Examining Kempe equivalence via commutative algebra

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Abstract

Kempe equivalence is a classical and important notion on vertex coloring in graph theory. In the present paper, we introduce several ideals associated with graphs and provide a method for determining whether two k-colorings are Kempe equivalent via commutative algebra. Moreover, we give a way to compute all k-colorings of a graph up to Kempe equivalence by virtue of the algebraic technique on Gröbner bases. As a consequence, the number of k-Kempe classes can be computed by using Hilbert functions. Finally, we introduce several algebraic algorithms related to Kempe equivalence.

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1 Introduction

A *k-coloring* f of a graph G on the vertex set $[d] := \{1, 2, \dots, d\}$ is a map from $[d]$ to $[k]$ such that $f(i) \neq f(j)$ for all $\{i, j\} \in E(G)$. The smallest integer $\chi(G)$ such that G has a $\chi(G)$-coloring is called the *chromatic number* of G. Given a k-coloring f of G, and integers $1 \leqslant i < j \leqslant k$, let H be a connected component of the induced subgraph of G on the vertex set $f^{-1}(i) \cup f^{-1}(j)$. Then we can obtain a new k-coloring g of G by setting

$$g(x) = \begin{cases} f(x) & x \notin H, \\ i & x \in H \text{ and } f(x) = j, \\ j & x \in H \text{ and } f(x) = i. \end{cases}$$

We say that g is obtained from f by a *Kempe switching*. Two k-colorings f and g of G are called *Kempe equivalent*, denoted by $f \sim_k g$, if there exists a sequence $f_0, f_1, \dots, f_s$ of k-colorings of G such that $f_0 = f$, $f_s = g$, and f_i is obtained from f_{i-1} by a Kempe

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switching. Let denote $\mathcal{C}_k(G)$ the set of all k-colorings of G. Then $\sim_k$ is an equivalence relation on $\mathcal{C}_k(G)$. The equivalence classes of $\mathcal{C}_k(G)$ by $\sim_k$ are called the k-Kempe classes. We denote $\mathrm{kc}(G,k)$ the quotient set $\mathcal{C}_k(G)/\sim_k$ and denote $\mathrm{Kc}(G,k)$ the number of k-Kempe classes of G, namely $\mathrm{Kc}(G,k) = |\mathrm{kc}(G,k)|$. Kempe switchings were introduced by Kempe in the false proof of the 4-Color Theorem. However, his idea is powerful in graph coloring theory. Recently, many researchers have studied Kempe switchings and Kempe equivalence. See [5] for an overview of Kempe equivalence.

Given a graph G, let $S(G)$ be the set of all stable sets of G. The *stable set ideal* I_G of G is a toric ideal arising from $S(G)$ of a polynomial ring $R[G] := \mathbb{K}[x_S \mid S \in S(G)]$ over a field $\mathbb{K}$. In [7], the authors showed that I_G is generated by binomials $\mathbf{x}_f - \mathbf{x}_g$ associated with k-colorings f and g of a replication graph of an induced subgraph of G, and found a relationship between Kempe equivalence on G and an algebraic property of I_G. In particular, by using the proof of [7, Theorem 1.3], we can examine if two k-colorings of G are Kempe equivalent by using I_G. However, I_G has too much information for this purpose. In the present paper, we introduce a simpler ideal J_G, which is generated by binomials $\mathbf{x}_f - \mathbf{x}_g$ associated with 2-colorings f and g of an induced subgraph of G, to determine Kempe equivalence on G. We call J_G the *2-coloring ideal* of G. Then our first main result is the following:

Theorem 1.1. *Let G be a graph on $[d]$ and let f, g be k-colorings of G. Then $f \sim_k g$ if and only if $\mathbf{x}_f - \mathbf{x}_g \in J_G$.*

Next, we compute all k-colorings of a graph G up to Kempe equivalence by virtue of the algebraic technique on Gröbner bases. Namely, a complete representative system for $\mathrm{kc}(G,k)$ is given. For this, we introduce another ideal K_G defined by

$$K_G := J_G + M_G,$$

where

$$M_G := \langle x_S x_T \mid S, T \in S(G), S \cap T \neq \emptyset \rangle.$$

The ideal K_G is called the *Kempe ideal* of G. Then our second main result is the following:

Theorem 1.2. *Let G be a graph on $[d]$ and $<$ a monomial order on $R[G]$, and let $\{\mathbf{x}_{f_1}, \ldots, \mathbf{x}_{f_s}\}$ be the set of all standard monomials of degree k with respect to the initial ideal $\mathrm{in}_<(K_G)$. Then*

$$\{f_1, \ldots, f_s\} \cap \mathcal{C}_k(G)$$

is a complete representative system for $\mathrm{kc}(G,k)$.

As a consequence, the number of k-Kempe classes $\mathrm{Kc}(G,k)$ can be computed by Hilbert functions (Corollary 6.9).

Finally, by using Theorems 1.1 and 1.2 and techniques on Gröbner bases, we introduce several algorithms related to Kempe equivalence. Specifically, our algorithms perform:

1. Determination of Kempe equivalence (Algorithm 7.2);

2. Computation of a complete representative system for $\mathrm{kc}(G, k)$ (Algorithm 7.3);
3. Enumeration of k-colorings that are Kempe equivalent (Algorithm 7.4);
4. Construction of a sequence of Kempe switchings between two Kempe equivalent k-colorings (Algorithm 7.6).

The present paper is organized as follows: In Section 2, we will recall the definition of stable set ideals and explain a relationship between stable set ideals and Kempe equivalence. Section 3 will give a brief introduction to Gröbner bases. In Section 4, we will define 2-coloring ideals and see their algebraic properties. In Section 5, a proof of Theorem 1.1 will be given. In Section 6, we will define Kempe ideals and prove Theorem 1.2. Finally, in Section 7, we will introduce several algorithms related to Kempe equivalence.

2 Stable set ideals

In this section, we define stable set rings and explain a relationship between stable set ideals and Kempe equivalence. Let G be a graph on the vertex set $[d]$ with the edge set $E(G)$. Given a subset $S \subset [d]$, let $G[S]$ denote the induced subgraph of G on the vertex set S. A subset $S \subset [d]$ is called a *stable set* (or an *independent set*) of G if $\{i, j\} \notin E(G)$ for all $i, j \in S$ with $i \neq j$. Namely, a subset $S \subset [d]$ is stable if and only if $G[S]$ is an empty graph. In particular, the empty set $\emptyset$ and any singleton $\{i\}$ with $i \in [d]$ are stable. Denote $S(G) = \{S_1, \ldots, S_n\}$ the set of all stable sets of G. Given a subset $S \subset [d]$, we associate the $(0, 1)$-vector $\rho(S) = \sum_{j \in S} \mathbf{e}_j$. Here $\mathbf{e}_j$ is the jth unit coordinate vector in $\mathbb{R}^d$. For example, $\rho(\emptyset) = (0, \ldots, 0) \in \mathbb{R}^d$. Let $\mathbb{K}[\mathbf{t}, s] := \mathbb{K}[t_1, \ldots, t_d, s]$ be the polynomial ring in $d+1$ variables over a field $\mathbb{K}$. Given a nonnegative integer vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geqslant 0}$, we write $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d} \in \mathbb{K}[\mathbf{t}, s]$. The *stable set ring* of G is

$$\mathbb{K}[G] := \mathbb{K}[\mathbf{t}^{\rho(S_1)} s, \ldots, \mathbf{t}^{\rho(S_n)} s] \subset \mathbb{K}[\mathbf{t}, s].$$

We regard $\mathbb{K}[G]$ as a homogeneous algebra by setting each $\deg(\mathbf{t}^{\rho(S_i)} s) = 1$. Note that $\mathbb{K}[G]$ is a toric ring. Let $R[G] = \mathbb{K}[x_{S_1}, \ldots, x_{S_n}]$ denote the polynomial ring in n variables over $\mathbb{K}$ with each $\deg(x_{S_i}) = 1$. The *stable set ideal* of G is the kernel of the surjective homomorphism $\pi : R[G] \to \mathbb{K}[G]$ defined by $\pi(x_{S_i}) = \mathbf{t}^{\rho(S_i)} s$ for $1 \leqslant i \leqslant n$. Note that I_G is a toric ideal, and hence a prime ideal generated by homogeneous binomials. The toric ring $\mathbb{K}[G]$ is called *quadratic* if I_G is generated by quadratic binomials. We say that "I_G is generated by quadratic binomials" even if $I_G = \{0\}$ (or equivalently, G is complete). It is easy to see that a homogeneous binomial $x_{S_{i_1}} \cdots x_{S_{i_r}} - x_{S_{j_1}} \cdots x_{S_{j_r}} \in R[G]$ belongs to I_G if and only if $\bigcup_{\ell=1}^{r} S_{i_\ell} = \bigcup_{\ell=1}^{r} S_{j_\ell}$ as multisets. See, e.g., [3] for details on toric rings and toric ideals.

We can describe a system of generators of I_G in terms of k-colorings. Given a graph G on the vertex set $[d]$, and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geqslant 0}$, let $G_{\mathbf{a}}$ be the graph obtained from G by replacing each vertex $i \in [d]$ with a complete graph $G^{(i)}$ of a_i vertices (if $a_i = 0$, then just delete the vertex i), and joining all vertices $x \in G^{(i)}$ and $y \in G^{(j)}$ such that

$\{i,j\}$ is an edge of G. In particular, if $\mathbf{a} = (1, \ldots, 1)$, then $G_\mathbf{a} = G$. If $\mathbf{a} = \mathbf{0}$, then $G_\mathbf{a}$ is the null graph (a graph without vertices). In addition, if $\mathbf{a}$ is a $(0,1)$-vector, namely, $\mathbf{a} \in \{0,1\}^d$, then $G_\mathbf{a}$ is an induced subgraph of G. If $\mathbf{a}$ is a positive vector, then $G_\mathbf{a}$ is called a *replication graph* of G. In general, $G_\mathbf{a}$ is a replication graph of an induced subgraph of G. Given a k-coloring f of $G_\mathbf{a}$ with $\mathbf{a} \in \mathbb{Z}^d_{\geqslant 0}$, we associate f with a monomial

$$\mathbf{x}_f := x_{S_{i_1}} \cdots x_{S_{i_k}} \in R[G],$$

where $S_{i_\ell} = \{j \in [d] \mid G^{(j)} \cap f^{-1}(\ell) \neq \emptyset\}$ for $\ell = 1, 2, \ldots, k$. Conversely, let $m = x_{S_{i_1}} \cdots x_{S_{i_k}} \in R[G]$ be a monomial of degree k. Then, for $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_p = |\{\ell : p \in S_{i_\ell}\}|$, there exists a k-coloring f of $G_\mathbf{a}$ such that $\mathbf{x}_f = m$ (see [7, Lemma 3.2]). For example, we consider the graphs G and $G_\mathbf{a}$ with $\mathbf{a} = (2,1,0,2)$ as follows:

(a) G (b) $G_\mathbf{a}$ with $\mathbf{a} = (2,1,0,2)$

We define a 4-coloring f of $G_\mathbf{a}$ by

$$f(i) = \begin{cases} 1 & i \in \{1_1, 4_1\}, \\ 2 & i \in \{2_1\}, \\ 3 & i \in \{1_2\}, \\ 4 & i \in \{4_2\}. \end{cases}$$

Then since $S_{i_1} = \{1,4\}, S_{i_2} = \{2\}, S_{i_3} = \{1\}, S_{i_4} = \{4\}$, one has

$$\mathbf{x}_f = x_{\{1,4\}} x_{\{2\}} x_{\{1\}} x_{\{4\}}.$$

On the other hand, we can obtain the 4-coloring f (up to exchanging colors and exchanging the coloring of vertices in each clique $G^{(i)}$ of $G_\mathbf{a}$) from the monomial $x_{\{1,4\}} x_{\{2\}} x_{\{1\}} x_{\{4\}}$ as follows: for each variable x_S, we assign one color to a single copy i_j of each vertex $i \in S$.

Note that, for k-colorings f and g of an induced subgraph of G, $\mathbf{x}_f = \mathbf{x}_g$ if and only if g is obtained from f by permuting colors. It is easy to see that $f \sim_k g$ if g is obtained from f by permuting colors. In this paper, we identify f and g if g is obtained from f by permuting colors. Then we can describe a system of generators of I_G as follows:

Proposition 2.1 ([7, Theorem 3.3])**.** *Let G be a graph on $[d]$. Then one has*

$$\mathbf{x}_f - \mathbf{x}_g \in I_G \iff f \text{ and } g \text{ are } k\text{-colorings of } G_\mathbf{a} \text{ with } \mathbf{a} \in \mathbb{Z}^d_{\geqslant 0} \text{ and } k \geqslant \chi(G_\mathbf{a})$$

and

$$\begin{aligned} I_G &= \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are } k\text{-colorings of } G_\mathbf{a} \text{ with } \mathbf{a} \in \mathbb{Z}^d_{\geqslant 0} \text{ and } k \geqslant \chi(G_\mathbf{a}) \rangle \\ &= \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are } k\text{-colorings of } G_\mathbf{a} \text{ with } \mathbf{a} \in \{0,1,\ldots,k\}^d \text{ and } k \geqslant \chi(G_\mathbf{a}) \rangle. \end{aligned}$$

Now, we explain a relationship between I_G and Kempe equivalence.

Proposition 2.2 ([7, Theorem 1.3])**.** *Let G be a graph on $[d]$. Then I_G is generated by quadratic binomials if and only if for any $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^d$ and $k \geqslant \chi(G_\mathbf{a})$, all k-colorings of $G_\mathbf{a}$ are Kempe equivalent, namely, one has*

$$\mathrm{Kc}(G_\mathbf{a}, k) = 1.$$

We consider the ideal $\langle [I_G]_2 \rangle$ of $R[G]$ generated by all quadratic binomials of I_G, namely,

$$\begin{aligned}\langle [I_G]_2 \rangle &= \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are 2-colorings of } G_\mathbf{a} \text{ with } \mathbf{a} \in \mathbb{Z}_{\geqslant 0}^d \rangle \\ &= \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are 2-colorings of } G_\mathbf{a} \text{ with } \mathbf{a} \in \{0,1,2\}^d \rangle \subset R[G].\end{aligned}$$

Proposition 2.3. *Let G be a graph on $[d]$ and f, g k-colorings of G. Then $f \sim_k g$ if and only if $\mathbf{x}_f - \mathbf{x}_g \in \langle [I_G]_2 \rangle$.*

We will see a proof of this proposition in Section 5 (Theorem 5.1) by using a similar discussion in the proof of Proposition 2.2 in [7].

3 Gröbner bases

In this section, we give a brief introduction to Gröbner bases. Let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ with $\deg(x_i) = 1$, and denote $\mathcal{M}_n$ the set of all monomials in the variables $x_1, \ldots, x_n$. A *monomial order* on R is a total order $<$ on $\mathcal{M}_n$ such that

1. $1 < u$ for all $1 \neq u \in \mathcal{M}_n$;
2. if $u, v \in \mathcal{M}_n$ and $u < v$, then $uw < vw$ for all $w \in \mathcal{M}_n$.

We give an example of a monomial order.

Example 3.1. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials in $\mathcal{M}_n$. We define the total order $<_{\mathrm{rev}}$ on $\mathcal{M}_n$ by setting $u <_{\mathrm{rev}} v$ if either (i) $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$, or (ii) $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the rightmost nonzero component of the vector $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$ is negative. It then follows that $<_{\mathrm{rev}}$ is a monomial order on R, which is called the (*graded*) *reverse lexicographic order* on R induced by the ordering $x_1 > x_2 > \cdots > x_n$. By reordering the variables, we can obtain another reverse lexicographic order on R. Hence, there are $n!$ reverse lexicographic orders on R.

Fix a monomial order $<$ on R. For a nonzero polynomial f of R, the *support* of f, denoted by $\mathrm{supp}(f)$, is the set of all monomials appearing in f and the *initial monomial* $\mathrm{in}_<(f)$ of f with respect to $<$ is the largest monomial belonging to $\mathrm{supp}(f)$ with respect to $<$. Let I be a nonzero ideal of R. Then the *initial ideal* $\mathrm{in}_<(I)$ of I with respect to $<$ is defined as follows:

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) \mid 0 \neq f \in I \rangle \subset R.$$

In general, even if $I = \langle f_1, \dots, f_s \rangle$, it is not necessarily true that

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_s) \rangle.$$

A finite set $\{g_1, \dots, g_s\}$ of nonzero polynomials belonging to I is called a *Gröbner basis* of I with respect to $<$ if $\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s) \rangle$. Note that if $\{g_1, \dots, g_s\}$ is a Gröbner basis of I, then $I = \langle g_1, \dots, g_s \rangle$. It is known that I always has a Gröbner basis. A Gröbner basis $\mathcal{G}$ for I with respect to $<$ is called *reduced* if the following conditions hold:

1. for all $p \in \mathcal{G}$, the coefficient of $\mathrm{in}_<(p)$ in p equals 1;
2. for all $p \in \mathcal{G}$, no monomial of p lies in $\langle \mathrm{in}_<(g) \mid g \in \mathcal{G} \setminus \{p\} \rangle$.

A nonzero ideal I has a unique reduced Gröbner basis with respect to a fixed monomial order $<$. In particular, given a Gröbner basis, we can easily get the reduced Gröbner basis from it.

Next we introduce a method for determining whether a finite system of generators of I is a Gröbner basis of I with respect to $<$. For two nonzero polynomials f and g in R, the polynomial

$$S(f, g) = \frac{m}{c_f \cdot \mathrm{in}_<(f)} \cdot f - \frac{m}{c_g \cdot \mathrm{in}_<(g)} \cdot g$$

is called the *S-polynomial* of f and g, where c_f is the coefficient of $\mathrm{in}_<(f)$ in f and c_g is that of $\mathrm{in}_<(g)$ in g, and m is the least common multiple of the initial monomials $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$.

Lemma 3.2 (Buchberger's Criterion [2, Chapter 2, §9, Theorem 3])**.** *Let I be a nonzero ideal of R, $\mathcal{G} = \{g_1, \dots, g_s\}$ a finite system of generators of I. Then $\mathcal{G}$ is a Gröbner basis of I with respect to a monomial order $<$ on R if and only if the remainder of S-polynomial $S(g_i, g_j)$ on division by $\mathcal{G}$ is* 0 *for all $i \neq j$.*

The algorithm called *Buchberger's Algorithm* ([2, Chapter 2, §7, Theorem 2]) is based on Buchberger's Criterion, and computes a Gröbner basis for I from a finite system of generators of I.

As applications of Gröbner bases, we can determine if a polynomial f belongs to an ideal. This problem is called the *ideal membership problem.* If $\mathcal{G}$ is a Gröbner basis for an ideal I of R with respect to a monomial order $<$, then every polynomial of $f \in R$ has a unique remainder on division by $\mathcal{G}$. The remainder is called the *normal form* of f with respect to $\mathcal{G}$.

Lemma 3.3 ([2, Chapter 2, §6, Corollary 2])**.** *Let I be a nonzero ideal of R and $\mathcal{G}$ a Gröbner basis of I with respect to a monomial order $<$ on R. Fix a polynomial f in R. Then $f \in I$ if and only if the normal form of f with respect to $\mathcal{G}$ equals* 0.

Next, we review a method for computing Hilbert functions. Let I be a graded ideal of R. The numerical function $H(R/I, -) : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ with $H(R/I, k) = \dim_{\mathbb{K}} R_k/I_k$ is called the *Hilbert function* of R/I, where R_k (resp. I_k) is the homogeneous part of degree k of R (resp. I) and $\dim_{\mathbb{K}} R_k/I_k$ is the dimension of R_k/I_k as $\mathbb{K}$-vector spaces. For a monomial order $<$ on R, a monomial $u \in \mathcal{M}_n$ is said to be *standard* with respect to $\mathrm{in}_<(I)$ if $u \notin \mathrm{in}_<(I)$.

Lemma 3.4 ([3, Theorem 1.19])**.** *Let I be a nonzero graded ideal of R and fix a monomial order $<$ on R. Let $\mathcal{B}_k$ denote the set of standard monomials of degree k with respect to $\mathrm{in}_<(I)$. Then $\mathcal{B}_k$ is a $\mathbb{K}$-basis of R_k/I_k as a $\mathbb{K}$-vector space. In particular, one has*

$$H(R/I,k) = |\mathcal{B}_k|.$$

Finally, we recall a property of the saturation of an ideal. For a nonzero ideal I of R and a polynomial $f \in R$, the *saturation* of I with respect to f is the ideal

$$I : f^\infty = \{g \in R \mid \text{there exists } i > 0 \text{ such that } f^i g \in I\}.$$

Lemma 3.5 ([3, Proposition 1.40])**.** *Let I be a nonzero graded ideal of R and let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to the reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$. Then*

$$\{g/x_n^k \mid g \in \mathcal{G},\ k \in \mathbb{Z}_{\geqslant 0},\ x_n^k \text{ divides } g,\ x_n^{k+1} \text{ does not divide } g\}$$

is a Gröbner basis of $I : x_n^\infty$.

4 2-coloring ideals

Given a graph G on $[d]$, we define the following two ideals of $R[G]$:

$$\begin{aligned} J_G :=& \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are 2-colorings of } G_{\mathbf{a}} \text{ with } \mathbf{a} \in \{0,1\}^d \rangle \\ =& \langle \mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are 2-colorings of an induced subgraph of } G \rangle, \\ L_G :=& \langle x_{S\setminus\{i\}} x_{\{i\}} - x_S x_\emptyset \mid i \in S \in S(G),\ |S| \geqslant 2 \rangle. \end{aligned}$$

We call J_G the 2*-coloring ideal* of G. Note that inclusions of ideals

$$L_G \subset J_G \subset \langle [I_G]_2 \rangle \subset I_G \tag{4.1}$$

hold. Moreover, from Proposition 2.1, for $I \in \{\langle [I_G]_2\rangle, J_G, L_G\}$, if $\mathbf{x}_f - \mathbf{x}_g \in I$, then f and g are k-colorings of $G_{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}^d_{\geqslant 0}$.

In this section, we discuss some properties of these ideals.

Lemma 4.1. *Let G be a graph on $[d]$. Then one has $L_G : \mathbf{x}_\emptyset^\infty = J_G : \mathbf{x}_\emptyset^\infty = \langle [I_G]_2 \rangle : \mathbf{x}_\emptyset^\infty = I_G$.*

Proof. Since $L_G \subset J_G \subset \langle [I_G]_2\rangle \subset I_G$, we have $L_G : \mathbf{x}_\emptyset^\infty \subset J_G : \mathbf{x}_\emptyset^\infty \subset \langle [I_G]_2\rangle : \mathbf{x}_\emptyset^\infty \subset I_G : \mathbf{x}_\emptyset^\infty$. In addition, since I_G is prime and does not contain $x_\emptyset^k$ for any k, we have $I_G : \mathbf{x}_\emptyset^\infty = I_G$. Thus it is sufficient to show that $I_G \subset L_G : \mathbf{x}_\emptyset^\infty$. Let

$$F = \prod_{i=1}^{s} x_{S_i} - \prod_{i=1}^{s} x_{S'_i} \in I_G,$$

where $S_i = \{k_1^{(i)}, \dots, k_{p_i}^{(i)}\}$, $S'_i = \{\ell_1^{(i)}, \dots, \ell_{q_i}^{(i)}\} \in S(G)$ for each i. Then

$$\begin{aligned} & x_{S_i} x_\emptyset^{p_i - 1} - \prod_{j=1}^{p_i} x_{\{k_j^{(i)}\}} \\ = \ & x_\emptyset^{p_i-2}\left(x_{S_i}x_\emptyset - x_{S_i \setminus \{k_1^{(i)}\}} x_{\{k_1^{(i)}\}}\right) + x_\emptyset^{p_i-3} x_{\{k_1^{(i)}\}}\left(x_{S_i\setminus\{k_1^{(i)}\}}x_\emptyset - x_{S_i\setminus\{k_1^{(i)},k_2^{(i)}\}}x_{\{k_2^{(i)}\}}\right) \\ & + \cdots + \left(\prod_{j=1}^{p_i-2} x_{\{k_j^{(i)}\}}\right)\left(x_{\{k_{p_i-1}^{(i)},k_{p_i}^{(i)}\}}x_\emptyset - x_{\{k_{p_i}^{(i)}\}}x_{\{k_{p_i-1}^{(i)}\}}\right) \end{aligned}$$

belongs to L_G. Note that if binomials $u_1 - v_1, \dots, u_s - v_s$ belong to L_G, then

$$\begin{aligned} u_1 \dots u_s - v_1 \dots v_s \ &= \ u_2 \cdots u_s(u_1 - v_1) + u_3 \cdots u_s v_1(u_2 - v_2) + u_4 \cdots u_s v_1 v_2 (u_3 - v_3) \\ & \quad + \cdots + v_1 \cdots v_{s-1}(u_s - v_s) \end{aligned}$$

belongs to L_G. Hence

$$G_1 = x_\emptyset^{\sum_{i=1}^s (p_i - 1)} \prod_{i=1}^s x_{S_i} - \prod_{i=1}^s \prod_{j=1}^{p_i} x_{\{k_j^{(i)}\}}$$

belongs to L_G. By the same argument,

$$G_2 = x_\emptyset^{\sum_{i=1}^s (q_i - 1)} \prod_{i=1}^s x_{S'_i} - \prod_{i=1}^s \prod_{j=1}^{q_i} x_{\{\ell_j^{(i)}\}}$$

belongs to L_G. Since F belongs to I_G, $\pi(\prod_{i=1}^s x_{S_i}) = \pi(\prod_{i=1}^s x_{S'_i})$. This implies that $\bigcup_{i=1}^s S_i = \bigcup_{i=1}^s S'_i$ as multisets. Hence one has $\sum_{i=1}^s p_i = \sum_{i=1}^s q_i$ and

$$\prod_{i=1}^s \prod_{j=1}^{p_i} x_{\{k_j^{(i)}\}} = \prod_{i=1}^s \prod_{j=1}^{q_i} x_{\{\ell_j^{(i)}\}}.$$

Thus

$$x_\emptyset^{\sum_{i=1}^s (p_i - 1)} F = G_1 - G_2$$

belongs to L_G. This implies that $F \in L_G : x_\emptyset^\infty$. Hence one has $I_G \subset L_G : x_\emptyset^\infty$. □

Therefore, combining Lemmata 3.5 and 4.1, we obtain the following proposition.

Proposition 4.2. *Let G be a graph on $[d]$ and let $I \in \{\langle [I_G]_2 \rangle, J_G, L_G\}$. If $\mathcal{G}$ is the reduced Gröbner basis of I with respect to a reverse lexicographic order such that $x_S \geqslant x_\emptyset$ for any $S \in S(G)$, then*

$$\{g/x_\emptyset^k \mid g \in \mathcal{G}, k \in \mathbb{Z}_{\geqslant 0}, x_\emptyset^k \text{ divides } g,\ x_\emptyset^{k+1} \text{ does not divide } g\}$$

is a Gröbner basis of I_G.

Next, we discuss when equality holds in the inclusions $L_G \subset J_G \subset \langle [I_G]_2 \rangle \subset I_G$. In particular, we characterize when these ideals are prime. Let $\overline{G}$ denote the complement graph of a graph G.

Proposition 4.3. *Let G be a graph on $[d]$. Then one has the following.*

(a) *Let $I \in \{\langle [I_G]_2 \rangle, J_G, L_G\}$. Then I is prime $\iff$ $I = I_G$.*

(b) *$I_G = \langle [I_G]_2 \rangle \iff I_G$ is generated by quadratic binomials.*

(c) *$\langle [I_G]_2 \rangle = J_G \iff \overline{G}$ has no 3-cycles.*

(d) *$J_G = L_G \iff G$ is a complete multipartite graph on the vertex set $V_1 \sqcup \cdots \sqcup V_t$ with $|V_j| \leqslant 3$ for each j.*

Proof. (a) Let $I \in \{\langle [I_G]_2 \rangle, J_G, L_G\}$. Recall that the toric ideal I_G is prime. Hence I is prime if $I_G = I$. Conversely, suppose that I is prime. Since $x_\emptyset^k \notin I$ holds for any k, it follows that $I = I : x_\emptyset^\infty$. From Lemma 4.1, we have $I = I : x_\emptyset^\infty = I_G$.

(b) Trivial.

(c) If $\overline{G}$ has a 3-cycle (i_1, i_2, i_3), then $x_{\{i_1,i_2,i_3\}} x_{\{i_1\}} - x_{\{i_1,i_2\}} x_{\{i_1,i_3\}} \in \langle [I_G]_2 \rangle \setminus J_G$. Suppose that $\overline{G}$ has no 3-cycles. Then $|S| \leqslant 2$ for all $S \in S(G)$. Let $h = x_{S_1} x_{S_2} - x_{S_3} x_{S_4} \in \langle [I_G]_2 \rangle$. Since h belongs to I_G, $S_1 \cup S_2$ coincides with $S_3 \cup S_4$ as multisets, and in particular, we have $S_1 \cap S_2 = S_3 \cap S_4$. Suppose that $S_1 \cap S_2$ is not empty. Let $i \in S_1 \cap S_2$. Then $(S_1 \setminus \{i\}) \cup (S_2 \setminus \{i\})$ coincides with $(S_3 \setminus \{i\}) \cup (S_4 \setminus \{i\})$ as multisets. Since $|S_j \setminus \{i\}| \leqslant 1$ for each $j = 1, 2, 3, 4$, it follows that (S_1, S_2) is equal to either (S_3, S_4) or (S_4, S_3). Then $h = 0$, a contradiction. Thus $S_1 \cap S_2 = \emptyset$, and hence $h \in J_G$. Therefore we have $\langle [I_G]_2 \rangle \subset J_G$, and hence $\langle [I_G]_2 \rangle = J_G$.

(d) Suppose that $J_G = L_G$. If $\overline{G}$ has a path of length two $P_3 = (i, j, k)$ as an induced subgraph, then it follows that $x_{\{i,j\}} x_{\{k\}} - x_{\{i\}} x_{\{j,k\}} \in J_G \setminus L_G$, a contradiction. Hence $\overline{G}$ has no P_3 as an induced subgraph. It is known that $\overline{G}$ has no P_3 as an induced subgraph if and only if G is a complete multipartite graph. Suppose that G has a part V_α with $|V_\alpha| \geqslant 4$. It then follows that $x_{\{i,j\}} x_{\{k,l\}} - x_{\{i,k\}} x_{\{j,l\}} \in J_G \setminus L_G$ where i, j, k, l are distinct vertices in V_α, a contradiction. Hence G is a complete multipartite graph on the vertex set $V_1 \sqcup \cdots \sqcup V_t$ with $|V_j| \leqslant 3$ for each j.

Suppose that G is a complete multipartite graph on the vertex set $V_1 \sqcup \cdots \sqcup V_t$ with $|V_j| \leqslant 3$ for each j. Then each $S \in S(G)$ is a subset of V_j for some j. It is enough to show that $J_G \subset L_G$. Let $h = x_{S_1} x_{S_2} - x_{S_3} x_{S_4}$ be a nonzero binomial in J_G. Then $S_1 \cap S_2 = S_3 \cap S_4 = \emptyset$ and $S_1 \cup S_2 = S_3 \cup S_4$. Suppose that $S_i \subset V_{j_i}$ for each $i = 1, 2, 3, 4$.

Case 1. ($S_1 = \emptyset$.) Then $h = x_\emptyset x_{S_2} - x_{S_3} x_{S_4}$, where $|S_3|, |S_4| \geqslant 1$ and $|S_3| + |S_4| = |S_2| \leqslant 3$. Since either $|S_3|$ or $|S_4|$ equals one, $h \in L_G$.

Case 2. ($S_j \neq \emptyset$ for each i.) Since $S_1 \cup S_2 = S_3 \cup S_4$, we may assume that $j_1 = j_3$ and $j_2 = j_4$. If $j_1 \neq j_2$, then $S_1 = S_3$ and $S_2 = S_4$, and hence $h = 0$. This is a contradiction. Thus we have $j_1 = j_2 = j_3 = j_4$. Since $2 \leqslant |S_1| + |S_2| = |S_3| + |S_4| \leqslant 3$, we may assume that $|S_1| = |S_3| = 1$. Then $h = (x_{S_1} x_{S_2} - x_\emptyset x_{S_1 \cup S_2}) + (x_\emptyset x_{S_3 \cup S_4} - x_{S_3} x_{S_4})$ belongs to L_G. □

Now, we return to the inclusions (4.1). If G is a complete multipartite graph, then I_G is generated by quadratic binomials ([6, Theorem 3.1]). Hence from Proposition 4.3 we know that $L_G = J_G$ implies $\langle [I_G]_2 \rangle = I_G$. Therefore, there are the following 6 cases:

(i) $L_G = J_G = \langle [I_G]_2 \rangle = I_G$.

(ii) $L_G = J_G \subsetneq \langle [I_G]_2 \rangle = I_G$.

(iii) $L_G \subsetneq J_G = \langle [I_G]_2 \rangle = I_G$.

(iv) $L_G \subsetneq J_G = \langle [I_G]_2 \rangle \subsetneq I_G$.

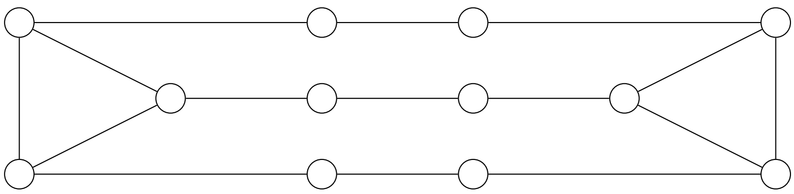
(v) $L_G \subsetneq J_G \subsetneq \langle [I_G]_2 \rangle = I_G$.

(vi) $L_G \subsetneq J_G \subsetneq \langle [I_G]_2 \rangle \subsetneq I_G$.

We give an example for each case.

Example 4.4. First, we recall that for any bipartite graph G, I_G is generated by quadratic binomials.

A complete graph K_d of d vertices satisfies the condition (i). In fact, one has $I_{K_d} = \{0\}$. For a $(3,3)$-complete bipartite graph $K_{3,3}$, since $\overline{K_{3,3}}$ has a 3-cycle, the condition (ii) holds. For a path of length three P_4, since $\overline{P_4}$ is also a path of length three, the condition (iii) holds. For $G = \overline{C_6}$, I_G is not generated by quadratic binomials from [4, Proposition 11]. Since $\overline{G} = C_6$ has no 3-cycles and G is not a complete multipartite graph, the condition (iv) is satisfied. For a 6-cycle C_6, since $\overline{C_6}$ has a 3-cycle and C_6 is not a complete multipartite graph, the condition (v) holds. Finally, we consider the following graph G:



It then follows from [6, Theorem 1.7] that I_G is not generated by quadratic binomials. Since $\overline{G}$ has a 3-cycle and G is not a complete multipartite graph, the condition (vi) holds.

5 Examining Kempe equivalence

In this section, we prove Theorem 1.1 and Proposition 2.3. In fact, we show the following.

Theorem 5.1. *Let G be a graph on $[d]$ and let f and g be k-colorings of an induced subgraph of G. Then the following conditions are equivalent:*

(i) $f \sim_k g$;

(ii) $\mathbf{x}_f - \mathbf{x}_g \in \langle [I_G]_2 \rangle$;

(iii) $\mathbf{x}_f - \mathbf{x}_g \in J_G$.

Proof. Suppose that f and g are k-colorings of an induced subgraph G_0 of G. Since $J_G \subset \langle [I_G]_2 \rangle$ holds, we have (iii) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). Suppose that $f \sim_k g$ and $\mathbf{x}_f - \mathbf{x}_g \notin J_G$. Let $f_0, f_1, \dots, f_s$ be a sequence of k-colorings of G_0 such that $f_0 = f$, $f_s = g$, and f_i is obtained from f_{i-1} by a Kempe switching. We may assume that $s \geqslant 1$ is minimal among all k-colorings f and g such that $f \sim_k g$ and $\mathbf{x}_f - \mathbf{x}_g \notin J_G$. Suppose that the Kempe switching from f to f_1 is obtained by a connected component H of the induced subgraph $G_0[f^{-1}(\mu) \cup f^{-1}(\eta)]$ by setting

$$f_1(x) = \begin{cases} f(x) & x \notin H, \\ \mu & x \in H \text{ and } f(x) = \eta, \\ \eta & x \in H \text{ and } f(x) = \mu. \end{cases}$$

Let $f' = f|_{G'}$ and $f'_1 = f_1|_{G'}$ be the restrictions of f and f_1 to $G' := G_0[f^{-1}(\mu) \cup f^{-1}(\eta)]$, respectively. Since f' and f'_1 are 2-colorings of G', $\mathbf{x}_{f'} - \mathbf{x}_{f'_1}$ belongs to J_G. Then

$$\mathbf{x}_f - \mathbf{x}_g = \frac{\mathbf{x}_f}{\mathbf{x}_{f'}}(\mathbf{x}_{f'} - \mathbf{x}_{f'_1}) + \left(\frac{\mathbf{x}_f}{\mathbf{x}_{f'}} \mathbf{x}_{f'_1} - \mathbf{x}_g \right) = \frac{\mathbf{x}_f}{\mathbf{x}_{f'}}(\mathbf{x}_{f'} - \mathbf{x}_{f'_1}) + \mathbf{x}_{f_1} - \mathbf{x}_g.$$

If $s = 1$, then $f_1 = g$ and hence

$$\mathbf{x}_f - \mathbf{x}_g = \frac{\mathbf{x}_f}{\mathbf{x}_{f'}}(\mathbf{x}_{f'} - \mathbf{x}_{f'_1}) \in J_G.$$

We may assume that $s \geqslant 2$. By the hypothesis on s, $\mathbf{x}_{f_1} - \mathbf{x}_g$ belongs to J_G. Hence $\mathbf{x}_f - \mathbf{x}_g$ belongs to J_G, a contradiction.

(ii) $\Rightarrow$ (i). Suppose that $f \not\sim_k g$ and $\mathbf{x}_f - \mathbf{x}_g \in \langle [I_G]_2 \rangle$. Since $\langle [I_G]_2 \rangle$ is a binomial ideal, it then follows from [3, Lemma 3.8] that there exists an expression

$$\mathbf{x}_f - \mathbf{x}_g = \sum_{r=1}^{s} \mathbf{x}_{w_r}(\mathbf{x}_{f_r} - \mathbf{x}_{g_r}), \tag{5.1}$$

where f_r and g_r are 2-colorings of $G_{\mathbf{a}_r}$ with $\mathbf{a}_r \in \mathbb{Z}^d_{\geqslant 0}$ for each r. We may assume that $s \geqslant 1$ is minimal among such f and g with $f \not\sim_k g$ and $\mathbf{x}_f - \mathbf{x}_g \in \langle [I_G]_2 \rangle$. Since $\mathbf{x}_f$ must appear in the right-hand side of (5.1), we may assume that $\mathbf{x}_f = \mathbf{x}_{w_1}\mathbf{x}_{f_1}$. Then $\mathbf{x}_f$ is divided by $\mathbf{x}_{f_1}$, and f_1 is the restriction of f to $G' = G_0[f^{-1}(\mu) \cup f^{-1}(\nu)]$ for some μ and ν. Since G' has a 2-coloring, it is a bipartite graph. Then f_1 and g_1 are Kempe equivalent. Let f' be a coloring of G_0 defined by

$$f'(x) = \begin{cases} g_1(x) & f(x) \in \{\mu, \nu\}, \\ f(x) & \text{otherwise}. \end{cases}$$

Then one obtains $f \sim_k f'$. Moreover, we have

$$\mathbf{x}_f - \mathbf{x}_{w_1}(\mathbf{x}_{f_1} - \mathbf{x}_{g_1}) = \mathbf{x}_{f'}. \tag{5.2}$$

If $s = 1$, then

$$\mathbf{x}_g = \mathbf{x}_f - \mathbf{x}_{w_1}(\mathbf{x}_{f_1} - \mathbf{x}_{g_1}) = \mathbf{x}_{f'},$$

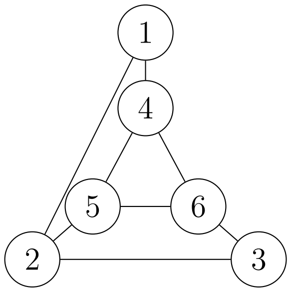
and hence $f \sim_k f' = g$, a contradiction. We may assume that $s \geqslant 2$. Since

$$\mathbf{x}_{f'} - \mathbf{x}_g = \mathbf{x}_f - \mathbf{x}_{w_1}(\mathbf{x}_{f_1} - \mathbf{x}_{g_1}) - \mathbf{x}_g = \sum_{r=2}^{s} \mathbf{x}_{w_r}(\mathbf{x}_{f_r} - \mathbf{x}_{g_r}) \in \langle [I_G]_2 \rangle,$$

$f' \sim_k g$ by the hypothesis on s. Thus $f \sim_k f' \sim_k g$, a contradiction. □

We see examples of Theorem 5.1.

Example 5.2. Let G be the graph as follows:



We consider two 3-colorings f and g of G defined by

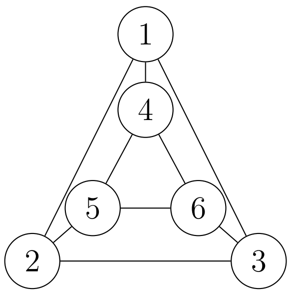
$$f(i) = \begin{cases} 1 & i \in \{1,5\}, \\ 2 & i \in \{2,6\}, \\ 3 & i \in \{3,4\} \end{cases} \text{ and } g(i) = \begin{cases} 1 & i \in \{1,3,5\}, \\ 2 & i \in \{2,6\}, \\ 3 & i = 4. \end{cases}$$

Then one has

$$\mathbf{x}_f - \mathbf{x}_g = x_{\{1,5\}}x_{\{2,6\}}x_{\{3,4\}} - x_{\{1,3,5\}}x_{\{2,6\}}x_{\{4\}} = x_{\{2,6\}}(x_{\{1,5\}}x_{\{3,4\}} - x_{\{1,3,5\}}x_{\{4\}}).$$

Since $x_{\{1,5\}}x_{\{3,4\}} - x_{\{1,3,5\}}x_{\{4\}} \in J_G$, it then follows from Theorem 5.1 that $f \sim_3 g$.

Example 5.3. Let G be the graph as follows:



We consider two 3-colorings f and g of G defined by

$$f(i) = \begin{cases} 1 & i \in \{1,5\}, \\ 2 & i \in \{2,6\}, \\ 3 & i \in \{3,4\} \end{cases} \text{ and } g(i) = \begin{cases} 1 & i \in \{1,6\}, \\ 2 & i \in \{2,4\}, \\ 3 & i \in \{3,5\}. \end{cases}$$

Then one has

$$\mathbf{x}_f - \mathbf{x}_g = x_{\{1,5\}}x_{\{2,6\}}x_{\{3,4\}} - x_{\{1,6\}}x_{\{2,4\}}x_{\{3,5\}}.$$

Since $\mathbf{x}_f - \mathbf{x}_g \notin J_G$, it then follows from Theorem 5.1 that $f \not\sim_3 g$.

See Section 7 for an algorithm to determine if $f \sim_k g$ by using the techniques on Gröbner bases.

6 Computing Kempe classes via Kempe ideals

In this section, we give a way to find all k-colorings of a graph G up to Kempe equivalence via commutative algebra. In particular, we prove Theorem 1.2. For a graph G on $[d]$, we define the ideal K_G of R by

$$K_G := J_G + M_G \subset R[G],$$

where

$$M_G := \langle x_S x_T \mid S, T \in S(G),\ S \cap T \neq \emptyset \rangle \subset R[G].$$

The ideal K_G is called the *Kempe ideal* of G.

The set of all monomials in M_G consists of all monomials associated with k-colorings of $G_{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^d$ such that $G_{\mathbf{a}}$ is not an induced subgraph of G.

Lemma 6.1. *Let G be a graph on $[d]$. Then $\mathbf{x}_f = x_{S_1}x_{S_2}\cdots x_{S_k} \notin M_G$ if and only if f is a k-coloring of an induced subgraph of G.*

Proof. Suppose that f is not a k-coloring of any induced subgraph of G. Then f is a k-coloring of $G_{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^d$ such that $a_j \geqslant 2$ for some j. Hence there exist integers $1 \leqslant a < b \leqslant k$ with $i \in S_a$ and $i \in S_b$. This implies that $\mathbf{x}_f$ can be divided by a monomial $x_{S_a}x_{S_b} \in M_G$, and hence $\mathbf{x}_f \in M_G$.

Suppose that $\mathbf{x}_f \in M_G$. There exist $S, T \in S(G)$ such that $S \cap T \neq \emptyset$ and $x_S x_T$ divides $\mathbf{x}_f$. Let $i \in S \cap T$. Then i appears at least twice in $S_1 \cup \cdots \cup S_k$. Thus f is not a k-coloring of any induced subgraph of G. □

As an easy consequence of this lemma, we have the following.

Corollary 6.2. *Let G be a graph on $[d]$. Then one has*

$$K_G = \langle [I_G]_2 \rangle + M_G.$$

The following lemma is useful to check if a homogeneous binomial in I_G belongs to M_G.

Lemma 6.3. *Let G be a graph on $[d]$ and let $\mathbf{x}_f - \mathbf{x}_g \in I_G$. Then the following conditions are equivalent*:

(i) $\mathbf{x}_f - \mathbf{x}_g \in M_G$;

(ii) *both $\mathbf{x}_f$ and $\mathbf{x}_g$ belong to M_G;*

(iii) *at least one of $\mathbf{x}_f$ and $\mathbf{x}_g$ belongs to M_G.*

Proof. First, (ii) $\Longrightarrow$ (iii) is trivial. Since M_G is a monomial ideal, the equivalence (i) $\Longleftrightarrow$ (ii) holds. We show that (iii) $\Longrightarrow$ (ii). Suppose that $\mathbf{x}_f \in M_G$, i.e., $\mathbf{x}_f$ is divided by $x_S x_T$ where $S, T \in S(G)$ and $S \cap T \neq \emptyset$. Let $i \in S \cap T$. Then $\pi(\mathbf{x}_f)$ is divided by t_i^2 since $\mathbf{x}_f$ is divided by $x_S x_T$. From $\mathbf{x}_f - \mathbf{x}_g \in I_G$, we have $\pi(\mathbf{x}_f) = \pi(\mathbf{x}_g)$. Hence if $\mathbf{x}_g = x_{S_1} x_{S_2} \cdots x_{S_k}$, then there exist $1 \leqslant a < b \leqslant k$ such that $i \in S_a \cap S_b$. Namely, $\mathbf{x}_g$ is divided by a monomial $x_{S_a} x_{S_b}$ such that $i \in S_a \cap S_b$. Thus $\mathbf{x}_g \in M_G$. □

Next, we discuss a Gröbner basis of K_G.

Proposition 6.4. *Let G be a graph on $[d]$, and let $\mathcal{G}_1$ be the reduced Gröbner basis of J_G with respect to a monomial order $<$ and set $\mathcal{G}_2 = \{x_S x_T \mid S, T \in S(G),\ S \cap T \neq \emptyset\}$. Then*

$$\mathcal{G} = (\mathcal{G}_1 \setminus M_G) \cup \mathcal{G}_2$$

is the reduced Gröbner basis of K_G with respect to $<$.

Proof. Since $\mathcal{G}_1$ is a Gröbner basis of J_G, we have $J_G = \langle \mathcal{G}_1 \rangle$. It follows from $M_G = \langle \mathcal{G}_2 \rangle$ that $\mathcal{G}$ is a set of generators of $K_G = J_G + M_G$. We apply Buchberger's Criterion to $\mathcal{G}$. If $p, q \in \mathcal{G}_1 \setminus M_G$, then the remainder of $S(p, q)$ on division by $\mathcal{G}$ is 0 since $\mathcal{G}_1$ is a Gröbner basis of J_G. If $p, q \in \mathcal{G}_2$, then both p and q are monomials, and hence $S(p, q) = 0$. Suppose that $p \in \mathcal{G}_2$ and $q \in \mathcal{G}_1 \setminus M_G$. Let $p = x_S x_T$ with $S, T \in S(G)$ and $S \cap T \neq \emptyset$, and let $q = \mathbf{x}_f - \mathbf{x}_g$ where f and g are k-colorings of $G_{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^d$ and $\mathrm{in}_<(q) = \mathbf{x}_f$.

Case 1. (p and $\mathbf{x}_f$ are relatively prime.) Then $S(p, q) = p\mathbf{x}_g$ is divided by $p \in \mathcal{G}_2$.

Case 2. ($\mathbf{x}_f$ is divided by x_S.) From Lemma 6.3, $\mathbf{x}_f$ is not divided by x_T since $q \notin M_G$. Then $S(p, q) = x_T \mathbf{x}_g$. Let $i \in S \cap T$. Since $\mathbf{x}_f$ is divided by x_S, $\pi(\mathbf{x}_f) = \pi(\mathbf{x}_g)$ is divided by t_i. Thus $\pi(x_T \mathbf{x}_g)$ is divided by t_i^2. Hence $x_T \mathbf{x}_g$ is divided by a monomial $x_{S'} x_T$ such that $i \in S' \cap T$. Then the remainder of $S(p, q) = x_T \mathbf{x}_g$ on division by $\mathcal{G}_2$ is 0.

Therefore the remainder of any S-polynomial on division by $\mathcal{G}$ is 0. By Buchberger's Criterion, $\mathcal{G}$ is a Gröbner basis of K_G. Since $\mathcal{G}_1$ is reduced, it is easy to see that $\mathcal{G}$ is reduced. □

Similarly to $\langle [I_G]_2 \rangle$ and J_G, we can determine if two k-colorings of G are Kempe equivalent by K_G.

Proposition 6.5. *Let G be a graph on $[d]$ and let f and g be k-colorings of an induced subgraph of G. Then $f \sim_k g$ if and only if $\mathbf{x}_f - \mathbf{x}_g \in K_G$.*

Proof. Let f and g be k-colorings of an induced subgraph G_0 of G. From Theorem 5.1 it is enough to show that $\mathbf{x}_f - \mathbf{x}_g \in J_G$ if and only if $\mathbf{x}_f - \mathbf{x}_g \in K_G$. Since $J_G \subset K_G$, $\mathbf{x}_f - \mathbf{x}_g \in K_G$ if $\mathbf{x}_f - \mathbf{x}_g \in J_G$.

Suppose that $\mathbf{x}_f - \mathbf{x}_g \in K_G$. We show that $\mathbf{x}_f - \mathbf{x}_g \in J_G$. Let $\mathcal{G}$ be the reduced Gröbner basis of K_G with respect to a monomial order $<$. From Proposition 6.4, $\mathcal{G} = (\mathcal{G}_1 \setminus M_G) \cup \mathcal{G}_2$ where $\mathcal{G}_1$ is the reduced Gröbner basis of J_G with respect to $<$ and $\mathcal{G}_2 = \{x_S x_T \mid S, T \in S(G),\ S \cap T \neq \emptyset\}$. Let $\mathbf{x}_{f'}$ (resp. $\mathbf{x}_{g'}$) denote the remainder of $\mathbf{x}_f$ (resp. $\mathbf{x}_g$) on division by $\mathcal{G}_1 \setminus M_G$. Suppose that $\mathbf{x}_{f'} \neq \mathbf{x}_{g'}$. Since f and g are k-colorings of G_0, we have $\mathbf{x}_f, \mathbf{x}_g \notin M_G$ from Lemma 6.1. From Lemma 6.3, $\mathbf{x}_{f'}, \mathbf{x}_{g'} \notin M_G$ since both $\mathbf{x}_f - \mathbf{x}_{f'}$ and $\mathbf{x}_g - \mathbf{x}_{g'}$ belong to J_G. Thus $\mathbf{x}_{f'} - \mathbf{x}_{g'}$ $(\neq 0)$ is the normal form of $\mathbf{x}_f - \mathbf{x}_g \in K_G$ with respect to $\mathcal{G}$. This contradicts that $\mathcal{G}$ is a Gröbner basis of K_G. Thus $\mathbf{x}_{f'} = \mathbf{x}_{g'}$. Then the normal form of $\mathbf{x}_f - \mathbf{x}_g$ with respect to $\mathcal{G}_1 \setminus M_G$ is zero, and hence $\mathbf{x}_f - \mathbf{x}_g \in J_G$. □

Now, we give a proof of Theorem 1.2. In fact, Theorem 1.2 follows from the following.

Theorem 6.6. *Let G be a graph on $[d]$ and $<$ a monomial order on $R[G]$, and let $\{\mathbf{x}_{f_1}, \dots, \mathbf{x}_{f_s}\}$ be the set of all standard monomials of degree k with respect to $\mathrm{in}_<(K_G)$. Then each f_i is a k-coloring of an induced subgraph of G. In addition, given an induced subgraph G' of G,*

$$\{f_1, \dots, f_s\} \cap \mathcal{C}_k(G')$$

is a complete representative system for $\mathrm{kc}(G', k)$.

Remark 6.7. If $\mathbf{x}_f = x_{S_1} x_{S_2} \cdots x_{S_k}$ is a standard monomial of degree k with respect to $\mathrm{in}_<(K_G)$, then f is a k-coloring of the induced subgraph of G on $\bigcup_{i=1}^k S_i$.

Proof of Theorem 6.6. Let $\mathcal{G}$ be the reduced Gröbner basis of K_G with respect to a monomial order $<$. From Proposition 6.4, $\mathcal{G} = (\mathcal{G}_1 \setminus M_G) \cup \mathcal{G}_2$ where $\mathcal{G}_1$ is the reduced Gröbner basis of J_G with respect to $<$ and $\mathcal{G}_2 = \{x_S x_T \mid S, T \in S(G),\ S \cap T \neq \emptyset\}$. Since each $\mathbf{x}_{f_i}$ is not divided by any monomial in $\mathcal{G}_2$, from Lemma 6.1, each f_i is a k-coloring of an induced subgraph of G.

Suppose that $f_i \sim_k f_j$ for some $1 \leqslant i < j \leqslant s$. From Theorem 5.1, $\mathbf{x}_{f_i} - \mathbf{x}_{f_j}$ belongs to $J_G (\subset K_G)$. Then $\mathrm{in}_<(\mathbf{x}_{f_i} - \mathbf{x}_{f_j})$ is not standard, a contradiction. Hence $f_i \not\sim_k f_j$ for any $1 \leqslant i < j \leqslant s$.

Let f be a k-coloring of G', and let $\mathbf{x}_{f'}$ be the remainder of $\mathbf{x}_f$ on division by $\mathcal{G}_1 \setminus M_G$. Then $\mathbf{x}_f - \mathbf{x}_{f'}$ belongs to J_G. Since $\mathbf{x}_f \notin M_G$, we have $\mathbf{x}_{f'} \notin M_G$ from Lemma 6.3. Thus $\mathbf{x}_{f'}$ is the normal form of $\mathbf{x}_f$ with respect to $\mathcal{G}$, and hence $\mathbf{x}_{f'} = \mathbf{x}_{f_j}$ for some j. Since $\mathbf{x}_f - \mathbf{x}_{f_j}$ belongs to J_G, we have $f \sim_k f_j$ from Theorem 5.1. □

We see an example of Theorem 6.6.

Example 6.8. Let G be the graph as in Example 5.3. We consider the reverse lexicographic order $<$ on $R[G]$ such that

$$x_\emptyset < x_{\{1\}} < \cdots < x_{\{6\}} < x_{\{1,5\}} < x_{\{1,6\}} < x_{\{2,4\}} < x_{\{2,6\}} < x_{\{3,4\}} < x_{\{3,5\}}.$$

Then there are 65 standard monomials of degree 3 with respect to $\text{in}_<(K_G)$. In particular, the standard monomials $x_{\{1,5\}}x_{\{2,6\}}x_{\{3,4\}}$ and $x_{\{1,6\}}x_{\{2,4\}}x_{\{3,5\}}$ correspond to 3-colorings of G. It then follows from Theorem 6.6 that the associated 3-colorings, which are the 3-colorings as in Example 5.3, form a complete representative system for $\text{kc}(G, k)$.

As a consequence of Theorem 6.6, the number of k-Kempe classes $\text{Kc}(G, k)$ can be computed by using Hilbert functions. We denote $\text{Ind}(G)$ the set of all induced subgraphs of G and denote $\text{Ind}_m(G)$ the set of all induced subgraphs of G with m vertices.

Corollary 6.9. *Let G be a graph on $[d]$. Then one has*

$$H(R[G]/K_G, k) = \sum_{G' \in \text{Ind}(G)} \text{Kc}(G', k).$$

In particular,

$$\text{Kc}(G, k) = \sum_{m=0}^{d} (-1)^{d-m} \sum_{G' \in \text{Ind}_m(G)} H(R[G']/K_{G'}, k).$$

Example 6.10. Let G be the graph as in Example 5.3. Then one has

$$H(R[G]/K_G, k) = \begin{cases} 1 & k = 0, \\ 13 & k = 1, \\ 49 & k = 2, \\ 65 & k = 3, \\ 64 & k \geqslant 4, \end{cases}$$

and

$$H(R[G']/K_{G'}, 3) = 32$$

for any $G' \in \text{Ind}_5(G)$. Note that $\text{Kc}(G', k) \geqslant 1$ for any $k \geqslant \chi(G)$ and $G' \in \text{Ind}(G)$. Moreover, one has $|\text{Ind}(G)| = 2^6 = 64$. Hence we have the following from $H(R[G]/K_G, k)$.

- Since $H(R[G]/K_G, 2) < 2^6$, one has $\chi(G) \geqslant 3$. In fact, $\chi(G) = 3$ in this case.
- Let $k \geqslant 4$. Since $H(R[G]/K_G, k) = 2^6$, $\text{Kc}(G', k) = 1$ for any $G' \in \text{Ind}(G)$.
- Since $H(R[G']/K_{G'}, 3) = 2^5$, $\text{Kc}(G'', 3) = 1$ for any $G'' \in \text{Ind}(G)$ with $G \neq G''$. Hence one has $\text{Kc}(G, 3) = 2$ from $H(R[G]/K_G, 3) = 65$.

Let I be a graded ideal of R. In general, the Hilbert function $H(R/I, k)$ is not always a polynomial. However, there exists a unique polynomial $P_{R/I} \in \mathbb{Q}[k]$ such that $H(R/I, k) = P_{R/I}(k)$ for k large enough. We call $P_{R/I}$ the *Hilbert polynomial* of R/I. We show that $P_{R[G]/K_G}$ is a constant which depends only on the number of vertices.

Proposition 6.11. *Let G be a graph on $[d]$. Then one has*

$$P_{R[G]/K_G}(k) = 2^d.$$

Proof. Let Δ be the maximum degree of G. Then it follows from [5, Corollary 2.5] that for any integer $k \geqslant \Delta + 1$ and for any induced subgraph G' of G, one has $\mathrm{Kc}(G', k) = 1$. Hence for any integer $k \geqslant \Delta + 1$, we obtain

$$H(R[G]/K_G, k) = \sum_{G' \in \mathrm{Ind}(G)} \mathrm{Kc}(G', k) = \sum_{G' \in \mathrm{Ind}(G)} 1 = |\mathrm{Ind}(G)| = 2^d.$$

This implies that

$$P_{R[G]/K_G}(k) = 2^d,$$

as desired. □

7 Algorithms

In this section, by using Theorems 1.1 and 1.2 and techniques on Gröbner bases, we introduce several algorithms related to Kempe equivalence.

First we give Algorithm 7.1 to compute the reduced Gröbner basis of K_G. From Proposition 4.2, $\mathcal{G}_2$ in Algorithm 7.1 is a Gröbner basis of I_G. It then follows that $\mathcal{G}_3$ in Algorithm 7.1 is a system of generators of J_G. Using the reduced Gröbner basis of K_G,

---

Algorithm 7.1 Computation of the reduced Gröbner basis of K_G

Input: The set of all stable sets of G, and a monomial order $<$ on $R[G]$.
Output: The reduced Gröbner basis of K_G with respect to $<$.

1: $\mathcal{F} := \{x_{S\setminus\{i\}} x_{\{i\}} - x_S x_\emptyset \mid i \in S \in S(G),\ |S| \geqslant 2\}$.
2: From $\mathcal{F}$, compute the reduced Gröbner basis $\mathcal{G}_1$ of L_G with respect to a reverse lexicographic order such that $x_S \geqslant x_\emptyset$ for any $S \in S(G)$.
3: $\mathcal{G}_2 := \{g/x_\emptyset^k \mid g \in \mathcal{G}_1, k \in \mathbb{Z}_{\geqslant 0},\ x_\emptyset^k \text{ divides } g,\ x_\emptyset^{k+1} \text{ does not divide } g\}$.
4: $\mathcal{G}_3 := \{x_{S_1}x_{S_2} - x_{S_3}x_{S_4} \in \mathcal{G}_2 \mid S_1, S_2, S_3, S_4 \in S(G),\ S_1 \cap S_2 = S_3 \cap S_4 = \emptyset\}$.
5: Compute the reduced Gröbner basis $\mathcal{G}$ of K_G with respect to $<$ from a system of generators $\mathcal{G}_3 \cup \{x_S x_T \mid S, T \in S(G), S \cap T \neq \emptyset\}$.
6: **return** $\mathcal{G}$.

Algorithm 7.2 determines whether $f \sim_k g$ or not. The correctness of Algorithm 7.2 is an immediate consequence of Lemma 3.3 and Proposition 6.5. From Theorem 5.1, it is possible to replace K_G with either $\langle [I_G]_2 \rangle$ or J_G in Algorithm 7.2. On the other hand, by the reduced Gröbner basis of K_G, we can compute the set of all standard monomials of degree k with respect to the initial ideal $\mathrm{in}_<(K_G)$. From Theorem 1.2, we have Algorithm 7.3 to compute a complete representative system for $\mathrm{kc}(G, k)$. Algorithm 7.4 enumerates all elements in a Kempe equivalent class. The correctness of Algorithm 7.4 is guaranteed by Theorem 6.6 together with the fact that, with respect to a Gröbner basis, the normal form of the monomials in the same residue class is unique. In addition, Algorithm 7.4 is based on [8, Algorithm 5.7] for enumeration of fibers using a Gröbner basis of a toric ideal. As

Algorithm 7.2 Determination of the Kempe equivalence

Input: k-colorings f and g of G, and the reduced Gröbner basis $\mathcal{G}$ of K_G.
Output: "$f \sim_k g$" or "$f \not\sim_k g$".

```
Compute the normal form m of x_f − x_g with respect to 𝒢.
if m = 0 then
    return "f ∼_k g"
else
    return "f ≁_k g"
end if
```

Algorithm 7.3 Computation of a complete representative system for $\mathrm{kc}(G, k)$

Input: The reduced Gröbner basis $\mathcal{G}$ of K_G.
Output: A complete representative system for $\mathrm{kc}(G, k)$.

```
From 𝒢, compute the set {x_{f_1}, ..., x_{f_s}} of all standard monomials of degree k with
    respect to the initial ideal in_<(K_G).
C := {}.
for i = 1, 2, ..., s do
    if x_{f_i} = x_{S_1} ··· x_{S_k} satisfies [d] = S_1 ∪ ··· ∪ S_k then
        C := C ∪ {f_i}.
    end if
end for
return C
```

stated in [8], "reverse search" technique is useful to improve the efficiency. Finally, we give an algorithm to find a sequence of Kempe switchings. We define a *Kempe basis* which is the set of sequences of colorings corresponding to the reduced Gröbner basis of K_G.

Definition 7.1. Work with the same notation as in Theorem 6.4, that is, $\mathcal{G} = (\mathcal{G}_1 \setminus M_G) \cup \mathcal{G}_2$ is the reduced Gröbner basis of K_G with respect to $<$. Let $\mathcal{G}_1 \setminus M_G = \{\mathbf{x}_{p_1} - \mathbf{x}_{q_1}, \dots, \mathbf{x}_{p_t} - \mathbf{x}_{q_t}\}$. From Theorem 1.1, p_j and q_j are Kempe equivalent for each j. Then

$$\left\{(f_1^{(1)}, \dots, f_{s_1}^{(1)}), \dots, (f_1^{(t)}, \dots, f_{s_t}^{(t)})\right\}$$

is called a *Kempe basis* of G with respect to $<$ if $f_1^{(j)} = p_j$, $f_{s_j}^{(j)} = q_j$, and $f_i^{(j)}$ is obtained from $f_{i-1}^{(j)}$ by a Kempe switching.

In order to give an algorithm to find a Kempe basis, the following three Procedures are important.

Algorithm 7.4 Enumeration of elements in a Kempe equivalent class

Input: k-coloring f of G, and the reduced Gröbner basis $\mathcal{G}$ of K_G.
Output: All k-colorings of G which are Kempe equivalent to f (up to permutations of colors).

```
Compute the normal form x_{f'} of x_f with respect to G.
A := {f'}, B := {}.
while A ≠ {} do
   Choose g ∈ A.
   for x_{g1} − x_{g2} ∈ G with x_{g1} > x_{g2} do
      if x_{g2} divides x_g and g' ∉ B where x_{g'} = x_g x_{g1}/x_{g2} then
         A := A ∪ {g'}.
      end if
   end for
   A := A \ {g}, B := B ∪ {g}.
end while
return B.
```

Procedure 1. ($\mathbf{x}_f - \mathbf{x}_g = x_{S_1}x_{S_2} - x_{S_3}x_{S_4} \in J_G \mapsto$ a sequence of Kempe switching from f to g.)

Suppose that f and g are 2-colorings of an induced subgraph G_0 of G. Let $\mathbf{x}_f - \mathbf{x}_g = x_{S_1}x_{S_2} - x_{S_3}x_{S_4} \in J_G$. Then $S_1 \cap S_2 = S_3 \cap S_4 = \emptyset$ and $S_1 \cup S_2 = S_3 \cup S_4$. Hence g is obtained from f by setting

$$g(x) = \begin{cases} f(x) & x \notin H, \\ 1 & x \in H \text{ and } f(x) = 2, \\ 2 & x \in H \text{ and } f(x) = 1, \end{cases}$$

where H is the induced subgraph of G_0 on the vertex set $(S_1 \setminus S_3) \sqcup (S_2 \setminus S_4)$. Suppose that H has p connected components $H_1, \dots, H_p$. For $l = 1, 2, \dots, p$, let $f_0 = f$ and let

$$f_l(x) = \begin{cases} f(x) & x \notin H, \\ 1 & x \in H_1 \cup \cdots \cup H_l \text{ and } f(x) = 2, \\ 2 & x \in H_1 \cup \cdots \cup H_l \text{ and } f(x) = 1. \end{cases}$$

Then $f_0, f_1, \dots, f_p$ is a sequence of 2-colorings of G_0 such that $f_0 = f$, $f_p = g$, and f_i is obtained from f_{i-1} by a Kempe switching.

Procedure 2. Let f' and g' be k-colorings of an induced subgraph G_1 of G. Suppose that $\mathbf{x}_{f'} - \mathbf{x}_{g'} = \mathbf{x}_h(\mathbf{x}_f - \mathbf{x}_g)$. Then f and g are the restrictions of f' and g' to some induced subgraph G_2 of G_1, respectively. Hence if $f_0, f_1, \dots, f_p$ is a sequence of k'-colorings of G_2 such that $f_0 = f$, $f_p = g$, and f_i is obtained from f_{i-1} by a Kempe switching, then

$f'_0, f'_1, \dots, f'_p$ satisfy $f'_0 = f'$, $f'_p = g'$, and f'_i is obtained from f'_{i-1} by a Kempe switching, where f'_i is a k-coloring of G_1 obtained by combining f_i with h.

Procedure 3. Let $\mathbf{x}_p - \mathbf{x}_q$ be a binomial in the reduced Gröbner basis $\mathcal{G}$ of K_G with respect to a monomial order $<$. From Proposition 6.4, any binomial $\mathbf{x}_p - \mathbf{x}_q \in \mathcal{G}$ is obtained from the reduced Gröbner basis of J_G with respect to $<$ which is computed from

$$\{\mathbf{x}_f - \mathbf{x}_g \mid f \text{ and } g \text{ are 2-colorings of an induced subgraph of } G\}$$

by Buchberger's Algorithm. By keeping track of the computation of Buchberger's Algorithm, we can compute an expression

$$\mathbf{x}_p - \mathbf{x}_q = \sum_{r=1}^{s} \mathbf{x}_{w_r}(\mathbf{x}_{f_r} - \mathbf{x}_{g_r}),$$

where f_r and g_r are 2-colorings of an induced subgraph of G for each r,

$$\mathbf{x}_{w_r}\mathbf{x}_{g_r} = \mathbf{x}_{w_{r+1}}\mathbf{x}_{f_{r+1}}$$

for each $r = 1, \dots, s-1$, and $\mathbf{x}_p = \mathbf{x}_{w_1}\mathbf{x}_{f_1}$, $\mathbf{x}_q = \mathbf{x}_{w_s}\mathbf{x}_{g_s}$. See [1, Chapter 2, §1] for details. Then we can compute a sequence of Kempe switchings from u_r to v_r where $\mathbf{x}_{u_r} = \mathbf{x}_{w_r}\mathbf{x}_{f_r}$ and $\mathbf{x}_{v_r} = \mathbf{x}_{w_r}\mathbf{x}_{g_r}$ by Procedures 1 and 2. Combining them, we have a sequence of Kempe switchings from p to q.

Using Procedures 1, 2, and 3, we have Algorithm 7.5 that computes a Kempe basis. Note that, if the reduced Gröbner basis of K_G consists of quadratic polynomials, then we can skip large part of the procedure in Algorithm 7.5.

We now use a Kempe basis to find a sequence of Kempe switchings between two colorings. Suppose that the normal form $\mathbf{x}_{f'}$ of $\mathbf{x}_f$ with respect to the reduced Gröbner basis $\mathcal{G}$ of K_G is given by

$$\mathbf{x}_f = \mathbf{x}_{f_0} \underset{h_1}{\rightarrow} \mathbf{x}_{f_1} \underset{h_2}{\rightarrow} \cdots \underset{h_t}{\rightarrow} \mathbf{x}_{f_t} = \mathbf{x}_{f'}, \quad (h_j \in \mathcal{G}).$$

By Procedure 2, we can construct a sequence of Kempe switchings from f_{i-1} to f_i by extending a sequence of Kempe switchings corresponding to h_{i-1} in a Kempe basis. Using this fact, we have Algorithm 7.6 to find a sequence of Kempe switchings.

Although Algorithm 7.1 requires the enumeration of all stable sets and the computation of a Gröbner basis, and thus is not computationally feasible for large graphs, its merit lies in providing a uniform algebraic framework. In particular, the method is mechanically applicable and conceptually clarifies the structure of the graph from an abstract algebraic viewpoint.

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Algorithm 7.5 Construction of a Kempe basis

Input: A monomial order $<$ on $R[G]$.
Output: A Kempe basis of G with respect to $<$.

Compute a system of (quadratic) generators

$$\mathcal{F} = \{\mathbf{x}_{f_1} - \mathbf{x}_{g_1}, \dots, \mathbf{x}_{f_t} - \mathbf{x}_{g_t}\} \cup \{x_S x_T \mid S, T \in S(G), S \cap T \neq \emptyset\}$$

of K_G.
for $r = 1, 2, \dots, t$ **do**
  By Procedure 1, compute a sequence F_r of 2-colorings corresponding to a sequence of Kempe switchings from f_r to g_r.
end for
Compute the reduced Gröbner basis $\mathcal{G}$ of K_G with respect to $<$ from $\mathcal{F}$.
$A := \{\}$.
for $\mathbf{x}_p - \mathbf{x}_q \in \mathcal{G}$ **do**
  By keeping track of the computation of $\mathcal{G}$ from $\mathcal{F}$, compute the expression

$$\mathbf{x}_p - \mathbf{x}_q = \sum_{r=1}^{s} \mathbf{x}_{w_r}(\mathbf{x}_{f_{i_r}} - \mathbf{x}_{g_{i_r}}),$$

  where $\mathbf{x}_{w_r}\mathbf{x}_{g_{i_r}} = \mathbf{x}_{w_{r+1}}\mathbf{x}_{f_{i_{r+1}}}$ for each $r = 1, \dots, s-1$, and $\mathbf{x}_p = \mathbf{x}_{w_1}\mathbf{x}_{f_{i_1}}$, $\mathbf{x}_q = \mathbf{x}_{w_s}\mathbf{x}_{g_{i_s}}$.
  for $r = 1, 2, \dots, s$ **do**
    $f := f'_{i_r}$ where $\mathbf{x}_{f'_{i_r}} = \mathbf{x}_{w_r}\mathbf{x}_{f_{i_r}}$, and $g := g'_{i_r}$ where $\mathbf{x}_{g'_{i_r}} = \mathbf{x}_{w_r}\mathbf{x}_{g_{i_r}}$.
    By extending F_{i_r}, find a sequence of colorings $h_1^{(r)}, \dots, h_{t_r}^{(r)}$ where $h_1^{(r)} = f$, $h_{t_r}^{(r)} = g$ and $h_j^{(r)}$ is obtained from $h_{j-1}^{(r)}$ by a Kempe switching.
  end for
  $A := A \cup \{(h_1^{(1)}, \dots, h_{t_1}^{(1)} (= h_1^{(2)}), h_2^{(2)}, \dots, h_{t_2}^{(2)} (= h_1^{(3)}), \dots, h_{t_s}^{(s)})\}$
end for
return A

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Algorithm 7.6 Construction of a sequence of Kempe switchings

Input: k-colorings f and g of G, the reduced Gröbner basis $\mathcal{G}$ of K_G, and a Kempe basis $\mathcal{K}$ of G.

Output: "a sequence of Kempe switchings from f to g" or "$f \not\sim_k g$".

Compute the normal form $\mathbf{x}_{f'}$ of $\mathbf{x}_f$ with respect to $\mathcal{G}$.
Compute the normal form $\mathbf{x}_{g'}$ of $\mathbf{x}_g$ with respect to $\mathcal{G}$.
if $\mathbf{x}_{f'} = \mathbf{x}_{g'}$ **then**
    By Procedure 2, from $\mathcal{K}$, find a sequence of colorings $f = f_1, \ldots, f_s, f'$, where f_i is obtained from f_{i-1} by a Kempe switching.
    By Procedure 2, from $\mathcal{K}$, find a sequence of colorings $g = g_1, \ldots, g_t, g'$ $(= f')$, where g_i is obtained from g_{i-1} by a Kempe switching.
    return $f_1, \ldots, f_s, f', g_t, g_{t-1}, \ldots, g_1$.
else
    return "$f \not\sim_k g$"
end if

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