

Single SEM Schubert Polynomials

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Submitted: March 9, 2025; Accepted: Jan 16, 2026; Published: Mar 27, 2026

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Abstract

We prove a pattern-avoidance characterization of $w \in S_n$ such that the Schubert polynomial \mathfrak{S}_w is a standard elementary monomial. This characterization tells us which quantum Schubert polynomials admit the simplest explicit formulas. We solve a similar pattern-avoidance problem for complete homogeneous monomials.

Mathematics Subject Classifications: 05E05, 05E14

1 Introduction

Schubert polynomials form an important basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$. For each permutation $w \in S_n$, there is a Schubert polynomial \mathfrak{S}_w in variables x_1, \dots, x_{n-1} . It is natural to wonder when a Schubert polynomial is just one monomial. This question has a concise answer in terms of *pattern avoidance*:

Proposition 1. *The Schubert polynomial \mathfrak{S}_w is a single monomial $x_1^{l_1} x_2^{l_2} \dots x_{n-1}^{l_{n-1}}$ if and only if the following equivalent conditions hold:*

1. *The Lehmer code of w is nonincreasing.*
2. *w avoids the pattern 132.*

Such permutations are called *dominant*. Our first goal is to give an analogous description of when a Schubert polynomial is a single *standard elementary monomial*.

Definition 2. A standard elementary monomial (SEM) is a product

$$e_{a_1}^1 e_{a_2}^2 e_{a_3}^3 \dots = e_{\vec{a}}$$

where e_j^i is the degree- j elementary symmetric polynomial in i variables, and only finitely many of the a_i s are nonzero.

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Fomin, Gelfand, and Postnikov [1] showed that SEMs form a basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$, and that this basis has deep connections with the basis of Schubert polynomials. Specifically, they studied SEMs in the context of *quantum Schubert polynomials*.

Theorem 3 ([1]). *If $\mathfrak{S}_w = \sum_{\bar{a}} c_{\bar{a}} e_{\bar{a}}$ is an expansion of \mathfrak{S}_w into SEMs, then the quantum Schubert polynomial \mathfrak{S}_w^q can be written as $\sum_{\bar{a}} c_{\bar{a}} E_{\bar{a}}$, where $E_{\bar{a}}$ is a quantum SEM.*

That is, if we can compute the SEM expansion of \mathfrak{S}_w , then we can also easily compute \mathfrak{S}_w^q . Asking which Schubert polynomials are single SEMs is thus equivalent to asking for the quantum Schubert polynomials which are easiest to compute.

Fomin, Gelfand, and Postnikov also studied the quantization of complete homogeneous monomials (CHMs). A CHM is analogously a product

$$h_{a_1}^1 h_{a_2}^2 \cdots = h_{\bar{a}}$$

where h_j^i is the degree- j homogeneous symmetric polynomial in i variables and only finitely many of the a_i s are nonzero. They showed that the quantization map sends a complete homogeneous polynomial to a certain determinant of quantum elementary symmetric polynomials, and more generally sends a CHM to a certain product of determinants of SEMs. Thus, CHMs are also relatively straightforward to quantize.

A fair amount of attention has already been given to SEM expansions of Schubert polynomials. Winkel [2] gave a determinantal formula for SEM expansions of Schur polynomials. He also observed interesting patterns in SEM expansions for Schubert polynomials experimentally; for instance, the coefficients tend to be small in absolute value. Hatam, Johnson, Liu, and Macaulay [3] gave a determinantal formula for the SEM expansion of \mathfrak{S}_w when w avoids a list of 13 patterns.

Despite this progress, SEM expansions for Schubert polynomials more generally are far from well understood: giving a cancellation-free formula for the expansion of a Schubert polynomial into SEMs remains an open problem, and many other properties of these expansions remain mysterious. Asking for the analogues of dominant permutations in the SEM basis is thus a natural question.

Theorem 4. *A Schubert polynomial \mathfrak{S}_w is a single standard elementary monomial if and only if w avoids the patterns 312 and 1432.*

We also consider the analogous question in the setting of CHMs. The relationship between Schubert polynomials and CHMs has been less explored so far.

Theorem 5. *A Schubert polynomial \mathfrak{S}_w is a complete homogeneous monomial if and only if w avoids the patterns 321 and 231.*

As a consequence, we notice that single-monomial, single-SEM, and single-CHM Schubert polynomials are all counted by nice enumerative sequences:

Corollary 6. *The number of $w \in S_n$ such that \mathfrak{S}_w is a single monomial, standard elementary monomial, and complete homogeneous monomial, is (respectively), the Catalan number C_n , the Fibonacci number F_{2n} , and 2^{n-1} .*

In section 2, we give some concise background on Schubert polynomials and pattern avoidance.

2 Background and preliminary lemmas

2.1 Permutation conventions

We will use the following notation throughout:

1. s_i is the simple transposition $(i, i + 1)$. Also, $s_{(i,j)}$ denotes the transposition (i, j) .
2. The *length* $\ell(w)$ of w is its number of inversions.
3. Multiplying w by a transposition always swaps positions, not values, of entries. So, if $w = w_1 w_2 \dots w_n$ in one-line notation, $s_i w = w_n \dots w_{i-1} w_{i+1} w_i \dots w_n$.
4. A *descent* (respectively *ascent*) of w is a position i such that $w_{i+1} < w_i$ (respectively, $w_{i+1} > w_i$).
5. If $u = u_1 \dots u_n$ is a permutation in one-line notation, then $w \in S_N$ is said to ‘contain u as a pattern’ if there exist indices $i_1 < i_2 \dots < i_n$ such that $w_{i_1}, w_{i_2} \dots w_{i_n}$ are in the same relative order as $u_1, u_2 \dots u_n$.

2.2 Standard elementary monomials and divided difference operators

Schubert polynomials can be defined in terms of *divided difference operators*. The operator ∂_i acts by

$$\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

where $(s_i \cdot f)(x_1 \dots x_n) = f(x_1 \dots x_{i+1}, x_i \dots x_n)$. Then, \mathfrak{S}_w may be defined recursively: for the longest element $w_0 \in S_n$, we let

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$$

Otherwise, define

$$\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{s_i w}$$

whenever $\ell(s_i w) = \ell(w) - 1$, and $\partial_i(\mathfrak{S}_w) = 0$ otherwise. Notice that $\partial_i(\mathfrak{S}_w) = 0$ if and only if i is an ascent of w (flipping an ascent increases $\ell(w)$ by 1, and flipping a descent decreases $\ell(w)$ by 1).

Both SEMs and CHMs interact nicely with divided difference operators:

Lemma 7. *If $i \neq k$, then $\partial_i(e_j^k) = 0$, and $\partial_i(e_j^i) = e_{j-1}^{i-1}$. Similarly, $\partial_i(h_j^k) = 0$ if $i \neq k$, and $\partial_i(h_j^i) = h_{j-1}^{i+1}$.*

The *twisted Leibniz rule* for divided difference operators says that

$$\partial_i(pq) = \partial_i(p)q + s_i(p)\partial_i(q)$$

Combining Lemma 7 and the twisted Leibniz rule, we can observe the following useful fact:

Lemma 8. *Suppose that \mathfrak{S}_w is a single SEM $e_{\bar{a}} = e_{a_1 a_2 \dots a_n}$. Then, $a_i = 0$ if and only if i is an ascent of w . Furthermore, if i is a descent of w and $i - 1$ is an ascent, then $\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{s_i w}$ is also a single SEM.*

There is an analogous fact in the case of CHMs:

Lemma 9. *Suppose that \mathfrak{S}_w is a single CHM, $h_{a_1 a_2 \dots a_n}$. Then, $a_i = 0$ if and only if i is an ascent of w . Furthermore, if i is a descent of w and $i + 1$ is an ascent, then $\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{s_i w}$ is also a single CHM.*

2.3 Permutations and Pipe Dreams

Pipe dreams are an important combinatorial model for Schubert polynomials, developed by Billey-Jockusch-Stanely [7] (see also [8] and [6]). Formally, a (reduced) pipe dream of w is a finite subset $D \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ satisfying the following condition: the ordered product of the generators s_{i+j-1} over all $(i, j) \in D$ (taken row by row from left to right, bottom to top) is a reduced word for w .

The set of pipe dreams of w is denoted $\text{RC}(w)$. Every pipe dream D of w can be assigned a *weight*:

$$\text{wt}(D) = \prod_{(i,j) \in D} x^i$$

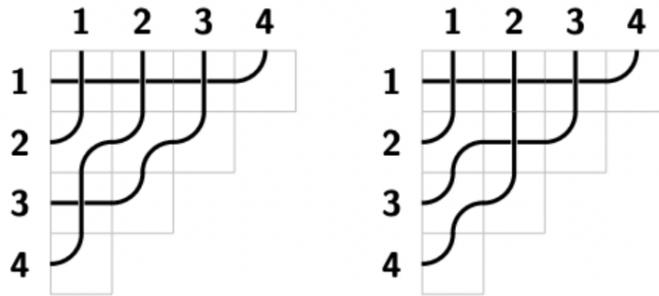
Pipe dreams for w correspond to monomials in \mathfrak{S}_w :

Theorem 10.

$$\mathfrak{S}_w = \sum_{D \in \text{RC}(w)} \text{wt}(D)$$

Pictorially, a pipe dream D can be viewed as a *strand diagram*: place a *crossing* at every $(i, j) \in D$ and a pair of *elbows* at every $(i, j) \notin D$. The *row* of a crossing is j , and its *column* is i . This process results in a pseudo-line arrangement, where each strands connects $(k, 0)$ to $(0, w(k))$ and no two strands intersect more than once.

Example 11. Below are the two pipe dreams for $w = 4132$. The weight of the first pipe dream is $x_1^3 x_3$ and the weight of the second is $x_1^3 x_2$, so, by Theorem 10, $\mathfrak{S}_{4132} = x_1^3 x_3 + x_1^3 x_2$.



Every permutation has a distinguished pipe dream, called the *bottom pipe dream*, also defined in [6]:

Definition 12. The *Lehmer code* $L(w) = (L_1, L_2 \dots L_n)$ is given by

$$L_i = |\{j > i | w(j) < w(i)\}|$$

The *bottom pipe dream* of w is the unique left-justified pipe dream for w where the number of crossings in row i is L_i .

For example, the bottom pipe dream of 4132 is given by the left pipe dream in Example 11.

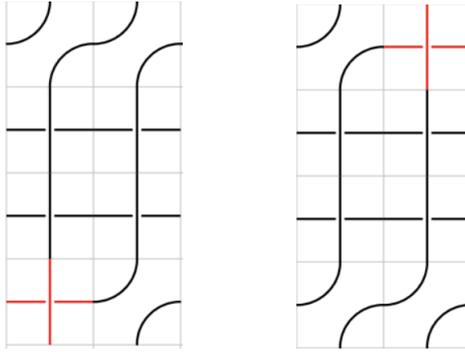
Bergeron-Billey [6] also gave a simple way to generate every pipe dream of w given its bottom pipe dream.

Definition 13. A *ladder move of order k* of $D \in RC(w)$ applied at a crossing $(i, j) \in D$ produces a new $D' \in RC(w)$ given by $D \setminus \{(i, j)\} \cup \{(i - k - 1, j + 1)\}$. A ladder move can only be applied at $(i, j) \in D$ if two conditions are satisfied:

1. $(i, j) \in D$, and $(i, j + 1), (i - k - 1, j), (i - k - 1, j + 1) \notin D$ and
2. For all $i - k \leq i' < i$, $(i', j), (i', j + 1) \in D$

Informally, the crossing (i, j) ‘climbs a ladder’ made of other crossings of D and arrives at a new square $(i - k - 1, j + 1)$.

Example 14. A ladder move of order 2 carries the bottom left crossing up by three rows and right by one column:



Theorem 15 ([6]). *Every reduced pipe dream for w can be obtained from the bottom pipe dream for w by a sequence of ladder moves.*

A ladder move of order 0 is also called a *simple ladder move*, only moving a crossing up one square and right one square. For example, the pipe dream on the right in Example 11 is obtained from the bottom pipe dream of 4132 by applying one simple ladder move.

To prove sufficiency in Theorems 4 and 5, we will make use of pipe dreams. To prove necessity, we will use 8 and *Monk's rule*. Monk's rule describes how to multiply \mathfrak{S}_w with an elementary symmetric polynomial e_1^k :

Theorem 16. *For any $w \in S_n$ and $k \in \mathbb{Z}^+$:*

$$e_1^k \mathfrak{S}_w = \sum_{(i_1, i_2)} \mathfrak{S}_{s_{i_1, i_2} w}$$

where the sum is over transpositions $s_{(i_1, i_2)}$ such that

1. $i_1 \leq k < i_2$ and
2. $s_{(i_1, i_2)} w$ covers w in the (strong) Bruhat order. (That is, $\ell(s_{(i_1, i_2)} w) = \ell(w) + 1$).

Example 17. Consider the product $e_1^2 = x_1 + x_2$ with \mathfrak{S}_w for $w = 3142$. The pairs of positions (1, 3), (2, 3) and (2, 4) are the only pairs satisfying both conditions in Monk's rule. Therefore:

$$e_1^2 \mathfrak{S}_{3142} = \mathfrak{S}_{4132} + \mathfrak{S}_{3412} + \mathfrak{S}_{3241}$$

3 Proofs of Theorems 4 and 5

3.1 Proof of Theorem 4

To prove Theorem 4, we first prove an equivalent characterization of 1432 and 312-avoiding permutations:

Lemma 18. *A permutation w avoids 1432 and 312 if and only if the Lehmer code $L(w) = (L_1 \dots L_n)$ satisfies the following three rules:*

1. *At every step, the Lehmer code changes by at most one. That is, $|L_i - L_{i+1}| \leq 1$.*
2. *Between any two increases in $L(w)$, there is at least one decrease. That is, if $L_i < L_{i+1}$ and $L_j < L_{j+1}$ for $i < j$, there is some k between i and j such that $L_k > L_{k+1}$.*

Proof. In one direction, first assume that w has a 1432 pattern. By definition, we can find indices $i < j < k < l$ with w_i, w_j, w_k, w_l in relative order 1432. Fixing j, k and l , choose i so that it is leftmost in w . Then, we claim that $L(i) + 2 \leq L(j)$. This follows because of the following three observations:

- $w_j > w_k, w_l > w_i$
- For all $i < i' < j$, $w_{i'} > w_i$ (no values between i and j contribute to $L(i)$) and
- for any $i' > j$ such that $w_{i'} < w_i$, $w_{i'}$ is also less than w_j (any value contributing to $L(i)$ also contributes to $L(j)$)

But if w follows the two Lehmer code rules above, it is impossible to have $i < j$ with $L(i) + 2 \leq L(j)$. Therefore, if w contains a 1432 pattern, it cannot satisfy both restrictions on $L(w)$.

Now, assume that w has a 312 pattern, so that we can find indices $i < j < k$ with w_i, w_j, w_k in relative order 312. We can assume that $j = i + 1$ (suppose there exists some i' between i and j . If $w_{i'} > w_k$, then the indices i', j, k also form a 312 pattern. If $w_{i'} < w_k$, then indices i, i', k form a 312 pattern. So, if $j \neq i + 1$, we can always choose the indices i, j closer together).

Then, for a 312 pattern $i, i + 1, k$, we have $L_i \geq L_{i+1} + 2$, since $w_i > w_{i+1}, w_k$ and $w_{i+1} < w_k$. So, if w contains a 312 pattern, then it does not satisfy the second Lehmer code restriction.

Now, we prove the other direction. First, suppose that we can find a pair of indices $i < j$ such that $L_i < L_j - 1$. Then, $w_i < w_j$, and there exist at least two indices $k > l > j$ such that

$$w_{i+1} > w_k, w_l > w_i$$

If $w_k > w_l$, then indices $i + 1, l, k$ give us a 312 pattern. If $w_k < w_l$, then indices $i, i + 1, l, k$ give us a 1432 pattern. Therefore, if $L_i - L_{i+1} \geq 2$, or there are two increases in the sequence $L(w)$ without any decreases between them, w contains either 1432 or 312.

Finally, suppose we can find an i such that $L_i > L_{i+1} + 1$. Then, $w_i > w_{i+1}$, and there is at least one index $j > i + 1$ with $w_i > w_j > w_{i+1}$. So, indices $i, i + 1, j$ give us a 312 pattern. □

Remark 19. The fact that the number of permutations in S_n avoiding 1432 and 312 is the Fibonacci number F_{2n} was proved by West [?], using generating functions and trees.

Proposition 18 gives us another simple proof of this fact. If $L(w)$ is the Lehmer code of w , then consider the lattice path P with vertices

$$(0, 0) = (0, L_n), (1, L_{n-1}) \dots (n-1, L_1)$$

The first condition on $L(w)$ in Lemma 18 implies that P is a *Motzkin meander*: a lattice path starting at $(0, 0)$ taking steps $D = (-1, 1), U = (1, 1), H = (1, 0)$ that never crosses under the x -axis. P satisfies the additional condition, coming from the second condition in Lemma 18, that between any two D steps there is at least one U step. Any such path corresponds to the Lehmer code of a unique permutation w , since such a P never increases above the line $y = x$.

To choose such a P , choose k steps in the lattice path to be H -steps. Of the remaining steps, assign the first one to be U (otherwise, the path goes below the x -axis). There are $n - k - 1$ steps left, and we can choose any non-adjacent subset of the remaining steps to be D . The number of such subsets is F_{n-k-1} . Thus, the number of such P is

$$\sum_{k=0}^n \binom{n}{k} F_{n-k-1} = F_{2n}$$

Lemma 20. *If $L(w)$ satisfies both conditions in 18, then \mathfrak{S}_w is a standard elementary monomial.*

Proof. If w is a dominant permutation, then the statement is clear: recall that if w is dominant, then $L(w)$ is nonincreasing and $\mathfrak{S}_w = x^{L(w)}$. Then, we have

$$\mathfrak{S}_w = x^{L(w)} = \prod_{i \in D(w)} e_i^i$$

where $D(w)$ is the set of descents of w .

Otherwise, consider the bottom pipe dream P of w . There is a unique *dominant* permutation w' with bottom pipe dream P' satisfying two conditions:

1. The Lehmer code of w' decreases by at most 1 at each step and
2. We can obtain P from P' by adding at most one crossing to the end of each row of crossings in P' . No crossing is added to the end of a row i where P has more crossings in row i than in row $i + 1$.

(That is, P is almost P' but has at most one extra crossing in each row). Call the crossings we add to P' the *outer crossings* of P . The outer crossings are grouped together in width one vertical columns, which we call *outer columns*. (See Example 20).

Theorem 15 tells us that every reduced pipe dream for w can be obtained from P by applying a sequence of ladder moves. Here, the only ladder moves that can ever be performed are *simple* ladder moves applied to the outer crossings. That is, if we imagine diagonal rails extending out from each outer crossing, the only pipe dreams for w are obtained by sliding each outer crossing along the rails.

Furthermore, suppose that in P , crossings in squares $(i, j), (i', j)$ with $i' < i$ lie in the same outer column. Then, the first crossing will always be in a row strictly higher than the row of the second crossing, even after applying any sequence of simple ladder moves. However, outer crossings that start in different outer columns do not interact with each other. Specifically, after any sequence of ladder moves, two crossings that started in different outer columns are never in adjacent squares.

From these observations, we see that each outer column contributes a factor of e_j^i , where i is the position of the lowest crossing in the outer column, and j is the number of crossings in the outer column. Since different outer columns do not interact with each other, simply multiplying these factors, along with the SEM associated to the dominant w' , gives us the SEM \mathfrak{S}_w . \square

Example 21. Below is the bottom pipe dream P for 35427861, with crossings denoted by $+$ and elbows denoted by empty squares. The outer crossings of P are shaded, and the associated dominant permutation is $w' = 34526781$. There are two outer columns, one with two crossings and one with one crossing.

+	+			
+	+	+		
+	+			
+				
+	+			
+	+			
+				

As described in the proof of 20, the first outer column contributes a factor of e_1^2 , the second contributes a factor of e_2^6 , and $\mathfrak{S}_{w'}$ contributes a factor of $x^{L(w')} = e_3^3 e_7^7$. Therefore,

$$\mathfrak{S}_{35427861} = e_1^2 e_3^3 e_2^6 e_7^7$$

Lemma 22. *Let $w \in S_n$, and suppose there exists j such that $L_{j+1} - L_j = k < -1$. Then, \mathfrak{S}_w cannot be a standard elementary monomial.*

Proof. Suppose that $\mathfrak{S}_w = e_{\vec{a}}$; then, consider the maximal monomials (in the reverse lexicographic ordering) of both sides. The maximal monomial of \mathfrak{S}_w is $x^{L(w)}$. Meanwhile, the maximal monomial of $e_{\vec{a}}$ is obtained by multiplying the maximal monomials of each $e_{a_i}^i$. For each j , there is at most one i where the maximal monomial of $e_{a_i}^i$ has a factor of x_j and not x_{j+1} , namely, $i = j$. Therefore, we must have $L_j \leq L_{j+1} + 1$, as desired. \square

Lemma 23. *Suppose that either $L_{i+1} - L_i = k > 1$, or there exist two increases in $L(w)$ with no decreases between them. Then, \mathfrak{S}_w cannot be a standard elementary monomial.*

Proof. We will induct on the length $\ell(w)$. By contradiction, first suppose that $L_{i+1} - L_i = k > 1$, but \mathfrak{S}_w is a single SEM $e_{\vec{a}}$. By Lemma 22, there must be at least k positions after position i where $L(w)$ decreases by 1. Furthermore, each such position is a descent of w .

First, suppose that $i+1, i+2, \dots, i+k$ are not all descents of w . Then, we can find some position $j > i+1$ such that j is a descent of w and $j-1$ is not. Then, $\ell(w) = \ell(s_j w) + 1$. Furthermore, by Lemma 8, $\partial_j(\mathfrak{S}_w) = \mathfrak{S}_{s_j w}$ is still a single SEM. However, since $j > i$, left-multiplying w by s_j does not affect L_i or L_{i+1} . We still have $L_i - L_{i+1} > 1$, which is a contradiction by inducting on $\ell(w)$.

Now, we may assume that $i+1, i+2, \dots, i+k$ are all descents of w , so that $L(w)$ has the form

$$(\dots, L_i, L_i + k, L_i + k - 1, L_i + k - 2 \dots L_i + 1, L_i \dots)$$

Let $w' = s_{i+1} w$. Since $i+1$ is a descent of w and i is not, $\mathfrak{S}_{w'}$ is a single SEM by Lemma 8. Moreover, if $\mathfrak{S}_w = e_{\vec{a}}$, we have $a_{i+1} = 1$ (by Lemma 7). We must therefore have

$$e_1^{i+1} \mathfrak{S}_{w'} = \mathfrak{S}_w \tag{1}$$

On the other hand, Monk's rule tells us that

$$e_1^{i+1} \mathfrak{S}_{w'} = \sum_{(i_1, i_2)} \mathfrak{S}_{s_{(i_1, i_2)} w'}$$

where the sum is over $i_1 \leq i+1 < i_2$ such that $s_{(i_1, i_2)} w'$ covers w' in the (strong) Bruhat order.

We claim that there are at least two summands when we expand e_1^{i+1} using Monk's rule, which will give us our contradiction. Indeed, we could take either $i_1 = i+1$ and $i_2 = i+2$, or $i_1 = i$ and $i_2 = i+k+1$. To see why $s_{(i, i+k+1)} w'$ covers w' in the strong Bruhat order, notice that $w_i < w_{i+k+1}$ since $L_i = L_{i+k+1}$. Furthermore, for all $i < j < i+k+1$, we have $w_j > w_{i+k+1}$ (since j is a descent of w for all such j), so that no w_j is between w_i and w_{i+k+1} . The pairs $(i+1, i+2)$ and $(i, i+k+1)$ therefore give us at least two summands in the Schubert expansion of $e_1^{i+1} \mathfrak{S}_{w'}$.

It remains to show that if there are two increases in $L(w)$ with no decreases in between, then \mathfrak{S}_w cannot be an SEM. The proof is very similar. Suppose that $L_i < L_{i+1}$, $L_j < L_{j+1}$ with $j > i$, and for all $i < k < j$, $L_k = L_{k+1}$. By the same reasoning as before, we may assume that $j+1$ and $j+2$ are both descents of w (otherwise, induct on $\ell(w)$ using Lemma 8). Now, consider $w' = s_{j+1} w$. Again by 8, $\mathfrak{S}_{w'}$ must be a single SEM, so that we must have

$$e_1^{j+1} \mathfrak{S}_{w'} = \mathfrak{S}_w \tag{2}$$

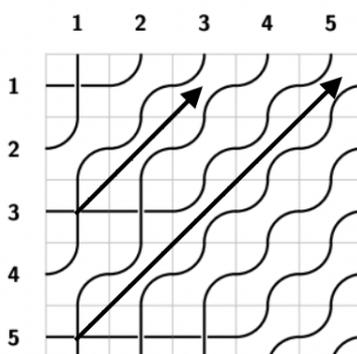
Here, when we apply Monk's rule to expand e_1^{j+1} , we could take either $(i_1, i_2) = (j+1, j+2)$ or $(i, j+3)$. This gives us our contradiction. □

Combining Lemmas 20, 22, and 23 completes the proof of Theorem 4.

3.2 Proof of Theorem 5

Lemma 24. *Suppose that w avoids patterns 321 and 231. Then, the bottom pipe dream P of w satisfies the following property: choose the leftmost crossing of any row of P and draw a diagonal ray pointing northeast from that crossing. Then, this ray never intersects a square containing a crossing in P , nor does any square immediately to the left of this ray contain a crossing in P .*

Example 25. Below is the bottom pipe dream for 1427356. If we draw a northeast ray out from the first crossing in row 3, then this diagonal line never intersects, neither does it pass through a square directly to the right of a crossing. However, the ray emitting from the first box in row 5 enters the square directly to the right of the rightmost crossing in row 3. Indeed, the subsequence 473 is a 231 pattern in 1427356.



Proof. For contradiction, let i be the largest row index contradicting the statement of Lemma 24. That is, the northeast ray coming from the leftmost crossing of row i intersects, or is directly right of, a crossing in some row $j < i$. Since $L_i > 0$ and i was the largest such row, we must have $L_{i+1} = 0$, and thus, since $L_i > L_{i+1}$, we have $w_i > w_{i+1}$. But then, we claim that $w_j > w_{i+1}$. Either $w_j > w_i$, in which case this claim follows by transitivity, or $w_j < w_{i+1}$, in which case, $L_{i+1} \geq L_j + (i - j) + 1 > 0$. Either way, we have either a 231 or a 321 pattern in w given by the indices $j, i, i + 1$. So, we have proved the claim. \square

Lemma 24 allows us to prove that 321 and 231-avoidance is sufficient for \mathfrak{S}_w to be a CHM:

Lemma 26. *If w avoids 231 and 321, then \mathfrak{S}_w is a single CHM. In particular, in this case $\mathfrak{S}_w = h_L(w)$.*

Proof. We use again the fact that all pipe dreams are obtained from the bottom pipe dream by ladder moves. Here, again all of the ladder moves we can perform are simple ladder moves that slide crossings northeast along their diagonals. Two crossings in the same row of the bottom pipe dream can never slide past each other (by the definition of ladder moves), and any two rows can slide independently of each other by Lemma 24.

Thus, row i contributes a factor $h_{L(i)}^i$, and multiplying these factors together gives us $\mathfrak{S}_w = \prod_i h_{L(i)}^i$. \square

Note that, unlike the case of SEMs and analogously to the case for usual monomials, the reverse-lexicographically maximal monomial in the CHM expansion of \mathfrak{S}_w is always $h_{L(w)}$.

Finally, we prove that 321 and 231 avoidance are necessary conditions in order for \mathfrak{S}_w to be a CHM.

Lemma 27. *If w contains a 321 pattern, then \mathfrak{S}_w is not a CHM.*

Proof. We induct on the length of w . First, suppose that w contains a 321 pattern and \mathfrak{S}_w is a single CHM. Let $i < j < k$ be indices such that $w_i > w_j > w_k$. First, we show how to reduce to the case where $i, j, k = i, i + 1, i + 2$.

If i is an ascent of w , choose new indices $i + 1, j, k$ which also give us a 321 pattern. If i and $i + 1$ are both descents of w , then we have a consecutive 321 pattern $i, i + 1, i + 2$. Otherwise, suppose that i is a descent of w and $i + 1$ is an ascent. We may apply Lemma 9 to conclude that $\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{s_i w}$ is also a single CHM. However, $s_i w$ contains either a 321 pattern (if $i + 1 < j$, then indices $i + 1, j, k$ give us a 321 pattern in $s_i w$) or a 231 pattern (if $i + 1 = j$, then $i, i + 1, k$ are the indices of a 231 pattern in $s_i w$). Inducting on the length of w , we get a contradiction.

Therefore, we may assume that we have a 321 pattern given by indices $i, i + 1, i + 2$. We may also assume that $i + 2$ is not a descent of w (otherwise, indices $i + 1, i + 2, i + 3$ also give a 321 pattern. Consider that pattern instead). By Lemma 9, $\partial_{i+1}(\mathfrak{S}_w) = \mathfrak{S}_{s_{i+1} w}$ is also a single CHM.

If $w_{i+3} < w_{i+1}$, then we can still find a 321 pattern in $s_j w$, given by indices $i, i + 2, i + 3$. By induction on the length of w , this is a contradiction. Otherwise, $i + 2$ is an ascent of $s_j w$. In this case, using Lemma 9 we deduce that we must have

$$h_1^{i+1} \mathfrak{S}_{s_j w} = \mathfrak{S}_w$$

However, similarly to before, this is impossible by Monk's Rule. Namely, in the expansion of $h_1^{i+1} \mathfrak{S}_{s_j w}$ into the Schubert basis, there is a summand corresponding to the pair $(i + 1, i + 2)$; there is also at least one more summand given by (i, i') , where i' is the first index greater than $i + 2$ such that $w_{i'} > w_i$. This gives us our contradiction. \square

Lemma 28. *If w contains a 231 pattern, then \mathfrak{S}_w is not a CHM.*

Proof. Let $i < j < k$ be indices such that $w_k < w_i < w_j$. By similar reasoning to before, we may assume that $j = i + 1$ and $k = i + 2$ by considering whether i is an ascent of w , $i, i + 1$ are both descents of w , or i is a descent of w and $i + 1$ is an ascent; we leave these details to the reader.

Now, given that $i, i + 1, i + 2$ form a 231 pattern, we may also assume that $i + 2$ is not a descent of w ; otherwise, $i + 1, i + 2, i + 3$ form a 321 pattern, which we have already ruled out. By Lemma 8, we deduce that we must have

$$h_1^{i+1} \cdot \mathfrak{S}_{s_{i+1} w} = \mathfrak{S}_w$$

But once again, we get a contradiction to Monk's Rule. Namely, w covers $s_{i+1}w$ in the strong Bruhat order, but so does $s_{i+1}s_{(i,i+2)}w$. Thus, the expansion of $h_1^{i+1} \cdot \mathfrak{S}_{s_{i+1}w}$ in the Schubert basis has at least two summands and cannot be equal to \mathfrak{S}_w . By contradiction, w cannot contain a 231 pattern. \square

Combining Lemmas 27, 28, and 26 proves Theorem 5.

4 Open Problems

Why does the pattern 1432 appear in Theorem 4? Recall that in our proof of Lemma 20, the only ladder moves we could apply to reduced pipe dreams of w were *simple* ladder moves. Gao [4] showed more generally that w avoids the pattern 1432 if and only if all reduced pipe dreams for w are related by simple ladder moves. Motivated by this fact, we ask the following questions:

1. Is there a cancellation-free formula for the SEM expansion of \mathfrak{S}_w , at least in the case where w is 1432-avoiding?
2. Is there a bound in terms of the number of 1432 patterns in w on the number of terms in the SEM expansion of w ? (Or in terms of 312 patterns in w ?)
3. In particular, is there a nice description of $w \in S_n$ such that the SEM expansion of \mathfrak{S}_w has only two terms?

Acknowledgements

We thank Ilani Axelrod-Freed, Matt Samuel, and Son Nguyen for interesting conversations about SEM expansions of Schubert polynomials.

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