. The acyclic orientation polynomial of Σ is therefore

$$a_\Sigma(t) = t^2 + 4t + 3,$$

which agrees with Eq. (2).

Example 17. For the signed graph Σ_1 given in Example 3, we similarly compute X_{Σ_1} as

$$\begin{aligned} X_{\Sigma_1} &= p^3_{\binom{1}{0}} - 2p_{\binom{1}{0}}p_{\binom{1}{1}} - p_{\binom{1}{0}}p_{\binom{2}{0}} + 2p_{\binom{2}{1}} \\ &= 2e_1e_2 + 2q_{\binom{1}{1}}e_1 + 2q_{\binom{2}{1}}. \end{aligned}$$

According to Theorem 15, this expansion shows that Σ_1 has:

- 2 acyclic orientations with 2 sinks (from $2e_1e_2$),
- 2 acyclic orientations with 1 sink (from $2q_{\binom{1}{1}}e_1$),
- 2 acyclic orientations with 0 sinks (from $2q_{\binom{2}{1}}$),

yielding a total of $2 + 2 + 2 = 6$ acyclic orientations. Hence, the acyclic orientation polynomial of Σ_1 is

$$a_{\Sigma_1}(t) = 2t^2 + 2t + 2,$$

as in Eq. (3).

With the augmented elementary B-symmetric basis, every B-symmetric function f can be expressed as a linear combination of $e_{\lambda,\mu,u}$:

$$f = \sum_{\lambda,u,\mu} c_{\lambda,\mu,u}\, e_{\lambda,\mu,u} \text{ with } c_{\lambda,\mu,u} \in \mathbb{Q},$$

Here, each basis element $e_{\lambda,\mu,u}$ is defined as follows. First, given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell},$$

Next, let μ be a two-line array of the form

$$\begin{pmatrix} a_1 & \ldots & a_r \\ b_1 & \ldots & b_r \end{pmatrix},$$

and define q_μ as the product

$$q_\mu = q_{\binom{a_1}{b_1}} \cdots q_{\binom{a_r}{b_r}}.$$

Finally, for an integer u, we define $e_{\lambda,\mu,u}$ as the product

$$e_{\lambda,\mu,u} = e_\lambda \, q_\mu \, z^u.$$

Let Λ_B denote the $\mathbb{Q}$-algebra of B-symmetric functions with $\mathbb{Q}$-coefficients. Define an algebra homomorphism $\varphi : \Lambda_B \to \mathbb{Q}[t]$, where $\mathbb{Q}[t]$ is the polynomial ring in the indeterminate t. This homomorphism is uniquely determined by its values on the basis elements given in Eq. (11) as follows:

- $\varphi(e_n) = t$ for all $n > 0$,
- $\varphi(q_{\binom{a}{b}}) = 1$ for all $a, b > 0$, and
- $\varphi(z) = 1$.

Due to the multiplicativity of φ, we have

$$\varphi(e_\lambda) = t^{\ell(\lambda)}, \tag{12}$$

where $\ell(\lambda)$ denotes the length of the partition λ. In addition,

$$\varphi(x_0) = -1, \text{ and} \tag{13}$$

$$\varphi(p_{\binom{a}{b}}) = (-1)^{a+b-1} \quad \text{for all } a, b \geqslant 1. \tag{14}$$

Proposition 18. *For the homomorphism φ, the image $\varphi(p_{\binom{n}{0}})$ is given by*

$$\varphi(p_{\binom{n}{0}}) = (-1)^{n-1}(1 - (1-t)^n). \tag{15}$$

Proof. The proof proceeds by induction. The base case $n = 1$ holds because $p_{\binom{1}{0}} = e_1$ and $\varphi(e_1) = t$. For $n > 1$, using Newton's identities, we get

$$\begin{aligned}
\varphi(p_{\binom{n}{0}}) &= (-1)^{n-1}\varphi(n \cdot e_n) + \sum_{i=1}^{n-1} (-1)^{i-1} \varphi(e_i) \cdot \varphi(p_{\binom{n-i}{0}}) \\
&= (-1)^{n-1}(nt) + \sum_{i=1}^{n-1} (-1)^{n-2} t \cdot \left(1 - (1-t)^{n-i}\right) \\
&= (-1)^{n-1} t \left(n - \sum_{i=1}^{n-1} \left(1 - (1-t)^{n-i}\right) \right) \\
&= (-1)^{n-1} \left(1 - (1-t)^n\right).
\end{aligned}$$

□

We are now ready to prove the signed graph analogue of Stanley's sink theorem.

Proof of Theorem 15. We begin by expressing the chromatic B-symmetric function X_Σ of the signed graph Σ in terms of the augmented elementary B-symmetric basis:

$$X_\Sigma = \sum_{\lambda,\mu,u} c_{\lambda,\mu,u} e_{\lambda,\mu,u}.$$

Applying the homomorphism φ to this equation, and using Eq. (12) together with the definition of φ, we obtain

$$\varphi(X_\Sigma) = \varphi(\sum_{\lambda,\mu,u} c_{\lambda,\mu,u} e_{\lambda,\mu,u}) = \sum_{\lambda,\mu,u} c_{\lambda,\mu,u} \varphi(e_\lambda)\varphi(q_\mu)\varphi(z^u) = \sum_{\lambda,\mu,u} c_{\lambda,\mu,u} t^{\ell(\lambda)}. \tag{16}$$

Now, consider a subset $S \subseteq E$ with type $\lambda(S)$ given by

$$\lambda(S) = \begin{pmatrix} |U| & a_1 & \dots & a_r \\ 0 & b_1 & \dots & b_r \end{pmatrix}.$$

According to Eq. (13), we have $\varphi(x_0^{|U|}) = (-1)^{|U|}$. Let C be the i-th balanced component of S. If C is positive, then $b_i = 0$ and hence, from Eq. (15),

$$\varphi\left(p_{\binom{a_i}{b_i}}\right) = (-1)^{a_i-1}(1-(1-t)^{a_i}) = (-1)^{|V(C)|-1} t_C,$$

and if C is negative, then $a_i, b_i \geqslant 1$ and hence by Eq. (14),

$$\varphi\left(p_{\binom{a_i}{b_i}}\right) = (-1)^{a_i+b_i-1} = (-1)^{|V(C)|+1} t_C.$$

Therefore,

$$\begin{aligned} \varphi(p_{\lambda(S)}) &= \varphi\left(x_0^{|U|} \prod_{1 \leqslant i \leqslant r} p_{\binom{a_i}{b_i}}\right) \\ &= (-1)^{|U|}\left(\prod_{C \in \mathcal{C}_b(S)} (-1)^{|V(C)|-1} t_C\right) \\ &= (-1)^{d-|\mathcal{C}_b(S)|} \prod_{C \in \mathcal{C}_b(S)} t_C, \end{aligned}$$

where the first equality follows from the definition of $p_{\lambda(S)}$, the second equality follows from the multiplicativity of φ, and the last follows from Eq. (9).

Thus, by Theorem 12, the definition $s_b(S) = |S| - d + |\mathcal{C}_b(S)|$ (Eq. (7)), and Eq. (8) in Theorem 8, we obtain

$$\varphi(X_\Sigma) = \varphi\left(\sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}\right) = \sum_{S \subseteq E} (-1)^{s_b(S)} \prod_{C \in \mathcal{C}_b(S)} t_C = a_\Sigma(t).$$

Finally, combining this with Eq. (16), we conclude that

$$a_\Sigma(t) = \sum_{\lambda,\mu,u} c_{\lambda,\mu,u}\, t^{\ell(\lambda)},$$

which completes the proof of the theorem. □

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