Suns in Triangle-Free Graphs of Large Chromatic Number

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Abstract

For an integer $t \geqslant 4$, a *t-sun* is a graph obtained from a t-vertex cycle C by adding a degree-one neighbor for each vertex of C. Trotignon asked whether every triangle-free graph of sufficiently large chromatic number has an induced subgraph that is a t-sun for some $t \geqslant 4$. This remains open, but we show that every triangle-free graph of chromatic number at least 48 has an induced subgraph that is either a t-sun for some $t \geqslant 5$, or a 4-sun with a single degree-one vertex deleted. In fact, we prove that for all $\ell \geqslant 5$, there exists $c = c(\ell) \in \mathbb{N}$ such that every triangle-free graph of chromatic number at least c has an induced subgraph that is either a t-sun for some $t \geqslant \ell$, or a 4-sun with a single degree-one vertex deleted.

Mathematics Subject Classifications: 05C15, 05C75

1 Introduction

The set of all positive integers is denoted by $\mathbb{N}$, and for integers k, k', we denote by $\{k, \dots, k'\}$ the set of all integers no smaller than k and no larger than k' (thus, $\{k, \dots, k'\} = \varnothing$ if and only if $k' < k$). Graphs in this paper have finite vertex sets, no loops, and no parallel edges. We use the same notation for induced subgraph and their vertex sets; in particular, for an induced subgraph H of a graph G, we also use H to denote the vertex set of H. The chromatic number and the clique number of a graph G are denoted, in order, by $\chi(G)$ and $\omega(G)$. Recall that a graph G is *triangle-free* if $\omega(G) \leqslant 2$.

Let $t \geqslant 4$ be an integer. A *t-sun* is the graph obtained from a t-vertex cycle by adding a degree-one neighbor for each vertex of the cycle. A graph G is *t-sun-free* if no induced

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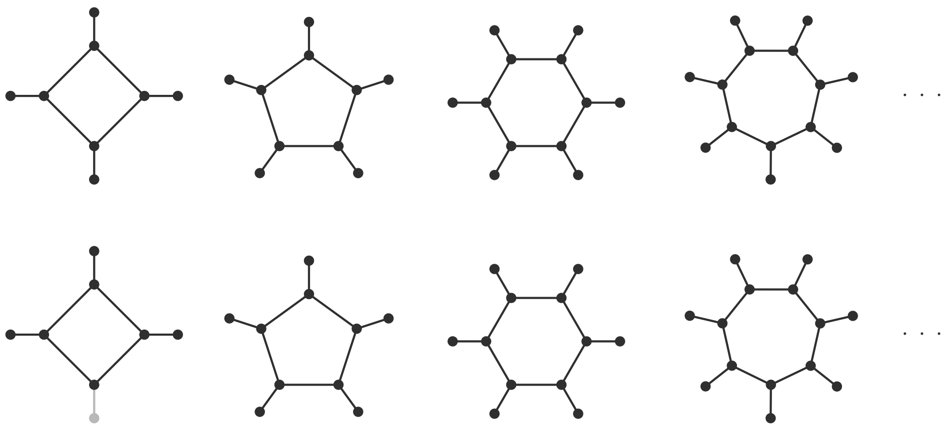

Figure 1: Suns (top) versus what we exclude in Theorem 2 (bottom).

subgraph of G is a t-sun, and G is *sun-free* if G is t-sun-free for all $t \geqslant 4$. Recently, Trotignon [5] proposed the following beautiful problem:

Problem 1 (Trotignon [5])**.** Is $\chi(G)$ bounded for every triangle-free sun-free graph G?

This remains open, but we come pretty close to a solution. Given an integer $t \geqslant 4$, a t-*sunspot* is the graph obtained from a t-sun by removing a single degree-one vertex. A graph G is t-*sunspot-free* if no induced subgraph of G is a t-sunspot (so, being t-sunspot-free implies being t-sun-free). We prove the following (see Figure 1):

Theorem 2. *Let G be a graph that is triangle-free, 4-sunspot-free, and t-sun-free for all $t \geqslant 5$. Then $\chi(G) \leqslant 47$.*

In particular, we will only work with 4-sunspots, so let us customize the notation and terminology in this specific case. Given a graph G, by a 4-*sunspot in G* we mean a 7-tuple $(x_1, x_2, x_3, x_4; y_1, y_2, y_3)$ of pairwise distinct vertices of G such that

$$E(G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}]) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1y_1, x_2y_2, x_3y_3\}.$$

It follows that G is 4-sunspot-free if and only if there is no 4-sunspot in G.

As pointed out in [5], it is not difficult check that "shift graphs" (see [4]) provide a construction of triangle-free graphs with arbitrarily large chromatic number and no induced subgraph that is a t-sun for any $t \geqslant 5$ (we omit the details). This means that excluding 4-suns is essential for the answer to Problem 1 to be affirmative. However, provided the 4-suns are excluded, it could be that the answer to Problem 1 is "yes" even if in addition we only exclude t-suns for all $t \geqslant \ell$, where $\ell \geqslant 5$ is a prescribed constant. At least the corresponding extension of Theorem 2 is true, and that is the main result of this paper:

Theorem 3. *For every integer $\ell \geqslant 6$, there is a constant $c_3 = c_3(\ell) \in \mathbb{N}$ such that $\chi(G) \leqslant c_3$ for every graph G that is triangle-free, 4-sunspot-free, and t-sun-free for all $t \geqslant \ell$. Moreover, $c_3(6) = 47$ works.*

The proof of Theorem 3 will be completed in the last section.

We conclude the introduction by discussing another problem of Trotignon [5] about extending Problem 1 to graphs of bounded clique number (for a general bound, instead of just triangle-free graphs). A graph class $\mathcal{C}$ is *χ-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ for every graph $G \in \mathcal{C}$. A *net* is the graph obtained from a 3-vertex cycle by adding a degree-one neighbor for each vertex of the cycle, and a graph G is *net-free* if no induced subgraph of G is a net.

Problem 4 (Trotignon [5])**.** Is the class of all net-free sun-free graphs χ-bounded?

(A remark: In [5], a net is called a "3-sun," and a net-free sun-free graph is said to be "sun-free." But we prefer our terminology, especially because the name net is pretty standard. Another reason is that we are not sure why the net needs to be excluded in Problem 4; it could be true that – again, in our terminology – the class of all sun-free graphs is already χ-bounded.) It turns out that a partial answer to Problem 4, analogous to Theorem 3, can in fact be derived from Theorem 3. A *bull* is the graph obtained from a net by removing a degree-one vertex, and a graph G is *bull-free* if no induced subgraph of G is a bull. The following was recently proved in [2] (the logarithm is base 2):

Theorem 5 (Hajebi [2])**.** *Let $t \in \mathbb{N}$ and let G be a bull-free graph in which every triangle-free induced subgraph has chromatic number at most t. Then we have $\chi(G) \leqslant \omega(G)^{4\log t+13}$.*

Since $\log 43 < 6$, from Theorems 3 and 5, it follows that:

Theorem 6. *Let G be a graph that is bull-free, 4-sunspot-free, and t-sun-free for all $t \geqslant 5$. Then $\chi(G) \leqslant \omega(G)^{37}$.*

In fact, for every integer $\ell \geqslant 6$, there exists $d = d(\ell) \in \mathbb{N}$ such that $\chi(G) \leqslant \omega(G)^d$ for every graph G that is bull-free, 4-sunspot-free, and t-sun-free for all $t \geqslant \ell$.

2 Flaps

Let G be a graph. For $x \in V(G)$ and $Y \subseteq V(G)$, we denote by $N_Y(x)$ the set of all vertices in Y that are adjacent to x in G. A *hole in G* is an induced subgraph H of G that is a cycle on four or more vertices. The *length* of H is the number of edges of H; if H has length $\ell \geqslant 4$, then we also say that H is an *ℓ-hole in G*, and we write $H = h_1\text{-}\cdots h_\ell\text{-}h_1$ to mean $V(H) = \{h_1, \ldots, h_\ell\}$ and $E(H) = \{h_i h_{i+1} : i \in \{1, \ldots, \ell-1\}\} \cup \{h_\ell h_1\}$. An *$H$-flap in G* is a 4-hole in G that shares at least one edge with H. We say that a graph G is

- *non-degenerate* if every vertex in G has degree at least $\chi(G) - 1$;
- *liberal* for all distinct and non-adjacent $x, y \in V(G)$, neither x nor y "dominates" the other; that is, we have $N_G(x) \setminus N_G(y) \neq \varnothing$ and $N_G(y) \setminus N_G(x) \neq \varnothing$; and
- *flapless* if for every hole H of length at least 6 in G, there is no H-flap in G.

Our goal in this section is to prove the following:

Theorem 7. *Let $c \in \mathbb{N}$ and let G be a triangle-free 4-sunspot-free graph such that $\chi(G) > 2c + 1$. Then G has an induced subgraph L with $\chi(L) = c + 1$ that is non-degenerate, liberal, and flapless.*

The proof is in several steps, starting with an observation (we omit the proof):

Observation 8. *Let $c \in \mathbb{N}$ and let G be a graph such that $\chi(G) > c$ and $\chi(G') \leqslant c$ for every induced subgraph G' of G other than G itself. Then $\chi(G) = c + 1$, and G is both non-degenerate and liberal.*

Let $r \in \mathbb{N} \cup \{0\}$ and let G be a graph. An *r-leveling in G* is an $(r+1)$-tuple $(L_0, \dots, L_r)$ of pairwise disjoint non-empty subsets of $V(G)$ such that:

- for every $i \in \{1, \dots, r\}$, every vertex in L_i has a neighbor in L_{i-1};
- for all distinct $i, j \in \{0, \dots, r\}$, if there is an edge in G with an end in L_i and an end in L_j, then $|i - j| = 1$.

We prove that:

Lemma 9. *Let $c \in \mathbb{N}$ and let G be a triangle-free graph with $\chi(G) > 2c + 1$. Then there is an r-leveling $(L_0, \dots, L_r)$ in G for some integer $r \geqslant 3$ such that $\chi(L_r) = c + 1$ and L_r is both non-degenerate and liberal.*

Proof. Let K be a connected component of G with $\chi(K) = \chi(G) > 2c + 1$. Let $x_0 \in K$ and let K_1 be a component of $G \setminus (N_K(x_0) \cup \{x_0\})$ with maximum chromatic number. Since $\chi(K) > 2c+1$ and $N_K(x_0)$ is a stable set in G (because G is triangle-free), it follows that $\chi(K_1) > 2c$. Since K is connected, it follows that there is a vertex $x_1 \in N_K(x_0)$ such that $K_1 \cup \{x_1\}$ is connected. For each $r \in \mathbb{N}$, let M_r be the set of all vertices of K_1 that are at distance exactly r from x_1 in the graph $K_1 \cup \{x_1\}$. Note that $M_1 = N_{K_1}(x_1)$ (and $M_r = \varnothing$ for sufficiently large r). It follows that $(M_1, \dots, M_r)$ is an $(r-1)$-leveling in K_1 (and so in G) for every $r \in \mathbb{N}$. Also, since $K_1 \cup \{x_1\}$ is connected, $K_1 = \bigcup_{r \in \mathbb{N}} M_r$ and by the second property in the definition of a leveling, $\chi(K_1) \leqslant 2 \max_{r \in \mathbb{N}} \chi(M_r)$. Note that $\chi(M_1) = 1$ (because G is triangle-free). We deduce that there exists $r_1 \in \mathbb{N}$ with $r_1 \geqslant 2$ such that $\chi(M_{r_1}) \geqslant \chi(K_1)/2 > c \geqslant 1$. Let K_2 be an induced subgraph of M_{r_1} with $V(K_2)$ minimal such that $\chi(K_2) > c$. By Observation 8, we have:

(1) $\chi(K_2) = c + 1$ *and K_2 is both non-degenerate and liberal.*

Now, for each $i \in \{0, \dots, r_1 + 1\}$, define L_i as follows: let $L_0 = \{x_0\}$, let $L_1 = \{x_1\}$, $L_i = M_{i-1}$ for every $i \in \{2, \dots, r_1\}$, and let $L_{r_1+1} = K_2 \subseteq M_{r_1}$. Then $(L_0, \dots, L_{r_1+1})$ is an $(r_1 + 1)$-leveling; in particular, we have $r_1 + 1 \geqslant 3$ since $r_1 \geqslant 2$. Furthermore, by (1), we have $\chi(L_{r_1+1}) = \chi(K_2) = c + 1$, and $L_{r_1+1} = K_2$ is both non-degenerate and liberal. This completes the proof of Lemma 9. □

Let G be a graph. A *path in* G is an induced subgraph P of G that is a path. We write $P = p_1\text{-}\cdots\text{-}p_\ell$, for $\ell \in \mathbb{N}$, to mean $V(P) = \{p_1, \ldots, p_\ell\}$ and $E(P) = \{p_i p_{i+1} : i \in \{1, \ldots, \ell - 1\}\}$. We call p_1, p_ℓ the *ends* of P, and the set $P \setminus \{p_1, p_\ell\}$ the *interior* of P. The *length* of P is the number of edges of P. Given a hole H in G and a vertex $x \in G \setminus H$, an *x-sector in H* is a path P of non-zero length in H such that x is adjacent in G to the ends of P and x is not adjacent in G to any vertex in the interior of P.

We need the next three lemmas:

Lemma 10. *Let $r \geqslant 2$ be an integer, let G be a triangle-free 4-sunspot-free graph, and let $(L_0, \ldots, L_r)$ be an r-leveling in G. Let H be a hole of length at least 6 in L_r and let $x \in L_{r-1}$. Then there is no x-sector of length at most 2 in H.*

Proof. There is no x-sector of length 1 in H because G is triangle-free. Assume that there is an x-sector of length 2 in H. Since H has length at least 6 and since G is triangle-free, there is a path $P = h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_5$ of length 4 in H such that $N_P(x) = \{h_2, h_4\}$. Since $r \geqslant 2$, it follows that x has a neighbor $y \in L_{r-2}$. But now $(h_2, x, h_4, h_3; h_1, y, h_5)$ is a 4-sunspot in G, a contradiction. This completes the proof of Lemma 10. □

Lemma 11. *Let $r \geqslant 3$ be an integer, let G be a triangle-free 4-sunspot-free graph, and let $(L_0, \ldots, L_r)$ be an r-leveling in G such that L_r is liberal. Let H be a hole of length at least 6 in L_r and let $x \in L_r \setminus H$. Then there is no x-sector of length at most 2 in H.*

Proof. There is no x-sector of length 1 in H because G is triangle-free. Assume that there is an x-sector of length 2 in H. Since H has length at least 6 and since G is triangle-free, there is a path $P = h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_5$ of length 4 in H such that $N_P(x) = \{h_2, h_4\}$. Since $r \geqslant 3$, it follows that h_3 has a neighbor $y \in L_{r-1}$ and y has a neighbor $z \in L_{r-2}$. By Lemma 10, there is no y-sector of length at most 2 in H, and so $N_P(y) = \{h_3\}$. Since $(h_2, h_3, h_4, x; h_1, y, h_5)$ is not a 4-sunspot in G, it follows that $xy \in E(G)$. Since L_r is liberal, there is a vertex $w \in N_{L_r}(x) \setminus N_{L_r}(h_3)$. In particular, we have $w \in L_r \setminus P$. Since G is triangle-free, it follows that $yw \notin E(G)$, and $N_P(w) \subseteq \{h_1, h_5\}$. Also, if $h_1 w \notin E(G)$, then $(h_2, x, y, h_3; h_1, w, z)$ is a 4-sunspot in G, and if $h_5 w \notin E(G)$, then $(h_4, x, y, h_3; h_5, w, z)$ is a 4-sunspot in G, a contradiction. We deduce that $N_P(w) = \{h_1, h_5\}$, and so $H' = w\text{-}h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_5\text{-}w$ is a hole of length 6 in L_r. Since $r \geqslant 3$, it follows that h_1 has a neighbor $y_1 \in L_{r-1}$, y_1 has a neighbor $y_2 \in L_{r-2}$, and y_2 has a neighbor $y_3 \in L_{r-3}$. By Lemma 10 applied to H' and y_1, we have $h_1 \in N_{H'}(y_1) \subseteq \{h_1, h_4\}$. Consequently, since $(h_2, h_1, w, x; h_3, y_1, h_5)$ is not a 4-sunspot in G, it follows that $xy_1 \in E(G)$, and so $yy_1 \notin E(G)$ (because G is triangle-free). Now, if $yy_2 \in E(G)$, then $(y_1, y_2, y, x; h_1, y_3, h_3)$ is a 4-sunspot in G, a contradiction. It follows that $yy_2 \notin E(G)$. But then $(w, x, y_1, h_1; h_5, y, y_2)$ is a 4-sunspot in G, again a contradiction. This completes the proof of Lemma 11. □

Lemma 12. *Let $r \geqslant 3$ be an integer, let G be a triangle-free 4-sunspot-free graph, and let $(L_0, \ldots, L_r)$ be an r-leveling in G such that L_r is liberal. Let H be a hole of length at least 6 in L_r and let $x_1\text{-}h_1\text{-}h_2\text{-}x_2\text{-}x_1$ be an H-flap in G where $h_1, h_2 \in H$ and $x_1, x_2 \in L_{r-1} \cup L_r$. Then $x_1, x_2 \in L_r \setminus H$.*

Proof. Suppose not. Assume that $x_1, x_2 \in L_r$. By symmetry, we may assume that $x_1 \in L_r \setminus H$ and $x_2 \in H$. But now h_1-h_2-x_2 is an x_1-sector of length 2 in H, contrary to Lemma 11. Therefore, one of x_1, x_2 belongs to L_{r-1}; say $x_1 \in L_{r-1}$. Let h_0 be the unique neighbor of h_1 in H other than h_2 and let h_3 be the unique neighbor of h_2 in H other than h_1. Since H has length at least 6, it follows that $P = h_0$-h_1-h_2-h_3 is a path of length 3 in H. By Lemma 10 applied to H, x_1, we have $N_P(x_1) = \{h_1\}$; in particular, $x_2 \neq h_3$ and so $x_2 \notin H$. By Lemma 10 and Lemma 11 applied to H, x_2 (depending on whether $x_2 \in L_{r-1}$ or $x_2 \in L_r \setminus H$), we have $N_P(x_2) = \{h_2\}$. Also, since $r \geqslant 2$, x_1 has a neighbor $y \in L_{r-2}$, and since G is triangle-free, $x_2 y \notin E(G)$. But now $(x_1, h_1, h_2, x_2; y, h_0, h_3)$ is a 4-sunspot in G, a contradiction. This completes the proof of Lemma 12. □

Let us now prove Theorem 7:

Proof of Theorem 7. By Lemma 9, there is an r-leveling $(L_0, \dots, L_r)$ in G for some integer $r \geqslant 3$ such that $\chi(L_r) = c + 1$ and L_r is both non-degenerate and liberal. We claim that $L = L_r$ is the desired induced subgraph of G. To see this, it suffices to show that L_r is flapless. Suppose for a contradiction that there is a hole H in L_r of length at least 6 for which there is an H-flap in L_r. By the definition, this means there is a 4-hole x_1-h_1-h_2-x_2-x_1 in L_r such that $h_1, h_2 \in H$.

By Lemma 12, we have $x_1, x_2 \in L_r \setminus H$. Let h_0 be the unique neighbor of h_1 in H other than h_2 and let h_3 be the unique neighbor of h_2 in H other than h_1. Since H has length at least 6, it follows that $P = h_0$-h_1-h_2-h_3 is a path of length 3 in H. By Lemma 11 applied to H, x_1 and x_2, we have $N_P(x_i) = \{h_i\}$ for $i \in \{1, 2\}$. Since $r \geqslant 3$, it follows that h_1 has a neighbor $y_1 \in L_{r-1}$ and x_2 has a neighbor $y_2 \in L_{r-1}$.

By Lemma 10 applied to H, y_1, we have $N_P(y_1) = \{h_1\}$. Since G is triangle-free, it follows that $x_1 y_1 \notin E(G)$. Also, by Lemma 12, y_1-h_1-h_2-x_2-y_1 is not an H-flap in G. Consequently, we have $x_2 y_1 \notin E(G)$; in particular $y_1 \neq y_2$. We deduce that h_1 is the only neighbor of y_1 in $P \cup \{x_1, x_2\}$. Since G is triangle-free, it follows that $x_1 y_2, h_2 y_2 \notin E(G)$. Also, by Lemma 12, neither y_2-h_1-h_2-x_2-y_2 nor x_2-h_2-h_3-y_2-x_2 is an H-flap in G. Thus, we have $h_1 y_2, h_3 y_2 \notin E(G)$. We deduce that x_2 is the only neighbor of y_2 in $\{h_1, h_2, h_3, x_1, x_2\}$. On the other hand, by Lemma 12, y_2-h_0-h_1-y_1-y_2 is not an H-flap in G, and so there is a vertex $z \in \{h_0, y_1\}$ for which $y_2 z \notin E(G)$. But now $(h_1, h_2, x_2, x_1; z, h_3, y_2)$ is a 4-sunspot in G, a contradiction. This completes the proof of Theorem 7. □

3 Flares

Let G be a graph. We say that $X, Y \subseteq V(G)$ are *anticomplete in* G if $X \cap Y = \varnothing$ and there is no edge in G with an end in X and an end in Y. If $X = \{x\}$, then we say that x is *anticomplete to* Y *in* G to mean X and Y are anticomplete in G.

Given a hole H in G, an *H-flare in G* is a map $\Phi : H \to 2^{G \setminus H}$ such that for every $h \in H$, we have $\Phi(h) \subseteq N_G(h) \setminus H$ with $|\Phi(h)| \leqslant 1$. In this case, we write $\Phi[h] = \Phi(h) \cup \{h\}$. The *order* of Φ is the number of vertices $h \in H$ with $|\Phi(h)| = 1$, and we say that Φ is *full* if the order of Φ is $|H|$. For $d \in \mathbb{N}$, we say that Φ is *d-safe* if for all vertices $h, h' \in H$ that are at distance at most d in H, the sets $\Phi(h)$ and $\Phi[h']$ are anticomplete in G.

In this section, we show that:

Theorem 13. *Let $d \in \mathbb{N}$ and let G be a triangle-free 4-sunspot-free graph of minimum degree at least $4d-1$ that is both liberal and flapless. Let H be a hole of length at least 6 in G. Then there is a full d-safe H-flare in G.*

First, we need a lemma:

Lemma 14. *Let G be a triangle-free 4-sunspot-free graph that is both liberal and flapless. Let H be a hole of length at least 6 in G and let Φ be an H-flare in G. Let $h, h' \in H$ be distinct and let $x \in \Phi[h]$. Then*

- *if h, h' are adjacent, then x, h' have no common neighbor in $G \setminus H$; and*
- *if h, h' are non-adjacent, then x, h' have at most one common neighbor in $G \setminus H$.*

Proof. Suppose not. Assume that h, h' are adjacent and x, h' have a common neighbor $x' \in G \setminus H$. Since $\{h, h', x'\}$ is not a triangle in G, it follows that $x \neq h$. But now x-h-h'-x'-x is an H-flap in G, a contradiction.

So, we may assume that h, h' are non-adjacent and h', x have two common neighbors $z, z' \in G \setminus H$. Since G is triangle-free, it follows that x-z-h'-z'-x is a 4-hole in G. We claim that:

(2) *There are vertices $y, y' \in H$ such that x, y, h', y' are all distinct, $xy, h'y' \in E(G)$, and $\{x, y\}$ and $\{h', y'\}$ are anticomplete in G.*

Since H has length at least 6, there are vertices $v, v' \in H$ such that h, v, h', v' are all distinct, $hv, h'v' \in E(G)$, and $\{h, v\}$ and $\{h', v'\}$ are anticomplete in G. Thus, if $x = h$, then (2) follows by choosing $y = v$ and $y' = v'$. Assume that $x \neq h$; thus, $x \in \Phi(h)$. In this case, let $y = h$ and let $y' = v'$. Then x, y, h', y' are all distinct (because $h, h', v' \in H$ are distinct, and $x \notin H$), $xy = xh \in E(G)$ (because $x \in \Phi(h)$), and $h'y' = h'v' \in E(G)$. Also, $xh' \notin E(G)$ (recall that x-z-h'-z'-x is a 4-hole in G), and $xy' \notin E(G)$ since otherwise z-h'-y'-x-z is an H-flap in G. Moreover, $y = h$ is anticomplete to $\{h', y'\} = \{h', v'\}$ in G. Hence, $\{x, y\}$ and $\{h', y'\}$ are anticomplete in G. This proves (2).

Let $y, y' \in H$ be as in (2). Since G is triangle-free, it follows that $\{y, y'\}$ and $\{z, z'\}$ are anticomplete in G. Since G is liberal and z, z' are distinct and non-adjacent, we may choose a vertex $w \in N_G(z) \setminus N_G(z')$ and a vertex $w' \in N_G(z') \setminus N_G(z)$. Since x, h' are common neighbors of z, z', it follows that $\{x, h'\} \cap \{w, w'\} = \varnothing$. In fact, since G is triangle-free, $\{x, h'\}$ and $\{w, w'\}$ are anticomplete in G, and $\{y, y'\} \cap \{w, w'\} = \varnothing$. Also, since z-h'-y'-w-z is not an H-flap, it follows that $wy' \notin E(G)$, and since z'-h'-y'-w'-z' is not an H-flap, it follows that $w'y' \notin E(G)$.

Now, since $(z, h', z', x; w, y', w')$ is not a 4-sunspot in G, we have $ww' \in E(G)$. Since G is triangle-free, we have $\{wy, w'y\} \not\subseteq E(G)$; say $wy \notin E(G)$. But then $(x, z, h', z'; y, w, y')$ is a 4-sunspot in G, a contradiction. This completes the proof of Lemma 14. $\square$

Theorem 13 is now almost immediate:

Proof of Theorem 13. Suppose not. Let Φ be a d-safe H-flare in G of maximum order. Since Φ is not full, there exists $h \in H$ such that $\Phi(h) = \varnothing$. Let D be the set of all vertices in $H \setminus \{h\}$ that are at distance at most d from h in H and let $U = \bigcup_{u \in D} \Phi[u]$; thus, we have $|D| \leqslant 2d$ and $|U| \leqslant 4d$. Since Φ is d-safe, it follows that $N_U(h) = N_H(h)$. Also, by Lemma 14, h has at most $4(d-1)$ neighbors in $G \setminus (H \cup U)$ with a neighbor in U. Since the degree of h in G is at least $4d - 1 > 4(d-1) + 2$, it follows that h has a neighbor x in $G \setminus H$ which is anticomplete to U in G. Define the H-flare $\Phi' : H \to 2^{G \setminus H}$ as follows: let $\Phi'(h) = \{x\}$, and let $\Phi'(h') = \Phi(h')$ for all $h' \in H \setminus \{h\}$. Since Φ is d-safe and x is anticomplete to U in G, it follows that Φ' is also d-safe. But the order of Φ' is strictly larger than the order of Φ, a contradiction. This completes the proof of Theorem 13. □

4 Part assembly

Here we complete the proof of Theorem 3. We need a couple of results from the literature, beginning with the following:

Theorem 15 (Chudnovsky, Scott, Seymour; see 1.4 in [1])**.** *For all $k, \ell \in \mathbb{N}$, there is a constant $c_{15} = c_{15}(k, \ell) \in \mathbb{N}$ such that for every graph G with $\omega(G) \leqslant k$ and $\chi(G) > c_{15}$, there is a hole of length at least ℓ in G.*

We also need two results from [3]. Let G be a graph. For a vertex $v \in V(G)$, we denote by $N^2_G(v)$ the set of all vertices that are at distance exactly two from v in G.

Theorem 16 (Scott and Seymour; see 3.1 in [3] and its proof)**.** *Let G be a triangle-free graph such that there is no 6-hole in G. Then we have $\chi(N^2_G(v)) \leqslant 2$ for every $v \in V(G)$.*

Theorem 17 (Scott and Seymour; see 3.10 in [3])**.** *Let $\kappa \in \mathbb{N}$ and let $\ell \geqslant 4$ be an integer. Let G be a graph with no hole of length more than ℓ such that $\chi(N_G(v)), \chi(N^2_G(v)) \leqslant \kappa$ for every vertex v of G. Then $\chi(G) \leqslant (2\ell - 2)\kappa$.*

Combining these three results yields the following:

Corollary 18. *For every integer $\ell \geqslant 6$, there exists $c_{18} = c_{18}(\ell) \in \mathbb{N}$ such that for every triangle-free G with $\chi(G) > c_{18}$, there is a hole of length at least ℓ in G. Moreover, $c_{18}(6) = 20$ works.*

Proof. The existence of c_{18} follows from Theorem 15 applied to $k = 2$ and ℓ. Also, by Theorems 16 and 17, we have

$$\chi(G) \leqslant 2(2 \times 6 - 2) = 20$$

for every triangle-free graph G with no hole of length at least 6. □

We are now ready to prove our main result, which we restate:

Theorem 3. *For every integer $\ell \geqslant 6$, there is a constant $c_3 = c_3(\ell) \in \mathbb{N}$ such that $\chi(G) \leqslant c_3$ for every graph G that is triangle-free, 4-sunspot-free, and t-sun-free for all $t \geqslant \ell$. Moreover, $c_3(6) = 47$ works.*

Proof. Let $\tau = c_{18}(\ell)$ be as in Corollary 18, where $c_{18}(6) = 20$. Let

$$c_3 = c_3(\ell) = 2\max\{\tau, 4\ell - 1\} + 1.$$

Note, in particular, that $c_3(6) = 47$. We will show that the above value of c_3 satisfies the theorem.

Suppose for a contradiction that there is a graph G with $\chi(G) > c_3$ which is triangle-free, 4-sunspot-free, and t-sun-free for all $t \geqslant \ell$. By Theorem 7, G has a non-degenerate, liberal, and flapless induced subgraph L with $\chi(L) = \max\{\tau, 4\ell - 1\} + 1$. Since $\chi(L) > \tau$, by Corollary 18, there is a hole of length at least $\ell \geqslant 6$ in L. Also, since L is non-degenerate and $\chi(L) > 4\ell - 1$, it follows that L has minimum degree at least $4\ell - 1$.

Let H be a shortest hole of length at least ℓ in L; say H has length $t \geqslant \ell$. Since L is both liberal and flapless, and the minimum degree of L is at least $4\ell - 1$, it follows from Theorem 13 that there is a full ℓ-safe H-flare Φ in L. For every $h \in H$, let $\Phi(h) = \{x_h\}$. We further claim that:

(3) *Let $h, h' \in H$ be distinct. Then x_h is anticomplete to $\{h', x_{h'}\}$ in L.*

Suppose not. Let $h, h' \in H$ be distinct such that x_h is not anticomplete to $\{h', x_{h'}\}$ in L, and subject to this property, the distance in H between h, h' is minimum. Let P_1 be a shortest path in H from h to h', and let P_2 be the other path in H from h to h'. Since Φ is ℓ-safe, it follows that both P_1 and P_2 have length at least $\ell + 1$. From the choice of h, h', it follows that x_h is anticomplete to $P_1 \setminus \{h'\}$ and $x_{h'}$ is anticomplete to $P_1 \setminus \{h\}$. Also, since x_h is not anticomplete to $\{h', x_{h'}\}$, it follows that there is a path Q in L from h to h' such that the interior of Q is contained in $\{x_h, x_{h'}\}$; in particular, Q has length 2 or 3. Consequently, $H' = h\text{-}P_1\text{-}h'\text{-}Q\text{-}h$ is a hole of length at least $\ell + 3$ in L. On the other hand, since P_2 has length at least $\ell + 1 \geqslant 7$ and Q has length at most 3, it follows that P_2 is strictly longer than Q. But then H' is a hole of length at least ℓ in L that is strictly shorter than H, a contradiction. This proves (3).

From (3), it follows that the induced subgraph of L with vertex set $H \cup \{x_h : h \in H\}$ is a t-sun, a contradiction because $t \geqslant \ell$. This completes the proof of Theorem 3. □

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