Monochromatic partitions in 2-edge-coloured bipartite graphs

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Abstract

We study two variations of the Gyárfás–Lehel conjecture on the minimum number of monochromatic components needed to cover an edge-coloured complete bipartite graph. Specifically, we show the following.

- For $p \gg (\log n/n)^{1/2}$, w.h.p. every 2-colouring of the random bipartite graph $G \sim G(n, n, p)$ admits a cover of all but $O(1/p)$ vertices of G using at most three vertex-disjoint monochromatic components.
- For every 2-colouring of a bipartite graph G with parts of size n and minimum degree $(13/16 + o(1))n$, the vertices of G can be covered using at most three vertex-disjoint monochromatic components.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

1.1 History of tree covers and partitions

An r-colouring of a graph G is a colouring of the edges of G with up to r different colours. We are interested in determining the smallest number $\mathrm{tp}_r(G)$ ($\mathrm{tc}_r(G)$) such that in any r-colouring of G, the vertex set of G can be partitioned (covered) using at most $\mathrm{tp}_r(G)$ ($\mathrm{tc}_r(G)$) monochromatic trees. There is a large amount of literature on this problem and its variants where trees are replaced by paths or cycles, and most of the attention has focused on the case when G is either the n-vertex complete graph K_n or the complete bipartite graph $K_{n,m}$ with parts of size n and m. We recommend the excellent surveys [7, 10] for further reading.

Determining $\mathrm{tc}_r(K_n)$ is related to the celebrated conjecture [12] of Ryser on matchings and transversals in hypergraphs (which was also conjectured by Lóvasz [17]). Ryser's

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conjecture can be equivalently formulated as follows: The vertex set of any r-coloured graph G can be covered by at most $(r-1)\alpha(G)$ monochromatic trees, where $\alpha(G)$ denotes the independence number of G [6, 9]. In particular, Ryser's conjecture would imply that $\mathrm{tc}_r(K_n) = r-1$ for all $n \geqslant 1$ and all $r \geqslant 2$, which is best possible for infinitely many values of r (see e.g. [6]) and has been confirmed for $r \leqslant 5$ and all $n \geqslant 1$ (see [5, 9, 18]). Further, it is easy to see that $\mathrm{tc}_r(K_n) \leqslant r$, since the set of all maximal monochromatic stars having a fixed vertex v as their centre clearly covers $V(K_n)$.

Erdős, Gyárfás, and Pyber [6] conjectured that the above bound holds even for tree *partitions*, that is, they conjectured that $\mathrm{tp}_r(K_n) = r-1$ for all $r \geqslant 2$. This conjecture has been confirmed for $r = 2, 3$ [6], and it is known that $\mathrm{tp}_r(K_n) \leqslant r$ if $n \geqslant n_0(r)$ is large enough [1, 11].

Let us now turn to the complete bipartite graph $K_{n,m}$. It is easy to see that $\mathrm{tc}_r(K_{n,m}) \leqslant 2r-1$, as we can fix any edge vw and consider the maximal monochromatic stars having v or w as their centre. Two of these stars are joined by the edge vw, and thus the total number of monochromatic trees needed to cover $K_{n,m}$ is at most $2r-1$. In [4], this observation and the following conjecture are attributed to Gyárfás [9] and Lehel [16].

Conjecture 1 (Gyárfás and Lehel [9, 16])**.** For all $n, m \geqslant 1$ and $r \geqslant 2$, $\mathrm{tc}_r(K_{n,m}) \leqslant 2r-2$.

The conjecture has been verified for $r \leqslant 5$ by Chen, Fujita, Gyárfás, Lehel, and Tóth [4]. Further, there are examples of r-colourings which show that the bound of $2r-2$ in Conjecture 1 is best possible [4, 9]. See Section 5.1 for more discussion of this conjecture.

1.2 Tree covering problems in random graphs

In 2017, Bal and DeBiasio [1] initiated the study of covering problems in edge-colourings of the binomial random graph $G(n,p)$. They showed that, for $r \geqslant 2$ and $p \ll (r \log n/n)^{1/r}$, w.h.p.[1] there is an r-colouring of $G(n,p)$ which does not admit a cover with monochromatic trees whose number is bounded by some function in r. In 2021, Bucić, Korándi, and Sudakov [3] showed that indeed $p = (\log n/n)^{1/r}$ is the threshold for the property of admitting a cover with $f(r)$ monochromatic trees for some function f.

In view of the results from the previous subsection, it seems natural to ask for the threshold of the property that $\mathrm{tc}_r(G(n,p)) \leqslant r$ or $\mathrm{tp}_r(G(n,p)) \leqslant r$. For $r = 1$, this corresponds to the threshold at which $G(n,p)$ becomes connected. Also note that, while in the deterministic setting it was conjectured that $\mathrm{tp}_r(K_n) \leqslant r-1$, it is not overly difficult to see that r trees are needed if $p \leqslant 1-\varepsilon$, for any constant $\varepsilon > 0$.

Bal and DeBiasio [1] conjectured that $\mathrm{tp}_r(G(n,p)) \leqslant r$ holds w.h.p. when $p \geqslant (1+\varepsilon)(r \log n/n)^{1/r}$ and $\varepsilon > 0$ is fixed. This was confirmed for $r = 2$ by Kohayakawa, Mota, and Schacht [15], and disproved for all $r \geqslant 3$ by Ebsen, Mota, and Schnitzer (see [15, Proposition 4.1]) who showed that $\mathrm{tp}_r(G(n,p)) \geqslant r+1$ holds w.h.p. when

[1] We say that $G(n,p)$ satisfies a property $\mathcal{P}$ *with high probability* (w.h.p.) if $\mathbb{P}(G(n,p) \in \mathcal{P}) = 1 - o(1)$ as n tends to infinity. We say that $\hat{p} = \hat{p}(n)$ is the *threshold* for the property $\mathcal{P}$, if $\mathbb{P}(G(n,p) \in \mathcal{P}) = 1 - o(1)$ when $p = \omega(\hat{p})$, and $\mathbb{P}(G(n,p) \in \mathcal{P}) = o(1)$ if $p = o(\hat{p})$.

$p \ll (\log n/n)^{1/(r+1)}$. For $r = 3$, Bradač and Bucić [2] showed that $p = (\log n/n)^{1/4}$ is indeed the threshold for $\mathrm{tc}_3(G(n, p)) \leqslant 3$, improving upon previous results from [1, 14]. However, as shown by Bucić, Korándi, and Sudakov [3], the threshold for $\mathrm{tc}_r(G(n, p)) \leqslant r$ is in general much higher, and lies somewhere between $(\log n/n)^{1/2^r}$ and $(\log n/n)^{\sqrt{r}/2^{r-2}}$.

A bipartite variant of these results, in the spirit of the Gyárfás–Lehel conjecture (Conjecture 1) is still missing. In this setting, the complete bipartite graph $K_{n,m}$ from Conjecture 1 should be replaced with a random bipartite graph $G(n, m, p)$, whose vertex set are disjoint copies of $[n]$ and $[m]$, respectively, with every possible edge between these two sets appearing independently with probability p.

We will focus here on the case $r = 2$ and $n = m$. (See Sections 5.2 and 5.3 for a discussion of other options.) Similarly, as for complete graphs and $G(n, p)$, the bound given by Conjecture 1 does not carry over to $G(n, n, p)$, even when $p = 1 - o(1)$. Indeed, we show that w.h.p. $\mathrm{tc}_2(G(n, n, p)) \geqslant 3$ when $1 - p$ exceeds $3 \log n/n$.

Proposition 2. *For $p \leqslant 1 - 3 \log n/n$, w.h.p. $\mathrm{tc}_2(G(n, n, p)) \geqslant 3$.*

Thus motivated, we are interested in the threshold of the property $\mathrm{tc}_2(G(n, n, p)) \leqslant 3$. We believe that this threshold should be the same as for the $r = 2$ case of the non-bipartite setting.

Conjecture 3. The threshold for $\mathrm{tc}_2(G(n, n, p)) \leqslant 3$ is $\hat{p} = (\log n/n)^{1/2}$.

Evidence for Conjecture 3 is given by the following theorem, which contains the 0-statement and an approximate form of the 1-statement.

Theorem 4. *There exist positive constants c and C such that the following holds.*

1. *If $p \leqslant c\,(\log n/n)^{1/2}$, then w.h.p. $\mathrm{tc}_2(G(n, n, p)) \geqslant 4$.*
2. *If $p \geqslant C\,(\log n/n)^{1/2}$, then w.h.p., in every 2-colouring of $G \sim G(n, n, p)$, all but at most $200/p$ vertices can be covered using at most three vertex-disjoint monochromatic trees.*

We prove Proposition 2 and Theorem 4 in Section 3.

1.3 Graphs of large minimum degree

In [1], Bal and DeBiasio asked whether a minimum degree version of the tree covering problem exists, and proved that for every $r \geqslant 2$ there is a constant $\alpha_r \in (0, 1)$ such that if G is an n-vertex graph with $\delta(G) \geqslant \alpha_r n$, then $\mathrm{tc}_r(G) \leqslant r$. They also found an r-coloured n-vertex graph with minimum degree roughly $rn/(r + 1)$ which cannot be covered with r monochromatic trees and conjectured that this would be the worst-case scenario.

Conjecture 5 (Bal and DeBiasio [1])**.** Let $n, r \geqslant 2$. If G is an n-vertex graph with

$$\delta(G) > \frac{r}{r+1}(n - r - 1), \tag{1}$$

then $\mathrm{tc}_r(G) \leqslant r$.

Shortly afterwards, Girão, Letzter, and Sahasrabudhe [8] proved a strengthening of Conjecture 5 for $r = 2$, showing that every n-vertex 2-edge-coloured graph of minimum degree exceeding $(2n-5)/3$ can be *partitioned* into at most two monochromatic components.

We propose a minimum degree version of the balanced case of the Gyárfás–Lehel conjecture (Conjecture 1). While in Conjecture 1, the bound on the number of monochromatic components is $2r-2$, we will need to work with the bound $2r-1$, since a non-complete bipartite host graph may need this many components, as witnessed by the random case (Proposition 2).

Question 6. What is the smallest number $\alpha_r > 0$ such that if G is a spanning subgraph of $K_{n,n}$ with minimum degree at least $\alpha_r n$, then $\mathrm{tc}_r(G) \leqslant 2r-1$?

The same question can be asked for partitioning.

Question 7. What is the smallest number $\beta_r > 0$ such that if G is a spanning subgraph of $K_{n,n}$ with minimum degree at least $\beta_r n$, then $\mathrm{tp}_r(G) \leqslant 2r-1$?

Note that $\beta_r \geqslant \alpha_r > 1/2$ for all $r \geqslant 2$. For this, it suffices to consider the graph G consisting of two disjoint copies of $K_{\frac{n}{2},\frac{n}{2}}$. Give each of these a colouring that cannot be covered by fewer than r components to see that $\mathrm{tc}_r(G) \geqslant 2r$. (Such a colouring can be obtained by taking a proper edge-colouring of $K_{r,r}$ and blowing up one of the edges).

For two colours, we show that the answer to Question 7 is at most $13/16 + o(1)$.

Theorem 8. *For every $\delta > 0$, there is n_0 such that for every $n \geqslant n_0$ the following holds. If G is a spanning subgraph of $K_{n,n}$ with minimum degree at least $(13/16+\delta)n$, then $\mathrm{tp}_2(G) \leqslant 3$.*

The constant 13/16 in Theorem 8 is possibly not tight, and we have no clue which constant should be the right answer. We prove Theorem 8 in Section 4.

2 Preliminaries

We collect some graph-theoretic notation, probabilistic inequalities and basic results on random bipartite graphs.

2.1 Basic notation

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A, B \subseteq V(G)$, $G[A]$ is the graph induced by A, and, if A, B are disjoint, $G[A, B]$ is the bipartite graph induced by A and B, that is, the bipartite graph with parts A and B, and all the edges $ab \in E(G)$ with $a \in A$ and $b \in B$. Write $e(G) := |E(G)|$, $e(A, B) := |E(G[A, B])|$ and $e(A) := |E(G[A])|$. For $x \in V(G)$ and $U \subseteq V(G)$, we write $N(x, U)$ for the set of neighbours of x in U and set $d(x, U) := |N(x, U)|$. If $U = V(G)$, we just write $N(x)$ and $d(x)$. When working with more than one graph, we add subscripts for the graph we are referring to, for example, $d_G(x)$ is the degree of a vertex x in the graph G.

Given a 2-colouring of G, with colours red and blue, we let G_R and G_B denote the subgraphs consisting of the red and blue edges, respectively. We will use the subscripts R and B instead of G_R and G_B when referring to the red and blue subgraph, respectively. For example, for a vertex $x \in V(G)$, we write $d_R(x)$ and $d_B(x)$ instead of $d_{G_R}(x)$ and $d_{G_B}(x)$.

For $a, b \in \mathbb{R}$ and $c > 0$, we write $a = b \pm c$ to denote that a satisfies $b - c \leqslant a \leqslant b + c$, while $a \ll b$ means that given b one can choose a sufficiently small so that all the following relevant statements hold. We omit floors and ceilings, if this does not affect the proofs.

2.2 Probabilistic inequalities

We will use the following standard probabilistic inequalities (see [13] for instance).

Lemma 9 (Markov's inequality)**.** *Let X be a non-negative random variable. Then, for any $\lambda > 0$, we have $\mathbb{P}(X \geqslant \lambda) \leqslant \mathbb{E}X/\lambda$.*

Lemma 10 (Chernoff's bound)**.** *Let X be a binomial random variable.*

(i) *If $0 < \delta < 1$, then $\mathbb{P}(X \leqslant (1-\delta)\mathbb{E}X) \leqslant e^{-\delta^2\mathbb{E}X/2}$.*

(ii) *If $0 < \delta \leqslant 3/2$, then $\mathbb{P}(|X - \mathbb{E}X| \geqslant \delta\mathbb{E}X) \leqslant 2e^{-\delta^2\mathbb{E}X/3}$.*

Lemma 11 (Paley–Zygmund inequality)**.** *Let X be a non-negative random variable with finite variance. Then, for any $0 < \delta < 1$, we have*

$$\mathbb{P}(X \geqslant \delta\mathbb{E}X) \geqslant (1-\delta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

2.3 Random graphs and graph properties

In the *binomial random graph* model $G(n, p)$ we consider a graph on the vertex set $[n]$ where every possible edge appears independently with probability p. By a *graph property* $\mathcal{P}$ we mean a collection of finite graphs, and we say that $\mathcal{P}$ is *monotone* increasing (resp. decreasing) if for any $G \in \mathcal{P}$, any graph obtained by adding (resp. deleting) edges to G also satisfies $\mathcal{P}$. Note that the property of tc_r being bounded from above by some fixed number k is a monotone graph property.

The *binomial random bipartite graph* model $G(n, m, p)$ is the bipartite graph whose vertex classes are disjoint copies of $[n]$ and $[m]$, respectively, and each possible edge appears independently with probability p. The terms *with high probability* and *threshold* are defined in the same way as for random graphs.

In the following lemma, we collect all the properties of random bipartite graphs that we will need for proving the approximate 1-statement of Theorem 4. This lemma is a bipartite version of Lemma 2.1 in [15], and most of the proofs are straightforward applications of Chernoff's bound. For the sake of completeness, we include its proof in the appendix.

Lemma 12. *For every $0 < \varepsilon < 1$, there exists a constant $C > 0$ such that for $p \geqslant C(\log n/n)^{1/2}$, if $G \sim G(n, n, p)$ has bipartition classes V_1 and V_2, then w.h.p. the following properties hold.*

(i) *For every* $v, w \in V_1$ *(resp.* $v, w \in V_2$*), we have* $d(v) = (1 \pm \varepsilon)pn$ *and* $|N(v) \cap N(w)| = (1 \pm \varepsilon)p^2 n$.

(ii) *For every* $v \in V_2$ *(resp.* $v \in V_1$*),* $U \subseteq N(v)$ *with* $|U| \geqslant pn/100$*, and* $W \subseteq V_2$ *(resp.* $W \subseteq V_1$*) with* $|W| \geqslant 100/p$*, we have* $e(U, W) \geqslant p|U||W|/2$.

(iii) *For every* $v \in V_2$ *(resp.* $v \in V_1$*) and* $U \subseteq N(v)$ *with* $|U| \geqslant pn/100$*, all but at most* $100/p$ *vertices* $v' \in V_2$ *(resp.* $v' \in V_1$*) satisfy* $d(v', U) \geqslant p^2 n/200$.

(iv) *Every subgraph* $H \subseteq G$ *with* $\delta(H) \geqslant (1/2 + \varepsilon)pn$ *is connected.*

We remark that for ensuring the degree condition $d(x) = (1 \pm \varepsilon)pn$ it suffices that $p \gg \log n/n$ rather than the stronger condition $p \gg (\log n/n)^{\frac{1}{2}}$.

The next lemma is crucial for the construction of a colouring that shows the 0-statement in Theorem 4.

Lemma 13. *Let* $c > 0$ *be sufficiently small and let* $p = c(\log n/n)^{1/2}$*. Let* $G \sim G(n, n, p)$ *with bipartition classes* V_1*,* V_2*. Then, for* $i \in [2]$*, w.h.p. there are at least* $e^{-3c^2 \log n}\binom{n}{2}$ *pairs* $\{u, v\}$ *of distinct vertices from* V_i *such that* u, v *have no common neighbours.*

Proof. The probability that two distinct vertices $u, v \in V_i$ have no common neighbours is $(1 - p^2)^n$. So, letting X denote the random variable counting the number of such $\{u, v\}$, we have

$$\mathbb{E}X = \binom{n}{2}(1 - p^2)^n \geqslant \binom{n}{2} e^{-2p^2 n},$$

where we used that $e^{-2x} \leqslant 1 - x$ holds for all $0 \leqslant x \leqslant 3/4$. On the other hand, we have

$$\begin{aligned}\mathbb{E}X^2 &= \binom{n}{2}\binom{n-2}{2}(1 - p^2)^{2n} + 6\binom{n}{3}(1 - 2p^2 + p^3)^n + \binom{n}{2}(1 - p^2)^n \\ &= (1 + o(1))(\mathbb{E}X)^2,\end{aligned}$$

where the second term comes from pairs $\{u, v\}$ and $\{u', v'\}$ that intersect in exactly one element. Therefore, by the Paley–Zygmund inequality (Lemma 11) we have

$$\mathbb{P}\left(X \geqslant e^{-3c^2 \log n}\tbinom{n}{2}\right) \geqslant \mathbb{P}(X \geqslant e^{-c^2 \log n}\mathbb{E}X) \geqslant (1 - e^{-c^2 \log n})^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} = 1 - o(1).$$

□

3 Tree covers in random bipartite graphs

In this section, we prove Proposition 2 and Theorem 4. We start with Proposition 2.

Proof of Proposition 2. Let $G \sim G(n, n, p)$ and let V_1 and V_2 be the bipartition classes of G. We first note that w.h.p. each vertex in V_i has at least two non-neighbours in V_{3-i}, for $i \in [2]$. Indeed, for a vertex $v \in V_1$ (say), the probability that v has less than 2 non-neighbours is

$$p^n + np^{n-1}(1-p) \leqslant e^{-3\log n} + ne^{-3\frac{n-1}{n}\log n} \leqslant 2n^{-2},$$

and thus by a union-bound the probability that there is a vertex with less than 2 non-neighbours is at most $2n \cdot 2n^{-2} = o(1)$. So, we may pick, arbitrarily, vertices $r \in V_1$ and $b \in V_2 \setminus N(r)$, and set $X = V_1 \setminus (N(b) \cup \{r\})$ and $Y = V_2 \setminus (N(r) \cup \{b\})$. Note that both X and Y are non-empty as both r and b have at least 2 non-neighbours.

We colour in red the edges from r to $N(r)$, the edges between X and $N(r)$, and the edges between Y and $N(b)$. We colour in blue the edges from b to $N(b)$, the edges between $N(r)$ and $N(b)$, and the edges between X and Y. In this colouring, no two components can cover all of $V(G)$ (the monochromatic components that cover r and b cannot cover Y), which proves that $\mathrm{tc}_2(G) \geqslant 3$. □

The following proposition proves Theorem 4 (i).

Proposition 14. *Let $c > 0$ be sufficiently small and let $p \leqslant c(\log n/n)^{1/2}$. Then w.h.p. $\mathrm{tc}_2(G(n,n,p)) \geqslant 4$.*

Proof. First, as the property $\mathcal{P} = \{H : \mathrm{tc}_2(H) \geqslant 4\}$ is monotone decreasing, we can assume that $p = c(\log n/n)^{1/2}$. Let $G \sim G(n,n,p)$ and let V_1 and V_2 be the bipartition classes of G. By the remark after Lemma 12, we may assume that for each $i \in [2]$, w.h.p. every vertex $v \in V_i$ satisfies $d(v) = (1 \pm 0.5)pn$. By Lemma 13, w.h.p. there are at least $e^{-3c^2\log n}\binom{n}{2}$ pairs of distinct vertices in V_i, for each $i \in [2]$, with no common neighbours.

We now construct an edge-colouring of G which can only be covered if at least four monochromatic components are used. We choose $u_1, v_1 \in V_1$ with no common neighbours. Then, we pick $u_2, v_2 \in V_2 \setminus (N(u_1) \cup N(v_1))$ with no common neighbours, which is possible because we have at least

$$e^{-3c^2\log n}\binom{n}{2} - 4pn^2 \geqslant e^{-3c^2\log n}\binom{n}{2} - 4c^2 n^{3/2}(\log n)^{1/2} \geqslant e^{-c^2\log n}\binom{n}{2}$$

options for $\{u_2, v_2\}$, provided c is small enough. We use red for all edges between u_i, v_i and their neighbours, for $i \in [2]$, and blue for all the rest. Since u_1, u_2, v_1, v_2 each belong to distinct singleton blue components and lie in four separate red components, we cannot cover G with fewer than four components. Therefore $\mathrm{tc}_2(G) \geqslant 4$. □

The proof of Theorem 4 (ii) is captured in the following theorem, whose proof is inspired by the approach of Kohayakawa, Mota, and Schacht [15].

Theorem 15. *There exists a constant $C > 0$ such that if $p \geqslant C\left(\log n/n\right)^{1/2}$, then w.h.p. in every 2-colouring of $G \sim G(n,n,p)$, all but at most $200/p$ vertices of G can be covered by at most three vertex-disjoint monochromatic trees.*

Proof. Let C be a sufficiently large constant. Since the property of being almost coverable by m monochromatic trees is monotone, we can assume $p = C(\log n/n)^{1/2}$. Let $G \sim G(n, n, p)$, with partition classes V_1 and V_2. Then, for $0 < \varepsilon \ll 1$, by Lemma 12 we know that w.h.p. G satisfies the following properties:

(B1) For $i \in [2]$ and $v, w \in V_i$, we have $d(v) = (1 \pm \varepsilon)pn$ and $|N(v) \cap N(w)| = (1 \pm \varepsilon)p^2 n$.

(B2) For $i \in [2]$, $v \in V_i$ and subsets $U \subseteq V_i$ and $W \subseteq N(v)$, with $|U| \geqslant 100/p$ and $|W| \geqslant pn/100$, we have $e(U, W) \geqslant p|U||W|/2$.

(B3) For $i \in [2]$, $v' \in V_{3-i}$ and a subset $W \subseteq N(v')$ with $|W| \geqslant pn/100$, all but at most $100/p$ vertices $v \in V_{3-i}$ satisfy $d(v, W) \geqslant p^2 n/200$.

(B4) Every subgraph $H \subseteq G$ with $\delta(H) \geqslant (1/2 + \varepsilon)pn$ is connected.

Moreover, as we may assume that n is sufficiently large, we have

$$\max\left\{(1+\varepsilon)p^2 n, 100/p\right\} \leqslant \frac{pn}{100}. \tag{2}$$

Suppose we are given a red and blue edge-colouring of G. If there is a monochromatic spanning component, we are done, so assume otherwise. Set $V_R = \{v \in V(G) : d_R(v) > \frac{1}{3}d(v)\}$ and $V_B = \{v \in V(G) : d_B(v) > \frac{1}{3}d(v)\}$. We claim that

$$V_R \neq \emptyset \neq V_B. \tag{3}$$

For contradiction, assume that $V_R = \emptyset$. Then (B1) implies that for every $v \in V(G)$,

$$d_B(v) \geqslant \frac{2}{3}(1-\varepsilon)pn \geqslant \left(\frac{1}{2} + \varepsilon\right)pn.$$

Therefore, by (B4), the blue graph G_B is connected and thus G has a monochromatic spanning tree. The same argument applies to V_B, which completes the proof of (3).

By (3), and after possibly swapping the names of V_1 and V_2, we can assume that there are vertices

$$r \in V_1 \cap V_R \text{ and } b \in V_2 \cap V_B.$$

(Indeed, if V_1 only meets V_R, say, then V_2 necessarily meets V_B, and the case that V_1 meets both sets is easy.) We will define a red tree T_1 and a blue tree T_2 having r and b as their respective roots (and later, depending on the structure of the colouring, we might define a third tree T_3). To define T_1 and T_2, we will define a vertex-colouring ρ, where v will have colour $\rho(v)$ if there are many monochromatic paths (in the edge-coloured graph) in colour $\rho(v)$ connecting v with the root of the tree in colour $\rho(v)$. Later, T_1 will consist of all vertices v with $\rho(v) =$ red and T_2 will consist of all vertices v with $\rho(v) =$ blue. To avoid any confusion, neighbourhoods will always be considered with respect to the edge colouring.

So let us define ρ. We set $\rho(r) =$ red and $\rho(b) =$ blue, and set for each $v \in N_R(r) \setminus \{b\}$

$$\rho(v) = \text{red}\,.$$

Since $r \in V_R$ and $b \in V_B$, we have $|N_R(r)|, |N_B(b)| \geqslant pn/100$, which, together with (B2), implies $e(N_R(r), N_B(b)) \geqslant p|N_R(r)||N_B(b)|/2$. By colour symmetry, we may assume without loss of generality that

$$e_R(N_R(r), N_B(b)) \geqslant \frac{p}{4}|N_R(r)||N_B(b)|. \tag{4}$$

Let

$$J_1 = \{v \in N_B(b) : |N_R(v) \cap N_R(r)| > p^2n/25\}.$$

The letter J stands for the word 'joker', as the vertices from J_1 can be used in both colours (each vertex from J_1 can be connected to r by a red path of length two and to b by a blue edge). We set $\rho(v) =$ blue for each $v \in N_B(b) \setminus J_1$ and claim that J_1 is large, namely,

$$|J_1| \geqslant pn/100. \tag{5}$$

To see (5), we start by setting $L := N_B(b) \setminus J_1$. Then

$$e_R(L, N_R(r)) \leqslant |L| \cdot \frac{p^2n}{25} \leqslant |N_B(b)| \cdot \frac{(1-\varepsilon)p^2n}{24} \leqslant \frac{p}{8}|N_R(r)||N_B(b)|,$$

where we used (B1) and the fact that $r \in V_R$, which implies that $|N_R(r)| \geqslant (1-\varepsilon)pn/3$. Thus, by (4), we have

$$e_R(J_1, N_R(r)) \geqslant p|N_R(r)||N_B(b)|/8.$$

As $|N(v) \cap N(r)| \leqslant (1+\varepsilon)p^2n$ for every $v \in V$, we conclude that

$$|J_1| \geqslant \frac{p|N_R(r)||N_B(b)|}{(1+\varepsilon)8p^2n} \geqslant \frac{pn}{100},$$

where in the last inequality we used that $d_R(r), d_B(b) \geqslant (1-\varepsilon)pn/3$. This proves (5).

Next, set

$$Z_2 = \{z \in V_2 \setminus (N_R(r) \cup \{b\}) : |N(z, J_1)| \geqslant p^2n/200\}$$

and set $K_2 = V_2 \setminus (N_R(r) \cup Z_2 \cup \{b\})$. Because of (B3) and (5), we have

$$|K_2| \leqslant 100/p. \tag{6}$$

For $z \in Z_2$, we set

$$\rho(z) = \begin{cases} \text{red} & \text{if} \quad |N_R(z, J_1)| \geqslant \dfrac{p^2n}{400}, \\ \text{blue} & \text{otherwise.} \end{cases}$$

Note that each vertex in Z_2 has at least $p^2n/400$ neighbours in J_1 in at least one of the colours. So, if $\rho(z) =$ blue, then $|N_B(z, J_1)| \geqslant p^2n/400$.

Up to this point, we have assigned a colour to every vertex in $V(G)$ except for those in $V_1 \setminus (N_B(b) \cup \{r\})$ and in K_2. Moreover, every vertex v sends an edge of colour $\rho(v)$ to either b or r (according to $\rho(v)$), or many edges of the same colour to J_1. In order to be able to define T_1 and T_2 as disjoint trees, we will need to assign colours to the vertices of

J_1 so that the vertices from Z_2 can all be connected with monochromatic paths to either r or b. For this reason, we will define $\rho(v)$ for the vertices of J_1 as follows:

> For each $v \in J_1$, define $\rho(v)$ by choosing a colour from $\{\text{red}, \text{blue}\}$ independently and uniformly at random.

Let Z_2^r be the set of all vertices $v \in Z_2$ with $\rho(v) = \text{red}$, and set $Z_2^b = Z_2 \setminus Z_2^r$. Note that for each $v \in Z_2^r$, the probability that $\rho(w) = \text{blue}$ for each vertex $w \in N_R(v, J_1)$ is

$$2^{-|N_R(v,J_1)|} \leqslant 2^{-p^2n/400} = o(n^{-1}).$$

The same argument holds for $v \in Z_2^b$, which proves that

$$\text{w.h.p. each } v \in Z_2 \text{ has at least one neighbour of colour } \rho(v) \text{ in } J_1. \tag{7}$$

To avoid any ambiguity, (7) means that each vertex in $v \in Z_2$ has at least one neighbour $v' \in N(v) \cap J_1$ such that $\rho(v) = \rho(v')$ and, by definition of ρ, the edge vv' has colour $\rho(v)$. Therefore, by (7), there is a red tree T_1 with vertex set $\rho^{-1}(\text{red})$ such that the vertices of Z_2^r are leaves of T_1. We define a blue tree T_2 analogously, with $V(T_2) = \rho^{-1}(\text{blue})$. Note that by our choice of ρ,

$$T_1 \text{ and } T_2 \text{ are disjoint and cover all of } (N_B(b) \cup \{r\}) \cup (V_2 \setminus K_2). \tag{8}$$

We still need to cover most of the vertices in $V_1 \setminus (N_B(b) \cup \{r\})$. For this, we will consider two cases.

Case 1. There is a vertex $\tilde{v} \in V_1 \setminus (N_B(b) \cup \{r\})$ such that

$$d_B(\tilde{v}, Z_2^r) \geqslant pn/100 \quad \text{or} \quad d_R(\tilde{v}, Z_2^b) \geqslant pn/100.$$

We will assume that $|N_B(\tilde{v}, Z_2^r)| \geqslant pn/100$, as the other case is completely analogous (with all colours switched). So for $J_2 = N_B(\tilde{v}, Z_2^r)$ we have

$$|J_2| \geqslant pn/100.$$

We will use $\tilde{v}$ as the root of a third monochromatic tree. This tree will be blue, and in order to define it, we will use a function ρ'.

Let $Z_1 = \{x \in V_1 \setminus (N_B(b) \cup \{r, \tilde{v}\}) : |N(x, J_2)| \geqslant p^2n/200\}$. We define, for each $x \in Z_1$,

$$\rho'(x) = \begin{cases} \text{red} & \text{if } |N_R(x) \cap J_2| \geqslant \frac{p^2n}{400}, \\ \text{blue} & \text{otherwise.} \end{cases}$$

As each vertex $x \in Z_1$ sends at least $\frac{p^2n}{400}$ edges in the same colour to J_2, vertices with blue assignment have at least $\frac{p^2n}{400}$ blue neighbours in J_2.

Now, for each $v \in J_2$, we choose $\rho'(v) \in \{\text{red}, \text{blue}\}$ uniformly at random, making all choices independently from each other. Similarly as above,

$$\text{w.h.p. each } x \in Z_1 \text{ has at least one neighbour of colour } \rho'(x) \text{ in } J_2. \tag{9}$$

As before, if $x \in Z_1$ has a neighbour $x' \in N(x) \cap J_2$ with $\rho(x) = \rho(x')$, then the edge xx' has colour $\rho(x)$. Let $K_1 = V_1 \setminus (N_B(b) \cup Z_1 \cup \{r, \tilde{v}\})$, and note that by (B3) we have $|K_1| \leqslant 100/p$. So, by (6), we know that

$$|K_1 \cup K_2| \leqslant \frac{200}{p}.$$

We let T_3 be a blue tree with vertex set $\{\tilde{v}\} \cup \{v \in J_2 \cup Z_1 : \rho'(v) = \text{blue}\}$. We let T_1' be the tree obtained from $T_1 - \{v \in J_2 : \rho'(v) = \text{blue}\}$ by adding as leaves all $x \in Z_1$ with $\rho'(x) = \text{red}$. Then the three trees T_1', T_2 and T_3 are vertex-disjoint and cover all of $V(G) \setminus (K_1 \cup K_2)$, that is, they cover all but $200/p$ vertices from G, which is as desired.

Case 2. For every vertex $v \in V_1 \setminus (N_B(b) \cup \{r\})$, we have

$$d_B(v, Z_2^r) \leqslant \frac{pn}{100} \quad \text{and} \quad d_R(v, Z_2^b) \leqslant \frac{pn}{100}.$$

Consider any vertex $v \in V_1 \setminus (N_b(b) \cup \{r\})$. Note that by (B1), (2) and (6),

$$d(v, Z_2) \geqslant d(v) - d(v, K_2) - d(v, N_R(r)) - 1 \geqslant \frac{pn}{2}, \tag{10}$$

and therefore, v either has at least $pn/10$ neighbours in Z_2^r or at least $pn/10$ neighbours in Z_2^b. In the former case, we have $d_R(v, Z_2^r) \geqslant pn/100 \geqslant 1$, as $d_B(v, Z_2^r) \leqslant pn/100$. In the latter case, we have $d_B(v, Z_2^b) \geqslant pn/100 \geqslant 1$.

So, we can add each vertex $v \in V_1 \setminus (N_b(b) \cup \{r\})$ as a leaf to one of the trees T_1, T_2, and the obtained two trees cover all but at most $100/p$ vertices of G. □

4 Graphs of large minimum degree

The whole section is devoted to the proof of Theorem 8.

Let n_0 be sufficiently large and let G be a bipartite graph with parts V_1 and V_2, each of size $n \geqslant n_0$, and with minimum degree $\delta(G) \geqslant (13/16 + \delta)n$. Let V_R and V_B be the set of vertices with at least $(9/16 + 3\delta/4)n$ neighbours in red and blue, respectively. If one of these sets is empty, say V_B, then the red graph has minimum degree at least

$$\left(\frac{13}{16} + \delta\right) n - \left(\frac{9}{16} + \frac{3\delta}{4}\right) n > n/4,$$

and thus G can be covered using at most 3 red components, and we are done. So we can assume both sets V_R, V_B are non-empty. This implies that there are vertices $r \in V_R$ and $b \in V_B$ from different partition classes of G (see the proof of Theorem 15 for an analogous argument).

Suppose without loss of generality that $r \in V_1$ and $b \in V_2$, and choose sets $X \subseteq N_R(r) \setminus \{b\}$ and $Y \subseteq N_B(b) \setminus \{r\}$ such that $|X| = |Y| = (9/16 + \delta/2)n$. Note that because of our condition on $\delta(G)$, for every vertex $v \in Y$ we have

$$d(v, X) \geqslant \left(\frac{9}{16} + \frac{\delta}{2}\right) n - \left(\frac{3}{16} - \delta\right) n \geqslant \left(\frac{3}{8} + \delta\right) n,$$

and thus we have $e(X,Y) \geqslant (3/8+\delta)n|Y|$. Therefore, there are at least $(3/16+\delta/2)n|Y|$ edges of the same colour between X and Y. Say this colour is red, and let $J_Y \subseteq Y$ be the set of all vertices $v \in Y$ satisfying $d_R(y,X) \geqslant \delta n/100$. (If the most popular colour between X and Y was blue, then we take $J_X \subseteq X$ instead, and accordingly change the rest of the proof.)

We claim that $|J_Y| \geqslant 3n/16$. Indeed, otherwise, we would have that

$$\left(\frac{3}{16}+\frac{\delta}{2}\right)n|Y| \leqslant e_R(X,Y) \leqslant \frac{3n}{16}\cdot|X| + |Y|\cdot\frac{\delta n}{100},$$

which is a contradiction as $|X| = |Y|$. Therefore, and because of our condition on $\delta(G)$, every vertex $v \in V_2$ satisfies

$$d(v,J_Y) \geqslant \frac{3n}{16} - \left(\frac{3n}{16}-\delta\right)n = \delta n. \tag{11}$$

Before starting to assign colours to vertices as in the proof of Theorem 4, we need to shrink the size of X for reasons that will become clear later in the proof. Let $p \in (0,1)$ be a small enough constant and take a subset $X' \subseteq X$ so that each vertex $x \in X$ is included in X' with probability p and all choices are made independently. Note that $\mathbb{E}|X'| = p|X| = (9/16+\delta/2)pn$ and, by definition of J_Y, each vertex $y \in J_Y$ satisfies $\mathbb{E}[d_R(y,X')] \geqslant \delta pn/100$. So, by Lemma 10, with probability $1-o(1)$ we have that

- $\frac{pn}{2} \leqslant |X'| \leqslant pn$, and
- for each vertex $y \in J_Y$, $d_R(y,X') \geqslant \delta pn/200$.

Let $W = V_2 \setminus (\{b\} \cup X')$. Now, as in the proof of Theorem 4, we will choose a preferred colour $\rho(v)$ for each vertex v. We set

$$\rho(v) = \begin{cases} \text{red} & \text{if } v \in X' \cup \{r\}, \text{ and} \\ \text{blue} & \text{if } v \in (Y \cup \{b\}) \setminus J_Y, \end{cases}$$

and for each $v \in J_Y$, we choose $\rho(v) \in \{\text{red}, \text{blue}\}$ independently uniformly at random. Let us next define $\rho(v)$ for each $v \in W$. Because of (11), for each $v \in W$ there is a colour $c_v \in \{\text{red}, \text{blue}\}$ such that v has at least $\delta n/2$ neighbours in J_Y in colour c_v, in which case we let $\rho(v) = c_v$.

Use Lemma 10 and a union bound to see that the following property holds with high probability.

> For each vertex $w \in W$ there are at least $\delta n/8$ vertices $v \in N(w,J_Y)$ in colour $\rho(w)$ such that $\rho(v) = \rho(w)$.

Set $W_B = \rho^{-1}(\text{blue}) \cap W$ and $W_R = \rho^{-1}(\text{red}) \cap W$, and observe that every vertex in W_B (resp. W_R) can be connected to b (resp. to r) by a path in colour blue (resp. red). As $|X'| \leqslant pn$ and p is sufficiently small, we have $|W| = n - |X'| - 1 \geqslant n - 2pn \geqslant 0.9n$, and thus, at least one of W_R, W_B has at least $0.4n$ vertices. We assume $|W_B| \geqslant 0.4n$ (the other case is analogous).

Let $U = V_1 \setminus (\{r\} \cup Y)$, and observe that every vertex $v \in U$ satisfies

$$d(v, W_B) \geqslant 0.4n - \left(\frac{3}{16} - \delta\right) n \geqslant \left(\frac{3}{16} + \delta\right) n. \tag{12}$$

If each vertex in U has at least one blue neighbour in W_B, we can assign $\rho(u) = \text{blue}$ for each $u \in U$. Then both $\rho^{-1}(\text{red})$, $\rho^{-1}(\text{blue})$ span vertex-disjoint monochromatic connected components, and we are done. So, we can assume that there is a vertex $u_0 \in U$ with no blue neighbour in W_B and thus $d_R(u_0, W_B) \geqslant (3/16 + \delta)n$ by (12).

Pick a set $J_X \subseteq N_R(u_0, W_B)$ of size $(3/16 + \delta)n$, and note that each vertex $u \in U$ satisfies

$$d(u, J_X) \geqslant (3/16 + \delta)n - (3/16 - \delta)n \geqslant 2\delta n.$$

In particular, for each vertex $u \in U$ there is a colour c_u such that u has at least δn neighbours in colour c_u in J_X. We set $\rho(u) = c_u$. Now, we randomly re-colour each vertex in J_X by choosing $\rho(v) \in \{\text{red}, \text{blue}\}$ uniformly at random for each $v \in J_X$. Since for each $u \in U$, the expected number of vertices in $N(u) \cap J_X$ with colour $\rho(u)$ is at least δn, by Lemma 10 we deduce that w.h.p. each vertex $u \in U$ has at least $\delta n/2$ neighbours $v \in J_X$ with $\rho(v) = \rho(u)$. So, the vertices with preference $\rho(v) = \text{red}$ span a red subgraph having at most two connected components, and the remaining vertices (those with preference $\rho(v) = \text{blue}$) span a blue connected subgraph, and we are done.

5 Concluding remarks

5.1 Variants of the Gyárfás-Lehel conjecture

Recall that in Conjecture 1, Gyárfás and Lehel conjectured that $\text{tc}_r(K_{n,m}) \leqslant 2r - 2$ for $n, m \geqslant 1$ and $r \geqslant 2$. In [4] and [9] one can find examples of r-colourings of $K_{n,m}$ with $n = r-1$ and $m = r!$ which show that the bound of $2r-2$ in Conjecture 1 is best possible.

We do not know of similar examples for balanced complete bipartite graphs. That is, we do not know whether there is, for some n, an r-colouring of $K_{n,n}$ which does not allow for a covering with fewer than $2r - 2$ trees. So, although we are inclined to believe the opposite, it is possible that $\text{tc}_r(K_{n,n})$ is lower than $2r - 2$. Thus motivated, we propose the following problem.

Problem 16. Determine $\text{tc}_r(K_{n,n})$ for all n, r.

Also, it seems that nothing is known about tree partitioning in complete bipartite graphs, which is why we suggest the following problem.

Problem 17. Determine $\text{tp}_r(K_{n,m})$ for all n, m, r.

We believe that, in analogy to the complete graph case, it could be true that

$$\text{tp}_r(K_{n,m}) = \text{tc}_r(K_{n,m}).$$

5.2 Unbalanced random bipartite host

While we believe the bipartite tree covering number $\mathrm{tc}_r(K_{n,m})$ should be the same if restricted to the instances where $n = m$, this might not be the case. Then, our main result, Theorem 4 corresponds to Problem 16, and the random version of Conjecture 1 should be more general. Nevertheless, our proof, up to minor modifications, yields the same conclusions for the host $G(n, m, p)$ when $m = \Theta(n)$.

5.3 More colours in the random setting

Given Theorem 4, a natural next step would be to find a random analogue of the Gyárfás–Lehel conjecture for more colours. In this direction, we propose the following question.

Question 18. For $r \geqslant 2$, determine the threshold for $\mathrm{tc}_r(G(n, n, p)) \leqslant 2r - 1$.

As in the two colour case, we believe that the answer to Question 18 should be the same as in the non-bipartite setting. In particular, given the result of Bradač and Bucić [2], we believe that the threshold for 3 colours should be $p = (\log n/n)^{1/4}$.

Conjecture 19. There exists a constant C such that if $p \geqslant C(\log n/n)^{1/4}$ and $G \sim G(n, n, p)$, then w.h.p. $\mathrm{tc}_3(G) \leqslant 5$.

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A Proof of Lemma 12

We let $C \gg \varepsilon^{-2}$.

(i): As $d(u) \sim \text{Bin}(n, p)$ for every $u \in V_1$, Chernoff's bound (Lemma 10) gives that the probability that there is a vertex u with $|d(u) - pn| \geqslant \varepsilon pn$ is bounded by $2ne^{-\varepsilon^2 pn/3} = o(1)$, as $pn \gg \log n$. Moreover, for distinct $u, v \in V_1$, we have that $|N(u) \cap N(v)| \sim \text{Bin}(n, p^2)$ and thus, by Chernoff's bound, the probability that are such u, v with $\Big||N(u) \cap N(v)| - p^2 n\Big| \geqslant \varepsilon pn^2$ is bounded by $2n^2 e^{-\varepsilon^2 p^2 n/3} = o(1)$, as $p^2 n \geqslant C \log n$ and $C \gg \varepsilon^{-2}$.

(ii): Fix $U \subseteq V_1$ and $W \subseteq V_2$, with $|U| \geqslant 100/p$ and $|W| \geqslant pn/100$. Note that $e(U, W) \sim \text{Bin}(|U||W|, p)$ and so, by Chernoff's bound (Lemma 10), we have

$$\mathbb{P}(e(U, W) < p|U||W|/2) \leqslant e^{-p|U||W|/8}.$$

Therefore, the probability that there exist a vertex $v \in V_2$, and subsets $U \subseteq N(v)$ and $W \subseteq V_2$, with $|U| \geqslant pn/100$ and $|W| \geqslant 100/p$, such that $e(U,W) < p|U||W|/2$, is bounded by

$$\begin{aligned}\sum_{u \geqslant \frac{pn}{100}, w \geqslant \frac{100}{p}} n\binom{n}{u}\binom{n}{w} p^u e^{-puw/8} &\leqslant \sum_{u \geqslant \frac{pn}{100}, w \geqslant \frac{100}{p}} n\left(\frac{enp}{u}\right)^u \left(\frac{en}{w}\right)^w e^{-puw/8} \\ &\leqslant \sum_{u \geqslant \frac{pn}{100}, w \geqslant \frac{100}{p}} ne^{6u + w\log n - puw/8} \\ &\leqslant \sum_{u \geqslant \frac{pn}{100}, w \geqslant \frac{100}{p}} ne^{-puw/200} \\ &\leqslant n^3 e^{-n/200} = o(1),\end{aligned}$$

where we used that $u \geqslant pn/100$ implies $(epn/u)^u \leqslant (100e)^u \leqslant e^{6u}$, and that $puw/15 \geqslant 6u$ and $puw/20 \geqslant pn^2w/2000 \geqslant w \log n$.

(iii): Let W be the set of those vertices $w \in V_2$ with $d(w,U) < p^2n/200$. As

$$e(U,W) = \sum_{w \in W} d(w,U) < pn|W|/200 \leqslant p|U||W|/2,$$

from the conclusion of (ii) we deduce that $|W| < 100/p$.

(iv): Fix a constant $0 < \delta \ll \varepsilon$. We first show that w.h.p. $e(A,B) \leqslant p|A||B| + \delta pn|A|$ for all sets A and B from opposite bipartition classes such that $|A|, |B| \geqslant pn/2$. Indeed, given such sets A and B, by Lemma 10 and since $p^2n \geqslant C^2 \log n$, we have

$$\mathbb{P}(e(A,B) \geqslant p|A||B| + \delta pn|A|) \leqslant e^{-(\delta n/|B|)^2 p|A||B|/3} \leqslant e^{-2\delta^2(pn)^2/3} = o(2^{2n}),$$

thus by a union bound over all such sets A, B we conclude.

Now let U be a component of H and set $X = V_1 \cap U$ and $Y = V_2 \cap U$. By (i), w.h.p. we have $|X|, |Y| \geqslant \delta(H) \geqslant pn/2$, and therefore

$$p|X||Y| + \delta pn|X| \geqslant e_H(X,Y) = \sum_{x \in X} d_H(x) \geqslant (1/2 + \varepsilon)pn|X|.$$

Hence $|Y| \geqslant (1/2 + \varepsilon/2)n$ as $\delta \ll \varepsilon$. Similarly, we have $|X| \geqslant (1/2 + \varepsilon/2)n$. As this is true for any component U of H, we conclude that H is connected.