Refined Product Formulas for Tamari Intervals

Alin Bostan[a] Frédéric Chyzak[b] Vincent Pilaud[c]

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Abstract

We provide surprising product formulas for the f-vectors of the canonical complexes of the Tamari lattices and of the cellular diagonals of the associahedra.

Mathematics Subject Classifications: 05A15, 05A19, 06A07

Introduction

Consider the *Tamari lattice* $\mathrm{Tam}(n)$, whose elements are the binary trees with n nodes, and whose cover relations are given by right rotations [Tam51]. An *interval* (or *relation*) is a pair $S \leqslant T$ in $\mathrm{Tam}(n)$. For a binary tree T, we denote by $\mathrm{des}(T)$ (resp. by $\mathrm{asc}(T)$) the number of binary trees covered by T (resp. covering T) in the Tamari lattice. In other words, $\mathrm{des}(T)$ (resp. $\mathrm{asc}(T)$) is the number of edges $i \to j$ in T with $i > j$ (resp. with $i < j$) when we orient the edges of T towards its root and label the nodes of T in inorder (meaning that we recursively label the left subtree of T, then the root, and then recursively label the right subtree of T). The purpose of this paper is to prove the following two surprising formulas, whose first few values are gathered in Tables 1 and 2.

Theorem 1. *For any $n, k \in \mathbb{N}$, the number $a_{n,k}$ of intervals $S \leqslant T$ of the Tamari lattice $\mathrm{Tam}(n)$ such that $\mathrm{des}(S) + \mathrm{asc}(T) = k$ is given by*

$$a_{n,k} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}.$$

Theorem 2. *For any $n, k \in \mathbb{N}$, the sum $b_{n,k}$ of the binomial coefficients $\binom{\mathrm{des}(S)+\mathrm{asc}(T)}{k}$ over all intervals $S \leqslant T$ of the Tamari lattice $\mathrm{Tam}(n)$ is given by*

$$b_{n,k} = \sum_{\ell=k}^{n-1} a_{n,\ell} \binom{\ell}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}.$$

[a] Inria, Sorbonne Université, LIP6, Paris, France (alin.bostan@inria.fr).
[b] Inria, Palaiseau, France (frederic.chyzak@inria.fr).
[c] Universitat de Barcelona & Centre de Recerca Matemàtica, Barcelona (vincent.pilaud@ub.edu).

$n\backslash k$	0	1	2	3	4	5	6	7	8	Σ
1	1									1
2	1	2								3
3	1	6	6							13
4	1	12	33	22						68
5	1	20	105	182	91					399
6	1	30	255	816	1020	408				2530
7	1	42	525	2660	5985	5814	1938			16965
8	1	56	966	7084	24794	42504	33649	9614		118668
9	1	72	1638	16380	81900	215280	296010	197340	49335	857956

Table 1: The first few values of $a_{n,k} = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}$. Note that the first column is 1, the second column is $n(n-1)$ [OEI10, A002378], the last three diagonals are [OEI10, A004321], [OEI10, A006630], and [OEI10, A000139], and the column sum is [OEI10, A000260]. The nth row gives the f-vector of the canonical complex of the Tamari lattice $\mathrm{Tam}(n)$.

$n\backslash k$	0	1	2	3	4	5	6	7	8
1	1								
2	3	2							
3	13	18	6						
4	68	144	99	22					
5	399	1140	1197	546	91				
6	2530	9108	12903	8976	3060	408			
7	16965	73710	131625	123500	64125	17442	1938		
8	118668	604128	1302651	1540770	1078539	446292	100947	9614	
9	857956	5008608	12660648	18086640	15958800	8898240	3058770	592020	49335

Table 2: The first few values of $b_{n,k} = \frac{2}{(3n+1)(3n+2)}\binom{n-1}{k}\binom{4n+1-k}{n+1}$. Note that the first column is [OEI10, A000260] while the diagonal is [OEI10, A000139]. The nth row gives the f-vector of the cellular diagonal of the $(n-1)$-dimensional associahedron.

These formulas are of interest for two main reasons. First, they count the faces of two complexes defined from the Tamari lattice and the associahedron: $(a_{n,k})_{0\leqslant k<n}$ is the f-vector of the *canonical complex* of the Tamari lattice [Rea15, Bar19, AP23], while $(b_{n,k})_{0\leqslant k<n}$ is the f-vector of the *cellular diagonal of the associahedron* [SU04, MS06, Lod11, MTTV21, LA22]. Second, they have relevant specializations: on the one hand, $\sum_{\ell=0}^{n-1} a_{n,\ell} = b_{n,0} = \frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1}$ counts all Tamari intervals [Cha07] and *rooted 3-connected planar triangulations with $2n+2$ faces* (see [BB09, FFN25] for bijections, and [OEI10, A000260] for more informations), and on the other hand, $a_{n,n-1} = b_{n,n-1} = \frac{2}{n(n+1)}\binom{3n}{n-1}$ counts *synchronized Tamari intervals*, *rooted non-separable planar maps with $n+1$ edges*, and *2-stack sortable permutations of $[n]$*, among others (see [PRV17, FPR17] and [OEI10, A000139] for more informations).

The paper is organized as follows. In Section 1, we present the connection between $a_{n,k}$

and the canonical complex of the Tamari lattice, and the connection between $b_{n,k}$ and the cellular diagonal of the associahedron. In Section 2, we show that the analytic approach of [Cha07, Cha18] can be adapted[1] to prove Theorems 1 and 2. Finally, in Section 3, we discuss bijective considerations[2] and note that bijective proofs of Theorems 1 and 2 can be derived from [FH24, FFN25].

1 Canonical complex of the Tamari lattice and diagonal of the associahedron

In this section, we interpret the numbers $a_{n,k}$ in terms of the canonical complex of the Tamari lattice (Section 1.1) and the numbers $b_{n,k}$ in terms of the cellular diagonal of the associahedron (Section 1.2). These two interpretations are our motivations to study $a_{n,k}$ and $b_{n,k}$, but are not used beyond this section. Rather than giving all details of the definitions of these objects, we thus prefer to refer to the original articles and only gather the essential material to make the connection.

1.1 Canonical complex of the Tamari lattice

A lattice $(L, \leqslant, \wedge, \vee)$ is *join semidistributive* when $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$. Any $x \in L$ then admits a *canonical join representation*, which is a minimal irredundant representation $x = \bigvee J$ (for the order $J \leqslant J'$ if for any $j \in J$, there is $j' \in J'$ with $j \leqslant j'$). The *canonical join complex* [Rea15, Bar19] of a join semidistributive lattice L is the simplicial complex of canonical join representations of the elements of L. Note that the dimension of the face of the canonical complex corresponding to an element x of L is the size of its canonical join representation, which is the number of elements covered by x in L. We define dually meet semidistributive lattices and their canonical meet complexes, and say that L is *semidistributive* when it is both join and meet semidistributive. The *canonical complex* [AP23] of a semidistributive lattice L is the simplicial complex whose faces are $J \sqcup M$ where $x = \bigvee J$ is the canonical join representation and $y = \bigwedge M$ is the canonical meet representation for an interval $x \leqslant y$ in L. Note that the dimension of the face of the canonical complex corresponding to an interval $x \leqslant y$ is the number of elements covered by x in L plus the number of elements covering y in L. Observe also that the canonical complex is flag, meaning that it is the clique complex of its graph.

Example 3. The Tamari lattice is semidistributive. Its join (resp. meet) irreducible elements are given by binary trees T with $\operatorname{des}(T) = 1$ (resp. with $\operatorname{asc}(T) = 1$), *i.e.* with a single right (resp. left) edge. Such a tree is made by glueing two left (resp. right) combs along a right (resp. left) edge, and can thus be encoded by an arc. The canonical join (resp. meet) representation of a binary tree T is a non-crossing arc diagram with one arc

[1] We note that several analytic approaches (via creative telescoping, via binomial sums, or via a holonomic recurrence system) are actually possible, as was discussed in a preliminary longer version of this paper [BCP24].

[2] Further considerations, in particular on the (im)possibility to refine the formulas of Theorems 1 and 2 by natural combinatorial parameters can be found in [BCP24].

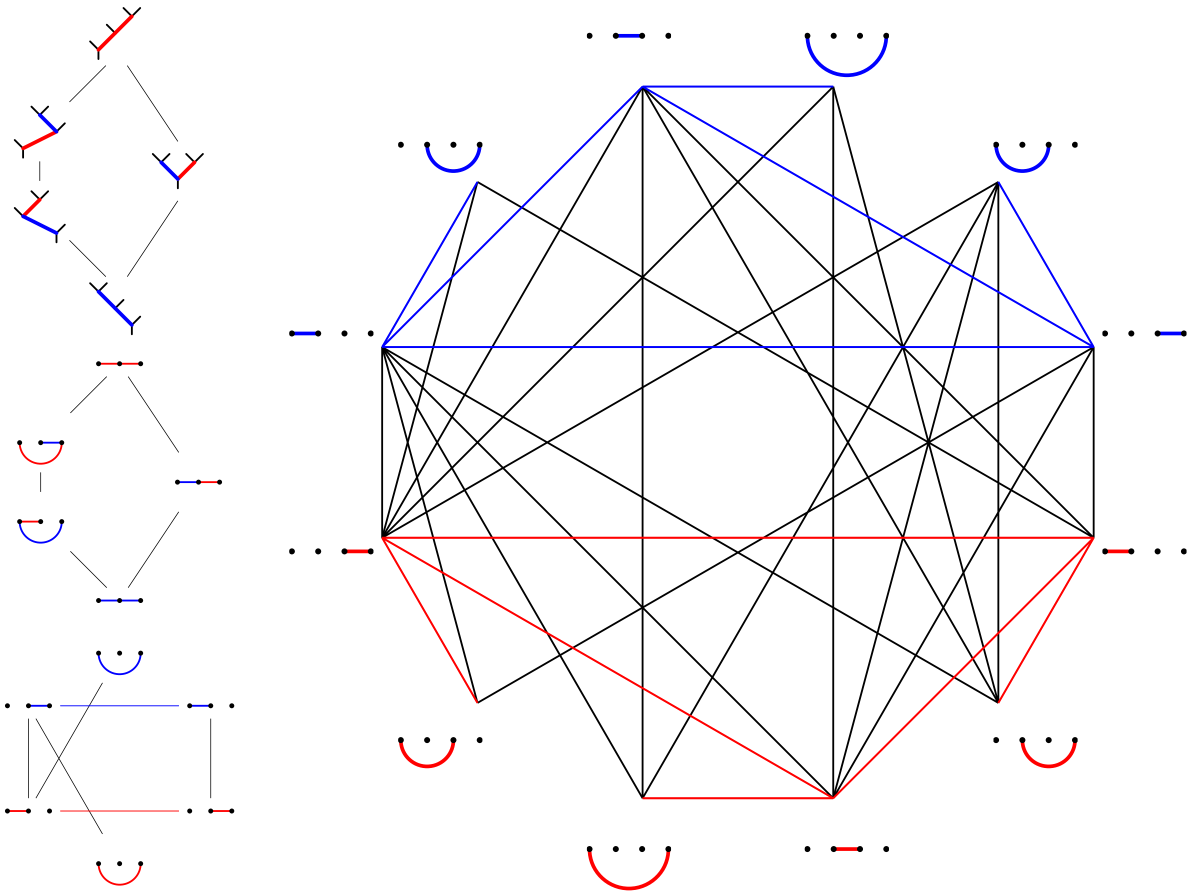


Figure 1: The canonical complex of the Tamari lattice. Left: The Tamari lattice Tam(2) seen on binary trees (top) and on semi-crossing arc bidiagrams (middle), and the canonical complex of Tam(2) (bottom), with f-vector $(1, 6, 6)$. Right: The canonical complex of Tam(3), with f-vector $(1, 12, 33, 22)$.

for each right (resp. left) edge of T, which is also known as the non-crossing partition corresponding to T. Moreover, for a Tamari interval $S \leqslant T$, an arc j of the canonical join representation of S can cross an arc m of the canonical meet representation of T only if j passes from above to below m. The canonical complex of the Tamari lattice is thus called the semi-crossing complex. This complex was extensively studied in [AP23] (note that the canonical complex of the Tamari lattice is just the restriction to down arcs of the canonical complex of the weak order, which was the one actually studied in [AP23]). It is illustrated in Figure 1. The top left picture shows the Tamari lattice where in each binary tree, the descents are colored red, and the ascents are colored blue. The middle left picture is the translation on arcs, obtained by flattening each tree to the horizontal line. The bottom left picture is the semi-crossing complex, thus the canonical complex of the Tamari lattice when $n = 3$ (note that it has indeed 13 faces: the empty set, 6 vertices, and 6 edges). The right picture is the semi-crossing complex, thus the canonical complex of the Tamari lattice when $n = 4$ (note that it has indeed 68 faces: the empty set, 12 vertices, 33 edges, and 22 triangles). Note that we only draw the graphs of the canonical complexes, since they are flag simplicial complexes.

We are now ready to observe the connection between the numbers $a_{n,k}$ of Theorem 1 and the f-vector of the canonical complex of the Tamari lattice. Recall that the *f-vector* of a d-dimensional polytopal complex of $\mathcal{C}$ is the vector $(f_0, f_1, \ldots, f_d)$ where f_i denotes the number of i-dimensional faces of $\mathcal{C}$.

Proposition 4. *The f-vector of the canonical complex of the Tamari lattice* $\mathrm{Tam}(n)$ *on binary trees with n nodes is given by $(a_{n,k})_{0 \leqslant k < n}$.*

Proof. The dimension of the face of the canonical complex of the Tamari lattice corresponding to an interval $S \leqslant T$ is the number of binary trees covered by S plus the number of binary trees covering T, which is precisely $\mathrm{des}(S) + \mathrm{asc}(T)$. Hence, the number of k-dimensional faces of the canonical complex of $\mathrm{Tam}(n)$ is given by $a_{n,k}$. □

1.2 Diagonal of the associahedron

The diagonal of a polytope P is the map $\delta : P \to P \times P$ defined by $x \mapsto (x, x)$. A *cellular approximation* of the diagonal of P (or just *cellular diagonal* of P for short) is a map $\tilde{\delta} : P \to P \times P$ homotopic to δ, which agrees with δ on the vertices of P, and whose image is a union of faces of $P \times P$. For a family of polytopes whose faces are products of polytopes in the family (like simplices, cubes, permutahedra or associahedra among others), some algebraic purposes additionally require the cellular diagonal to be compatible with the face structure. Finding cellular diagonals in such families of polytopes is a difficult and important challenge at the crossroad of operad theory, homotopical algebra, combinatorics and discrete geometry, see [SU04, MS06, Lod11, MTTV21, LA22] and the references therein.

Here, we focus on the associahedra. Algebraic diagonals for the associahedra were found in [SU04] and later in [MS06, Lod11]. The first topological diagonal for the associahedra, as defined above, was given in [MTTV21] for the realizations of the associahedra of [Lod04, SS93]. It recovers, at the cellular level, all the previous formulas [SU24, DOLAPS23]. We simply denote by Δ_d the cellular diagonal of the d-dimensional associahedron of [Lod04, SS93] constructed in [MTTV21]. The faces of Δ_d are given by the following description, called the *magical formula*.

Proposition 5 ([MTTV21, Thm. 2])**.** *The k-dimensional faces of the cellular diagonal Δ_d correspond to the pairs (F, G) of faces of the associahedron with*

$$\dim(F) + \dim(G) = k \qquad \text{and} \qquad \max(F) \leqslant \min(G)$$

where $\leqslant$, max *and* min *refer to the order given by the Tamari lattice.*

The method of [MTTV21], fully developed in [LA22] relies on the theory of fiber polytopes of [BS92]. It enables to see the cellular diagonal of the associahedron as a polytopal complex refining the associahedron, a point of view we shall adopt in our figures for the rest of the paper.

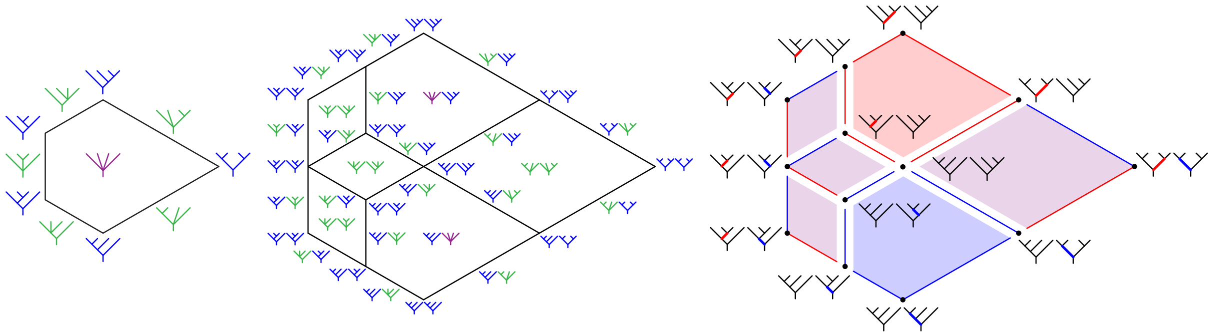


Figure 2: Left: The 2-dimensional associahedron with its faces labeled by Schröder trees with 4 leaves (in particular, its vertices correspond to binary trees). Middle: The cellular diagonal Δ_2 with its faces labeled by pairs of Schröder trees given by the magical formula (in particular, its vertices correspond to Tamari intervals). Right: The decomposition of the cellular diagonal Δ_2 obtained by associating each face (F, G) to the Tamari interval $\max(F) \leqslant \min(G)$. The f-vector is $(13, 18, 6)$.

Example 6. The cellular diagonal Δ_2 is illustrated in Figure 2. The left picture is the 2-dimensional associahedron, with faces labeled by Schröder trees (the colors depend on the dimension), and in particular with vertices labeled by binary trees. The middle picture is the cellular diagonal Δ_2 seen as a polyhedral complex refining the 2-dimensional associahedron, with faces labeled by pairs (F, G) of Schröder trees, and in particular with vertices labeled by Tamari intervals. The right picture is a decomposition of Δ_2, where each face (F, G) is associated to the Tamari interval $\max(F) \leqslant \min(G)$. In other words, the Tamari interval associated to a pair (F, G) of Schröder trees is obtained by replacing each p-ary node of F (resp. of G) by a right (resp. left) comb with p leaves. For each Tamari interval $S \leqslant T$, we have colored in red (resp. blue) the edges of S (resp. of T) corresponding to descents of S (resp. to ascents of T).

We are now ready to observe the connection between the numbers $b_{n,k}$ of Theorem 2 and the f-vector of the cellular diagonal of the $(n-1)$-dimensional associahedron.

Proposition 7. *The f-vector of the cellular diagonal Δ_{n-1} of the $(n-1)$-dimensional associahedron is given by $(b_{n,k})_{0 \leqslant k < n}$.*

Proof. For each binary tree T, there are precisely $\binom{\operatorname{des}(T)}{\ell}$ (resp. $\binom{\operatorname{asc}(T)}{\ell}$) ℓ-dimensional faces of the associahedron whose maximal (resp. minimal) vertex is T, because the associahedron is a simple polytope. We thus directly derive from the magical formula of Proposition 5 that the number of k-dimensional faces of Δ_{n-1} is

$$\sum_{S \leqslant T} \sum_{0 \leqslant \ell \leqslant k} \binom{\operatorname{des}(S)}{\ell} \binom{\operatorname{asc}(T)}{k-\ell} = \sum_{S \leqslant T} \binom{\operatorname{des}(S) + \operatorname{asc}(T)}{k} = b_{n,k}. \qquad \square$$

Remark 8. This proof can also be interpreted on Figure 2. Namely, by attaching each face (F, G) to the Tamari interval $\max(F) \leqslant \min(G)$, we have partitioned the face poset

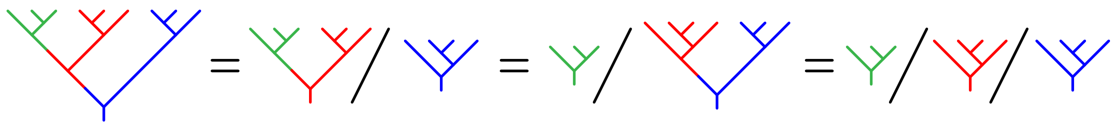


Figure 3: All grafting decompositions of a binary tree.

of Δ_{n-1} into boolean lattices based at its vertices. As the boolean lattice attached to a Tamari interval $S \leqslant T$ has rank $\mathrm{des}(S) + \mathrm{asc}(T)$, we obtain that the number of k-dimensional faces in this part of the face poset is $\binom{\mathrm{des}(S)+\mathrm{asc}(T)}{k}$.

2 Analytic proof

We provide an analytic proof of Theorems 1 and 2 following the approach of F. Chapoton [Cha07, Cha18].

2.1 Grafting decompositions

We first obtain a polynomial equation satisfied by the generating function $A(t, z) := \sum a_{n,k} t^n z^k$, that will be exploited in Sections 2.2 and 2.3 to derive Theorems 1 and 2. Following the approach of [Cha07, Cha18], we use a standard decomposition of Tamari intervals that naturally introduces an additional catalytic variable.

We denote by S/S' (resp. by $S'\backslash S$) the binary tree obtained by grafting the root of S on the leftmost (resp. rightmost) leaf of S'. A grafting decomposition of S is an expression $S = S_0/S_1/\dots/S_k$ where S_i is a binary tree with at least a node. In other words, a grafting decomposition of S is obtained by cutting some of the edges of S along the path from its root to its leftmost leaf. See Figure 3. For a binary tree T, we denote by $n(T)$ the number of nodes of T and by $\ell(T)$ the number of edges along the path from its root to its leftmost leaf (here, we only count edges between two nodes). To fix the ideas, $n(Y) = 1$ and $\ell(Y) = 0$ for the unique binary tree Y with a single node (and thus two leaves). The following observations were made in [Cha07, Sect. 3] and [Cha18, Sect. 3.1], and are illustrated in Figure 4.

Lemma 9 ([Cha07, Cha18])**.**

(i) Assume that $S = S_0/S_1/\dots/S_k$ and $T = T_0/T_1/\dots/T_k$ are such that $n(S_i) = n(T_i)$ for all $i \in [k]$. Then $S \leqslant T$ if and only if $S_i \leqslant T_i$ for all $i \in [k]$.

(ii) If $S \leqslant T$, then we can write $S = S_0/S_1/\dots/S_\ell$ and $T = T_0/T_1/\dots/T_\ell$ where $\ell = \ell(T)$ and $n(S_i) = n(T_i)$ for all $i \in [\ell]$.

Consider now the generating function

$$\mathbb{A}(u, v, t, z) := \sum_{S \leqslant T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)},$$

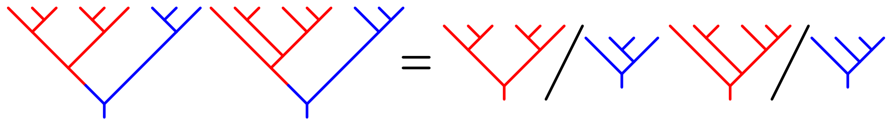
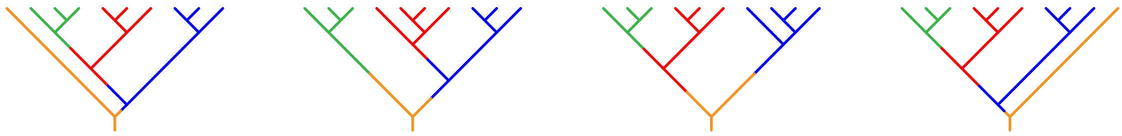


Figure 4: A grafting decomposition of a Tamari interval.



$S'_0 = Y\backslash(S_0/S_1/S_2)$ $S'_1 = S_0/Y\backslash(S_1/S_2)$ $S'_2 = (S_0/S_1)/Y\backslash S_2$ $S'_3 = (S_0/S_1/S_2)/Y$

Figure 5: The binary trees S'_k for $0 \leqslant k \leqslant 3$ obtained from the binary tree S of Figure 3 in the proof of Proposition 10.

where the sum ranges over all Tamari intervals (with arbitrary many nodes). To simplify notations, we abbreviate $A_u := A_u(t,z) := \mathbb{A}(u,1,t,z)$ and $A_u^\circ := A_u^\circ(t,z) := \mathbb{A}(u,0,t,z)$. Note that

$$A_1(t,z) := \mathbb{A}(1,1,t,z) = A(t,z).$$

Observe also that $A_u^\circ(t,z)$ is the generating function of indecomposable Tamari intervals, *i.e.* of Tamari intervals $S \leqslant T$ where $\ell(T) = 0$ so that the decomposition of Lemma 9 (ii) is trivial. Lemma 9 leads to the following functional equation connecting A_u and A_1.

Proposition 10. *The generating functions* $A_u := \mathbb{A}(u,1,t,z)$ *and* $A_1 := \mathbb{A}(1,1,t,z)$ *satisfy the quadratic functional equation*

$$(u-1)A_u = t\big(u - 1 + u(u+z-1)A_u - zA_1\big)\big(1 + uzA_u\big).$$

Proof. This statement could be directly deduced by substituting $x = 1$ and $y = \bar{y} = z$ in the equation given in [Cha18, Prop. 1]. For completeness, we prefer to transpose the proof as a simpler version of the proof of [Cha18, Prop. 1] is sufficient for our purpose.

By definition, any Tamari interval $S \leqslant T$ is either indecomposable or can be decomposed as $S = S'/S''$ and $T = T'/T''$ for an indecomposable Tamari interval $S' \leqslant T'$ and an arbitrary Tamari interval $S'' \leqslant T''$. Since $\ell(S) = \ell(S') + \ell(S'') + 1$, $n(S) = n(S') + n(S'')$, $\operatorname{des}(S) = \operatorname{des}(S') + \operatorname{des}(S'')$, and $\operatorname{asc}(T) = \operatorname{asc}(T') + \operatorname{asc}(T'') + 1$, we obtain

$$A_u = A_u^\circ + uzA_u^\circ A_u. \tag{1}$$

Now from any Tamari interval (S,T) where $S = S_0/S_1/\ldots/S_{\ell(S)}$, we can construct $\ell(S)+2$ indecomposable Tamari intervals (S'_k, T') for $0 \leqslant k \leqslant \ell(S)+1$, where

$$S'_k = \big(S_0/\ldots/S_{k-1}\big)/Y\backslash\big(S_k/\ldots/S_{\ell(S)}\big) \qquad \text{and} \qquad T' = Y\backslash T$$

(recall that Y denotes the unique binary tree with a single node). See Figure 5. For the extreme values of k, we have $S'_0 = Y\backslash S$ and $S'_{\ell(S)} = S/Y$. Moreover, any indecomposable

Tamari interval (S', T') with $n(S') = n(T') > 1$ is obtained in a single way by this procedure. Since $\ell(S'_k) = k$, $n(S') = n(S) + 1$, $\operatorname{des}(S') = \operatorname{des}(S) + 1$ when $k \leqslant \ell(S)$ while $\operatorname{des}(S'_{\ell(S)+1}) = \operatorname{des}(S)$, and $\operatorname{asc}(T') = \operatorname{asc}(T)$, we obtain

$$A_u^{\circ} = t\Big(1 + z\frac{uA_u - A_1}{u - 1} + uA_u\Big). \tag{2}$$

Combining Equations (1) and (2), we finally get

$$A_u = t\Big(1 + z\frac{uA_u - A_1}{u - 1} + uA_u\Big)\big(1 + uzA_u\big),$$

which rewrites as

$$(u - 1)A_u = t\big(u - 1 + u(u + z - 1)A_u - zA_1\big)\big(1 + uzA_u\big). \qquad \square$$

We are now ready to derive our functional equation on A using the quadratic method [GJ04].

Proposition 11. *The generating function $A = A(t, z)$ is a root of the polynomial $P(t, z, X)$ of $\mathbb{Q}[t, z, X]$ given by*

$$\begin{gathered} t^3z^6X^4 \\ + t^2z^4(tz^2 + 6tz - 3t + 3)X^3 \\ + tz^2(6t^2z^3 + 9t^2z^2 - 12t^2z + 2tz^2 + 3t^2 - 6tz + 21t + 3)X^2 \\ + (12t^3z^4 - 4t^3z^3 - 9t^3z^2 - 10t^2z^3 + 6t^3z + 26t^2z^2 - t^3 + 6t^2z + tz^2 + 3t^2 - 12tz - 3t + 1)X \\ + t(8t^2z^3 - 12t^2z^2 + 6t^2z - tz^2 - t^2 + 10tz + 2t - 1). \end{gathered}$$

Proof. We simply apply the quadratic method [GJ04]. The quadratic equation of Proposition 10 can be rewritten as $\alpha A_u^2 + \beta A_u + \gamma = 0$, where

$$\alpha = tu^2z(u+z-1), \quad \beta = tu(u+z-1) + tuz(u-1) - tuz^2A_1 - u + 1, \quad \gamma = t(u-1) - tzA_1.$$

The discriminant $\Delta := \beta^2 - 4\alpha\gamma$ must have multiple roots, which implies that its own discriminant in u vanishes. Removing clearly non-vanishing factors, this leads to the equation of the statement. Note that Δ having only degree 4 in u, the formula for the discriminant could be worked out by hand. $\square$

Remark 12. When specialized at $z = 0$, Proposition 11 shows that $A(t, 0)$ is a root of the polynomial

$$P(t, 0, X) = -(t - 1)^3X - t(t - 1)^2$$

which recovers the fact that $A(t, 0) = t/(1 - t) = t + t^2 + t^3 + \cdots$.

Remark 13. When specialized at $z = 1$, Proposition 11 shows that $A(t, 1)$ is a root of the polynomial

$$P(t, 1, X) = t^3X^4 + t^2(4t+3)X^3 + t(6t^2+17t+3)X^2 + (4t^3+25t^2-14t+1)X + t^3 + 11t^2 - t.$$

This is the classical functional equation for the generating function of Tamari intervals (see *e.g.* [Cha07, Eq. (5)]). The curve defined by $P(t,1,X)$ has genus zero and admits the rational parametrization

$$t = \frac{s}{(s+1)^4}, \quad X = s - s^2 - s^3. \tag{3}$$

As a consequence, the unique root $A = A(t,1) = t + 3t^2 + 13t^3 + 68t^4 + 399t^5 + 2530t^6 + \cdots$ in $\mathbb{Q}[[t]]$ of the polynomial $P(t,1,X)$ can be written as

$$A = S - S^2 - S^3, \tag{4}$$

where $S = t + 4t^2 + 22t^3 + 140t^4 + \cdots$ is the unique solution in $\mathbb{Q}[[t]]$ of

$$t = \frac{S}{(S+1)^4}.$$

From this equation, the coefficients of S, S^2 and S^3 can be computed via Lagrange inversion. More precisely, for $r \geqslant 1$, Lagrange inversion gives

$$[t^n]S^r = \frac{1}{n}[s^{n-1}]\, rs^{r-1}\phi(s)^n = \frac{r}{n}[s^{n-r}]\phi(s)^n,$$

where $\phi(s) := (s+1)^4$. Since

$$[s^a]\phi(s)^n = [s^a](s+1)^{4n} = \binom{4n}{a},$$

we obtain that, for $r \in \{1,2,3\}$,

$$[t^n]S^r = \frac{r}{n}[s^{n-r}]\phi(s)^n = \frac{r}{n}\binom{4n}{n-r}.$$

Hence, Equation (4) implies that

$$[t^n]A = [t^n]S - [t^n]S^2 - [t^n]S^3$$

is given by

$$\frac{1}{n}\left(\binom{4n}{n-1} - 2\binom{4n}{n-2} - 3\binom{4n}{n-3}\right) = \frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1},$$

as proved in [Cha07, Thm. 2.1].

2.2 Theorem 1 by Lagrange inversion

We will now mimic the approach in Remark 13, and extract the coefficients of $A(t,z)$ to obtain Theorem 1. The starting point is that the curve in t, X defined by the polynomial $P(t,z,X) \in \mathbb{Q}(z)[t,X]$ from Proposition 11 still has genus zero and admits the following rational parametrization:

$$t = \frac{s}{(s+1)(sz+1)^3}, \quad X = s - zs^2 - zs^3, \tag{5}$$

which lifts the parametrization (3). As a consequence, the unique root A in $\mathbb{Q}[[t,z]]$ of the polynomial $P(t,z,X)$ can be written

$$A = S - zS^2 - zS^3, \tag{6}$$

where $S = t + (3z+1)\,t^2 + (12z^2+9z+1)\,t^3 + \cdots$ is the unique solution in $\mathbb{Q}[z][[t]]$ of

$$t = \frac{S}{(S+1)(Sz+1)^3}. \tag{7}$$

There exist infinitely many rational parametrizations of P, but the one in Equation (5) has a double advantage: on the one hand, Equation (7) is under a form amenable to Lagrange inversion, and therefore allows to express the coefficient of $z^k t^n$ in S and in its powers; on the other hand, the simple form of Equation (6) allows to easily extract the coefficient of $t^n z^k$ in A as a sum of similar coefficients of S, S^2 and S^3. Putting together Equations (6) and (7) enables us to express the coefficient of $t^n z^k$ in A as a binomial sum. Let us give a few more details.

For $r \geqslant 1$ Lagrange inversion gives

$$[t^n z^k] S^r = \frac{1}{n}[s^{n-1} z^k] r s^{r-1} \phi(s)^n = \frac{r}{n}[s^{n-r} z^k]\phi(s)^n,$$

where $\phi(s) := (s+1)(sz+1)^3$. We have that

$$[s^a]\phi(s)^n = [s^a](s+1)^n (sz+1)^{3n} = \sum_{i+j=a} \binom{n}{i}\binom{3n}{j} z^j,$$

and therefore

$$[s^a z^k]\phi(s)^n = \binom{n}{a-k}\binom{3n}{k}.$$

It follows that, for $r \in \{1,2,3\}$,

$$[t^n z^k] S^r = \frac{r}{n}[s^{n-r} z^k]\phi(s)^n = \frac{r}{n}\binom{n}{n-r-k}\binom{3n}{k} = \frac{r}{n}\binom{n}{k+r}\binom{3n}{k},$$

Hence, Equation (6) implies that

$$a_{n,k} = [t^n z^k] A = [t^n z^k] S - [t^n z^{k-1}] S^2 - [t^n z^{k-1}] S^3$$

is given by

$$\frac{1}{n}\left(\binom{n}{k+1}\binom{3n}{k} - 2\binom{n}{k+1}\binom{3n}{k-1} - 3\binom{n}{k+2}\binom{3n}{k-1}\right) = \frac{2}{n(n+1)}\binom{3n}{k}\binom{n+1}{k+2},$$

which proves Theorem 1.

2.3 Theorem 2 by a binomial identity

We now simply derive Theorem 2 from Theorem 1, which amounts to checking the following binomial identity.

Proposition 14. *For any $n, k \in \mathbb{N}$,*

$$\sum_{\ell=k}^{n-1} \frac{2}{n(n+1)} \binom{n+1}{\ell+2} \binom{3n}{\ell} \binom{\ell}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}.$$

We shall actually prove the following generalization.

Proposition 15. *For any $n, k, r \in \mathbb{N}$,*

$$\sum_{\ell=k}^{n-1} \binom{n+1}{\ell+2} \binom{r}{\ell} \binom{\ell}{k} = \frac{n(n+1)}{(r+1)(r+2)} \binom{n-1}{k} \binom{r+n+1-k}{n+1}.$$

Proof. Using the identity

$$\binom{r}{\ell}\binom{\ell}{k} = \binom{r}{k}\binom{r-k}{r-\ell}$$

this amounts to showing that

$$\binom{r}{k} \sum_{\ell \geqslant 0} \binom{n+1}{\ell+2} \binom{r-k}{r-\ell} = \frac{n(n+1)}{(r+1)(r+2)} \binom{n-1}{k} \binom{r+n+1-k}{n+1}.$$

This is in turn equivalent to

$$\sum_{\ell \geqslant 0} \binom{n+1}{\ell+2} \binom{r-k}{r-\ell} = \binom{r+n+1-k}{r+2},$$

which is a particular case of the classical Chu–Vandermonde identity. □

3 Bijections

In this section, we present some bijective considerations on Theorems 1 and 2. We first describe some statistics equivalent to $\operatorname{des}(S)$ and $\operatorname{asc}(T)$ (Section 3.1), expressed in terms of canopy agreements in binary trees (Section 3.1.1), of valleys and double falls in Dyck paths (Section 3.1.2), and of internal degree of Schnyder woods in planar triangulations (Section 3.1.3). We then use bijective results of [FH24] to provide a more bijective proof of Theorem 1 (Section 3.2).

3.1 Equivalent statistics

Transporting the ascent and descent statistics, we can interpret the formulas of Theorems 1 and 2 on other combinatorial families encoding Tamari intervals. Here, we provide three alternative interpretations which seem particularly relevant to us.

3.1.1 Canopy agreements

Recall that the *canopy* of a binary tree T with n nodes is the vector $\mathrm{can}(T)$ of $\{-,+\}^{n-1}$ whose jth coordinate is $-$ if and only if the following equivalent conditions are satisfied:

(i) the $(j+1)$st leaf of T is a right leaf,

(ii) there is an oriented path joining its jth node to its $(j+1)$st node,

(iii) the jth node of T has an empty right subtree,

(iv) the $(j+1)$st node of T has a non-empty left subtree,

(v) the cone corresponding to T is located in the halfspace $x_j \leqslant x_{j+1}$.

(In all these conditions, recall that T is labeled in inorder and oriented towards its root.) We need the following three immediate observations, illustrated in Figures 6 and 7.

Lemma 16. *For any binary trees S and T,*

(i) the number of $-$ (resp. $+$) entries in the canopy of T is $\mathrm{asc}(T)$ (resp. $\mathrm{des}(T)$),

(ii) if $S \leqslant T$ in Tamari order, then the canopy of S is componentwise smaller than the canopy of T for the natural order $- \leqslant +$,

(iii) if $S \leqslant T$, then the number of positions where the entries of the canopies of both S and T are $-$ (resp. $+$) is given by $\mathrm{asc}(T)$ (resp. by $\mathrm{des}(S)$).

Proof. (i) By the characterization (iv) of the canopy above, $\mathrm{can}(T)_j = -$ if and only if there is an edge $i \to j+1$ for some $i \leqslant j$, which thus defines an ascent of T. Hence, the number of $-$ entries in $\mathrm{can}(T)$ is $\mathrm{asc}(T)$. By symmetry, the number of $+$ entries in $\mathrm{can}(T)$ is $\mathrm{des}(T)$

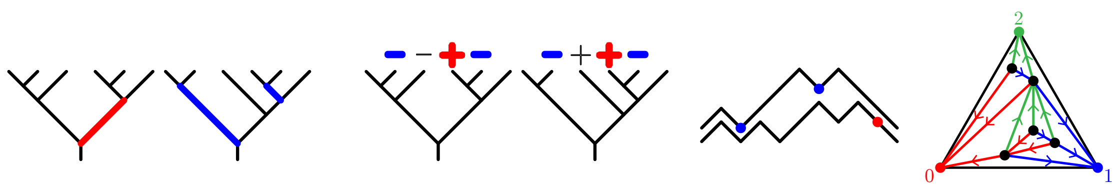


Figure 6: Connections between equivalent statistics. The descents of S (resp. ascents of T) on the left correspond to the positions where the canopies of S and T are both positive (resp. negative) in the middle left, to the double falls of $\pi(S)$ (resp. the valleys of $\pi(T)$) in the middle right, and to the intermediate nodes of the tree T_0 (resp. T_1) on the right.

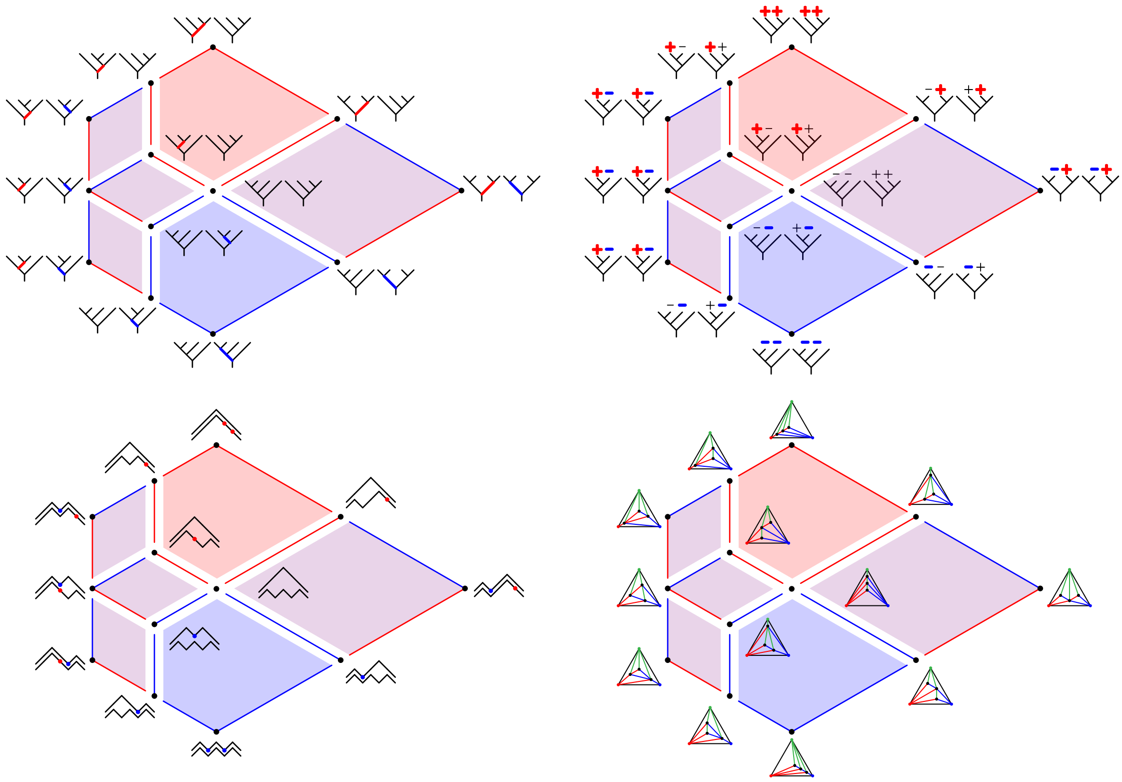


Figure 7: The decomposition of the cellular diagonal Δ_2 of Figure 2, labeled using the equivalent statistics of Figure 6.

(ii) It is sufficient to prove (ii) for a cover relation in the Tamari order. If the edge $i \to j$ with $i < j$ is rotated, then the canopy is unchanged, except maybe its ith entry, which changes from $-$ to $+$ when $j = i + 1$. An alternative global argument is to observe that if $S \leqslant T$, then any linear extension of S is smaller than any linear extension of T, so that there cannot be both oriented paths from $i+1$ to i in S and from i to $i+1$ in T, and to use the characterization (ii) of the canopy above.

(iii) We have $\mathrm{can}(S)_j = \mathrm{can}(T)_j = -$ if and only if $\mathrm{can}(T)_j = -$ (by (ii)), so that the number of such positions is $\mathrm{asc}(T)$ by (i). By symmetry, the number of positions j with $\mathrm{can}(S)_j = \mathrm{can}(T)_j = +$ is $\mathrm{des}(S)$. ☐

Using Lemma 16, we can transpose Theorems 1 and 2 in terms of canopy. We denote by $\mathrm{agr}(S, T)$ the number of *canopy agreements* between two binary trees S and T (*i.e.* of positions where the entries of the canopies of S and T agree).

Corollary 17. *For any $n, k \in \mathbb{N}$, we have*

$$|\{S \leqslant T \mid \mathrm{agr}(S,T) = k\}| = |\{S \leqslant T \mid \mathrm{des}(S) + \mathrm{asc}(T) = k\}| = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k},$$

where $S \leqslant T$ are intervals of the Tamari lattice $\mathrm{Tam}(n)$ *on binary trees with n nodes.*

Corollary 18. *For any $n, k \in \mathbb{N}$, we have*

$$\sum_{S \leqslant T} \binom{\operatorname{agr}(S,T)}{k} = \sum_{S \leqslant T} \binom{\operatorname{des}(S) + \operatorname{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1},$$

where the sums range over the intervals $S \leqslant T$ of the Tamari lattice $\mathrm{Tam}(n)$ *on binary trees with n nodes.*

Remark 19. For $k = n-1$ in both Corollaries 17 and 18, we recover that the number of synchronized Tamari intervals (*i.e.* with $\operatorname{agr}(S,T) = n-1$) is given by

$$\frac{2}{n(n+1)} \binom{3n}{n-1} = \frac{2}{(n+1)(2n+1)} \binom{3n}{n} = \frac{2}{(3n+1)(3n+2)} \binom{3n+2}{n+1}.$$

See [PRV17, FPR17].

Remark 20. Note that the first equalities of Corollaries 17 and 18 follow from [Cha18, Sect. 5]. The approach of [Cha18, Sect. 5] is however a bit of a detour as it passes again through generating functions, when the simple observation of Lemma 16 (iii) suffices.

3.1.2 Dyck paths

Recall that a *Dyck path* of semilength n is a path from $(0,0)$ to $(2n,0)$ using n up steps $(1,1)$ (denoted U) and n down steps $(1,-1)$ (denoted D) and never passing below the horizontal axis. We denote by π the standard bijection from binary trees to Dyck paths. Namely, the Dyck path $\pi(T)$ corresponding to a binary tree T is obtained by walking clockwise around the contour of T and marking an U step when finding a leaf and a D step when walking back an edge $j \to i$ with $i < j$. Note that π transports the rotation on binary trees to the Tamari shift on Dyck paths, which exchanges a D step preceding an U step with the corresponding excursion (meaning the longest subpath which stays above this U step). See Figures 6 and 7 for illustrations. The following lemma is classical and immediate.

Lemma 21. *The bijection π from binary trees to Dyck path sends:*

- *the ascents of T to the* valleys *of $\pi(T)$ (a D step followed by an U step),*
- *the descents of T to the* double falls *of $\pi(T)$ (two consecutive D steps),*
- *the edges on the left branch of T to the* contacts *of $\pi(T)$ (its points on the horizontal axis).*

Using Lemma 21, we can transpose Theorems 1 and 2 in terms of Dyck paths. We denote by $\operatorname{val}(P)$ (resp. $\operatorname{df}(P)$) the number of valleys (resp. of double falls) of a Dyck path P.

Corollary 22. *For any $n, k \in \mathbb{N}$, we have*

$$| \{ P \leqslant Q \mid \mathrm{df}(P) + \mathrm{val}(Q) = k \} | = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k},$$

where $P \leqslant Q$ are intervals of the Tamari lattice $\mathrm{Tam}(n)$ *on Dyck paths of semilength n.*

Corollary 23. *For any $n, k \in \mathbb{N}$, we have*

$$\sum_{P \leqslant Q} \binom{\mathrm{df}(P) + \mathrm{val}(Q)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1},$$

where the sum ranges over the intervals $P \leqslant Q$ of the Tamari lattice $\mathrm{Tam}(n)$ *on Dyck paths of semilength n.*

3.1.3 Triangulations and minimal realizers

We now consider the bijection of [BB09] from Tamari intervals to rooted triangulations using Schnyder woods. Schnyder woods were introduced in [Sch89] for straightline embedding purposes, and the structure of Schnyder woods was investigated in particular in [OdM94, Pro97, Fel04b]. We refer to [Fel04a, Chap. 2] for a nice pedagogical presentation of Schnyder woods and their applications.

Recall that a *planar map* M is an embedding of a planar graph on the sphere, considered up to continuous deformations. A *face* of M is a connected component of the complement of M, and a *corner* is a pair of consecutive edges around a vertex. A *rooted* map is a map where a root corner is marked. The face containing this corner is then considered as the *external* face, and the vertices and edges of this external face are the external vertices and edges. A *triangulation* is a map where all faces have degree 3. Euler formula implies that a rooted triangulation with n internal vertices has $3n$ internal edges and $2n + 1$ internal triangles.

Consider a rooted triangulation M and denote by v_0, v_1, v_2 the external vertices of M counterclockwise around the external face, and by U the internal vertices of M. A *realizer* (or *Schnyder wood* [Sch89]) of M is an orientation and coloring with colors $\{0, 1, 2\}$ of the edges of M such that

- for each $i \in \{0, 1, 2\}$, the i-edges form a tree with vertices $U \cup \{v_i\}$ oriented towards v_i,
- counterclockwise around each internal vertex, we see a 0-source, some 2-targets, a 1-source, some 0-targets, a 2-source, and some 1-targets. (Note that some means possibly none.)

(An i-edge is an edge colored i, and an i-source or i-target is the source or target of and i-edge.) A realizer is *minimal* (resp. *maximal*) if it contains no clockwise (resp. counterclockwise) cycle. It was observed in [OdM94, Pro97, Fel04b] that the Schnyder woods on a given triangulation M have the structure of a distributive lattice, where the cover relations correspond to reorientation of certain clockwise cycles. This has the following immediate consequence.

Theorem 24 ([OdM94, Pro97, Fel04b]). *Every triangulation has a unique minimal (resp. maximal) realizer.*

Consider now a realizer (T_0, T_1, T_2) of a rooted triangulation M. Walking clockwise around T_0, we define two Dyck paths P and Q as follows:

- P has an U (resp. D) step each time we move farther from v_0 (resp. closer to v_0),
- Q has an U step each time we move farther from v_0 (except the first step), and a D step each time we pass a 1-target.

See Figures 6 and 7 for illustrations. This map was defined in [BB09], where it is proved that it behaves very nicely with respect to three lattice structures on Dyck paths (the Stanley lattice, the Tamari lattice and the Kreweras lattice). Here, we will use only the connection to the Tamari lattice, but we first make an immediate observation. We call *intermediate nodes* of a rooted tree T the nodes which are neither the root, nor the leaves of T.

Lemma 25. *Consider the pair (P, Q) of Dyck paths obtained from a realizer (T_0, T_1, T_2). Then*

- *the double falls of P correspond to the intermediate nodes of T_0,*
- *the valleys of Q correspond to the intermediate nodes of T_1,*
- *the contacts of P correspond to the corners of edges of T_0 incident to v_0.*

We now restrict to minimal realizers to obtain a bijection between rooted triangulations and Tamari intervals, as described in [BB09]. We denote by $\mathrm{bb}(M)$ the pair of Dyck paths (P, Q) obtained from the minimal realizer of M.

Theorem 26 ([BB09]). *The map* bb *is a bijection from rooted triangulations with n internal vertices to the intervals of the Tamari lattice on Dyck paths of semilength n.*

Using Lemma 25 and Theorem 26, we can transpose Theorems 1 and 2 in terms of maps. For a rooted triangulation M, with minimal realizer (T_0, T_1, T_2), we denote by $\mathrm{inodes}(M)$ the number of intermediate nodes of T_0 plus the number of intermediate nodes of T_1.

Corollary 27. *For any $n, k \in \mathbb{N}$, we have*

$$|\{M \mid \mathrm{inodes}(M) = k\}| = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k},$$

where the M's are the rooted triangulations with n internal vertices.

Corollary 28. *For any $n, k \in \mathbb{N}$, we have*

$$\sum_{M} \binom{\mathrm{inodes}(M)}{k} = \frac{2}{(3n+1)(3n+2)}\binom{n-1}{k}\binom{4n+1-k}{n+1},$$

where the sums range over all rooted triangulations M with n internal vertices.

3.2 Theorem 1 from triangulations

We now derive Theorem 1 from triangulations using the following result of [FH24]. It was obtained via a bijection from planar triangulations endowed with their minimal realizers to planar mobiles. We state it here in terms of canopies of binary trees.

Theorem 29 ([FH24, Coro. 2]). *Let $f_{i,j,k}$ denote the number of Tamari intervals $S \leqslant T$ with i positions p where $\mathrm{can}(S)_p = \mathrm{can}(T)_p = -$, with j positions p where $\mathrm{can}(S)_p = \mathrm{can}(T)_p = +$, and with k positions p where $\mathrm{can}(S)_p = -$ while $\mathrm{can}(T)_p = +$. Then the corresponding generating function $F := F(u,v,w) := \sum_{i,j,k} f_{i,j,k} u^i v^j w^k$ is given by*

$$uvF = uU + vV + wUV - \frac{UV}{(1+U)(1+V)},$$

where the series $U := U(u,v,w)$ and $V := V(u,v,w)$ satisfy the system

$$\begin{aligned} U &= (v + wU)(1+U)(1+V)^2 \\ V &= (u + wV)(1+V)(1+U)^2. \end{aligned}$$

Corollary 30. *The generating function $A := A(t,z) := \sum a_{n,k} t^n z^k$ is given by*

$$tz^2 A = 2tzS + tS^2 - \frac{S^2}{(1+S)^2}, \tag{8}$$

where the series $S := S(t,z)$ satisfies

$$S = t(z+S)(1+S)^3. \tag{9}$$

Proof. By Corollary 17, we have $A(t,z) = tF(tz,tz,t)$. Specializing $u = v = tz$ and $w = t$ in Theorem 29, we thus obtain the expression for $A(t,z)$ by observing that the series $U(tz,tz,t)$ and $V(tz,tz,t)$ coincide and denoting $S(t,z) := U(tz,tz,t) = V(tz,tz,t)$. This concludes the proof. □

Differentiating Equation (8) with respect to the variable t, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(tz^2 A) &= 2zS + 2tz\frac{\partial S}{\partial t} + S^2 + 2tS\frac{\partial S}{\partial t} - \frac{2S}{(1+S)^2}\frac{\partial S}{\partial t} + \frac{2S^2}{(1+S)^3}\frac{\partial S}{\partial t} \\ &= 2zS + S^2 + \frac{2}{(1+S)^3}\frac{\partial S}{\partial t}\Big(t(z+S)(1+S)^3 - S(1+S) + S^2\Big) \\ &= 2zS + S^2, \end{aligned} \tag{10}$$

where the last equality follows from Equation (9).

We obtain by Lagrange inversion in Equation (9) that for $r \geqslant 1$,

$$[t^n z^k] S^r = \frac{r}{n}[s^{n-r} z^k]\phi(s)^n,$$

where $\phi(s) := (z+s)(1+s)^3$. Thus

$$[t^n z^k]S^r = \frac{r}{n}[s^{n-r}z^k](z+s)^n(1+s)^{3n} = \frac{r}{n}\binom{n}{k}\binom{3n}{k-r}.$$

Hence, Equation (10) implies that

$$a_{n,k} = [t^n z^k]A = \frac{1}{n+1}[t^n z^{k+2}]\frac{\partial}{\partial t}(tz^2 A) = \frac{1}{n+1}\left(2[t^n z^{k+1}]S + [t^n z^{k+2}]S^2\right)$$

is given by

$$\frac{2}{n(n+1)}\left(\binom{n}{k+1} + \binom{n}{k+2}\right)\binom{3n}{k} = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}.$$

Remark 31. In fact, the recent direct bijection of [FFN25] between Tamari intervals and blossoming trees enables to obtain Theorem 1 in an even simpler way. Details can be found in [FFN25].

4 *q*-analogues

We conclude this paper with an insightful observation due to a referee, which opens the door to a promising and exciting direction for future research.

Recall that *q-analogues* in combinatorics are generalizations of classical numbers and formulas in which integers are replaced by polynomials in a variable q, in such a way that the original formulas are recovered when $q \to 1$. They typically arise by refining a counting problem with an additional statistic (such as inversions), so that the answer becomes a generating function rather than a single number.

A basic building block is the *q-integer* $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, and from it the *q-factorial* $[n]_q! := [1]_q[2]_q \cdots [n]_q$. The *q-binomial coefficients* (or *Gaussian binomial coefficients*) are then defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

As $q \to 1$, one recovers the ordinary integer n, factorial $n!$, and binomial coefficient $\binom{n}{k}$. Combinatorially, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts k-dimensional subspaces of $\mathbb{F}_q^n$ and also serves as a generating function for statistics such as inversions in binary words with k ones and $n-k$ zeros.

Theorem 1 states that

$$\sum_{\ell=0}^{n-1} \binom{n+1}{\ell+2}\binom{3n}{\ell} \Big/ \binom{n+1}{2} = \binom{4n+1}{n+1} \Big/ \binom{3n+2}{2}$$

counts all Tamari intervals [Cha07]. The following q-analogue was conjectured by an anonymous referee.

Proposition 32. *For any* $n, k \in \mathbb{N}$,

$$\sum_{\ell=0}^{n-1} q^{\ell(\ell+2)} \begin{bmatrix} n+1 \\ \ell+2 \end{bmatrix}_q \begin{bmatrix} 3n \\ \ell \end{bmatrix}_q \Big/ \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = \begin{bmatrix} 4n+1 \\ n+1 \end{bmatrix}_q \Big/ \begin{bmatrix} 3n+2 \\ 2 \end{bmatrix}_q.$$

Further experiments lead us to extend Proposition 32. Theorem 2 states that

$$\sum_{\ell=k}^{n-1} \binom{n+1}{\ell+2} \binom{3n}{\ell} \binom{\ell}{k} \Big/ \binom{n+1}{2} = \binom{n-1}{k} \binom{4n+1-k}{n+1} \Big/ \binom{3n+2}{2}.$$

We observed the following q-analogue.

Proposition 33. *For any* $n, k \in \mathbb{N}$,

$$\sum_{\ell=k}^{n-1} q^{(\ell-k)(\ell+2)} \begin{bmatrix} n+1 \\ \ell+2 \end{bmatrix}_q \begin{bmatrix} 3n \\ \ell \end{bmatrix}_q \begin{bmatrix} \ell \\ k \end{bmatrix}_q \Big/ \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 4n+1-k \\ n+1 \end{bmatrix}_q \Big/ \begin{bmatrix} 3n+2 \\ 2 \end{bmatrix}_q.$$

Propositions 32 and 33 can be proved for instance using computer algebra tools, such as creative telescoping [Zei91, Zei90, PWZ96]. We leave their combinatorial interpretations for future research.

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