Treewidth of generalized Hamming graph, bipartite Kneser graph and generalized Petersen graph

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Abstract

Let t, q and n be positive integers. Write $[q] = \{1, 2, \ldots, q\}$. The generalized Hamming graph $H(t, q, n)$ is the graph whose vertex set is the cartesian product of n copies of $[q]$ ($q \geqslant 2$), where two vertices are adjacent if their Hamming distance is at most t. In particular, $H(1, q, n)$ is the well-known Hamming graph and $H(1, 2, n)$ is the hypercube. In 2006, Chandran and Kavitha described the asymptotic value of $tw(H(1, q, n))$, where $tw(G)$ denotes the treewidth of G. In this paper, we give the exact pathwidth of $H(t, 2, n)$ and show that $tw(H(t, q, n)) = \Theta(tq^n/\sqrt{n})$ when n goes to infinity. Based on those results, we show that the treewidth of the bipartite Kneser graph $BK(n, k)$ is $\binom{n}{k} - 1$ when n is sufficient large relative to k and the bounds of $tw(BK(2k + 1, k))$ are given. Moreover, we present the bounds of the treewidth of generalized Petersen graph.

Mathematics Subject Classifications: 05C75, 05D05

1 Introduction

Treewidth is a well-studied parameter in graph theory. Many NP-complete problem can be solved in polynomial time on graphs of bounded treewidth [36, 7, 11]. Besides, treewidth is also useful in structural graph theory. For example, Robertson and Seymour used it to prove the Graph Minor Theorem [30, 31, 32]. In the past few decades, there has been much literature investigating the treewidth of certain graphs (see for example, [17, 18, 22, 25, 35]). However, it is difficult to estimate the treewidth even asymptotically.

Throughout this paper, graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The degree of a vertex $v \in V(G)$

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in G is denoted by $d_G(v)$, and we ignore the subscript G in case of no ambiguity. The maximum and minimum degree of G are denoted as $\Delta(G)$ and $\delta(G)$, respectively. Write $[n] = \{1, 2, \ldots, n\}$ and denote by $\binom{[n]}{k}$ the family of all k-subsets of $[n]$. The treewidth, pathwidth and bandwidth of graph G are denoted by $tw(G)$, $pw(G)$ and $bw(G)$, respectively. It is worth mentioning that the isoperimetric problem of the hypercube is an important topic which provides a foundation for our subsequent proof. The readers may refer to [15, 14, 29, 16, 3] for more details.

The *Hamming graph* $H(q, n)$ is the graph on q^n vertices, which correspond to all n-vectors whose components are from $[q]$ $(q \geqslant 2)$. When $q = 2$, it is more common to treat each vertex of $H(2, n)$ as a binary n-vector, that is, an n-vector whose components are from $\{0, 1\}$. Two of the vertices in $H(q, n)$ are adjacent only when they differ in just one coordinate. The *hypercube graph* is a special case of the Hamming graph when $q = 2$ which is well-studied in parallel computing, coding theory and many other areas[1, 19, 20].

The Hamming distance of two n-vectors is the number of coordinates where one differs from the other. Two vertices in a Hamming graph are adjacent if and only if their Hamming distance is no more than 1. If two vertices in $H(q, n)$ is adjacent if and only if their Hamming distance is no more than t, then we denote the *generalized Hamming graph* as $H(t, q, n)$. Then $H(1, q, n) = H(q, n)$ and $H(1, 2, n)$ is the hypercube graph. We call $H(t, 2, n)$ the *generalized hypercube graph* and we will give the exact value of its pathwidth and bandwidth as Theorem 1. When $t = 1$, our result covers the previous results of hypercube graph. It is interesting to see that in the generalized form of hypercube graph, its treewidth still can be described by an exact expression. Having exact form of treewidth is not a common phenomenon, especially with such a complicated form.

Theorem 1. *We have*

$$\begin{aligned} & pw(H(t,2,n)) = bw(H(t,2,n)) \\ = & \sum_{k=\lfloor (n-t)/2 \rfloor}^{\lfloor (n-t)/2 \rfloor + t - 1} \binom{n}{k} + \sum_{a=0}^{\lfloor (n-t-1)/2 \rfloor} \left(\binom{t+2a}{t+a-1} - \binom{t+2a}{a-1} \right). \end{aligned} \tag{1}$$

The treewidth of $H(q, n)$ is asymptotically $\Theta(q^n/\sqrt{n})$ from [9]. To be more specific, there exists constants c_1 and c_2 not depending on q, such that for sufficiently large n, $c_1 q^n/\sqrt{n} \leqslant tw(H(q, n)) \leqslant c_2 q^n/\sqrt{n}$. In this paper, we generalize this result to generalized Hamming graph $H(t, q, n)$ as Theorem 2. The previous result about $tw(H(q, n))$ can be derived from our result by letting $t = 1$. It is interesting to see that the distance variable t in the generalized Hamming graph $H(t, q, n)$ has linear impact on its treewidth.

Theorem 2. *There exists constant c_1 and c_2 not depending on t or q, such that for any positive integers t and q. When n is sufficiently large, we have that*

$$c_1 t q^n/\sqrt{n} \leqslant tw(H(t, q, n)) \leqslant c_2 t q^n/\sqrt{n}.$$

Kneser graph $K(n,k)$ is the graph whose vertex set is $\binom{[n]}{k}$ and where two vertices are adjacent if and only if the two corresponding sets are disjoint. The treewidth of Kneser graph is studied by Harvey in 2014 [17]. *Generalized Kneser graph*, *q-Kneser graph* and *generalized q-Kneser graph* are three derived class from Kneser graph and their treewidth can also be exactly described when n is large enough [8, 26, 27]. *Bipartite Kneser graph* is also an important variant of Kneser graph whose vertex set is $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$ denoted by $BK(n,k)$ where $2k \leqslant n$. A k-subset A and a $(n-k)$-subset B in bipartite Kneser graph is adjacent if $A \subseteq B$. We will give the exact value of $tw(BK(n,k))$ when n is sufficient large relative to k as Theorem 3. When $n = 2k+1$, we give the bounds of the treewidth of the bipartite Kneser graph as Theorem 4.

Theorem 3. *If* $3\binom{n-k}{k} \geqslant 2\binom{n}{k}$ *and* $k \geqslant 2$*, then* $tw(BK(n,k)) = \binom{n}{k} - 1$.

Theorem 4. *There exists two constants* c_1 *and* c_2 *such that for any positive integer* k*,*

$$c_1 \frac{1}{k}\binom{2k+1}{k} \leqslant tw(BK(2k+1,k)) \leqslant tw(J(2k+1,k)) \leqslant c_2\binom{2k+1}{k}.$$

Petersen graph is a well-studied Kneser graph $K(5,2)$. *Generalized Petersen graph* is an extension of Petersen graph denoted by $G_{n,k}$ whose vertex set and edge set are

$$V(G_{n,k}) = \{v_1, \ldots, v_n, u_1, \ldots, u_n\},$$
$$E(G_{n,k}) = \{v_i u_i\} \cup \{v_i v_{i+1}\} \cup \{u_i u_{i+k}\}, i = 1, 2, \ldots, n,$$

where subscripts are to be read modulo n and $k < n/2$. We will show the treewidth of $G_{n,k}$ when n is sufficient large relative to k as Theorem 5.

Theorem 5. *Let* n *and* k *be positive integers satisfying that* $n \geqslant 8(2k+2)^2$*. Then we have*

$$2k+1 \leqslant tw(G_{n,k}) \leqslant pw(G_{n,k}) \leqslant 2k+2.$$

Theorem 3, 4 and 5 can be viewed as different generalizations of Kneser graph. And in Theorem 4, it is interesting to see that the treewidth of the bipartite Kneser graph has a close relationship with that of the well-studied Johnson graph. The treewidth of Johnson graph $J(n,k)$ is also an interesting topic. When $k = 2$, $tw(J(n,2))$ and $pw(J(n,2))$ have exact formulas while for other k it remains unknown [12]. Our result may help to have a deeper understanding of the treewidth of Johnson graph.

The rest of this paper is organized as follows. In Section 3, we give a proof of the treewidth of generalized Hamming graph (Theorem 1 and Theorem 2). In this part, we mainly use properties of bandwidth and techniques of Hales numbering and the isoperimetric problems by Harper [15, 14].

In Section 4, we study the treewidth of the bipartite Kneser graph and Johnson graph (Theorem 3 and Theorem 4). In this part, we mainly use the techniques of separators, properties of cross-intersecting families and chordal completions. Since the Johnson graph can be viewed as a slice of generalized Hamming graph, we need the results in Section 3 to prove our results.

In Section 5, we study the treewidth of generalized Petersen graph (Theorem 5). In this part, we mainly use brambles and path-decomposition constructions.

2 Preliminaries and definitions

In this section, we give definitions involving in treewidth, pathwidth and bandwidth of a graph $G(V, E)$.

Definition 6. A tree-decomposition of a graph $G(V, E)$ is a pair (X, T), where $T(I, F)$ is a tree with vertex set I and edge set F, and $X = \{X_i \mid i \in I\}$ is a family of subsets of V, one for each node of T, such that:

- $\bigcup_{i \in I} X_i = V$.
- For each edge $uv \in E$, there exists an $i \in I$ such that $u, v \in X_i$.
- For all $i, j, k \in I$, if j is on the path from i to k in T, then $X_i \cap X_k \subset X_j$.

The width of a tree-decomposition (X, T) is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G is the minimum treewidth over all possible tree-decompositions of G and denoted by $tw(G)$. The problem of deciding whether a graph has tree decomposition of treewidth at most k is NP-complete [2]. However, there is an exact algorithm finding treewidth of given graph G when taking $tw(G)$ as a constant [5, 23]. A path decomposition of G is a tree decomposition (X, T) in which T is required to be a path. The pathwidth of G is defined to be the minimum width over all path decompositions of G and is denoted by $pw(G)$.

A bijection $\phi : V \to \{1, 2, \ldots, n\}$ is called an ordering of the vertices of G (in short, an ordering of G). Then for any edge $e = \{u, v\} \in E$, let $\Delta(e, \phi) = |\phi(u) - \phi(v)|$.

Definition 7. A bandwidth of a graph $G(V, E)$, denoted by $bw(G)$, is the minimum over all possible orderings ϕ of V of the maximum value of $\Delta(e, \phi)$ over all edges $e \in E$. That is,

$$bw(G) = \min_{\phi} \max_{e \in E} \Delta(e, \phi).$$

There are important inequalities between treewidth, pathwidth and bandwidth as following.

Proposition 8 ([6]). *For any graph G,*

$$tw(G) \leqslant pw(G) \leqslant bw(G). \tag{2}$$

Let $X \subseteq V(G)$ be a subset of vertices and $G[X]$ be the subgraph induced in G by X. Define $G-X = G[V(G)-X]$. Given $p \in (0,1)$, define the *p-separator* of G to be a subset $X \subseteq V(G)$ such that no component of $G-X$ contains more than $p|V(G)-X|$ vertices. Proposition 9 describes the relationship between treewidth and separator.

Proposition 9 ([32]). *For any graph G, there exists a 1/2-separator of G with at most $tw(G)+1$ vertices.*

Corollary 10 is directly from Proposition 9.

Corollary 10. For any graph G, there exists a separator X of G with at most $tw(G)+1$ vertices. And there exists a partition of $V(G)-X$ into sets A and B such that

$$|V(G)-X|/3 \leqslant |A|, |B| \leqslant 2|V(G)-X|/3.$$

Proposition 9 and Corollary 10 are useful tools for estimating the lower bound on treewidth.

3 Treewidth of generalized Hamming graph

3.1 Bandwidth of Hamming graph $H(t,2,n)$

The pathwidth and bandwidth of hypercubes can be exactly calculated as following.

Proposition 11 ([9]). *We have*

$$pw(H(1,2,n)) = bw(H(1,2,n)) = \sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor}. \tag{3}$$

In this section, we intend to prove Theorem 1. When $t=1$, Eq 1 is exactly the same as Eq 3. Thus, we can view Theorem 1 as a generalization of Proposition 11. Using the following techniques, we will derive some recursion formulas (Proposition 19 and Proposition 20) and use induction to prove Theorem 1. However, Eq 1 is much more complicated and once we know its formula, it is always "easy" to verify it when we have some recursion formulas using induction. The exact expression of Eq 10 and 12 actually come from our elegant observation from some instances with computer assistance.

To prove Theorem 1, we need some preparation. In [15], Harper showed that if an ordering φ of $G(V,E)$ is in Hales order (i.e., a Hales numbering), then $\max_{e\in E} \Delta(e,\varphi)$ takes minimum over all numbering, that is, $bw(G) = \min_{\phi} \max_{e\in E} \Delta(e,\phi) = \max_{e\in E} \Delta(e,\varphi)$.

Definition 12 (Hales numbering [15]). If there exists an ordering ϕ all of whose beginning segments obey the following two conditions, then we call such orderings *Hales numberings*. Note that Hales numbering does not always exists and it is not unique.

1. For a set of l vertices, let Φ_l be the number of vertices in the set having neighbors not in the set. Φ_l must be minimized for all beginning segments $S_l = \{v \in V \mid \phi(v) \leqslant l\}$.

2. The $\Phi'_l = l - \Phi_l$ "interior vertices" of S_l must be numbered $1, 2, \ldots, \Phi'_l$, i.e., have the lowest possible numbers on them.

Harper [15] also give a sample of Hales numbering of $H(t, 2, n)$[1]. From [34], we can build such Hales numbering φ in the following way.

Define an $\binom{n}{k} \times n$ matrix $A_k^{(n)}$ as following. Let $A_n^{(n)}$ be $(\underbrace{1, 1, \ldots, 1}_{n \text{ factors}})$ and let $A_0^{(n)}$ be $(\underbrace{0, 0, \ldots, 0}_{n \text{ factors}})$ for any positive integer n. When $0 < k < n$, $A_k^{(n)}$ is defined recursively as Eq 4.

$$A_k^{(n)} \triangleq \begin{bmatrix} A_{k-1}^{(n-1)} & \mathbf{1} \\ A_k^{(n-1)} & \mathbf{0} \end{bmatrix}, \tag{4}$$

where $\mathbf{1}$ (resp. $\mathbf{0}$) is an all one (resp. all zero) $\binom{n-1}{k-1}$ (resp. $\binom{n-1}{k}$) column vector. Clearly, the rows of $A_k^{(n)}$ enumerates all binary vectors of length n whose number of '1' is k. Define $S^{(n)}$ as following:

$$S^{(n)} \triangleq \begin{bmatrix} A_0^{(n)} \\ A_1^{(n)} \\ \vdots \\ A_n^{(n)} \end{bmatrix}. \tag{5}$$

Then $S^{(n)}$ is an $2^n \times n$ matrix. Each row of $S^{(n)}$ exactly corresponds to an n-vector (a vertices of $H(t, 2, n)$) and vice versa. Let $\eta^{(n)}$ be the ordering of $H(t, 2, n)$ defined by row order of $S^{(n)}$, that is, if $v \in V(H(t, 2, n))$ corresponds to the i-th row of $S^{(n)}$, then let $\eta^{(n)}(v) = i$.

$\eta^{(n)}$ is a Hales numbering of $H(t, 2, n)$ (see [34] and [15]) and, hence, we have the following Proposition 13.

Proposition 13. $bw(H(t, 2, n)) = \max_{e \in E} \Delta(e, \eta^{(n)})$.

In order to calculate $bw(H(t, 2, n))$, we need some more definitions.

[1] In [15], Harper did not prove this statement. Harper first gave a Hales numbering of hypercube and, then claimed that the numbering is also in Hales order for the distance generalized graph (that is, $H(t, 2, n)$) in comments (III (b)).

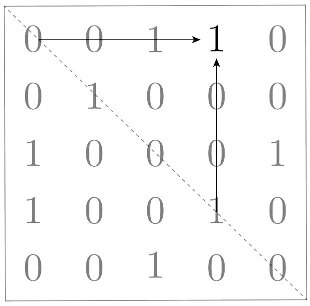

Figure 1: Bandwidth of a square matrix.

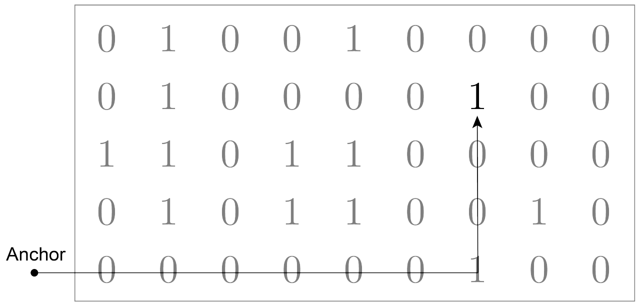

Figure 2: Manhattan radius and the imaginary anchor.

Definition 14. Given a graph $G(V, E)$, for two vertex subsets $V_1 \subseteq V$ and $V_2 \subseteq V$ numbered by ordering η_1 and η_2 respectively, the adjacency matrix of V_1 and V_2 is defined to be a $|V_1| \times |V_2|$ matrix M such that for any $u \in V_1$ and $v \in V_2$,

$$M(\eta_1(u), \eta_2(v)) = \begin{cases} 1 & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacent matrix of $G(V, E)$ with ordering η is the adjacent matrix of V and V ordered by η as defined above.

Definition 15. The bandwidth of an $s \times s$ non-zero square matrix M denoted by $bw(M)$ is the maximum absolute value of the difference between the row and column indices of a nonzero element of that matrix, i.e.,

$$bw(M) = \max_{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant s} \{|i - j| \mid M(i, j) \neq 0\}, \tag{6}$$

which is exactly the maximum Manhattan distance from a nonzero element to the main diagonal of the matrix (see Figure 1).

Remark 16. Given an ordering η of a graph G, $\max_{e \in E} \Delta(e, \eta)$ is equal to the bandwidth of the adjacency matrix of G ordered by η.

Remark 16 is directly from definitions. Let $M^{(t,n)}$ be the adjacency matrix of $H(t, 2, n)$ ordered by $\eta^{(n)}$. From Remark 16 and Proposition 13, we have Lemma 17.

Lemma 17. $bw(H(t, 2, n)) = bw(M^{(t,n)})$.

Now our aim is to give exact value of $bw(M^{(t,n)})$. To achieve this, we need to define the Manhattan radius of matrix.

Definition 18. For an $s \times t$ non-zero matrix M, its Manhattan radius $r(M)$ is defined by

$$r(M) = \max_{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t} \{s - i + j \mid M(i,j) \neq 0\}, \tag{7}$$

which is exactly the maximum Manhattan distance from a nonzero element of M to the position immediately to the left of the bottom-left corner of M (an imaginary element $M(s,0)$). The imaginary element is called the anchor of M (see Figure 2). For the convenience of the proof, let $r(M) = -\infty$ if M is a zero matrix or an empty matrix.

Note that $r(M) > 0$ if and only if M is non-zero and non-empty. If M is an $s \times s$ symmetric matrix, then by definition,

$$r(M) = bw(M) + s. \tag{8}$$

Let $V_k^{(t,n)} \subseteq V(H(t,2,n))$ be a vertex set containing vertices of $H(t,2,n)$ whose corresponding n-vector has exactly k ones, where $0 \leqslant k \leqslant n$. Then $\{V_0^{(t,n)}, V_1^{(t,n)}, \ldots, V_n^{(t,n)}\}$ forms a partition of $V(H(t,2,n))$. Recall that each row of $A_k^{(n)}$ correspond to a vertex of $V_k^{(t,n)}$ and vice versa. Let $\eta_k^{(n)} = \eta^{(n)}|_{V_k^{(t,n)}}$. Let $M_{k,k'}^{(t,n)}$ be the adjacent matrix of $V_k^{(t,n)}$ and $V_{k'}^{(t,n)}$ ordered by $\eta_k^{(n)}$ and $\eta_{k'}^{(n)}$ respectively.

For the convenience of proof, let $M_{k,k'}^{(t,n)}$ be empty matrix if either k or k' is larger than n or less than zero. It is easy to verify that $r(M_{k,k'}^{(t,n)}) > 0$ if and only if $|k - k'| \leqslant t$ and $0 \leqslant k, k' \leqslant n$. Then from definition, we have

$$M^{(t,n)} = \begin{bmatrix} M_{0,0}^{(t,n)} & M_{0,1}^{(t,n)} & M_{0,2}^{(t,n)} & \cdots & M_{0,n-1}^{(t,n)} & M_{0,n}^{(t,n)} \\ M_{1,0}^{(t,n)} & M_{1,1}^{(t,n)} & M_{1,2}^{(t,n)} & \cdots & M_{1,n-1}^{(t,n)} & M_{1,n}^{(t,n)} \\ M_{2,0}^{(t,n)} & M_{2,1}^{(t,n)} & M_{2,2}^{(t,n)} & \cdots & M_{2,n-1}^{(t,n)} & M_{2,n}^{(t,n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n-1,0}^{(t,n)} & M_{n-1,1}^{(t,n)} & M_{n-1,2}^{(t,n)} & \cdots & M_{n-1,n-1}^{(t,n)} & M_{n-1,n}^{(t,n)} \\ M_{n,0}^{(t,n)} & M_{n,1}^{(t,n)} & M_{n,2}^{(t,n)} & \cdots & M_{n,n-1}^{(t,n)} & M_{n,n}^{(t,n)} \end{bmatrix}. \tag{9}$$

Note that $M^{(t,n)}$ is an $2^n \times 2^n$ matrix. If $t \geqslant n$, then $M^{(t,n)}$ is an all one matrix except for the elements in main diagonal. Hence $bw(M^{(t,n)}) = 2^n - 1$. In the following, we only consider the situation when $t < n$. Since the bandwidth of $M^{(t,n)}$ is the maximum Manhattan distance from a non-zero element to the main diagonal and $M^{(t,n)}$ is symmetric, we only need to consider non-zero element from submatrices in Eq 9 on the diagonal or on the top right of the diagonal, that is, non-zero elements from $M_{k,k'}^{(t,n)}$ with $k' \geqslant k$. Since $M_{k,k'}^{(t,n)}$ is all zero when $|k - k'| > t$, we only need to consider non-zero element from $M_{k,k'}^{(t,n)}$ with $k \leqslant k' \leqslant k + t$. By the definition of bandwidth and Manhattan radius, we have Proposition 19.

Proposition 19. *We have*

$$bw(M^{(t,n)}) = \max_{\substack{k=0,\ldots,n-1,\\ 1\leqslant p\leqslant t,\\ p+k\leqslant n}} \left\{ \sum_{q=1}^{p-1} \binom{n}{k+q} + r(M^{(t,n)}_{k,k+p}), bw(M^{(t,n)}_{k,k}) \right\}. \tag{10}$$

Note that $M^{(t,n)}_{k,k}$ is a symmetric matrix. By Eq 8, we have $bw(M^{(t,n)}_{k,k}) = r(M^{(t,n)}_{k,k}) + \binom{n}{k}$. If we can calculate all $r(M^{(t,n)}_{k,k'})$ with $k \leqslant k' \leqslant k+t$, then we can calculate $bw(M^{(t,n)})$ via Eq 10 and, hence, obtain $bw(H(t,2,n))$ by Lemma 17. Now, our aim is to calculate all $r(M^{(t,n)}_{k,k'})$ with $k \leqslant k' \leqslant k+t$.

From Eq 4, we have that if t,n,k,p are non-negative intergers satisfying that $k+p \leqslant n$ and $n \geqslant 1$, then

$$M^{(t,n)}_{k,k+p} = \begin{pmatrix} M^{(t,n-1)}_{k-1,k+p-1} & M^{(t-1,n-1)}_{k-1,k+p} \\ M^{(t-1,n-1)}_{k,k+p-1} & M^{(t,n-1)}_{k,k+p} \end{pmatrix}. \tag{11}$$

Let $r^{(1)}_{t,n,k,p} = r(M^{(t,n-1)}_{k-1,k+p-1}) + \binom{n-1}{k}$, $r^{(2)}_{t,n,k,p} = r(M^{(t-1,n-1)}_{k-1,k+p}) + \binom{n-1}{k} + \binom{n-1}{k+p-1}$, $r^{(3)}_{t,n,k,p} = r(M^{(t-1,n-1)}_{k,k+p-1})$ and $r^{(4)}_{t,n,k,p} = r(M^{(t,n-1)}_{k,k+p}) + \binom{n-1}{k+p-1}$. From the definition of $r(M^{(t,n)}_{k,k+p})$ and Eq 11, we have Proposition 20.

Proposition 20. *For non-negative integers t,n,k,p satisfying that $k+p \leqslant n$ and $n \geqslant 1$, we have*

$$r(M^{(t,n)}_{k,k+p}) = \max\left\{ r^{(1)}_{t,n,k,p}, r^{(2)}_{t,n,k,p}, r^{(3)}_{t,n,k,p}, r^{(4)}_{t,n,k,p} \right\}. \tag{12}$$

Considering $M^{(t,n)}_{k_1,k_2}$ with $k_1 \leqslant k_2 \leqslant k_1+t$, it is easy to verify that if the parity of k_2-k_1 and t is different, then $M^{(t,n)}_{k_1,k_2} = M^{(t-1,n)}_{k_1,k_2}$. Hence, we only need to calculate all $M^{(t,n)}_{k_1,k_2}$ with $k_1 \leqslant k_2 \leqslant k_1+t$ and $k_2-k_1 \equiv t \pmod 2$. In this case, assume $t-(k_2-k_1) = 2s$. Then $M^{(t,n)}_{k,k+t-2s}$ is non-zero. If $t \geqslant n-1$, then $M^{(t,n)}_{k,k+t-2s}$ is an all-one matrix (except the main diagonal when $t=2s$) and, thus, $r(M^{(t,n)}_{k,k+t-2s}) = \binom{n}{k} + \binom{n}{k+t-2s} - 1$. When $1 \leqslant t < n-1$, we let

$$A^{(1)}_{t,n,k,s} = \sum_{a=0}^{k-s-1} \left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1} \right),$$

$$A^{(2)}_{t,n,k,s} = \sum_{m=t-3s+1+2k}^{n-s} \left(\binom{m-1}{k+t-2s-1} - \binom{m-1}{k-s-1} \right),$$

$$B_{t,n,k,s} = \sum_{a=0}^{n-t-k+s-1} \left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1} \right),$$

$$
\begin{aligned}
C_{t,n,k,s} &= \sum_{m=n-s+1}^{n} \binom{m-1}{k+t-2s-1} = \binom{n}{k+t-2s} - \binom{n-s}{k+t-2s}, \\
D_{t,n,k,s} &= \sum_{m=k+t-2s+1}^{n} \binom{m-1}{k+t-2s-1}.
\end{aligned}
$$

Here we define $\binom{x}{y} = 0$ if $y < 0$ or $y > x$. Note that all $A^{(1)}, A^{(2)}, B, C, D$ terms are non-negative. It is easy to verify that if $k + t - 2s = n$, then $D_{t,n,k,s} = 0$, otherwise $k + t - 2s < n$, then $D_{t,n,k,s} = \binom{n}{k+t-2s} - \binom{k+t-2s}{k+t-2s} = \binom{n}{k+t-2s} - 1$. Then we have the following results.

Lemma 21. *Let t, n, k, s be non-negative integers satisfy that $t \geqslant 2s, n \geqslant 1, t \geqslant 1$ and $k + t - 2s \leqslant n$. Then*

$$
r(M^{(t,n)}_{k,k+t-2s}) = \begin{cases} \binom{n}{k} + \binom{n}{k+t-2s} - 1 & \text{if } t \geqslant n-1, \\ \binom{n}{k} + A^{(1)}_{t,n,k,s} + A^{(2)}_{t,n,k,s} + C_{t,n,k,s} & \text{if } t < n-1, 0 \leqslant k-s \leqslant \lfloor (n-t)/2 \rfloor, \\ \binom{n}{k} + B_{t,n,k,s} + C_{t,n,k,s} & \text{if } t < n-1, \lfloor (n-t)/2 \rfloor \leqslant k-s \leqslant n-t, \\ \binom{n}{k} + D_{t,n,k,s} & \text{if } t < n-1,\ k-s \leqslant 0 \text{ or } k-s \geqslant n-t. \end{cases} \tag{13}
$$

Lemma 22. *We have*

$$
bw(M^{(t,n)}) = \sum_{k=\lfloor (n-t)/2 \rfloor}^{\lfloor (n-t)/2 \rfloor + t-1} \binom{n}{k} + \sum_{a=0}^{\lfloor (n-t-1)/2 \rfloor} \left(\binom{t+2a}{t+a-1} - \binom{t+2a}{a-1} \right). \tag{14}
$$

From Eq 10 and 12, we can prove Lemmas 21 and 22 by induction. The complete proofs of Lemmas 21 and 22 can be find in **Appendix A** and **Appendix B**, respectively. Lemmas 22 and 17 show the exact bandwidth of $H(t, 2, n)$.

For a set $S \subseteq V$, let $N(S) = \{v \in V - S \mid \exists u \in S, uv \in E\}$, $\Phi(S) = |N(S)|$ and $b_v(l, G) = \min_{S \subseteq V, |S| = l} \Phi(S)$. Harper [15] showed that $bw(G) = \max_{1 \leqslant s \leqslant |V|} b_v(s, G)$ if G admits a Hales numbering. Therefore, $bw(H(t, 2, n)) = \max_{1 \leqslant s \leqslant 2^n} b_v(s, H(t, 2, n))$.

Theorem 23 (Theorem 1 of [9])**.** *Let $G(V, E)$ be any graph on n vertices, and let $1 \leqslant s \leqslant n$. Then $pw(G) \geqslant b_v(s, G)$.*

Lemma 24. *We have $pw(H(t, 2, n)) = bw(H(t, 2, n))$.*

Proof. By Theorem 23, we have $pw(H(t, 2, n)) \geqslant b_v(s, H(t, 2, n))$ for all $1 \leqslant s \leqslant 2^n$. Then $pw(H(t, 2, n)) \geqslant \max_{1 \leqslant s \leqslant 2^n} b_v(s, H(t, 2, n)) = bw(H(t, 2, n))$. Combining Eq 2, we have $pw(H(t, 2, n)) = bw(H(t, 2, n))$. □

Theorem 1 can be derived from Lemmas 17, 22 and 24.

3.2 Treewidth of $H(t, q, n)$

In this subsection we analyze the asymptotic behavior of $tw(H(t,q,n))$ when n goes to infinity and give the proof of Theorem 2.

We first prove the lower bound of $tw(H(t,q,n))$ by Proposition 25.

Proposition 25 (Lemma 7 of [9])**.** *Let $G(V,E)$ be a graph with n vertices. If for each subset X of V with $n/4 \leqslant |X| \leqslant n/2$, $\Phi(X) \geqslant k$, then $tw(G) \geqslant k-1$.*

Lemma 26. *$tw(H(t,q,n)) \geqslant c_1 tq^n/\sqrt{n}$ for some constant c_1 not depending on t or q when n is sufficiently large.*

Proof. By Proposition 25, we have $tw(H(t,q,n)) \geqslant \min b_v(m, H(t,q,n)) - 1$ over integers m in the range $q^n/4 \leqslant m \leqslant q^n/2$. So it is sufficient to give a lower bound for $b_v(m, H(t,q,n))$ over $m \in [q^n/4, q^n/2]$.

In [14], Harper showed that[2] if

$$m = q^n \sum_{i=0}^{r} \binom{n}{i} x^{n-i}(1-x)^i \text{ for some } x, r,\ 0 < x < 1, \tag{15}$$

then

$$b_v(m, H(t,q,n)) \geqslant q^n \min_{x,r} \left\{ \sum_{i=1}^{t} \left(\binom{n}{r+i} x^{n-r-i}(1-x)^{r+i} \right) \right\}, \tag{16}$$

where the minimum is taken over all x, r satisfying Eq 15.

In [9], it is proved that when $m = q^n \sum_{i=0}^{r} \binom{n}{i} x^{n-i}(1-x)^i$, we have

$$n(1-x) - \sqrt{4nx(1-x)} < r < n(1-x) + \sqrt{4nx(1-x)}.$$

By Stirling's approximation, it can be shown that for all r in the above range, we have

$$q^n \binom{n}{r+i} x^{n-r-i}(1-x)^{r+i} \geqslant c_1(q^n/\sqrt{n}), 1 \leqslant i \leqslant t$$

for some constant $c_1 > 0$ not depending on t and q. Then, $b_v(m, H(t,q,n)) \geqslant c_1 tq^n/\sqrt{n}$. □

Then we intend to estimate the upper bound via bandwidth.

Lemma 27. *$tw(H(t,q,n)) \leqslant pw(H(t,q,n)) \leqslant c_2 tq^n/\sqrt{n}$ for some constant c_2 not depending on t or q when n is sufficiently large.*

[2]This statement is not explicitly stated in [14], but can be easily inferred from Theorem 3 on pp. 302.

Proof. For convenience, we first assume q is even. The case when q is odd can be handled similarly. Let f be a function from $[q]$ to $\{0,1\}$ as follows:

$$f(i) = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant q/2, \\ 1 & \text{if } q/2 < i \leqslant q. \end{cases}$$

Suppose that $(a_1, a_2, \ldots, a_n) \in [q]^n$ is an n-vector corresponding to a vertex x of $H(t,q,n)$. Define a function g from $V(H(t,q,n))$ to $V(H(t,2,n))$ that maps $x \in V(H(t,q,n))$ to the vertex $g(x) \in V(H(t,2,n))$ which corresponds to the vector $(f(a_1), f(a_1), \ldots, f(a_n))$. Note that g maps exactly $(q/2)^n$ vertices of $H(t,q,n)$ to a given vertex of $H(t,2,n)$.

Let $H(t,2,n)$ have a path decomposition whose bags are $\{P_i\}$. By replacing each P_i with $P_i' = \bigcup_{y \in P_i} g^{-1}(y)$, it is easy to show that $\{P_i'\}$ is a path decomposition of $H(t,q,n)$. Therefore, $pw(H(t,q,n)) \leqslant pw(H(2,q,n)) \cdot \left(\frac{q}{2}\right)^n$.

From Theorem 1 and Stirling's approximation, we have $bw(H(t,2,n)) \leqslant c_2 t 2^n/\sqrt{n}$ for some constant c_2. Therefore, we have $tw(H(t,q,n)) \leqslant pw(H(t,q,n)) \leqslant c_2 t q^n/\sqrt{n}$. $\square$

Combining Lemmas 26 and 27, we can derive Theorem 2.

4 Treewidth of bipartite Kneser graph and Johnson graph

For positive integers n and k satisfying $n \geqslant 2k+1$, the bipartite Kneser graph $BK(n,k)$ has all subsets of $[n]$ with k or $n-k$ elements as vertices and an edge between any two vertices when one is a subset of the other. It is also called *middle cube graph*. In the following, we will use k-subsets and $(n-k)$-subsets of $[n]$ to represent the vertices of bipartite Kneser graph. We call vertices that are k-subsets of $[n]$ the *left part* denoted by V_L, and the rest is called the *right part* denoted by V_R. V_L and V_R are two parts of bipartite Kneser graph $BK(n,k)$.

For positive integers n and k satisfying $n > k$, the *Johnson graph* $J(n,k)$ has all subsets of $[n]$ with k elements and an edge between any two vertices when their intersection has exactly $(k-1)$ elements. Since we have a bijection between all k-subsets of $[n]$ and all binary n-vectors with exactly k ones, we can also treat a vertex of $J(n,k)$ as an n-vector with exactly k ones. From this point of view, two vertices are adjacent iff their the Hamming distance of their corresponding n-vectors is no more than 2. Therefore, $J(n,k)$ is the k-th slice of $H(2,2,n)$, that is, $J(n,k)$ is a subgraph of $H(2,2,n)$ induced by vertices from $H(2,2,n)$ corresponding to n-vector with exactly k ones.

Let $G(V,E)$ be a graph and $S \subseteq V$ a vertex subset of G. Denote the subgraph of G induced by S as $G[S]$. Recalling the definition of $V_k^{(t,n)}$ in subsection 3.1, we have Proposition 28.

Proposition 28. *We have* $J(n,k) \cong H(2,2,n)[V_k^{(2,n)}]$.

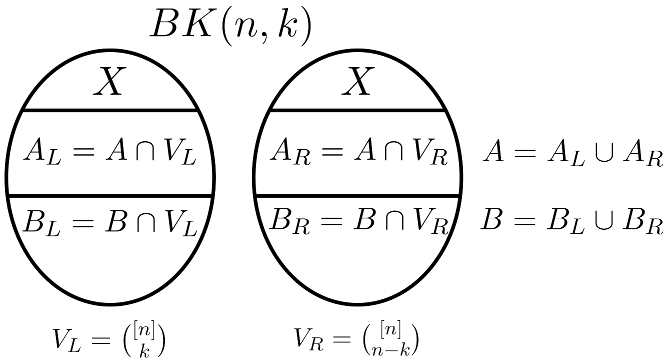

Figure 3: Sketch graph of $BK(n,k)$. The condition that there is no edge between A and B equals to there is no edge between A_L and B_R, and between A_R and B_R.

4.1 Treewidth of $BK(n,k)$ when n is large enough

In this section, we focus on the treewidth of $BK(n,k)$ when n is large enough and give the proof of Theorem 3. Before proof, we need the following proposition.

Proposition 29 ([17]). *For any graph G, $tw(G) \leqslant \max\{\Delta(G), |V(G)| - \alpha(G) - 1\}$, where $\alpha(G)$ is the independent number of G.*

Lemma 30. *We have $tw(BK(n,k)) \leqslant \binom{n}{k} - 1$.*

Proof. From [21], we have that bipartite Kneser graph has a perfect matching. Consequently, $\alpha(BK(n,k)) = \binom{n}{k}$. Note that $BK(n,k)$ is a regular graph with order $2\binom{n}{k}$ and $\Delta(BK(n,k)) = \binom{n-k}{k}$. By Proposition 29, we have $tw(BK(n,k)) \leqslant \binom{n}{k} - 1$. □

Lemma 31. *When $k \geqslant 2$ and $3\binom{n-k}{k} \geqslant 2\binom{n}{k}$, $tw(BK(n,k)) \geqslant \binom{n}{k} - 1$.*

Proof. Denote $BK(n,k)$ by G and $V(G) = V_L \cup V_R$, where $V_L = \binom{[n]}{k}$ and $V_R = \binom{[n]}{n-k}$. Suppose $tw(G) < \binom{n}{k} - 1$. From Corollary 10, there exists a separator X of G with $|X| < \binom{n}{k}$ such that there exists non-empty vertex set A and B with $A \cup B = V(G) - X$, $A \cap B = \emptyset$, $|V(G) - X|/3 \leqslant |A|, |B| \leqslant 2|V(G) - X|/3$ and there is no edge between A and B. Let $A_L = A \cap V_L$, $A_R = A \cap V_R$, $B_L = B \cap V_L$ and $B_R = B \cap V_R$ (see Figure 3).

Since A, B are nonempty and $|X| < \binom{n}{k}$, we assume without loss of generality that A_L and B_R are nonempty. Let

$$\mathcal{A} = \left\{ S \in \binom{[n]}{k} \mid S \in A_L \right\},$$
$$\mathcal{B} = \left\{ S \in \binom{[n]}{n-k} \mid S \in B_R \right\}.$$

Let $\mathcal{C} = \{[n] - S \mid S \in \mathcal{B}\} \subseteq \binom{[n]}{k}$. Since there is no edge between A_L and B_R, for any $S_1 \in \mathcal{A}$ and $S_2 \in \mathcal{B}$, $S_1 \nsubseteq S_2$ which implies $S_1 \cap ([n] - S_2) \neq \emptyset$.

Definition 32 (Cross-intersecting families)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of a finite set X. We say that $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting* if for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $A \cap B \neq \emptyset$.

Hence, $\mathcal{A}$ and $\mathcal{C}$ is cross-intersecting. By the properties of cross-intersecting families from [13], we have $|\mathcal{A}|+|\mathcal{C}| \leqslant \binom{n}{k}-\binom{n-k}{k}+1$. Notice that $|\mathcal{A}| = |A_L|$ and $|\mathcal{C}| = |\mathcal{B}| = |B_R|$. Then we have $|A_L| + |B_R| \leqslant \binom{n}{k} - \binom{n-k}{k} + 1$.

If A_R, B_L are both nonempty, then, similarly, we have $|A_R|+|B_L| \leqslant \binom{n}{k}-\binom{n-k}{k}+1$. Hence, $|A| + |B| \leqslant 2\left(\binom{n}{k} - \binom{n-k}{k} + 1\right)$ and $|X| = |V(G)| - |A| - |B| \geqslant 2\binom{n-k}{k} - 2$. Since $k \geqslant 2$ and $3\binom{n-k}{k} \geqslant 2\binom{n}{k}$, we have $2\binom{n-k}{k} - 2 \geqslant \binom{n}{k}$ which derives a contradiction with $|X| < \binom{n}{k}$.

If A_R, B_L are both empty, then we have $|A| + |B| = |A_L| + |B_R| \leqslant \binom{n}{k} - \binom{n-k}{k} + 1$. Hence, $|X| = |V(G)| - |A| - |B| \geqslant \binom{n}{k} + \binom{n-k}{k} - 1 \geqslant \binom{n}{k}$, a contradiction.

If there is only one empty set in $\{A_R, B_L\}$, say $A_R \neq \emptyset$ and $B_L = \emptyset$, then we have $|B| = |B_R| \leqslant \binom{n}{k}-\binom{n-k}{k}$ by $|A_L|+|B_R| \leqslant \binom{n}{k}-\binom{n-k}{k}+1$ and $A_L \neq \emptyset$. Since $|V(G)-X|/3 \leqslant |A|, |B| \leqslant 2|V(G) - X|/3$, we have $|A| + |B| \leqslant 3|B| \leqslant 3\left(\binom{n}{k} - \binom{n-k}{k}\right)$. Since $3\binom{n-k}{k} \geqslant 2\binom{n}{k}$, we have $|X| = |V(G)|-|A|-|B| \geqslant 3\binom{n-k}{k}-\binom{n}{k} \geqslant \binom{n}{k}$, a contradiction. □

Theorem 3 can be easily derived from Lemmas 30 and 31.

4.2 Treewidth of $BK(2k+1,k)$ and $J(2k+1,k)$

When n is large enough, the treewidth of the bipartite Kneser graph can be exactly calculated by Theorem 3. Now we focus on the treewidth of the bipartite Kneser graph when n is small and give the proof of Theorem 4.

In order to prove our reslt, we need more definitions. A graph G is chordal if and only if, in any cycle of length larger than 3 in G, there exists a chord connecting two nonadjacent vertices of the cycle. Given a graph G, define $\omega(G)$, the clique number of G to be the number of vertices of the largest clique in G. The treewidth of a graph G has a close relationship with its chordal supergraph as Proposition 33 shows.

Proposition 33 ([32])**.** *Given a graph G, $tw(G) = \min\{\omega(H)-1 \mid G \subseteq H, H \text{ is chordal}\}$.*

Using Proposition 33, we can prove the relationship between the treewidth of the bipartite Kneser graph and Johnson graph as Lemma 34. We first explain the proof ideas. Note that $BK(n,k)$ is a bipartite graph with two parts $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$. And the vertex set of $J(n,k)$ is $\binom{[n]}{k}$ which is the same as one part of $BK(n,k)$. Thus, we can embed $J(n,k)$ into $BK(n,k)$ by letting the $\binom{[n]}{k}$ part of $BK(n,k)$ is isomorphic to $J(n,k)$. We then will show the result graph is chordal. After calculating its clique number and then using Proposition 33, we can derive our result.

Lemma 34. *For any positive integer k, $tw(BK(2k+1,k)) \leqslant tw(J(2k+1,k))$.*

Proof. Let $n = 2k+1$. By Proposition 33, there exists a chordal graph H such that $J(n, k)$ is a subgraph of H and $\omega(H) - 1 = tw(J(n, k))$. Add edges to the left part of $BK(n, k)$ such that the left part is isomorphic to H and denote the result graph as $BK'(n, k)$. That is, $BK'(n, k)[V_L] \cong H$.

We first claim that $BK'(n, k)$ is chordal. Suppose C is a cycle in $BK'(n, k)$ with length larger than 3. We first consider the case that there is a vertex $v \in V(C) \cap V_R$. Let u_1 and u_2 be the two neighbors of v on C. Then $u_1, u_2 \in V_L$. Let $A_v \in \binom{[n]}{k+1}$ be the corresponding set of v and $A_{u_1}, A_{u_2} \in \binom{[n]}{k}$ the corresponding sets of u_1 and u_2 respectively. Since $u_1v, u_2v \in E(BK'(n, k))$, we have $A_{u_1} \subseteq A_v$ and $A_{u_2} \subseteq A_v$. Then $|A_{u_1} \cap A_{u_2}| = k - 1$. Hence, $u_1u_2 \in E(BK'(n, k))$. Now assume all vertices of C are from the left side. Since H is chordal, there must exist a chord in C. From above all, we have that $BK'(n, k)$ is chordal.

Let W be a clique of $BK'(n, k)$. If there exists a vertex $v \in W$ from the right side, then $|W| \leqslant k + 1 + 1$ by $d(v) = k + 1$. If all vertices of W are from the left part, then $|W| \leqslant \omega(H) = tw(J(n, k)) + 1$. Hence

$$tw(BK(n, k)) \leqslant \omega(BK'(n, k)) - 1 \leqslant \max\{tw(J(n, k)), k + 1\}.$$

Notice that $J(2k+1, k)$ is $k(k+1)$-regular. Since $tw(G) \geqslant \delta(G)$ [24] for any graph G, we have $tw(J(2k+1, k)) \geqslant k(k+1) \geqslant k+1$. Then $tw(BK(2k+1, k)) \leqslant tw(J(2k+1, k))$. □

Lemma 35. *For positive integers n and k satisfying $n > k$, $tw(J(n, k)) \leqslant bw(M_{k,k}^{(2,n)}) = r(M_{k,k}^{(2,n)}) - \binom{n}{k}$.*

Proof. Let $G = H(2, 2, n)[V_k^{(2,n)}]$ for short. By Proposition 28, we have $G \cong J(n, k)$.

Let $t = 2$ in subsection 3.1. Recalling that $\eta' \triangleq \eta_k^{(n)}$ is an ordering of $V_k^{(2,n)}$, then η' is also an ordering of G. Note that the adjacency matrix of G with the ordering η' is exactly $M_{k,k}^{(2,n)}$. By Remark 16 and the definition of bandwidth, we have

$$bw(M_{k,k}^{(2,n)}) = \max_{e \in E(G)} \Delta(e, \eta') \geqslant \min_{\eta} \max_{e \in E(G)} \Delta(e, \eta) = bw(G) = bw(J(n, k)). \qquad (17)$$

Combining Eq 2 and Eq 8 with $s = |V_k^{(2,n)}| = \binom{n}{k}$, we can derive the lemma. □

Specifically, take $n = 2k + 1$, and then from Lemma 21, we can derive the asymptotic behavior of $bw(M_{k,k}^{(2,2k+1)})$ by calculating:

$$\lim_{k \to +\infty} bw(M_{k,k}^{(2,2k+1)}) / \binom{2k+1}{k} = 1/2. \qquad (18)$$

Therefore, $tw(J(2k + 1, k)) = O(\binom{2k+1}{k})$.

Proposition 36 ([10])**.** *Suppose G is a k-regular graph with n vertices and $A(G)$ is the adjacency matrix of G. Let $\mu(G) = k - \lambda(G)$ where $\lambda(G)$ is the second-largest eigenvalue of $A(G)$. Then*

$$tw(G) \geqslant \left\lfloor \frac{3n}{4} \frac{\mu(G)}{\Delta(G) + 2\mu(G)} \right\rfloor - 1.$$

Proposition 37 ([28])**.** *The characteristic polynomial of $BK(2k+1, k)$ is*

$$\prod_{i=1}^{k+1} (\lambda + i)^{\binom{n}{k+1-i} - \binom{n}{k-i}} (\lambda - i)^{\binom{n}{k+1-i} - \binom{n}{k-i}}.$$

Lemma 38. *We have*

$$tw(BK(2k+1, k)) \geqslant \left\lfloor \frac{3}{2} \binom{2k+1}{k} \frac{1}{k+3} \right\rfloor - 1.$$

Lemma 38 can be derived from Propositions 36 and 37. Theorem 4 can be derived by Lemmas 34, 35 and 38.

5 Treewidth of generalized Petersen graph

In this section, we determine the treewidth of generalized Petersen graph. The vertex set and edge set of generalized Petersen graph $G_{n,k}$ are

$$V(G_{n,k}) = \{v_1, \ldots, v_n, u_1, \ldots, u_n\},$$
$$E(G_{n,k}) = \{v_i u_i\} \cup \{v_i v_{i+1}\} \cup \{u_i u_{i+k}\}, i = 1, 2, \ldots, n,$$

where subscripts are to be read modulo n and $k < n/2$. Let G be a graph and X and Y are two connected subgraphs of G. We say X *touches* Y when $V(X) \cap V(Y) \neq \emptyset$ or there exists an edge between X and Y. A *bramble* of G is a family of connected subgraphs of G that all touch each other. Let S be a subset of $V(G)$. S is said to be a *hitting set* of bramble B if S has nonempty intersection with each of the subgraphs in B. The order of a bramble is the smallest size of a hitting set. Brambles may be used to characterize the treewidth of a given graph.

Proposition 39 ([33])**.** *Let G be a graph. Then $tw(G) \geqslant k$ if and only if G contains a bramble of order at least $k+1$.*

With the help of Proposition 39, we now can give the proof of Theorem 5.

Proof of Theorem 5: First, we intend to prove the lower bound. Construct a bramble $B = \{B_i\}$ of $G_{n,k}$ as

$$V_i = \{v_i, v_{i+1}, \ldots, v_{i+t}, u_{i+t}, u_{i+t+k}, u_{i+t+2k}, \ldots, u_{i+t+tk}\},$$
$$B_i = G_{n,k}[V_i],$$

where $t = \lceil \frac{n}{2k+2} \rceil, i = 1, 2, \ldots, n$. Then we have $|V_i| = 2t + 2$ and B_i is connected. For each pair i and j, we intend to prove that B_i touches B_j. Without loss of generality, assume $1 \leqslant i < j \leqslant n$.

- If $j \leqslant i + t$, then $v_j \in V_i \cap V_j$.
- If $i + t < j \leqslant i + t + tk$, let r be the minimum integer in $\{0, 1, \ldots, k-1\}$ such that $j + r \equiv i + t \pmod k$. Noticing that $t \geqslant k$ when $n \geqslant 8(2k+2)^2$. Hence, $v_{j+r} \in V_j, u_{j+r} \in V_i$, and $u_{j+r}v_{j+r} \in E(G_{n,k})$.
- If $i+t+tk < j < n+i-t$, then $j+t < i+n < j+n-t-tk \leqslant j+t+tk$. The last inequality comes from $2kt+2t \geqslant n$ by $t = \lceil \frac{n}{2k+2} \rceil$ and then we have $t+tk \geqslant n-t-tk$ which implies $j + t < i + n \leqslant j + t + tk$. The next proof is the same as that in the second situation.
- If $n + i - t \leqslant j \leqslant n$, the proof is the same as that in the first situation.

Then B is a bramble. Let S be a hitting set of B. We construct a hypergraph H with vertex set $V(H) = V(G_{n,k})$ and hyperedge set $\{V_i\}_{1 \leqslant i \leqslant n}$.

Definition 40 (Transversal)**.** Let H be a hypergraph on a set X with edges $E_1, \ldots, E_m$. A set $T \subseteq X$ is called a transversal of H if T intersects every edge of H, that is,

$$T \cap E_i \neq \emptyset, \forall i = 1, 2, \ldots, m.$$

Then S is a transversal of H and thus, $\min |S| = \tau(H)$ where $\tau(H)$ is the transversal number of H. Since $\tau(H) \geqslant \max\limits_{H' \subseteq H} \frac{m(H')}{\Delta(H')}$ [4], where $m(H')$ is the number of edges in H' and $\Delta(H')$ is the maximum degree of H', we have that the order of B is

$$\min |S| \geqslant \frac{m(H)}{\Delta(H)} = \frac{n}{t+1}.$$

Since $t = \lceil \frac{n}{2k+2} \rceil$, we have $t - 1 \leqslant \frac{n}{2k+2}$ and

$$2k + 2 - \frac{n}{t+1} \leqslant \frac{n}{t-1} - \frac{n}{t+1} = \frac{2n}{t^2 - 1} < 1.$$

Therefore, the order of B is no less than $2k + 2$. From Proposition 39, we can derive that $tw(G_{n,k}) \geqslant 2k + 1$.

The upper bound can be proved via construction. Construct a path-decomposition of $G_{n,k}$ as following (see Figure 4 and Figure 5).

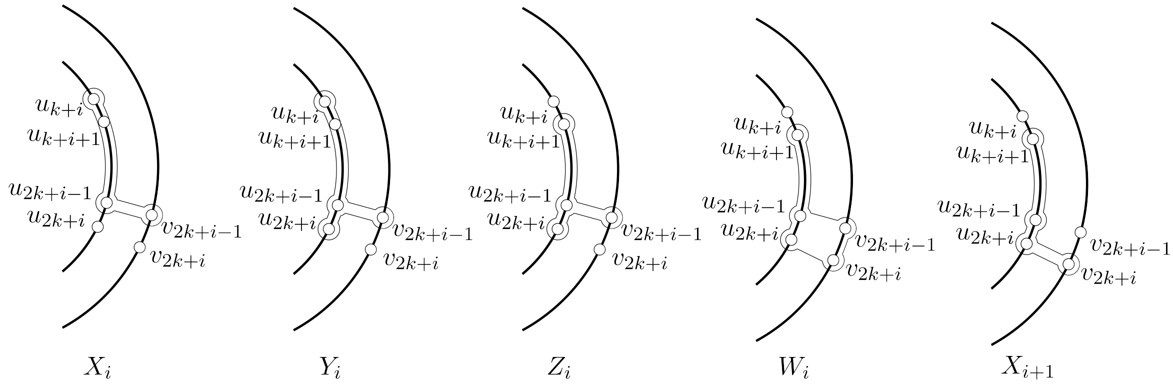

Figure 4: The vertex sets X_i, Y_i, Z_i and W_i.

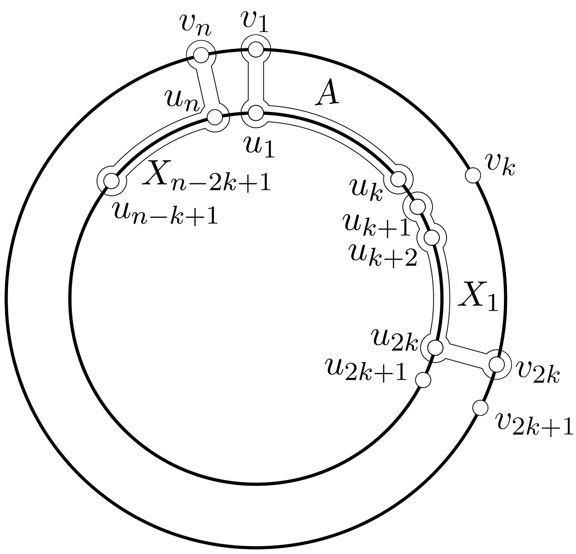

Figure 5: The defined sets on generalized Petersen graph.

Step 1. Let $A = \{v_1, u_1, \ldots, u_k\}$, $B_1 = \{v_1, \ldots, v_k\}$ and $B_2 = \{v_k, \ldots, v_{2k}\}$.

Step 2. Let $X_1 = \{u_{k+1}, \ldots, u_{2k}, v_{2k}\}$.

Step 3. Recursively define $Y_i = X_i \cup \{u_{2k+i}\}$, $Z_i = Y_i - \{u_{k+i}\}$, $W_i = Z_i \cup \{v_{2k+i}\}$ and $X_{i+1} = W_i - \{v_{2k+i-1}\}$ for $1 \leqslant i \leqslant n-2k$. It is easy to verify that $X_i = \{u_{k+i}, \ldots, u_{2k+i-1}, v_{2k+i-1}\}$.

Step 4. Define $X_{n-2k+1} = W_{n-2k} - \{v_{n-1}\}$.

Step 5. Define a path decomposition $\mathcal{P}_1$ of $G_{n,k} - A$ by successively connect

$$(B_1, B_2, X_1, Y_1, Z_1, W_1, X_2, \ldots, W_{n-2k}, X_{n-2k+1}).$$

Step 6. Add each vertex in A to all bags of $\mathcal{P}_1$, then we obtain a path decomposition $\mathcal{P}_2$ of $G_{n,k}$.

Here we explain why we construct the path-decomposition in this way. For a cycle $v_1v_2\dots v_k$, the way to build its path-decomposition is to first delete any vertex, say v_1, then the left part is a path $v_2\dots v_k$. Then build a path-decomposition like $\{v_2v_3\} - \{v_3v_4\} - \dots - \{v_{k-1}v_k\}$ and then add v_1 to all bags. Here the generalized Petersen graph behaves like a "double-cycle". We first delete A, and the rest part behaves like a "path". Then the sequence of $X_i, Y_i, Z_i, W_i, X_{i+1}$ is like what we do in the path-decomposition of a cycle. Once we move one vertex so that the width do not increase too much. B_1 and B_2 are designed to cover the left vertices. Finally we add A to all bags just like for the cycle we add v_1 back.

It is easy to verify $\mathcal{P}_1$ is a path-decomposition of $G_{n,k} - A$ by checking the three properties which implying that $\mathcal{P}_2$ is a path-decomposition of $G_{n,k}$.

Since $|X_i| = |Z_i| = k+1, |Y_i| = |W_i| = k+2, |B_1| = k, |B_2| = k+1$ and $|A| = k+1$, the width of $\mathcal{P}_2$ is $2k+3$ and, hence, $tw(G_{n,k}) \leqslant pw(G_{n,k}) \leqslant 2k+2$. □

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Appendix A: Proof of Lemma 21

Before the proof, we need several definitions. We say the parameter tuple (t, n, k, s) is *valid* if $t \geqslant 2s, n \geqslant 1, 1 \leqslant t < n-1$ and $k + t - 2s \leqslant n$, that corresponds to the

$r^{(i)}$-term	matrix	t'	n'	k'	s'
$r^{(1)}_{t,n,k,t-2s}$	$r(M^{(t,n-1)}_{k-1,k+t-2s-1})$	t	$n-1$	$k-1$	s
$r^{(2)}_{t,n,k,t-2s}$	$r(M^{(t-1,n-1)}_{k-1,k+t-2s})$	$t-1$	$n-1$	$k-1$	$s-1$
$r^{(3)}_{t,n,k,t-2s}$	$r(M^{(t-1,n-1)}_{k,k+t-2s})$	$t-1$	$n-1$	k	s
$r^{(4)}_{t,n,k,t-2s}$	$r(M^{(t,n-1)}_{k,k+t-2s})$	t	$n-1$	k	s

Table 1: Parameter tuple of the matrix in the Eq 13 of $r^{(i)}_{t,n,k,t-2s}$.

non-trivial situation of Lemma 21. The parameter tuple of matrix in the formula of $r^{(i)}_{t,n,k,t-2s}$ ($i = 1, 2, 3, 4$) is shown as Table 1.

First, it is easy to verify that $D_{t,n,k,s} = A^{(1)}_{t,n,k,s} + A^{(2)}_{t,n,k,s} + C_{t,n,k,s}$ holds when $k - s = 0$ and, $D_{t,n,k,s} = B_{t,n,k,s} + C_{t,n,k,s}$ when $k - s = n - t$ and $A^{(1)}_{t,n,k,s} + A^{(2)}_{t,n,k,s} = B_{t,n,k,s}$ hold when $k - s = \lfloor (n - t)/2 \rfloor$. Hence, in those situations, we can calculate $r(M^{(t,n)}_{k,k+t-2s})$ in both ways.

In the following, we prove the Lemma by induction on $t + n + k + s$. From Table 1, when we calculate $\{r^{(i)}_{t,n,k,t-2s}\}_{1\leqslant i\leqslant 4}$, the term always obtains a smaller $t' + n' + k' + s'$, which means we can use Eq 13 to calculate those terms by induction.

Step 1: verify Lemma 21 when either $t \geqslant n - 1$, or $t = 1$.

When $t \geqslant n - 1$, it is trivial to verify $M^{(t,n)}_{k,k+t-2s}$ is an all-one matrix and, thus, $r(M^{(t,n)}_{k,k+t-2s}) = \binom{n}{k} + \binom{n}{k+t-2s} - 1$ from definition.

When $t = 1$, then $s = 0$. In this case $r^{(2)}_{1,n,k,1} = -\infty$ and $r^{(3)}_{1,n,k,1} = -\infty$ and, hence, $r(M^{(1,n)}_{k,k+1}) = \max\{r^{(1)}_{1,n,k,1}, r^{(4)}_{1,n,k,1}\}$. We can prove Eq 13 holds for $t = 1$ by induction on n. The proof is omitted.

Then, we give a proof of Lemma 21 via Eq 12 by induction. Suppose Eq 13 holds for $t + n + k + s \leqslant N - 1$. Now consider when $t + n + k + s = N$. In the following steps, we only need to consider the case $t \geqslant 2$ and $t < n - 1$.

Step 2: verify Eq 13 when $k = 0$ and $1 < t < n - 1$.

If $k = 0$, then $k + t - 2s < n$ and $r^{(1)}_{t,n,k,t-2s} = -\infty$ and $r^{(2)}_{t,n,k,t-2s} = -\infty$.

Case 2.1: If $t = 2s$, then $r(M^{(t,n)}_{k,k+t-2s}) = r(M^{(t,n)}_{0,0}) = 1$ from definition. It is easy to verify Eq 13 holds in this case.

Case 2.2: If $t > 2s$, then $(t - 1, n - 1, k, s)$ are valid and, hence, $r^{(3)}_{t,n,k,t-2s} = r(M^{(t-1,n-1)}_{k,k+(t-1)-2s})$ can be calculated via Eq 13 by induction.

Since $k-s \leqslant 0$, we have

$$\begin{aligned}
r^{(3)}_{t,n,k,t-2s} &= r(M^{(t-1,n-1)}_{k,k+(t-1)-2s}) \\
&= \binom{n-1}{k} + D_{t-1,n-1,k,s} \\
&\leqslant \binom{n}{k} + \sum_{m=k+t-2s+1}^{n} \binom{m-1}{k+t-2s-1} = \binom{n}{k} + D_{t,k,n,s},
\end{aligned}$$

where the second equality comes from $k-s \leqslant 0$ and Eq 13 by induction.

Subcase 2.2.1: If $t = n-2$, then

$$\begin{aligned}
r^{(4)}_{t,n,k,t-2s} &= r(M^{(t,n-1)}_{k,k+t-2s}) + \binom{n-1}{k+t-2s-1} \\
&= \binom{n-1}{k} + \binom{n-1}{k+t-2s} - 1 + \binom{n-1}{k+t-2s-1} = \binom{n}{n-2s-2} \\
&= \binom{n}{k} + D_{t,n,k,s},
\end{aligned}$$

where the second equality comes from the trivial situation of Lemma 21 which we have proved before Therefore, we have verified Eq 13 in this subcase.

Subcase 2.2.2: If $t < n-2$, then $(t, n-1, k, s)$ are valid and, hence, $r^{(4)}_{t,n,k,t-2s} = r(M^{(t,n-1)}_{k,k+t-2s})$ can be calculated via Eq 13 by induction.

$$\begin{aligned}
r^{(4)}_{t,n,k,t-2s} &= r(M^{(t,n-1)}_{k,k+t-2s}) + \binom{n-1}{k+t-2s-1} \\
&= \binom{n-1}{k} + D_{t,n-1,k,s} + \binom{n-1}{k+t-2s-1} \\
&= \binom{n}{k} + D_{t,n,k,s},
\end{aligned}$$

where the second equality comes from Step 1 and the third equality comes from $k = 0$.

From above, combining Eq 12, we can verify that Eq 13 holds when $k = 0$.

Step 3: verify Eq 13 when $k+t-2s = n$ and $1 < t < n-1$.

In this case, $r^{(2)}_{t,n,k,t-2s} = -\infty$ and $r^{(4)}_{t,n,k,t-2s} = -\infty$.

Case 3.1: If $t = 2s$, then $k = n$ and $r(M^{(t,n)}_{k,k+t-2s}) = r(M^{(t,n)}_{n,n}) = 1$ from definition. It is easy to verify Eq 13 holds in this case.

Case 3.2: If $t < 2s$, then $(t-1, n-1, k, s)$ is valid and, hence, $r^{(3)}_{t,n,k,p} = r(M^{(t-1,n-1)}_{k,k+p-1})$

can be calculated via Eq 13 by induction. Since $k-s \geqslant n-t$, we have

$$\begin{aligned} r^{(3)}_{t,n,k,t-2s} &= r(M^{(t-1,n-1)}_{k,k+(t-1)-2s}) \\ &= \binom{n-1}{k} + D_{t-1,n-1,k,s} \\ &= \binom{n-1}{k} \\ &\leqslant \binom{n}{k} = \binom{n}{k} + D_{t,k,n,s}, \end{aligned}$$

where the second equality comes from Eq 13 by induction.

Subcase 3.2.1: If $t = n-2$, then

$$\begin{aligned} r^{(1)}_{t,n,k,t-2s} &= r(M^{(t,n-1)}_{k-1,k-1+t-2s}) + \binom{n-1}{k} \\ &= \binom{n-1}{k} + \binom{n-1}{k-1} + \binom{n-1}{k-1+t-2s} - 1 \\ &= \binom{n}{k} = \binom{n}{k} + D_{t,n,k,s}, \end{aligned}$$

where the second equality comes from Step 1 and the following equalities come from $k+t-2s=n$.

Subcase 3.2.2: If $t < n-2$, then $(t, n-1, k-1, s)$ is valid and, hence, $r^{(1)}_{t,n,k,t-2s} = r(M^{(t,n-1)}_{k-1,k-1+t-2s}) + \binom{n-1}{k}$ can be calculated via Eq 13 by induction. That is

$$\begin{aligned} r^{(1)}_{t,n,k,t-2s} &= r(M^{(t,n-1)}_{k-1,k-1+t-2s}) + \binom{n-1}{k} \\ &= \binom{n-1}{k-1} + D_{t,n-1,k-1,s} + \binom{n-1}{k} \\ &= \binom{n}{k} + D_{t,n,k,s}, \end{aligned}$$

where the second equality comes from Eq 13 by induction, and the third equality comes from $k+t-2s=n$.

From above, combining Eq 12, we can verify that Eq 13 holds when $k+t-2s=n$.

Step 4: verify Eq 13 when $k > 0$, $k+t-2s < n$, $1 < t < n-1$ and $s > 0$.

In this case, $\{r^{(i)}_{t,n,k,t-2s}\}_{1 \leqslant i \leqslant 4}$ are all positive. By the definition, we have that $r^{(3)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ always holds. Hence, we only need to consider $r^{(1)}_{t,n,k,t-2s}$, $r^{(2)}_{t,n,k,t-2s}$ and $r^{(4)}_{t,n,k,t-2s}$ and their value can be calculated from Eq 13 by induction.

Step 4.1: verify that $r^{(2)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13.

Note that $(t-1, n-1, k-1, s-1)$ is valid and, hence, we can calculate $r^{(2)}_{t,n,k,t-2s}$ via Eq 13 by induction.

Case 4.1.1: If $k - s < 0$ or $k - s > n - t$, then $(k-1)-(s-1) < 0$ or $(k-1)-(s-1) > (n-1)-(t-1)$.

Therefore,

$$\begin{aligned} r^{(2)}_{t,n,k,t-2s} &= r(M^{(t-1,n-1)}_{k-1,(k-1)+(t-1)-2(s-1)}) + \binom{n-1}{k} + \binom{n-1}{k+t-2s-1} \\ &= \binom{n-1}{k-1} + \binom{n-1}{k} + \binom{n-1}{k+t-2s-1} + D_{t-1,n-1,k-1,s-1} \\ &= \binom{n}{k} + \sum_{m=k+t-2s+1}^{n} \binom{m-1}{k+t-2s-1} = \binom{n}{k} + D_{t,n,k,s}, \end{aligned}$$

where the second equality comes from Eq 13 by induction.

Case 4.1.2: If $0 \leqslant k - s \leqslant \lfloor (n-t)/2 \rfloor$, then $0 \leqslant (k-1)-(s-1) \leqslant \lfloor ((n-1)-(t-1))/2 \rfloor$.

Therefore,

$$\begin{aligned} & r^{(2)}_{t,n,k,t-2s} \\ &= r(M^{(t-1,n-1)}_{k-1,(k-1)+(t-1)-2(s-1)}) + \binom{n-1}{k} + \binom{n-1}{k+t-2s-1} \\ &= \binom{n}{k} + \sum_{a=0}^{k-s-1} \left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1} \right) \\ &\quad + \sum_{m=t-3s+1+2k}^{n-s} \left(\binom{m-1}{k+t-2s-1} - \binom{m-1}{k-s-1} \right) + \sum_{m=n-s+1}^{n} \binom{m-1}{k+t-2s-1} \\ &= \binom{n}{k} + A^{(1)}_{t,n,k,s} + A^{(2)}_{t,n,k,s} + C_{t,n,k,s}, \end{aligned}$$

where he second equality comes from Eq 13 by induction.

Case 4.1.3: If $\lfloor (n-t)/2 \rfloor \leqslant k - s \leqslant n - t$, then $\lfloor ((n-1)-(t-1))/2 \rfloor \leqslant (k-1)-(s-1) \leqslant (n-1)-(t-1)$.

Therefore,

$$\begin{aligned}
&r^{(2)}_{t,n,k,t-2s}\\
&= r(M^{(t-1,n-1)}_{k-1,(k-1)+(t-1)-2(s-1)}) + \binom{n-1}{k} + \binom{n-1}{k+t-2s-1}\\
&= \binom{n-1}{k-1} + \binom{n-1}{k} + \binom{n-1}{k+t-2s-1} + B_{t-1,n-1,k-1,s-1} + C_{t-1,n-1,k-1,s-1}\\
&= \binom{n}{k} + \sum_{a=0}^{n-t-k+s-1}\left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1}\right) + \sum_{m=n-s+1}^{n}\binom{m-1}{k+t-2s-1}\\
&= \binom{n}{k} + B_{t,n,k,s} + C_{t,n,k,s},
\end{aligned}$$

where he second equality comes from Eq 13 by induction.

Consequently, $r^{(2)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13. Hence, we only need to prove $r^{(1)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ and $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$. Then by Eq 12, we can finally prove Lemma 21.

Step 4.2: verify that $\boldsymbol{r^{(1)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}}$.

First, it is easy to verify the inequality when $t = n-2$, then we assume $t < n-2$ in the following.

Case 4.2.1: If $\boldsymbol{(k-1)-s<0}$, then $(k-1)-(s-1) = k-s \leqslant 0$ and

$$\begin{aligned}
&D_{t,n-1,k-1,s} - D_{t-1,n-1,k-1,s-1}\\
&= \sum_{m=k+t-2s}^{n-1}\binom{m-1}{k+t-2s-2} - \sum_{m=k+t-2s+1}^{n-1}\binom{m-1}{k+t-2s-1}\\
&= \binom{n-2}{k+t-2s-2} - \sum_{m=k+t-2s+1}^{n-1}\binom{m-2}{k+t-2s-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&r^{(1)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s}\\
=&r(M^{(t,n-1)}_{k-1,k-1+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k+t-2s-1}\\
=&-\binom{n-2}{k+t-2s-1} - \sum_{m=k+t-2s+1}^{n-1}\binom{m-2}{k+t-2s-1} \leqslant 0.
\end{aligned} \tag{19}$$

Case 4.2.2: If $\boldsymbol{0 \leqslant (k-1)-s < \lfloor((n-1)-t)/2\rfloor}$, then $0 \leqslant 1 \leqslant k-s \leqslant$

$\lfloor (n-t-1)/2 \rfloor \leqslant \lfloor (n-t)/2 \rfloor$ and noting that $t-3s+2k-1 \leqslant t-s-1+(n-t-1) \leqslant n-s-2$

$$
\begin{aligned}
&A^{(1)}_{t,n-1,k-1,s} - A^{(1)}_{t-1,n-1,k-1,s-1} = -\left(\binom{t+2k-3s-2}{t+k-2s-2} - \binom{t+2k-3s-2}{k-s-2}\right),\\
&A^{(2)}_{t,n-1,k-1,s} - A^{(2)}_{t-1,n-1,k-1,s-1}\\
=&\sum_{m=t-3s+2k-1}^{(n-1)-s}\left(\binom{m-1}{k+t-2s-2} - \binom{m-1}{k-s-2}\right)\\
&-\sum_{m=t-3s+2k+1}^{n-s}\left(\binom{m-1}{k+t-2s-1} - \binom{m-1}{k-s-1}\right)\\
=&\left(\binom{t-2s+2k-2}{k+t-2s-2} - \binom{t-3d+2k-2}{k-s-2}\right)\\
&-\sum_{m=t-3s+2k}^{n-s-1}\left(\binom{m-1}{k+t-2s-1} - \binom{m-1}{k-s-1}\right),\\
&C_{t,n-1,k-1,s} - C_{t-1,n-1,k-1,s-1}\\
=&\sum_{m=n-s}^{n-1}\binom{m-1}{k+t-2s-2} - \sum_{m=n-s+1}^{n-1}\binom{m-1}{k+t-2s-1}\\
\leqslant&\binom{n-2}{k+t-2s-2} - \sum_{m=n-s}^{n-2}\binom{m-1}{k+t-2s-1}.
\end{aligned}
$$

From above three terms, it is easy to get

$$
\begin{aligned}
&r^{(1)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s}\\
=&r(M^{(t,n-1)}_{k-1,k-1+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k-1+t-2s}\\
=&\sum_{i=1}^{2}(A^{(i)}_{t,n-1,k-1,s} - A^{(i)}_{t-1,n-1,k-1,s-1}) + (C_{t,n-1,k-1,s} - C_{t-1,n-1,k-1,s-1}) - \binom{n-1}{k-1+t-2s}\\
\leqslant&0.
\end{aligned}
$$

Case 4.2.3: If $\lfloor ((n-1)-t)/2 \rfloor \leqslant (k-1)-s \leqslant (n-1)-t$, then $\lfloor (n-t)/2 \rfloor \leqslant \lfloor (n-t+1)/2 \rfloor \leqslant k-s \leqslant n-t$.

Since $B_{t,n-1,k-1,s} - B_{t-1,n-1,k-1,s-1} = 0$, we have,

$$\begin{aligned}
& r^{(1)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s} \\
=& r(M^{(t,n-1)}_{k-1,k-1+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k-1+t-2s} \\
=& (B_{t,n-1,k-1,s} - B_{t-1,n-1,k-1,s-1}) + (C_{t,n-1,k-1,s} - C_{t-1,n-1,k-1,s-1}) - \binom{n-1}{k-1+t-2s} \\
=& C_{t,n-1,k-1,s} - C_{t-1,n-1,k-1,s-1} - \binom{n-1}{k-1+t-2s} \\
\leqslant& \binom{n-2}{k+t-2s-2} - \binom{n-1}{k-1+t-2s} - \sum_{m=n-s}^{n-2} \binom{m-1}{k+t-2s-1} \\
\leqslant& 0.
\end{aligned}$$

Case 4.2.4: If $(k-1)-s > (n-1)-t$, then $k-s > n-t$.
Same as Eq 19, we can verify $r^{(1)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ in this case.
From above all, we have verified $r^{(1)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ in all cases.

Step 4.3: verify that $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$.

First, if $t = n-2$, then

$$\begin{aligned}
r^{(4)}_{t,n,k,t-2s} &= \binom{n-1}{k+t-2s-1} + r(M^{(t,n-1)}_{k,k+t-2s}) \\
&= \binom{n-1}{k+t-2s-1} + \binom{n-1}{k} + \binom{n-1}{k+t-2s} - 1.
\end{aligned}$$

Moreover, we have

$$r^{(2)}_{t,n,k,t-2s} = \binom{n-1}{k} + \binom{n-1}{k+t-2s-1} + r(M^{(t-1,n-1)}_{k-1,k+t-2s})$$

If $k-s <= 0$ or $k-s >= n-t = 2$, then $r(M^{(t-1,n-1)}_{k-1,k+t-2s}) \geqslant D_{t-1,n-1,k-1,s-1} = \binom{n-1}{k+t-2s} - 1$. Otherwise, $k-s = 1$ and then $r(M^{(t-1,n-1)}_{k-1,k+t-2s}) \geqslant \binom{n-1}{k-1} = \binom{n-1}{k+t-2s-1}$. In both cases, we can derive $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$. Thus, in the following, we assume $t < n-2$.

Case 4.3.1: If $k-s < 0$, then $(k-1)-(s-1) < 0$.

Therefore,

$$
\begin{aligned}
& r^{(4)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s} \\
=& r(M^{(t,n-1)}_{k,k+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k} \\
=& \binom{n-1}{k} - \binom{n-1}{k-1} - \binom{n-1}{k} + (D_{t,n-1,k,s} - D_{t-1,n-1,k-1,s-1}) \\
\leqslant & \sum_{m=k+t-2s+1}^{n-1} \binom{m-1}{k+t-2s-1} - \sum_{m=k+t-2s+1}^{n-1} \binom{m-1}{k+t-2s-1} = 0.
\end{aligned}
\tag{20}
$$

Case 4.3.2: If $0 \leqslant k-s \leqslant \lfloor ((n-1)-t)/2 \rfloor$, then $0 \leqslant k-s \leqslant \lfloor (n-t-1)/2 \rfloor \leqslant \lfloor (n-t)/2 \rfloor$ and $2(k-s) \leqslant n-t-1$. We can derive that

$$
\begin{aligned}
& A^{(1)}_{t,n-1,k,s} - A^{(1)}_{t-1,n-1,k-1,s-1} = 0, \\
& A^{(2)}_{t,n-1,k,s} - A^{(2)}_{t-1,n-1,k-1,s-1} = -\left(\binom{n-s-1}{k+t-2s-1} - \binom{n-s-1}{k-s-1}\right), \\
& C_{t,n-1,k,s} - C_{t-1,n-1,k-1,s-1} \\
=& \sum_{m=n-s}^{n-1} \binom{m-1}{k+t-2s-1} - \sum_{m=n-s+1}^{n-1} \binom{m-1}{k+t-2s-1} \\
=& \binom{n-s-1}{k+t-2s-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& r^{(4)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s} \\
=& r(M^{(t,n-1)}_{k,k+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k} \\
=& -\binom{n-1}{k-1} + \sum_{i=1}^{2} (A^{(i)}_{t,n-1,k,s} - A^{(i)}_{t-1,n-1,k-1,s-1}) + (C_{t,n-1,k,s} - C_{t-1,n-1,k-1,s-1}) \\
=& \binom{n-s-1}{k-s-1} - \binom{n-1}{k-1} \leqslant 0.
\end{aligned}
$$

Case 4.3.3: If $\lfloor ((n-1)-t)/2 \rfloor < k-s \leqslant (n-1)-t$, then $\lfloor (n-t)/2 \rfloor \leqslant \lfloor (n-t+1)/2 \rfloor \leqslant k-s \leqslant n-t$.

We have

$$\begin{aligned}
&B_{t,n-1,k,s} - B_{t-1,n-1,k-1,s-1}\\
=&\sum_{a=0}^{n-t-k+s-2}\left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1}\right)\\
&-\sum_{a=0}^{n-t-k+s-1}\left(\binom{t-s+2a}{t-s+a-1} - \binom{t-s+2a}{a-1}\right)\\
=&-\binom{2n-2k-t+s-2}{n-k-2} + \binom{2n-2k-t+s-2}{n-t-k+s-2}.
\end{aligned}$$

Write $\lambda \triangleq k-s-\frac{n-t}{2}$, then $\lambda \geqslant 0$ by the condition. Therefore,

$$\begin{aligned}
&r^{(4)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s}\\
=&r(M^{(t,n-1)}_{k,k+t-2s}) - r(M^{(t-1,n-1)}_{k-1,k-1+t-1-2(s-1)}) - \binom{n-1}{k}\\
=&-\binom{n-1}{k-1} + (B_{t,n-1,k,s} - B_{t-1,n-1,k-1,s-1}) + (C_{t,n-1,k,s} - C_{t-1,n-1,k-1,s-1})\\
\leqslant&\binom{n-s-1}{k+t-2s-1} - \binom{n-1}{k-1} - \binom{2n-2k-t+s-2}{n-k-2} + \binom{2n-2k-t+s-2}{n-t-k+s-2}\\
\leqslant&\binom{n-s-1}{k-s-2\lambda} - \binom{n-1}{k-1} - \binom{n-s-2-2\lambda}{k-s-2\lambda} + \binom{n-s-2-2\lambda}{k-s-2\lambda-2}.
\end{aligned}$$

If $\lambda=0$, it is easy to verify $r^{(4)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s} \leqslant 0$ with the above equation. If $\lambda \geqslant 1$, then we first claim that $\binom{n-s-1}{k-s-2\lambda} \leqslant \binom{n-s-1}{k-s-1}$. If $k-s-1 \leqslant \frac{n-s-1}{2}$, it holds naturally. Otherwise, when $k-s-1 > \frac{n-s-1}{2}$, note that $k-s-2\lambda \leqslant \frac{n-s-1}{2}$ holds, since $t \geqslant 2s \geqslant 2$. It suffices to prove that $k-s-1-\frac{n-s-1}{2} \leqslant \frac{n-s-1}{2} - (k-s-2\lambda)$ which is easy to verify. As a consequence, we have

$$\begin{aligned}
&r^{(4)}_{t,n,k,t-2s} - r^{(2)}_{t,n,k,t-2s}\\
\leqslant&\binom{n-s-1}{k+t-2s-1} - \binom{n-1}{k-1}\\
=&\binom{n-s-1}{k-s-2\lambda} - \binom{n-1}{k-1}\\
\leqslant&\binom{n-s-1}{k-s-1} - \binom{n-1}{k-1} \leqslant 0.
\end{aligned}$$

Case 4.3.4: If $\boldsymbol{k-s > (n-1)-t}$, then $k-s \geqslant n-t$.

Same as Eq 20, we can verify $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ in this case.

From above all, we have verified $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(2)}_{t,n,k,t-2s}$ in all cases.

Step 5: verify Eq 13 when $\boldsymbol{k>0}$, $\boldsymbol{k+t-2s<n}$ and $\boldsymbol{s=0}$. In this case, $r^{(2)}_{t,n,k,t-2s} = -\infty$ and $\{r^{(i)}_{t,n,k,t-2s}\}_{1=1,3,4}$ are positive.

Case 5.1: If $1 \leqslant k < \lfloor (n-t)/2 \rfloor$, then we have $k \leqslant \lfloor ((n-1)-t)/2 \rfloor$ and $2k \leqslant n-t-1$. We first verify that $r^{(4)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13.

$$\begin{aligned} r^{(4)}_{t,n,k,t-2s} &= \binom{n-1}{k+t-2s-1} + r(M^{(t,n-1)}_{k,k+t-2s}) \\ &= \binom{n-1}{k+t-2s-1} + \binom{n-1}{k} + A^{(1)}_{t,n-1,k,s} + A^{(2)}_{t,n-1,k,s} + C_{t,n-1,k,s}. \end{aligned}$$

Note that $A^{(1)}_{t,n-1,k,s} = A^{(1)}_{t,n,k,s}$, $C^{(1)}_{t,n-1,k,s} = 0 = C_{t,n,k,s}$ when $s = 0$, and

$$A^{(2)}_{t,n-1,k,s} = A^{(2)}_{t,n,k,s} - \left(\binom{n-1}{k+t-2s-1} - \binom{n-1}{k-s-1} \right).$$

Then we can derive that

$$r^{(4)}_{t,n,k,t-2s} = \binom{n}{k} + A^{(1)}_{t,n,k,s} + A^{(2)}_{t,n,k,s} + C_{t,n,k,s}.$$

Consequently, $r^{(4)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13 and we then only need to prove $r^{(1)}_{t,n,k,t-2s} \leqslant r^{(4)}_{t,n,k,t-2s}$ and $r^{(3)}_{t,n,k,t-2s} \leqslant r^{(4)}_{t,n,k,t-2s}$. We omit the details here.

Case 5.2: If $\lfloor (n-t)/2 \rfloor < k \leqslant n-t$, then $\lfloor (n-1-t)/2 \rfloor \leqslant k-1 \leqslant (n-1)-t$. We first verify that $r^{(1)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13.

$$\begin{aligned} r^{(1)}_{t,n,k,t-2s} &= \binom{n-1}{k} + r(M^{(t,n-1)}_{k-1,k+t-2s-1}) \\ &= \binom{n-1}{k} + \binom{n-1}{k-1} + B_{t,n-1,k-1,s} + C_{t,n-1,k-1,s} \\ &= \binom{n}{k} + B_{t,n,k,s} + C_{t,n,k,s}. \end{aligned}$$

Consequently, $r^{(1)}_{t,n,k,t-2s}$ is always equal to RHS of Eq 13 and we then only need to prove $r^{(3)}_{t,n,k,t-2s} \leqslant r^{(1)}_{t,n,k,t-2s}$ and $r^{(4)}_{t,n,k,t-2s} \leqslant r^{(1)}_{t,n,k,t-2s}$. We omit the details here.

Step 6: verify the base case of induction.

Consider when $t+n+k+s = 2$ and show that (t,n,k,s) is valid. The only case is $t = 1, n = 1, k = 0, s = 0$ and, then, by Step 1, we can verify Eq 13 in this case.

By induction, we have finally proved Lemma 21. □

Appendix B: Proof of Lemma 22

We give the proof of Lemma 22 via Eq 10. Note that when $t = 1$, $bw(M^{(1,n)}) = \sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor}$ [15].

If $n = 2l + 1$ for some integer l, then

$$\begin{aligned}
&\sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor} \\
=&\sum_{a=0}^{l} \binom{2a}{a} + \sum_{a=0}^{l-1} \binom{2a+1}{a} \\
=&\binom{2l+1}{l} + \sum_{a=0}^{l-1} \left(\binom{2a+1}{a} - \binom{2a+1}{a-1} \right) \\
=&\sum_{k=\lfloor (n-t)/2 \rfloor}^{\lfloor (n-t)/2 \rfloor + t-1} \binom{n}{k} + \sum_{a=0}^{\lfloor (n-t-1)/2 \rfloor} \left(\binom{t+2a}{t+a-1} - \binom{t+2a}{a-1} \right),
\end{aligned}$$

where the second equality can be proved by induction on l.

If $n = 2l$ for some integer l, then

$$\begin{aligned}
&\sum_{m=0}^{n-1} \binom{m}{\lfloor m/2 \rfloor} \\
=&\sum_{a=0}^{l-1} \binom{2a}{a} + \sum_{a=0}^{l-1} \binom{2a+1}{a} \\
=&\binom{2l}{l} + \sum_{a=0}^{l-1} \left(\binom{2a+1}{a} - \binom{2a+1}{a-1} \right) \\
=&\sum_{k=\lfloor (n-t)/2 \rfloor}^{\lfloor (n-t)/2 \rfloor + t-1} \binom{n}{k} + \sum_{a=0}^{\lfloor (n-t-1)/2 \rfloor} \left(\binom{t+2a}{t+a-1} - \binom{t+2a}{a-1} \right),
\end{aligned}$$

where the second equality can be proved by induction on l as well.

Therefore, when $t = 1$, Eq 14 holds. Then we intend to prove Lemma 22 by induction on t. Suppose Eq 14 holds for $t < T$, now consider when $t = T$.

First consider the situation when $k = \lfloor (n-t)/2 \rfloor$ and $p = t$. Then we have

$$\sum_{q=1}^{t-1} \binom{n}{k+q} + r(M_{k,k+t}^{(t,n)}) = \sum_{q=0}^{t-1} \binom{n}{k+q} + \sum_{a=0}^{\lfloor (n-t-1)/2 \rfloor} \left(\binom{t+2a}{t+a-1} - \binom{t+2a}{a-1} . \right) \tag{21}$$

The result matches the RHS of Eq 14. In the following, we only need to prove other term is no more than this value.

For convenience, define $\widetilde{r}(M_{k,k+p}^{(t,n)}) = \sum_{q=1}^{p-1} \binom{n}{k+q} + r(M_{k,k+p}^{(t,n)})$ when $1 \leqslant p \leqslant t$ and $\widetilde{r}(M_{k,k}^{(t,n)}) = bw(M_{k,k}^{(t,n)})$. Actually, $\widetilde{r}(M_{k,k+p}^{(t,n)})$ where $0 \leqslant k \leqslant k+p \leqslant n$ is exactly the maximum Manhattan distance from a nonzero element of $M_{k,k+p}^{(t,n)}$ in M to the main diagonal of the matrix M. There exists a consist expression for $\widetilde{r}(M^{(t,n)}_{k,k+p})$ as following.

$$\widetilde{r}(M^{(t,n)}_{k,k+p}) = \sum_{q=0}^{p-1} \binom{n}{k+q} - \binom{n}{k} + r(M^{(t,n)}_{k,k+p}).$$

Then, our purpose is to prove $\widetilde{r}(M^{(t,n)}_{k,k+p})$ is no larger than Eq 21 for any integer k, p satisfying $0 \leqslant k \leqslant k+p \leqslant n$ and $0 \leqslant p \leqslant t$.

From definition, when $k = 0$, we have that $\widetilde{r}(M^{(t,n)}_{k,k+p})$ reach its maximal when $p = t$. Similarly, when $k+p = n$, $\widetilde{r}(M^{(t,n)}_{k,k+p})$ reach its maximal when $k = n-t$, that is, $p = t$. When $k > 0$ and $k+p < n$, we have $\widetilde{r}(M^{(t,n)}_{k,k+p}) \leqslant \widetilde{r}(M^{(t,n)}_{k-1,k+p+1})$ when $p+2 \leqslant t$. Hence, we only need to prove $\widetilde{r}(M^{(t,n)}_{k,k+p})$ is no larger than Eq 21 for the following two cases:

(1) $p = t-1$,

(2) $p = t$.

Case 1: $p = t-1$. In this case, we have $\widetilde{r}(M^{(t,n)}_{k,k+p}) = \widetilde{r}(M^{(t-1,n)}_{k,k+p}) \leqslant bw(M^{(t-1,n)})$ by definition. Since the value of $bw(M^{(t-1,n)})$ can be calculated from Lemma 22 by induction, and it is not hard to verify the RHS of Eq. 14 is increasing with respect to t. Then we can reach our conclusion in this case.

Case 2: $p = t$.

If $k \leqslant \lfloor (n-t)/2 \rfloor$, then

$$\begin{aligned}
&\widetilde{r}(M^{(t,n)}_{k,k+t}) - \widetilde{r}(M^{(t,n)}_{k-1,k+t-1}) \\
=&\binom{n}{k+t-1} - \binom{n}{k} + \binom{n}{k} - \binom{n}{k-1} + \left(\binom{2k+t-2}{k+t-2} - \binom{2k+t-2}{k-2}\right) \\
&+ \sum_{m=t+1+2k}^{n-s} \left(\binom{m-1}{k+t-1} - \binom{m-1}{k-1}\right) - \sum_{m=t-1+2k}^{n-s} \left(\binom{m-1}{k+t-2} - \binom{m-1}{k-2}\right) \\
=&\binom{n}{k+t-1} - \binom{n}{k-1} + \sum_{m=t+2k}^{n-1} \left(\binom{m-1}{k+t-2} - \binom{m-1}{k-2}\right) - \left(\binom{n-1}{k+t-2} - \binom{n-1}{k-2}\right) \\
=&\binom{n-1}{k-t-1} - \binom{n-1}{k-1} + \sum_{m=t+2k}^{n-1} \left(\binom{m-1}{k+t-2} - \binom{m-1}{k-2}\right) \\
\geqslant&0.
\end{aligned}$$

It shows that when $k \leqslant \lfloor (n-t)/2 \rfloor$, $\widetilde{r}(M^{(t,n)}_{k,k+t})$ reaches its maximum at $k = \lfloor (n-t)/2 \rfloor$. If $k \geqslant \lfloor (n-t)/2 \rfloor$, then

$$\begin{aligned}
&\widetilde{r}(M^{(t,n)}_{k,k+t}) - \widetilde{r}(M^{(t,n)}_{k+1,k+t+1}) \\
=&\binom{n}{k+1} - \binom{n}{k+t} + \binom{n}{k} - \binom{n}{k+1} + \left(\binom{2n-t-2k-4}{n-k-3} - \binom{2n-t-2k-4}{n-t-k-3}\right) \\
\geqslant&0.
\end{aligned}$$

Combining the above situations, we have proved that $\widetilde{r}(M_{k,k+t}^{(t,n)})$ reaches its maximum at $k = \lfloor (n-t)/2 \rfloor$. That is, the maximum value of is $\widetilde{r}(M_{k,k+t}^{(t,n)})$ as Eq 21.

From above all, we have proven Lemma 22. □