On 102-avoiding inversion sequences

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Abstract

In this article, we provide a bijection between the set of inversion sequences avoiding the pattern 102 and the set of 2-Schröder paths having neither peaks nor valleys and ending with a diagonal step. To achieve this, we introduce two intermediate objects, called UVD paths and labeled F-paths, and establish bijections among all four families. For each of these combinatorial objects, we define a natural statistic and enumerate the corresponding structures with respect to this statistic. In addition, we study inversion sequences avoiding 102 and another pattern of length 3, providing refined enumerations according to the same statistic.

Mathematics Subject Classifications: 05A19, 05A05, 05A15

1 Introduction

An integer sequence $e = (e_1, e_2, \ldots, e_n)$ is called an *inversion sequence of length* n if $0 \leqslant e_i < i$ for all $i \in [n]$. While pattern avoidance in permutations has been extensively studied (see Kitaev [6] for the survey), the study of pattern avoidance in inversion sequences is a more recent development, initiated independently by Corteel et al. [1], and Mansour and Shattuck [8]. They studied inversion sequences avoiding patterns of length 3. Following the initial work, Martinez and Savage [9] reframed the notion of length-three pattern from a word of length 3 to a triple of binary relations. Yan and Lin [13] further investigated Wilf-equivalences among pairs of such patterns. Recently, Testart [12] completed the enumeration of inversion sequences avoiding one or two patterns of length 3. Meanwhile, Hong and Li [4] nearly completed the Wilf classification for patterns of length four.

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Mansour and Shattuck [8] found that the generating function $A(x)$ for the number of 102-avoiding inversion sequences satisfies the functional equation

$$A(x) = 1 + \left(x - x^2\right) A(x)^3.$$

Seo and Shin [11] showed that the generating function for the number of 2-Schröder paths having neither peaks nor valleys and ending with a diagonal step satisfies the same equation, demonstrating that the two sets are equinumerous. They posed the open problem of finding a bijection between the two sets. The number of inversion sequences avoiding 102 and another pattern of length 3 was studied by several authors [1, 7, 12, 13].

In this paper, we resolve this problem by introducing two new objects: UVD paths and labeled F-paths. We construct bijections among the set of UVD paths, the set of labeled F-paths, and the set of 102-avoiding inversion sequences. As a consequence, we obtain a bijection between 102-avoiding inversion sequences and 2-Schröder paths having neither peaks nor valleys and ending with a diagonal step, thus answering the question of Seo and Shin [11].

We further define a statistic called *rank* on 102-avoiding inversion sequences extending the notion introduced for $(102, 101)$-avoiding sequences in [3]. Analogous statistics are also defined for UVD paths and labeled F-paths. We then enumerate 102-avoiding inversion sequences with a fixed rank and study the doubly-avoiding cases where a second pattern $\tau \in \{001, 011, 012, 021, 110, 120, 201, 210\}$ is also avoided. Recently, Kim, Seo, and Shin [5] showed that the generating function for $(102, 000)$-avoiding inversion sequences with a fixed rank is algebraic and provided its minimal polynomial. Testart [12] independently gave essentially the same result using the statistic *sites*. The cases involving patterns $010, 100$ remain open.

The rest of the paper is organized as follows. Section 2 introduces definitions and background. In Section 3, we construct bijections among 102-avoiding inversion sequences, UVD paths, and labeled F-paths. Section 4 provides refined enumerations based on the rank statistic, while Section 5 explores the doubly-avoiding cases for various patterns τ.

2 Preliminaries

In this section, we provide definitions for the objects we discuss in this article. Let n be a positive integer.

2.1 Inversion sequences

An integer sequence $e = (e_1, e_2, \ldots, e_n)$ is called an *inversion sequence of length n* if $0 \leqslant e_j \leqslant j - 1$ for all $j \in [n] := \{1, 2, \ldots, n\}$. We denote the largest integer in e by $\max(e)$. We also denote by $\mathrm{fdes}(e)$ the position p of the first descent, i.e.,

$$e_1 \leqslant e_2 \leqslant \ldots \leqslant e_p > e_{p+1}$$

with $e_{n+1} = -1$.

Pattern avoidance in inversion sequences can be defined in a similar way to that in permutations. Given a word $w \in \{0, 1, \ldots, k-1\}^k$, let the word obtained by replacing the i-th smallest entry in w with $i-1$ be called the *reduction* of w. We say that an inversion sequence $e = (e_1, e_2, \ldots, e_n)$ *contains* the pattern w if there exist some indices $i_1 < i_2 < \cdots < i_k$ such that the reduction of $e_{i_1}e_{i_2}\ldots e_{i_k}$ is w. Otherwise, e is said to *avoid* the pattern w. Let $\mathcal{IS}_n$ denote the set of inversion sequences of length n and $\mathcal{IS}_n(w_1, \ldots, w_r)$ denote the set of inversion sequences of length n avoiding all the patterns $w_1, \ldots, w_r$.

For $e = (e_1, e_2, \ldots, e_n) \in \mathcal{IS}_n(102)$, if $p = \mathrm{fdes}(e)$, then $e_p = \max(e)$. In this case, we define

$$\mathrm{rank}(e) := \mathrm{fdes}(e) - \max(e) - 1 = (p-1) - e_p \geqslant 0.$$

2.2 2-Schröder paths and UVD paths

A 2-*Schröder path of semilength* n is a lattice path from $(0,0)$ to $(n, 2n)$ that does not go below the line $y = 2x$ and consists of north steps $N = (0,1)$, east steps $E = (1,0)$, and diagonal steps $H = (1,1)$.

Let $\mathcal{SP}_n$ denote the set of 2-Schröder paths of semilength n having neither peaks NE nor valleys EN and ending with a diagonal step $H = (1,1)$. Note that $\mathcal{SP}_n$ is the set $\mathcal{DP}^{(2)}_{n,2n}(NE, EN)$ introduced in [11]. A diagonal step that touches the line $y = 2x$ is called a *return* in a 2-Schröder path. The number of returns of a 2-Schröder path P is denoted by $\mathrm{block}(P)$.

Define a *UVD path* S *of semilength* n to be a lattice path from $(0,0)$ to $(2n, 0)$ that stays weakly above the x-axis, consists of up steps $u = (1,1)$, down steps $d = (1,-1)$, and vertical steps $v = (0,-2)$, avoids the consecutive patterns uv and vu, and ends with a d step. Note that a UVD path of semilength n can be expressed as a word $S = s_1s_2\ldots s_{2n+r}$ consisting of $n+r$ up steps, $n-r$ down steps, and r vertical steps, for some nonnegative integer r.

Let $\mathcal{UVD}_n$ denote the set of UVD paths of semilength n. By the linear transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ -2 & 1 \end{smallmatrix}\right)$, a path P in $\mathcal{SP}_n$ corresponds to a UVD path S in $\mathcal{UVD}_n$, which is denoted by $S = MP$. Therefore, we can regard M as a map from $\mathcal{SP}_n$ to $\mathcal{UVD}_n$. In fact, M is a bijection that sends N to u, E to v, and H to d.

Given a UVD path S, a down step in S that touches the x-axis is called a *return*. Let $\mathrm{vox}(S)$ denote the number of valleys du on the x-axis of S, and set $\mathrm{vox}(\emptyset) := -1$ by convention. Then the number of returns of S is clearly equal to $\mathrm{vox}(S) + 1$.

Therefore, if $S = MP$, then

$$\mathrm{vox}(S) = \mathrm{block}(P) - 1,$$

i.e., M is a bijection from $\mathcal{SP}_n$ to $\mathcal{UVD}_n$ with $\mathrm{block}(P) = \mathrm{vox}(MP) + 1$.

2.3 Labeled *F*-paths

In [3], Huh et al. introduced *F-paths* and constructed a bijection between $(102, 101)$-avoiding inversion sequences and F-paths. An *F-path of length* ℓ is a lattice path in $\mathbb{Z}^2$

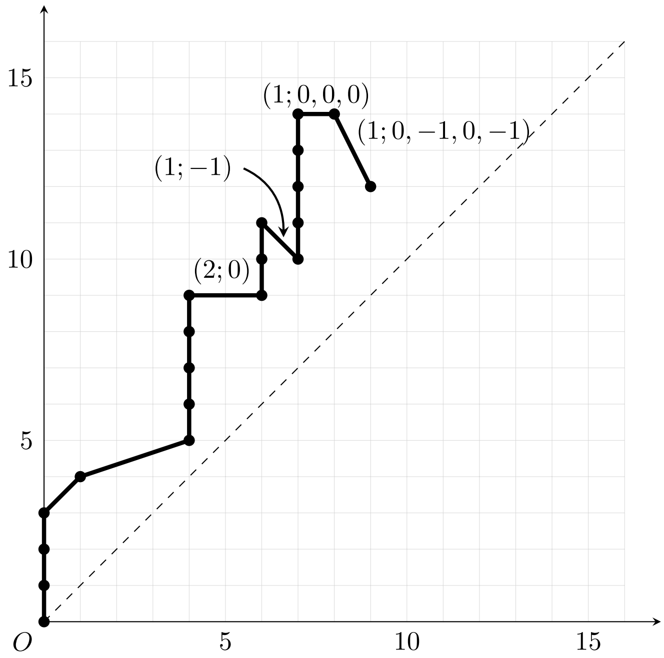


Figure 1: A labeled F-path Q of semilength 24 and height 3, with the labels $(a; 1)$ are omitted for brevity.

that starts at the origin and does not go below the line $y = x$ as a sequence of lattice points

$$(x_0, y_0), (x_1, y_1), \dots, (x_\ell, y_\ell)$$

of which every step $(x_j - x_{j-1}, y_j - y_{j-1})$, denoted by s_j, is in the set

$$F := \{(a, 1) : a \geqslant 0\} \cup \{(a, b) : a \geqslant 1, b \leqslant 0\},$$

for $j = 1, 2, \dots, \ell$.

Here, we extend the family of F-paths by assigning a label to each step, in order to accommodate the superset $\mathcal{IS}_n(102)$ of $\mathcal{IS}_n(102, 101)$. A *labeled F-path* is an F-path where every step $(a, 1)$ is assigned a label $(a; 1)$ and every other step (a, b) with $b \leqslant 0$ is assigned a label $(a; b_1, \dots, b_k)$ for some nonpositive integers $b_1, \dots, b_k$ with $k \geqslant 1$ such that $b_1 + \cdots + b_k = b$. We say that a step with a label $(a; b_1, \dots, b_k)$ has the *semilength* k. Define the semilength of a labeled F-path as the sum of the semilengths of its steps. In Figure 1, the labeled F-path Q has exactly 19 steps, but the semilength of Q is equal to 24, since the semilengths of the last two steps of Q are 3 and 4. Let $\mathcal{LF}_n$ be the set of labeled F-paths of semilength n.

Given a labeled F-path $Q = (0, 0), (x_1, y_1), \dots, (x_\ell, y_\ell)$, define the *height* of Q as the value $y_\ell - x_\ell$ of the last lattice point (x_ℓ, y_ℓ) of Q and denote it by $\mathrm{ht}(Q)$.

3 Bijections

In this section, we give bijections among the sets $\mathcal{LF}_n$, $\mathcal{IS}_{n+1}(102)$, and $\mathcal{UVD}_{n+1}$.

3.1 A bijection ϕ from $\mathcal{LF}_n$ to $\mathcal{IS}_{n+1}(102)$

We construct a map from labeled F-paths to 102-avoiding inversion sequences. Define

$$\phi : \bigcup_{n\geqslant 0} \mathcal{LF}_n \to \bigcup_{n\geqslant 0} \mathcal{IS}_{n+1}(102)$$

recursively, with the property that $\mathrm{ht}(Q) = \mathrm{rank}(\phi(Q))$ as follows:

For the initial case $n = 0$, we define $\phi((x_0, y_0)) := (0)$, ensuring that $\mathrm{ht}((x_0, y_0)) = \mathrm{rank}((0)) = 0$. For the recursive step $n \geqslant 1$, consider a labeled F-path

$$Q = (x_0, y_0), (x_1, y_1), \ldots, (x_\ell, y_\ell)$$

in $\mathcal{LF}_n$ with $\ell \geqslant 1$. Let

$$\hat{Q} := (x_0, y_0), (x_1, y_1), \ldots, (x_{\ell-1}, y_{\ell-1}) \in \mathcal{LF}_{n-k},$$

where k is the semilength of the last step of Q with $0 < k \leqslant n$. We define

$$\hat{e} := (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{n-k+1}) = \phi(\hat{Q}) \in \mathcal{IS}_{n-k+1}(102)$$

so that $\mathrm{ht}(\hat{Q}) = \mathrm{rank}(\hat{e})$ and $\hat{e}_{\hat{p}} = \max(\hat{e})$, where $\hat{p} = \mathrm{fdes}(\hat{e})$. Depending on the label of the last step (a, b) of Q, we define $\phi(Q) := e$ differently. Let $\max(\hat{e}) + a = m$.

(1) If the label of the last step of Q is $(a; 1)$ with $a \geqslant 0$, then $b = k = 1$ and $\hat{e} = (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$. We define e by inserting an entry m into the sequence $\hat{e}$ after the entry $\hat{e}_{\hat{p}}$, that is,

$$e := (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{\hat{p}}, m, \hat{e}_{\hat{p}+1}, \ldots, \hat{e}_n).$$

It is straightforward to verify that e avoids the pattern 102. Moreover, we have

$$\begin{aligned} \max(e) &= \max(\hat{e}) + a = m, \\ \mathrm{fdes}(e) &= \mathrm{fdes}(\hat{e}) + 1 = \hat{p} + 1, \\ \mathrm{rank}(e) &= \mathrm{rank}(\hat{e}) + (1 - a) = \mathrm{ht}(\hat{Q}) + (1 - a) = \mathrm{ht}(Q). \end{aligned}$$

(2) If the label of the last step (a, b) of Q is $(a; b_1, b_2, \ldots, b_k)$ with $a \geqslant 1$ and $b_1, \ldots, b_k \leqslant 0$ so that $b \leqslant 0$, we define e by inserting k instances of m into the sequence $\hat{e} = (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{n-k+1})$, after the entries $\hat{e}_{j_1}, \hat{e}_{j_2}, \ldots, \hat{e}_{j_k}$, where

$$j_1 = \hat{p} + b_1 + b_2 + \cdots + b_k - 1 = \hat{p} + b - 1$$

and

$$j_i = \hat{p} + b_i + b_{i+1} + \cdots + b_k \quad \text{for } i = 2, \ldots, k.$$

Since $j_1 < j_2 \leqslant j_3 \leqslant \ldots \leqslant j_k$ the sequence e can be formally written as

$$e := (\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{j_1}, m, \hat{e}_{j_1+1}, \ldots, \hat{e}_{j_2}, m, \hat{e}_{j_2+1}, \ldots, \hat{e}_{j_k}, m, \hat{e}_{j_k+1}, \ldots, \hat{e}_{n-k+1}).$$

If the values of some j_i's are the same, then m is inserted multiple times at the same position after the entry $\hat{e}_{j_i}$. It is straightforward to verify that $e \in \mathcal{IS}_{n+1}(102)$ with

$$\begin{aligned}
\max(e) &= \max(\hat{e}) + a = m, \\
\operatorname{fdes}(e) &= j_1 + 1 = \operatorname{fdes}(\hat{e}) + b, \\
\operatorname{rank}(e) &= \operatorname{rank}(\hat{e}) + (b - a) = \operatorname{ht}(\hat{Q}) + (b - a) = \operatorname{ht}(Q).
\end{aligned}$$

Thus, $\phi(Q)$ can be defined by e in $\mathcal{IS}_{n+1}(102)$ in both cases.

Let us apply the map ϕ to the labeled F-path Q of semilength 24 in Figure 1, in order to obtain a 102-avoiding inversion sequence $\phi(Q)$. Let $Q = (x_0, y_0), (x_1, y_1), \ldots, (x_{19}, y_{19}) \in \mathcal{LF}_{24}$, and denote $Q^{(j)} = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j)$ for $j = 0, \ldots, 19$. It is clear that $Q^{(0)} = (x_0, y_0)$ and $Q^{(19)}$ represents the entire path Q. Defining $e^{(j)} = \phi(Q^{(j)})$, we obtain the following inversion sequences for some values of j:

$$\begin{aligned}
e^{(3)} &= 000\dot{0}, & e^{(12)} &= 000014444666\dot{6}4, \\
e^{(4)} &= 0000\dot{1}, & e^{(13)} &= 0000144446\dot{7}664, \\
e^{(5)} &= 00001\dot{4}, & e^{(17)} &= 00001444467777\dot{7}664, \\
e^{(9)} &= 000014444\dot{4}, & e^{(18)} &= 00001444467777\dot{8}788664, \\
e^{(10)} &= 00001444\dot{6}4, & e^{(19)} &= 000014444677\dot{9}797998788664 = \phi(Q).
\end{aligned}$$

Here, we write $e_1 e_2 \ldots e_\ell$ to denote the sequence $(e_1, e_2, \ldots, e_\ell)$, and a dot placed above a number indicates the value of $\operatorname{fdes}(e^{(j)})$.

Theorem 1. *The map*

$$\phi : \bigcup_{n \geqslant 0} \mathcal{LF}_n \to \bigcup_{n \geqslant 0} \mathcal{IS}_{n+1}(102)$$

is a bijection with $\operatorname{ht}(Q) = \operatorname{rank}(\phi(Q))$.

Proof. Note that for a given $Q = (x_0, y_0), (x_1, y_1), \ldots, (x_\ell, y_\ell)$ in $\mathcal{LF}_n$, it is straightforward to verify that $x_\ell = \max(\phi(Q))$ and $y_\ell = \operatorname{fdes}(\phi(Q)) - 1$. To prove the bijectivity of ϕ, it suffices to show that, given a 102-avoiding inversion sequence e, one can uniquely reconstruct a labeled F-path Q such that $e = \phi(Q)$, by determining the sequence $\hat{e} = \phi(\hat{Q})$ and the last step $(a, b) = (x_\ell - x_{\ell-1}, y_\ell - y_{\ell-1}) = (\max(e) - \max(\hat{e}), \operatorname{fdes}(e) - \operatorname{fdes}(\hat{e}))$ of Q, along with its label.

For $n \geqslant 1$, let e be a 102-avoiding inversion sequence in $\mathcal{IS}_{n+1}(102)$ with $\operatorname{fdes}(e) = p$ and $\max(e) = m$. From the definition of $\operatorname{fdes}(e)$, it follows that $e_{p-1} \leqslant m$ and $e_{p+1} < m$, where we adopt the convention $e_{n+2} = -1$. Hence, we consider the following two cases:

1. If $e_{p+1} < e_{p-1} \leqslant m$, then we must have $e_j \leqslant e_{p-1} \leqslant m$ for all j with $p+2 \leqslant j \leqslant n+1$; otherwise, $e_{p-1}e_{p+1}e_j$ would form the pattern 102. Hence, the sequence

$$\hat{e} = (e_1, e_2, \ldots, e_{p-1}, e_{p+1}, \ldots, e_{n+1})$$

belongs to $\mathcal{IS}_n(102)$ with $\max(\hat{e}) = e_{p-1}$ and $\mathrm{fdes}(\hat{e}) = \mathrm{fdes}(e) - 1$. Thus, we recover $(a, b) = (a, 1)$ with label $(a; 1)$, where

$$a = \max(e) - \max(\hat{e}) \geqslant 0.$$

2. If $e_{p-1} \leqslant e_{p+1} < m$, let the indices of the entries equal to m in e be denoted by $i_1, i_2, \ldots, i_k$, satisfying

$$p = i_1 \leqslant i_2 - 2 \leqslant i_3 - 3 \leqslant \ldots \leqslant i_k - k,$$

for some $k \geqslant 1$. By removing all the entries equal to m from e, we obtain the sequence

$$\hat{e} = (e_1, e_2, \ldots, e_{i_1-1}, e_{i_1+1}, \ldots, e_{i_2-1}, e_{i_2+1}, \ldots\ldots, e_{i_k-1}, e_{i_k+1}, \ldots, e_{n+1})$$

which belongs to $\mathcal{IS}_{n-k+1}(102)$. Let $\mathrm{fdes}(\hat{e}) = \hat{p}$. Then we recover (a, b) with label $(a; b_1, b_2, \ldots, b_k)$, where $a = \max(e) - \max(\hat{e}) \geqslant 1$ and $b = b_1 + b_2 + \cdots + b_k \leqslant 0$ as follows: We consider two subcases:

   - If $k = 1$, let $b_1 = p - \hat{p}$. Since e avoids the pattern 102 and $e_{p-1} \leqslant e_{p+1} < m$, it follows that $\hat{p} \geqslant i_1 = p$ so that $b = b_1 \leqslant 0$.
   - If $k \geqslant 2$, let
   $$b_j = \begin{cases} i_1 - i_2 + 2 & \text{for } j = 1, \\ i_j - i_{j+1} + 1 & \text{for } j = 2, \ldots, k-1, \\ i_k - k - \hat{p} & \text{for } j = k. \end{cases}$$
   Since $\hat{p} \geqslant i_k - k$, we have $b_j \leqslant 0$ for all j, so $b \leqslant 0$.

   In both cases, we have

$$\mathrm{fdes}(\hat{e}) = \hat{p} = p - (b_1 + b_2 + \cdots + b_k) = \mathrm{fdes}(e) - b.$$

Thus, the labeled F-path $\phi^{-1}(e)$ can be recovered recursively by attaching the step (a, b) with its label to the end of $\phi^{-1}(\hat{e})$. Therefore, ϕ is bijective. □

3.2 A bijection ψ from $\mathcal{LF}_n$ to $\mathcal{UVD}_{n+1}$

We construct a map from labeled F-paths to UVD paths. Define

$$\psi : \bigcup_{n \geqslant 0} \mathcal{LF}_n \to \bigcup_{n \geqslant 0} \mathcal{UVD}_{n+1}$$

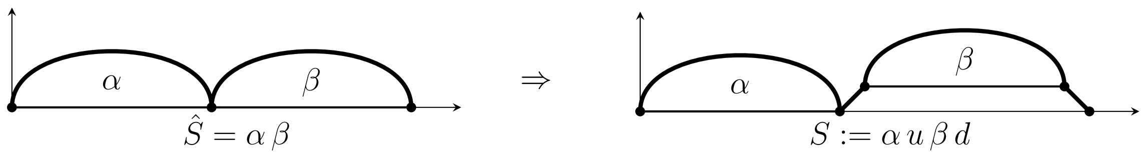


Figure 2: The construction of S from $\hat{S}$ in Case (1) of ψ.

recursively, with the property that $\mathrm{ht}(Q) = \mathrm{vox}(\psi(Q))$ as follows:

For the initial case $n = 0$, we define $\psi((x_0, y_0)) := ud$, ensuring that $\mathrm{ht}((x_0, y_0)) = \mathrm{vox}(ud) = 0$. For the recursive step $n \geqslant 1$, consider a labeled F-path

$$Q = (x_0, y_0), (x_1, y_1), \dots, (x_\ell, y_\ell)$$

in $\mathcal{LF}_n$ with $\ell \geqslant 1$. Let

$$\hat{Q} := (x_0, y_0), (x_1, y_1), \dots, (x_{\ell-1}, y_{\ell-1}) \in \mathcal{LF}_{n-k},$$

where k is the semilength of the last step of Q with $0 < k \leqslant n$. We define

$$\hat{S} := \hat{s}_1 \hat{s}_2 \dots \hat{s}_{2(n-k+1)+\hat{r}} = \psi(\hat{Q}) \in \mathcal{UVD}_{n-k+1}$$

where $\hat{r}$ is the number of vertical steps in $\hat{S}$. Thus, $\mathrm{ht}(\hat{Q}) = \mathrm{vox}(\hat{S})$, which we denote by $\hat{h}$. Recall that $\hat{S}$ has exactly $\hat{h}+1$ returns. Depending on the label of the last step (a, b) of Q, we define $\psi(Q) := S$ differently.

(1) If the label of the last step of Q is $(a; 1)$, then $k = 1$. Let p be the index of the $(\hat{h} + 1 - a)$-th return in $\hat{S} = \hat{s}_1 \hat{s}_2 \dots \hat{s}_{2n+\hat{r}}$. Note that a can be at most $\hat{h} + 1$; if $a = \hat{h} + 1$, we set $p = 0$ by convention. We decompose $\hat{S}$ as

$$\alpha = \hat{s}_1 \hat{s}_2 \dots \hat{s}_p \quad \text{and} \quad \beta = \hat{s}_{p+1} \dots \hat{s}_{2n+\hat{r}},$$

so that $\mathrm{vox}(\alpha) = \hat{h} - a$ and $\mathrm{vox}(\beta) = a - 1$, where α or β may be empty. Define

$$S := \alpha\, u\, \beta\, d = \hat{s}_1 \hat{s}_2 \dots \hat{s}_p u \hat{s}_{p+1} \dots \hat{s}_{2n+\hat{r}}\, d.$$

See Figure 2 for an illustration of $\hat{S}$ and S. Since $\hat{s}_p = d$ and $\hat{s}_{p+1} = u$, the path S contains neither uv nor vu and hence belongs to $\mathcal{UVD}_{n+1}$. Moreover, the subpath $u\beta d$ has no valleys on the x-axis, so

$$\mathrm{vox}(S) = \mathrm{vox}(\hat{S}) + (1 - a) = \mathrm{ht}(\hat{Q}) + (1 - a) = \mathrm{ht}(Q).$$

(2) If the label of the last step (a, b) of Q is $(a; b_1, b_2, \dots, b_k)$ for $a \geqslant 1$ and $b = b_1 + b_2 + \dots + b_k \leqslant 0$, then we define j_i as the index of the $(\hat{h} + 1 + b_{i+1} + b_{i+2} + \dots + b_k - a)$-th return for $i = 0, \dots, k$. (In particular, j_k is the index of the $(\hat{h} + 1 - a)$-th return of $\hat{S}$.) Let p be the index satisfying that

$$\hat{s}_p \neq v, \quad \hat{s}_{p+1} = \dots = \hat{s}_{j_0 - 1} = v, \quad \text{and} \quad \hat{s}_{j_0} = d.$$

Since

$$1 \leqslant p < j_0 \leqslant j_1 \leqslant \ldots \leqslant j_k < 2(n-k+1) + \hat{r},$$

the path $\hat{S}$ can be decomposed into $k+3$ paths based on the indices p, j_0, …, j_k as

$$\begin{aligned}
\alpha &= \hat{s}_1 \hat{s}_2 \ldots \hat{s}_p, \\
\beta &= \hat{s}_{p+1} \ldots \hat{s}_{j_0} (= v \ldots vd), \\
\sigma_i &= \hat{s}_{j_{i-1}+1} \ldots \hat{s}_{j_i} \quad \text{for } i = 1, \ldots, k, \\
\tau &= \hat{s}_{j_k+1} \ldots \hat{s}_{2(n-k+1)+\hat{r}},
\end{aligned}$$

with $\mathrm{vox}(\alpha\beta) = \hat{h} + (b-a)$, $\mathrm{vox}(\sigma_j) = -b_j - 1$ for $j = 1, \ldots, k$ and $\mathrm{vox}(\tau) = a - 1$. Note that α, β, and τ are nonempty and σ_i is empty if $j_{i-1} = j_i$. From $\hat{S}$, we construct a UVD path S by inserting $u\sigma_i u$'s in order after α, followed by τ, k instances of v, and β:

$$\begin{aligned}
S :=& \alpha\ u\sigma_1 u\ u\sigma_2 u \ldots u\sigma_k u\ \tau\ v^k\ \beta \\
=& \hat{s}_1 \hat{s}_2 \ldots \hat{s}_p\, u\hat{s}_{j_0+1} \ldots \hat{s}_{j_1} u\, u\hat{s}_{j_1+1} \ldots \hat{s}_{j_2} u\ \ldots \\
& \ldots u\hat{s}_{j_{k-1}+1} \ldots \hat{s}_{j_k} u\, \hat{s}_{j_k+1} \ldots \hat{s}_{2(n-k+1)+\hat{r}}\, v \ldots v\, \hat{s}_{p+1} \ldots \hat{s}_{j_0}.
\end{aligned}$$

See Figure 3 for $\hat{S}$ and S. It is easy to check that the last lattice point of S is $(2n+2, 0)$ and S has $(\hat{r} + k)$ vertical steps. Letting $r := \hat{r} + k$, the number of all steps of S is $2(n+1) + r$. It is obvious that S has neither uv nor vu, and S is in $\mathcal{UVD}_{n+1}$ with

$$\mathrm{vox}(S) = \mathrm{vox}(\hat{S}) + (b_1 + b_2 + \cdots + b_k - a) = \mathrm{ht}(\hat{Q}) + (b - a) = \mathrm{ht}(Q).$$

Thus, $\psi(Q)$ can be defined by S in $\mathcal{UVD}_{n+1}$ in both cases.

Continuing with the labeled F-path Q from Figure 1 and the labeled F-paths $Q^{(0)}$, …, $Q^{(19)}$ introduced in Subsection 3.1, we now apply the map ψ recursively to obtain the corresponding UVD path $\psi(Q)$. Defining $S^{(j)} = \psi(Q^{(j)})$, in the process of finding $\psi(Q)$, we obtain some $S^{(j)}$'s as follows.

$$\begin{aligned}
S^{(3)} &= ud\ ud\ ud\ ud, \\
S^{(4)} &= ud\ ud\ ud\ uudd \\
S^{(5)} &= ud\ uududuuddd, \\
S^{(9)} &= ud\ uududuuddd\ ud\ ud\ ud\ ud, \\
S^{(10)} &= ud\ uududuuddd\ ud\ uuuududvd, \\
S^{(12)} &= ud\ uududuuddd\ ud\ uuuududvd\ ud\ ud, \\
S^{(13)} &= ud\ uududuuddd\ ud\ uuuududuuduudvvd, \\
S^{(17)} &= ud\ uududuuddd\ ud\ uuuududuuduudvvd\ ud\ ud\ ud\ ud, \\
S^{(18)} &= ud\ uududuuddd\ ud\ uuuududuuduudvvd\ ud\ ud\ uuuuuuuudvvvd, \\
S^{(19)} &= ud\ uududuuddd\ ud\ uuuududuuduuduuuduuuuduuuuuuuudvvvdvvvvvvd.
\end{aligned}$$

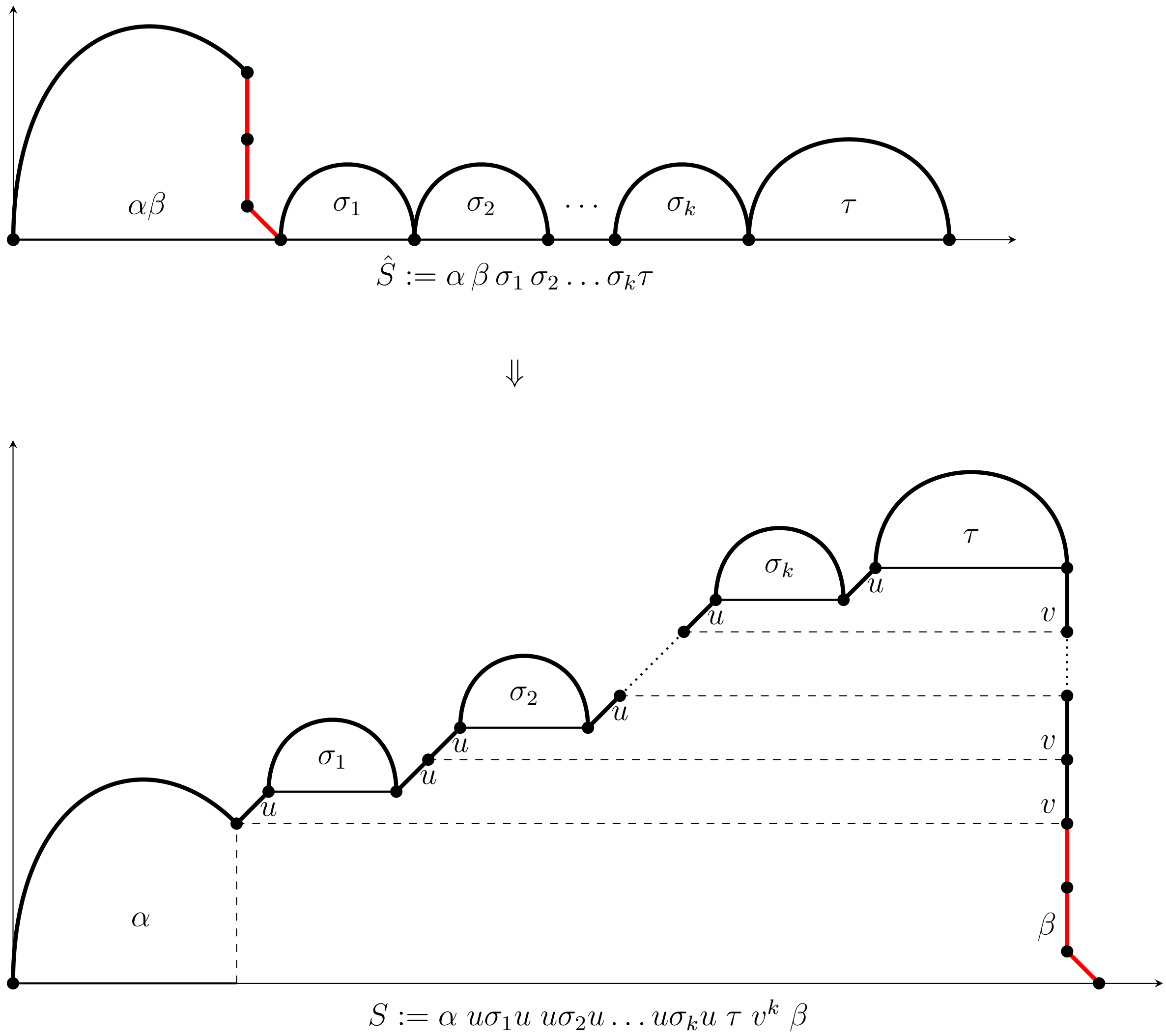


Figure 3: The construction of S from $\hat{S}$ in Case (2) of ψ. The red steps indicate the steps in β.

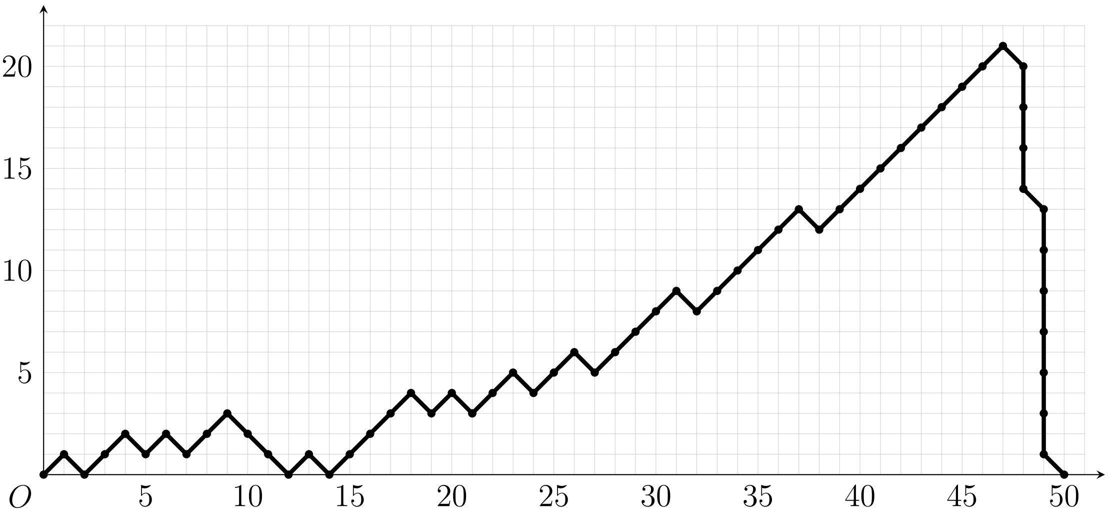


Figure 4: The UVD path $\psi(Q)$ corresponding to Q in Figure 1.

Here, a valley du on the x-axis is indicated by a small gap between d and u so that valleys can be visually distinguished. See Figure 4 for the UVD path $\psi(Q)$.

Theorem 2. *The map*

$$\psi : \bigcup_{n\geqslant 0} \mathcal{LF}_n \to \bigcup_{n\geqslant 0} \mathcal{UVD}_{n+1}$$

is a bijection with $\mathrm{ht}(Q) = \mathrm{vox}(\psi(Q))$.

Proof. As in the case of 2-Schröder paths, we denote the number of returns of a UVD path S by $\mathrm{block}(S)$. For a given $Q = (x_0, y_0), (x_1, y_1), \ldots, (x_\ell, y_\ell)$ in $\mathcal{LF}_n$, one can easily verify that

$$\mathrm{block}(\psi(Q)) = \mathrm{vox}(\psi(Q)) + 1 = \mathrm{ht}(Q) + 1.$$

To prove the bijectivity of ψ, it suffices to show that, given a UVD path S, one can uniquely reconstruct a labeled F-path Q such that $S = \psi(Q)$, by determining the labeled F-path $\hat{S} = \psi(\hat{Q})$ and the last step (a, b) of Q, along with its label.

For $n \geqslant 1$, let S be a UVD path in $\mathcal{UVD}_{n+1}$. There are two cases to consider depending on the second-to-last step of S.

(1) If the second-to-last step of S is not vertical, then S can be decomposed as $\alpha\, u\, \beta\, d$, where both α and β are (possibly empty) UVD paths. In this case, we recover $\hat{S} = \alpha\beta$, which is in $\mathcal{UVD}_n$ with

$$\mathrm{block}(\hat{S}) = \mathrm{block}(\alpha) + \mathrm{block}(\beta) = \mathrm{block}(S) - 1 + \mathrm{block}(\beta).$$

Thus, we recover $(a, b) = (a, 1)$ with label $(a; 1)$, where $a = \mathrm{block}(\beta)$.

(2) If the second-to-last step of S is vertical, then there exists a unique positive integer k such that S can be expressed as

$$S = \alpha\ u\sigma_1 u\ u\sigma_2 u \dots u\sigma_k u\ \tau\ v^k\ \beta,$$

where $\alpha\beta$, σ_1, $\dots$, σ_k, and τ are UVD paths, β is of the form $v \dots vd$, and σ_i may be empty. To see why this decomposition is always possible and unique, let j be the number of consecutive vertical steps immediately preceding the final d step. We examine these v steps one by one, starting from the leftmost (or topmost as shown in Figure 3) one. For each v step, there is a corresponding subpath to its left at the same height level. We define k $(0 \leqslant k \leqslant j)$ to be the maximum number of such v steps for which the corresponding subpath can be uniquely written in the form $u\sigma_i u$, where σ_i is a UVD path. This process uniquely determines k and the subpaths $\sigma_1, \dots, \sigma_k$. The remaining parts of the path, after identifying these components, uniquely determine α, τ, and β. Hence, the decomposition is well-defined and unique.

In this case, we recover
$$\hat{S} = \alpha\,\beta\,\sigma_1\,\sigma_2 \dots \sigma_k \tau,$$
which is in $\mathcal{UVD}_{n-k+1}$ with

$$\begin{aligned}\operatorname{block}(\hat{S}) &= \operatorname{block}(\alpha\beta) + \operatorname{block}(\sigma_1) + \cdots + \operatorname{block}(\sigma_k) + \operatorname{block}(\tau) \\ &= \operatorname{block}(S) - (b_1 + \cdots + b_k - a),\end{aligned}$$

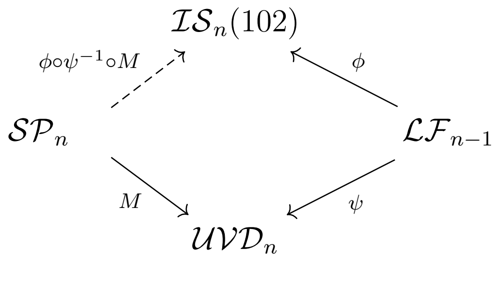
where $a = \operatorname{block}(\tau) \geqslant 1$ and $b_j = -\operatorname{block}(\sigma_j) \leqslant 0$ for $j = 1, \dots, k$. Thus, we recover (a, b) with label $(a; b_1, \dots, b_k)$.

Therefore, the labeled F-path $\psi^{-1}(S)$ can be recovered recursively by attaching the step (a, b) with its label to the end of $\psi^{-1}(\hat{S})$. Thus, ψ is bijective. $\square$

Combining the three maps M, ϕ, and ψ, for a positive integer n, the map

$$\phi \circ \psi^{-1} \circ M : \mathcal{SP}_n \to \mathcal{IS}_n(102)$$

is a bijection from the set of 2-Schröder paths having neither peaks NE nor valleys EN ending with a diagonal step D to the set of 102-avoiding inversion sequences. In particular, $\operatorname{block}(P) = \operatorname{rank}(\phi \circ \psi^{-1}(P))$ for a path $P \in \mathcal{SP}_n$. The following diagram visually represents the connections and mappings between the sets $\mathcal{SP}_n$, $\mathcal{LF}_{n-1}$, $\mathcal{UVD}_n$, and $\mathcal{IS}_n(102)$, along with the labels for each map.



4 Refined enumeration of 102-avoiding inversion sequences

In this section, we provide the number of 102-avoiding inversion sequences with a fixed rank using the bijection $\phi\psi^{-1}$ and the Lagrange inversion formula. For nonnegative integers n and t, let $\mathcal{IS}_{n,t}(102)$ denote the set of $e \in \mathcal{IS}_n(102)$ with $\mathrm{rank}(e) = t$ and $\mathcal{UVD}_{n,t}$ denote the set of $S \in \mathcal{UVD}_n$ with $\mathrm{vox}(S) = t$. Let

$$D = D(x) := \sum_{n\geqslant 1} |\mathcal{UVD}_n|\, x^n, \qquad D_t = D_t(x) := \sum_{n\geqslant 1} |\mathcal{UVD}_{n,t}|\, x^n.$$

By decomposition of UVD paths in $\cup_{n\geqslant 1}\mathcal{UVD}_n$, we have

$$D_t = D_0^{t+1}, \qquad D = \sum_{t\geqslant 0} D_t = \frac{D_0}{1-D_0}.$$

Recall that for a UVD path S, S has the semilength n if and only if the sum of the numbers of d and v in S equals to n. Given a UVD path S, S ends with $v^k d$ for some nonnegative integer k. If S ends with $v^0 d$, then S can be written as $S = S_1\, u\, S_2\, d$, where S_1 and S_2 are UVD paths. If S ends with $v^k d$ for some positive integer k, then S can be written as follows:

$$S = S_1\, u\, S_2\, u\, \ldots S_{2k+1}\, u\, S_{2k+2}\, v^k\, d,$$

where each S_i is a UVD path, and in addition, S_{2k+2} is nonempty. In the former case, the path S contributes to $(1+D)(1+D)x = x(1+D)^2$, whereas in the latter cases, the path S contributes to

$$(1+D)\cdots(1+D)\cdot D\cdot x^k\cdot x = x^{k+1}(1+D)^{2k+1}D.$$

So we get

$$\begin{aligned} D &= x(1+D)^2 + \sum_{k\geqslant 1} x^{k+1}(1+D)^{2k+1}D \\ &= x(1+D)^2 + \frac{x^2(1+D)^3 D}{1-x(1+D)^2}. \end{aligned}$$

Multiplying both sides by $1 - x(1+D)^2$, we obtain

$$D = (x - x^2)(1+D)^3.$$

Since $D = \frac{D_0}{1-D_0}$, we get

$$D_0 = (x-x^2)\frac{1}{(1-D_0)^2}$$

and letting $y = x - x^2$, we have

$$E(y) = y\frac{1}{(1-E(y))^2}, \tag{1}$$

where the power series $E(y)$ satisfies $E(x-x^2) = D_0(x)$. By applying the Lagrange inversion formula [2, eq. (2.2.1)] to (1), we obtain

$$[y^n]\, E(y)^{t+1} = \frac{t+1}{n} \left[z^{n-t-1}\right] \left(\frac{1}{(1-z)^2}\right)^n = \frac{t+1}{n}\binom{3n-t-2}{n-t-1}.$$

Thus we have

$$D_0(x)^{t+1} = \sum_{n \geqslant t+1} \frac{t+1}{n}\binom{3n-t-2}{n-t-1}\left(x-x^2\right)^n. \tag{2}$$

Since $D_t = D_0^{t+1}$, we obtain the following theorem from Theorems 1 and 2 and (2).

Theorem 3. *Let n and t be integers such that $n \geqslant 1$ and $0 \leqslant t \leqslant n-1$. Then we have*

$$|\mathcal{IS}_{n,t}(102)| = |\mathcal{UVD}_{n,t}| = \sum_{j=t+1}^{n} (-1)^{n-j} \frac{t+1}{j}\binom{3j-t-2}{j-t-1}\binom{j}{n-j}.$$

5 $(102, \tau)$-avoiding inversion sequences with a fixed rank

The enumeration of inversion sequences avoiding the pattern 102 and another pattern τ of length 3 has been previously studied by Corteel et al. [1], Huh et al. [3], Kotsireas et al. [7], Testart [12], and Yan and Lin [13]. Among these, the $(102, 101)$-avoiding case has been particularly well-studied in [3], where several statistics on $\mathcal{IS}_n(102, 101)$ were investigated and explicitly enumerated. In particular, our rank statistic appears naturally in that context as one of the statistics considered. One of their results can be restated as follows (compare with $a_n(*, *, m)$ in Corollary 6.3 of [3]).

Proposition 4 ([3])**.** *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of $(102, 101)$-avoiding inversion sequences e of length n with $\operatorname{rank}(e) = t$ is given by*

$$\frac{t+1}{n}\sum_{i=1}^{n-t-1}\binom{n}{i}\binom{n-t+i-2}{2i-1}.$$

Motivated by this, we extend the study to the enumeration of inversion sequences of length n that avoid both patterns 102 and $\tau \in \{001, 011, 012, 021, 110, 120, 201, 210\}$, and have fixed rank t. Here, we exclude the patterns 000, 010, and 100 from the set of candidates for τ, as these cases do not easily admit a generalizable description in terms of the rank statistic.

We denote the set of such sequences by

$$\mathcal{IS}_{n,t}(102, \tau) := \{e \in \mathcal{IS}_n(102, \tau) : \operatorname{rank}(e) = t\}.$$

Note that $\mathcal{IS}_{n,n-1}(102, \tau)$ is the singleton set $\{(0, 0, \ldots, 0)\}$ or the empty set, depending on the choice of τ and n. Therefore, we restrict our attention to the range $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$.

5.1 (102, 001)-avoiding inversion sequences

It was shown in [1] that $e \in \mathcal{IS}_n(001)$ if and only if e is of the form

$$e_1 < e_2 < \cdots < e_k \geqslant e_{k+1} \geqslant \cdots \geqslant e_n \tag{3}$$

for some k, and furthermore, that every 001-avoiding inversion sequence also avoids the pattern 102. Therefore, we conclude that $e \in \mathcal{IS}_n(102, 001)$ if and only if e is of the form (3).

Proposition 5. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of $(102, 001)$-avoiding inversion sequences e of length n with $\mathrm{rank}(e) = t$ is given by 2^{n-t-2}.*

Proof. Consider inversion sequences $e = (e_1, e_2, \ldots, e_n)$. By (3), we see that $e \in \mathcal{IS}_{n,t}(102, 001)$ if and only if e is of the form

$$e_i = i - 1 \text{ for } 1 \leqslant i \leqslant m, \qquad m = e_{m+1} = \cdots = e_{m+t+1} > e_{m+t+2} \geqslant \cdots \geqslant e_n$$

for some m. The number of such sequences is given by

$$\sum_{m=1}^{n-t-1} \binom{n-t-2}{m-1} = 2^{n-t-2}.$$

□

5.2 (102, 011)-avoiding inversion sequences

We begin with the following lemma, which provides a finer understanding of the structure of $(102, 011)$-avoiding inversion sequences.

Lemma 6. *For positive integers n and t with $t \leqslant n - 1$, we have*

$$|\mathcal{IS}_{n+1,t}(102, 011)| = |\mathcal{IS}_{n,t-1}(102, 011)|.$$

Proof. We prove the lemma by exhibiting a bijection.

Let $e = (e_1, e_2, \ldots, e_{n+1}) \in \mathcal{IS}_{n+1,t}(102, 011)$ with $t > 0$. Define a map $\varphi(e) := (e_2, \ldots, e_{n+1})$. To show that $\varphi(e) \in \mathcal{IS}_{n,t-1}(102, 011)$, observe that if $e_i = i - 1$ for some $i \geqslant 2$, then by the structure of $(102, 011)$-avoiding sequences, we must have $e_j = j - 1$ for all $j \geqslant i$ up to the maximum value, which forces $\mathrm{rank}(e) = 0$, contradicting the assumption $t > 0$. Hence, $e_i < i - 1$ for all $i \geqslant 2$, and the tail sequence lies in $\mathcal{IS}_n(102, 011)$ with rank $t - 1$. The map φ is clearly bijective, completing the proof. □

Yan and Lin [13, Theorem 3.1] showed that the total number of $(102, 011)$-avoiding inversion sequences of length n is given by

$$|\mathcal{IS}_n(102, 011)| = F_{2n-1}, \tag{4}$$

where $\{F_n\}$ is the Fibonacci sequence, defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with initial values $F_0 = 0$ and $F_1 = 1$.

As a direct consequence of Lemma 6, we now refine this result by enumerating the number of such sequences according to their rank.

Proposition 7. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of* $(102, 011)$*-avoiding inversion sequences e of length n with* $\operatorname{rank}(e) = t$ *is given by $F_{2n-2t-2}$, where F_k denotes the k-th Fibonacci number.*

Proof. We proceed by induction on $n \geqslant 2$. For the base case $n = 2$ and $t = 0$, we have

$$|\mathcal{IS}_{2,0}(102, 011)| = |\{01\}| = 1 = F_2.$$

Now assume that the proposition holds for some $n \geqslant 2$ and all $0 \leqslant t \leqslant n-2$. By Lemma 6 and the induction hypothesis, for $1 \leqslant t \leqslant n-1$, we have

$$|\mathcal{IS}_{n+1,t}(102, 011)| = |\mathcal{IS}_{n,t-1}(102, 011)| = F_{2n-2(t-1)-2} = F_{2(n+1)-2t-2}.$$

Moreover, by (4), it follows that

$$\sum_{t=1}^{n} |\mathcal{IS}_{n+1,t}(102, 011)| = \sum_{t=1}^{n} |\mathcal{IS}_{n,t-1}(102, 011)| = F_{2n-1},$$

and

$$|\mathcal{IS}_{n+1,0}(102, 011)| = F_{2n+1} - F_{2n-1} = F_{2n} = F_{2(n+1)-2}.$$

Therefore, the proposition holds for $n + 1$, completing the induction step. □

5.3 (102, 012)-avoiding inversion sequences

In this subsection, we are effectively studying 012-avoiding inversion sequences, as avoidance of the pattern 012 automatically implies avoidance of the pattern 102. We nevertheless keep the notation $(102, 012)$-avoiding to remain consistent with the general framework of the paper, which focuses on inversion sequences avoiding 102 together with an additional pattern.

The Fibonacci numbers F_{n+1} are well known to count the number of tilings of a $1 \times n$ board using unit squares S and dominoes D, where each domino covers two adjacent cells.

In light of the identity

$$|\mathcal{IS}_n(102, 012)| = F_{2n-1}, \tag{5}$$

established by Yan and Lin [13, Theorem 3.1], we construct a bijection between $(102, 012)$-avoiding inversion sequences of length n and domino tilings of a board of length $2n - 2$. This bijection allows a natural refinement by the rank statistic as follows.

Proposition 8. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of* $(102, 012)$*-avoiding inversion sequences e of length n with* $\operatorname{rank}(e) = t$ *is given by $(t+1)F_{2n-2t-3}$, where F_k denotes the k-th Fibonacci number.*

Proof. It is straightforward to verify that $e = (e_1, e_2, \ldots, e_n) \in \mathcal{IS}_n(102, 012)$ if and only if e is of the form

$$m = e_{i_1} \geqslant e_{i_2} \geqslant \cdots \geqslant e_{i_\ell} \geqslant 1 \quad \text{for some } m < i_1 < i_2 < \cdots < i_\ell,$$

with $e_j = 0$ for all other indices. Here, $m = \max(e)$.

We define a map $\varphi : \mathcal{IS}_n(102, 012) \to \mathcal{T}_{2n-2}$, where $\mathcal{T}_{2n-2}$ denotes the set of tilings of a $1 \times (2n-2)$ board with squares and dominoes.

We set $\varphi((0, 0, \ldots, 0)) := D^{n-1}$, which corresponds to a tiling composed entirely of dominoes.

For an inversion sequence e with $\max(e) = m > 0$, let $e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}$ be the nonzero entries of e, forming a weakly decreasing sequence. For convenience, we set $b_j := e_{i_j} - e_{i_{j+1}}$ for $1 \leqslant j \leqslant \ell - 1$, and $b_\ell := e_{i_\ell} - 1$.

Now define

$$\varphi(e) := \xi(e_{m+1})\,\xi(e_{m+2})\,\ldots\,\xi(e_n),$$

where

$$\xi(e_k) := \begin{cases} SD^{b_j}S & k = i_j \text{ for some } j, \\ D & \text{otherwise.} \end{cases}$$

For example, when $n = 3$, the following correspondences hold:

$$000 \mapsto DD, \quad 001 \mapsto DSS, \quad 011 \mapsto SSSS, \quad 010 \mapsto SSD, \quad 002 \mapsto SDS.$$

In general, the resulting tiling $\varphi(e)$ consists of 2ℓ squares and $n - \ell - 1$ dominoes, totaling $2n - 2$ units in length, as desired. Hence, it is easy to recover ℓ, m, and the original sequence from the tiling, confirming that the map is invertible.

Next, we examine how the rank of e corresponds to the structure of the tiling $\varphi(e)$. Let $\mathrm{rank}(e) = t$.

We first treat the case $t = n - 2$. In this case, the inversion sequences are precisely those of the form $(0^{n-i}, 1^i)$ for some $1 \leqslant i \leqslant n - 1$. Thus, they correspond to tilings of the form $D^{n-i-1}S^{2i}$, and the total number of such tilings is $n - 1$. This agrees with the value of $(t + 1)F_{2n-2t-3}$ when $t = n - 2$, since $F_1 = 1$.

Now assume that $0 \leqslant t < n-2$, and let $m = \max(e) \geqslant 1$. Then the entries in positions from $m + 1$ to $m + t + 1$ $(= \mathrm{fdes}(e))$ must be of the form

$$(e_{m+1}, e_{m+2}, \ldots, e_{m+t+1}) = (0^i, m^{t-i+1}),$$

for some $0 \leqslant i \leqslant t$. Under the bijection φ, this corresponds to a tiling whose initial segment is of the form

$$\begin{cases} D^iS^{2t-2i+1}D & \text{if } b_{t-i+1} > 0, \\ D^iS^{2t-2i+2}D & \text{if } b_{t-i+1} = 0, \end{cases}$$

since $b_1 = \cdots = b_{t-i} = 0$ and $t < n - 2$, in which case the condition $b_{t-i+1} = 0$ implies that the index $m + t + 2$ necessarily exists and satisfies $e_{m+t+2} = 0$. These two types of initial segments account for all possible configurations of inversion sequences with rank $t < n - 2$, and their contributions sum to

$$(t + 1)(F_{2n-2t-5} + F_{2n-2t-4}) = (t + 1)F_{2n-2t-3},$$

as desired. Here, the factor $t + 1$ corresponds to the number of possible choices for i, and $F_{2n-2t-5}$ (respectively $F_{2n-2t-4}$) counts the number of tilings of a $1 \times (2n - 2t - 6)$ board (respectively $1 \times (2n - 2t - 5)$ board). The proof is complete. □

5.4 (102, 021)-avoiding inversion sequences

A *Dyck path of semilength* n is a UVD path of semilength n that consists only of up steps u and down steps d, with no vertical steps. The following is a well-known result on Dyck paths.

Lemma 9. *Let k and n be positive integers with $k \leqslant n$. The number of Dyck paths of semilength n ending with the subpath ud^k is equal to the number of Dyck paths P of semilength n with k returns. The number is given by*

$$\frac{k}{n}\binom{2n-k-1}{n-1}.$$

The sequence in Lemma 9 appears in OEIS [10] as entry A33184. We now apply Lemma 9 to enumerate $(102, 021)$-avoiding inversion sequences according to their rank.

Proposition 10. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of $(102, 021)$-avoiding inversion sequences e of length n with $\operatorname{rank}(e) = t$ is given by*

$$(t+1)\left(2^{n-t-2} - (n-t-1) + \sum_{m=1}^{n-t-1} \frac{1}{m+t+1}\binom{2m+t}{m}\right).$$

Proof. We begin by considering inversion sequences $e = (e_1, e_2, \ldots, e_n)$ with $\max(e) = m > 0$.

It is straightforward to verify that $e \in \mathcal{IS}_{n,t}(102, 021)$ if and only if it is of one of the following two forms:

$$e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{m+t+1} = m, \quad e_{m+t+2} = e_{m+t+3} = \cdots = e_n = 0, \tag{6}$$

$$\begin{aligned} &e_1 = \cdots = e_{m+k} = 0, \quad e_{m+k+1} = \cdots = e_{m+t+1} = m, \quad e_{m+t+2} = 0, \\ &\quad e_{m+t+3}, \ldots, e_n \in \{0, m\} \text{ with at least one entry equal to } m, \end{aligned} \tag{7}$$

for some $0 \leqslant k \leqslant t$.

In the first case (6), such weakly increasing inversion sequences $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{m+t+1} = m$ are in bijection with Dyck paths of semilength $m+t+1$ via the standard encoding in which e_i counts the number of down steps preceding the i-th up step. The condition $e_{m+t+1} = m$ is then equivalent to the Dyck path ending with the subpath ud^{t+1}. Such Dyck paths are counted by

$$\frac{t+1}{m+t+1}\binom{2m+t}{m}$$

by Lemma 9.

For the second case (7), the sequences are counted by $(t+1)(2^{n-t-m-2} - 1)$. Here, the factor $t+1$ counts the number of possible choices for k, and $2^{n-t-m-2} - 1$ counts the number of possible assignments for $e_{m+t+3}, \ldots, e_n$, each of which can be either 0 or

m excluding the case with $e_{m+t+2} = \cdots = e_n = 0$. Hence, the total number of inversion sequences $e \in \mathcal{IS}_{n,t}(102, 021)$ is given by

$$(t+1)\left(\sum_{m=1}^{n-t-1} \frac{1}{m+t+1}\binom{2m+t}{m} + \sum_{m=1}^{n-t-2} (2^{n-t-m-2}-1)\right),$$

which completes the proof. □

5.5 (102, 120)-avoiding inversion sequences

Yan and Lin [13, Theorem 4.10] showed that the total number of $(102, 120)$-avoiding inversion sequences of length n is given by

$$|\mathcal{IS}_n(102, 120)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}. \tag{8}$$

We first establish a lemma that characterizes a subclass of $(102, 120)$-avoiding inversion sequences.

Lemma 11. *Let $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, and define*

$$A_{n,t} := \{e \in \mathcal{IS}_{n,t}(102, 120) : e_{\max(e)+t} < \max(e)\}.$$

Then the following statements hold:

1. *For integers n and t with $n \geqslant 2$ and $1 \leqslant t \leqslant n-1$,*

$$|A_{n+1,t}| = \sum_{i=t-1}^{n-2} |A_{n,i}|.$$

2. *For integers n and t with $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, $|A_{n,t}| = \binom{2n-t-3}{n-1}$.*

Proof. 1. Let $e = (e_1, e_2, \ldots, e_{n+1}) \in A_{n+1,t}$ with $\max(e) = m > 0$. Since e avoids the pattern 120 and $e_{m+t} < e_{m+t+1} = m$, the sequence satisfies $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{m+t}$ and $e_{m+t} \leqslant e_j$ for $m+t+2 \leqslant j \leqslant n+1$.

We define a map

$$\rho : A_{n+1,t} \to \bigcup_{a=0}^{n-t-1} A_{n,t+a-1}$$

as $\rho(e) := (e_1, \ldots, e_{m+t-1}, e_{m+t+1} - a, e_{m+t+2} - a, \ldots, e_{n+1} - a)$, where $a = e_{m+t} - e_{m+t-1} \geqslant 0$. Since $e_{m+t+1} - a = e_{m+t+1} - e_{m+t} + e_{m+t-1} > e_{m+t-1}$, it follows that $\max(\rho(e)) = e_{m+t+1} - a = m - a$ and $\mathrm{rank}(\rho(e)) = t + a - 1$. Moreover, $\rho(e)$ preserves the weakly increasing property up to index $m+t-1$, and the suffix beginning at $e_{m+t+1} - a$ avoids the patterns 102 and 120, since this property is inherited from e. Thus, $\rho(e) \in A_{n,t+a-1}$ for $0 \leqslant a \leqslant n-t-1$.

Conversely, given $1 \leqslant t \leqslant n-2$ and any $\rho(e) = (f_1, f_2, \ldots, f_n) \in A_{n,t+a-1}$, we can recover the original sequence $e \in A_{n+1,t}$ as

$$(f_1, f_2, \ldots, f_{k-2}, f_{k-2}+a, f_{k-1}+a, \ldots, f_n+a),$$

where $k = \max(\rho(e)) + \operatorname{rank}(\rho(e)) + 2$. This shows that ρ is a bijection.

2. We prove that $|A_{n,t}| = \binom{2n-t-3}{n-1}$ for all $0 \leqslant t \leqslant n-2$ by induction on $n \geqslant 2$. The base case holds since $|A_{2,0}| = |\{(0,1)\}| = \binom{1}{1}$. Now assume the statement holds for some $n \geqslant 2$; we aim to prove it for $n+1$. By part (1) and the induction hypothesis, for $1 \leqslant t \leqslant n-1$, we have

$$|A_{n+1,t}| = \sum_{i=t-1}^{n-2} |A_{n,i}| = \sum_{i=t-1}^{n-2} \binom{2n-i-3}{n-1} = \binom{2n-t-1}{n} = \binom{2(n+1)-t-3}{(n+1)-1}. \tag{9}$$

It remains to compute $|A_{n+1,0}|$. To this end, observe that any sequence $e \in \mathcal{IS}_{n+1}(102, 120)$ satisfying $e_{\max(e)+\operatorname{rank}(e)} = \max(e)$ corresponds bijectively to a sequence in $\mathcal{IS}_n(102, 120)$ by simply removing the entry $e_{\max(e)+\operatorname{rank}(e)}$, and vice versa. Thus,

$$|\{e \in \mathcal{IS}_{n+1}(102, 120) : e_{\max(e)+\operatorname{rank}(e)} = \max(e)\}| = |\mathcal{IS}_n(102, 120)|.$$

Hence, the number of sequences in $\mathcal{IS}_{n+1}(102, 120)$ with $e_{\max(e)+\operatorname{rank}(e)} < \max(e)$ is given by

$$\begin{aligned} |\mathcal{IS}_{n+1}(102, 120)| - |\mathcal{IS}_n(102, 120)| &= \left(1 + \sum_{i=1}^{n} \binom{2i}{i-1}\right) - \left(1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}\right) \\ &= \binom{2n}{n-1} \end{aligned}$$

by (8). Therefore, by (9),

$$|A_{n+1,0}| = \binom{2n}{n-1} - \sum_{t=1}^{n-1} |A_{n+1,t}| = \binom{2n}{n-1} - \sum_{t=1}^{n-1} \binom{2n-t-1}{n} = \binom{2(n+1)-3}{(n+1)-1},$$

which completes the inductive step.

□

Proposition 12. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of $(102, 120)$-avoiding inversion sequences e of length n with $\operatorname{rank}(e) = t$ is given by*

$$\binom{2n-t-2}{n-t-1} - \binom{2n-2t-3}{n-t-1}.$$

Proof. Since

$$\sum_{i=0}^{t} \binom{2n-t-i-3}{n-t-2} = \binom{2n-t-2}{n-t-1} - \binom{2n-2t-3}{n-t-1}$$

it suffices to show that, for each i, the inversion sequences

$$e = (e_1, e_2, \ldots, e_n) \in \mathcal{IS}_{n,t}(102, 120)$$

with $t \leqslant n-2$ satisfying

$$e_1, e_2, \ldots, e_{\max(e)+t-i} < e_{\max(e)+t-i+1} = \cdots = e_{\max(e)+t+1} = \max(e)$$

is counted by $\binom{2n-t-i-3}{n-t-2}$. Clearly, these sequences are in bijection with the $(102, 120)$-avoiding inversion sequences in $A_{n-i,t-i}$ by removing i consecutive $\max(e)$ elements. By Lemma 11 (2), we obtain

$$|A_{n-i,t-i}| = \binom{2(n-i)-(t-i)-3}{(n-i)-1} = \binom{2n-t-i-3}{n-t-2}.$$

This completes the proof. □

5.6 (102, 201)-avoiding inversion sequences

We begin by enumerating $(102, 201, 101)$-avoiding inversion sequences e with $\text{rank}(e) = t$ and $\max(e) = m$.

Proposition 13. *Let m and n be positive integers. For $0 \leqslant t \leqslant n-2$ and $1 \leqslant m \leqslant n-t-1$, the number of inversion sequences $e \in \mathcal{IS}_n(102, 201, 101)$ with $\text{rank}(e) = t$ and $\max(e) = m$ is given by*

$$\frac{t+1}{m+t+1}\binom{2m+t}{m}\binom{n-t-2}{m-1}.$$

Proof. Consider $(102, 201, 101)$-avoiding inversion sequences $e = (e_1, \ldots, e_n)$. Note that e contains no triple $1 \leqslant i < j < k \leqslant n$ such that $e_i > e_j < e_k$. Thus, for $0 \leqslant t \leqslant n-2$ and $1 \leqslant m \leqslant n-t-1$, $e \in \mathcal{IS}_n(102, 201, 101)$ with $\text{rank}(e) = t$ and $\max(e) = m$ if and only if e is of the form

$$e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{m+t+1} = m > e_{m+t+2} \geqslant \cdots \geqslant e_n. \tag{10}$$

As shown in the proof of Proposition 5.7, the possible subsequences $(e_1, \ldots, e_{m+t+1})$ are in bijection with Dyck paths of semilength $m+t+1$ ending with ud^{t+1}, there are

$$\frac{t+1}{m+t+1}\binom{2m+t}{m}$$

such subsequences by Lemma 9. The proof follows since there are $\binom{n-t-2}{m-1}$ ways to make the subsequence $(e_{m+t+2}, \ldots, e_n)$ such that $m > e_{m+t+2} \geqslant \cdots \geqslant e_n \geqslant 0$. □

Now we investigate $(102, 201)$-avoiding inversion sequences of length n that contain the pattern 101. For an inversion sequence e, let $\hat{e}$ denote the sequence obtained from e by removing all the entries $e_i = \max(e)$. We denote by $\mathcal{IS}_n(102, 201 : 101)$ the set of inversion sequences of length n that avoid both patterns 102 and 201, but contain the pattern 101.

Lemma 14. *Let m, n, and t be integers such that $n \geqslant 4$, $0 \leqslant t \leqslant n-4$, and $1 \leqslant m \leqslant n-t-3$. For inversion sequences $e = (e_1, e_2, \ldots, e_n)$, $e \in \mathcal{IS}_n(102, 201)$ with $\operatorname{rank}(e) = t$ and $\max(e) = m$ contains the pattern 101 if and only if e is of the form*

$$\begin{aligned} e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{m+s} \leqslant \hat{m}, \quad e_{m+s+1}, \ldots, e_{m+t+1} = m, \quad e_{m+t+2} = \hat{m}, \\ e_{m+t+3}, \ldots, e_{m+t+k+3} \in \{\hat{m}, m\}, \quad \hat{m} > e_{m+t+k+4} \geqslant \cdots \geqslant e_n, \end{aligned} \tag{11}$$

with $(e_{m+t+3}, \ldots, e_{m+t+k+3}) \neq (\hat{m}, \ldots, \hat{m})$ for some integers k, $\hat{m}$, and s such that $0 \leqslant \hat{m} \leqslant m-1$, $0 \leqslant s \leqslant t$, and $0 \leqslant k \leqslant n-m-t-3$.

Consequently, $\hat{e}$ is a $(102, 201, 101)$-avoiding inversion sequence.

Proof. Consider $(102, 201)$-avoiding inversion sequences $e = (e_1, e_2, \ldots, e_n)$. Clearly, if e is of the form (11), then $e \in \mathcal{IS}_n(102, 201 : 101)$ with $\operatorname{rank}(e) = t$ and $\max(e) = m$.

Now suppose that $e \in \mathcal{IS}_n(102, 201)$ with $\operatorname{rank}(e) = t$ and $\max(e) = m$ contains the pattern 101, i.e., it satisfies that $e_i = e_k > e_j$ for some $i < j < k$. If $e_i < m$, then e has the subsequences (m, e_j, e_k) or (e_i, e_j, m), which contradicts that e avoids both patterns 102 and 201. Hence, $e_i = e_k = m$, and it follows that $e_j = \hat{m}$, where $\hat{m} = \max(\hat{e})$. This yields that e contains the subsequence $(m, \hat{m}, m)$ and $\hat{e}$ is a $(102, 201, 101)$-avoiding inversion sequence. Then, as in the proof of Proposition 13, $\hat{e}$ is of the form described in (10), namely,

$$\hat{e}_1 \leqslant \hat{e}_2 \leqslant \cdots \leqslant \hat{e}_r = \hat{m} > \hat{e}_{r+1} \geqslant \cdots \geqslant \hat{e}_\ell$$

for some r and ℓ. Since e avoids both patterns 102 and 201 and having $\operatorname{rank}(e) = t$ and $\max(e) = m$, e needs to satisfy (11). Lastly, $(e_{m+t+3}, \ldots, e_{m+t+k+3}) \neq (\hat{m}, \ldots, \hat{m})$ since e contains $(m, \hat{m}, m)$ as a subsequence. This completes the proof. ☐

Proposition 15. *Let m, n, and t be integers such that $n \geqslant 4$, $0 \leqslant t \leqslant n-2$, and $1 \leqslant m \leqslant n-t-3$. The number of inversion sequences $e \in \mathcal{IS}_n(102, 201 : 101)$ with $\operatorname{rank}(e) = t$ and $\max(e) = m$ is given by*

$$\begin{aligned} &(t+1)\left(2^{n-m-t-2} - 1\right) \\ &+ \sum_{j=1}^{m-1} \sum_{s=0}^{t} \sum_{k=0}^{n-m-t-3} (2^{k+1} - 1) \cdot \frac{m+s-j+1}{m+s+1} \binom{m+j+s}{j} \binom{n+j-m-t-k-4}{j-1}. \end{aligned}$$

Proof. By Lemma 14, we need to count the inversion sequences of the form (11) with the additional property $(e_{m+t+3}, \ldots, e_{m+t+k+3}) \neq (j, \ldots, j)$, for some integers k, j, and s, such that $0 \leqslant j \leqslant m-1$, $0 \leqslant s \leqslant t$, and $0 \leqslant k \leqslant n-m-t-3$.

In the special case $j = 0$, the entries of e are restricted to the set $\{0, m\}$. In this case, we have $k = n - m - t - 3$, so there are $(t+1)(2^{n-m-t-2} - 1)$ inversion sequences of the form

$$e_1 = \cdots = e_{m+s} = 0, \quad e_{m+s+1} = \cdots = e_{m+t+1} = m, \quad e_{m+t+2} = 0,$$
$$e_{m+t+3}, \ldots, e_n \in \{0, m\} \text{ with at least one entry equal to } m$$

with $0 \leqslant s \leqslant t$.

Now consider the case $j > 0$. The possible weakly increasing subsequences $(e_1, \ldots, e_{m+s})$ satisfying $e_{m+s} \leqslant j$ are in bijection with Dyck paths of semilength $m+s+1$ ending with $ud^{m+s-j+1}$. Hence, by Lemma 9, their count is

$$\frac{m+s-j+1}{m+s+1}\binom{m+j+s}{j}.$$

The subsequence $(e_{m+t+3}, \ldots, e_{m+t+k+3}) \in \{j, m\}^{k+1} \setminus \{j\}^{k+1}$ contributes $2^{k+1} - 1$ choices. Finally, the number of weakly decreasing sequences $(e_{m+t+k+4}, \ldots, e_n)$ with entries at most $j-1$ is given by $\binom{n+j-m-t-k-4}{j-1}$. Combining these counts completes the proof. □

By summing over both classes of sequences—those avoiding and those containing the pattern 101-we obtain the total count of $(102, 201)$-avoiding inversion sequences of given rank.

Corollary 16. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-2$, the number of $(102, 201)$-avoiding inversion sequences e of length n with $\operatorname{rank}(e) = t$ is given by*

$$\sum_{m=1}^{n-t-1} a(n,t,m) + \sum_{m=1}^{n-t-3} b(n,t,m),$$

where

$$a(n,t,m) = \frac{t+1}{m+t+1}\binom{2m+t}{m}\binom{n-t-2}{m-1},$$
$$b(n,t,m) = (t+1)\big(2^{n-m-t-2} - 1\big) + \sum_{j=1}^{m-1}\sum_{s=0}^{t}\sum_{k=0}^{n-m-t-3} (2^{k+1}-1)\cdot\frac{m+s-j+1}{m+s+1}\binom{m+j+s}{j}\binom{n+j-m-t-k-4}{j-1}.$$

5.7 (102, 210)-avoiding inversion sequences

In this subsection, we identify the labeled F-paths corresponding to $(102, 210)$-avoiding inversion sequences with a fixed rank, and subsequently enumerate these paths.

For a nonnegative integer n, recall $\mathcal{LF}_n$ denotes the set of labeled F-paths of semilength n. Let $\mathcal{LF}_{n,t}$ denote the set of Q in $\mathcal{LF}_n$ with $\operatorname{ht}(Q) = t$. For each step (a, b) in a labeled F-path Q, we call a step (a, b)

- *north* if $a = 0$ and $b = 1$,
- *up* if $a \geqslant 1$ and $b = 1$,
- *down* if $a \geqslant 1$ and $b \leqslant 0$.

Now, consider a down step (a, b) with an associated label $(a; b_1, \ldots, b_k)$. We call a down step (a, b)

- *pure* if $k = 1$,
- 0-*tailed* if $k \geqslant 2$ and $b_2 = \cdots = b_k = 0$,
- *complex* if $k \geqslant 2$ and there exists at least one $i \in \{2, \ldots, k\}$ such that $b_i < 0$.

To analyze a particular subclass of labeled F-paths related to $(102, 210)$-avoiding inversion sequences, we define two disjoint subsets of $\mathcal{LF}_{n,t}$ based on the nature and placement of the (at most one) down step: Let $\mathcal{A}_{n,t}$ be the set of $Q \in \mathcal{LF}_{n,t}$ such that Q contains at most one down step, which (if present) is either pure or 0-tailed; and let $\mathcal{B}_{n,t}$ be the set of $Q \in \mathcal{LF}_{n,t}$ such that Q contains exactly one down step, which is complex, and all steps following it (if any) are north steps. Their union defines the subclass of interest:

$$\mathcal{LF}_{n,t}(210) := \mathcal{A}_{n,t} \cup \mathcal{B}_{n,t}.$$

Lemma 17. *For nonnegative integers n and t, we have*

$$\phi(\mathcal{LF}_{n,t}(210)) = \mathcal{IS}_{n+1,t}(102, 210).$$

Proof. First, suppose $Q \in \mathcal{A}_{n,t}$, meaning that it contains at most one down step, which is either pure or 0-tailed. Then $\phi(Q)$ has at most one descent, and hence it avoids the 210 pattern. Therefore, $\phi(Q) \in \mathcal{IS}_{n+1,t}(102, 210)$.

Next, suppose $Q \in \mathcal{B}_{n,t}$, so that Q has a unique down step which is complex, and all subsequent steps (if any) are north steps. These north steps cannot contribute to a 210 pattern in $\phi(Q)$, and although the unique complex down step may introduce one or more descents due to the insertion of a maximum value, the resulting sequence still avoids the 210 pattern. Therefore, $\phi(Q) \in \mathcal{IS}_{n+1,t}(102, 210)$ in this case as well.

On the other hand, suppose $Q \notin \mathcal{LF}_{n,t}(210)$. If Q contains two or more down steps, then by construction $\phi(Q)$ contains the pattern 210. If Q has a unique down step that is complex and followed by an up step, then again $\phi(Q)$ contains 210. Hence, in both cases, $\phi(Q) \notin \mathcal{IS}_{n+1,t}(102, 210)$.

Combining the above observations, we conclude that

$$\phi(\mathcal{LF}_{n,t}(210)) = \mathcal{IS}_{n+1,t}(102, 210). \qquad \square$$

For a nonnegative integer t, let

$$\begin{aligned} A_t = A_t(x) &:= \sum_{n\geqslant 0} |\mathcal{A}_{n,t}|\, x^n, \\ A(u,x) &:= \sum_{n,t\geqslant 0} |\mathcal{A}_{n,t}|\, u^t x^n = \sum_{t\geqslant 0} A_t(x)\, u^t, \\ B(u,x) &:= \sum_{n,t\geqslant 0} |\mathcal{B}_{n,t}|\, u^t x^n. \end{aligned}$$

Recall that $C = C(x)$ denotes the generating function for the Catalan numbers C_n, which counts the number of labeled F-paths in $\mathcal{A}_{n,0}$ consisting only of north and up steps. This yields that $A_0 - C$ is the generating function for the number of labeled F-paths in $\mathcal{A}_{n,0}$ containing a unique down step, which is either pure or 0-tailed. Given a labeled F-path Q in $\cup_{n\geqslant 1}\mathcal{A}_{n,t}$, Q can be written as follows:

$$Q = Q_0(0,1)Q_1 \ldots (0,1)Q_t,$$

where each Q_i is in $\cup_{n\geqslant 0}\mathcal{A}_{n,0}$. Moreover, either none of the paths $Q_0, Q_1, \ldots, Q_t$ contains a down step, or exactly one of them contains a unique down step, which is either pure or 0-tailed. In the former case, the path Q contributes to $CxC\cdots xC = x^tC^{t+1}$, whereas in the latter cases, the path Q contributes to

$$(A_0 - C)xC\cdots xC + Cx(A_0-C)\cdots xC + \cdots + CxC\cdots x(A_0-C) = (t+1)(A_0-C)x^tC^t.$$

Thus we have the following equation.

$$A_t = x^tC^{t+1} + (t+1)\,(A_0 - C)\,x^tC^t = ((t+1)A_0 - tC)\,(xC)^t. \tag{12}$$

To find the generating function A_0, let us consider the last step of $Q \in \mathcal{A}_{n,0}$ with $n \geqslant 1$.

- If the last step is an up step of the form $(t+1, 1)$, then this case contributes $x \cdot A_t$.
- If the last step is a pure down step $(a, a-t)$ for $1 \leqslant a \leqslant t$, then this case contributes $tx \cdot x^tC^{t+1}$ since the last step is the only down step. Here, the factor t accounts for the number of choices for a.
- If the last step is a 0-tailed down step $(a, a-t)$ labeled as $(a; a-t, 0, \ldots, 0)$ for $1 \leqslant a \leqslant t$, then this case contributes

  $$tx \cdot (x + x^2 + x^3 + \cdots) \cdot x^tC^{t+1},$$

  since the last step is again the only down step, and the length of its 0-tail can be any positive integer. Here, the generating function $x + x^2 + x^3 + \cdots$ accounts for the number of ways to assign a positive length 0-tail.

Thus, we get the following equation for A_0:

$$\begin{aligned}
A_0 - 1 &= \sum_{t\geqslant 0} xA_t + \sum_{t\geqslant 1} t(xC)^{t+1} + \sum_{t\geqslant 1} t\frac{x}{1-x}(xC)^{t+1} \\
&= xA_0 \sum_{t\geqslant 0}(t+1)(xC)^t + \frac{x}{1-x}\sum_{t\geqslant 1} t(xC)^{t+1} && (\because (12)) \\
&= \frac{xA_0}{(1-xC)^2} + \frac{x}{1-x}\frac{x^2C^2}{(1-xC)^2} \\
&= xC^2A_0 + \frac{x^3C^4}{1-x}. && (\because C = \tfrac{1}{1-xC})
\end{aligned}$$

Solving the equation for A_0, we have

$$\begin{aligned}
A_0 &= \frac{1 + x^3C^4/(1-x)}{1-xC^2} \\
&= C + \frac{xC^3 - xC^2 + x^3C^4/(1-x)}{1-xC^2} && (\because 1 - C = -xC^2) \\
&= C + \frac{1}{1-x}\frac{x^2C^4}{1-xC^2} && (\because C^3 - C^2 = xC^4) \\
&= C + \frac{1}{1-x}\frac{x^2C^3}{\sqrt{1-4x}}. && (\because \tfrac{C}{1-xC^2} = \tfrac{1}{\sqrt{1-4x}})
\end{aligned}$$

Substituting this expression for A_0 into the identity in (12), we obtain the following closed form for $A(u,x)$.

$$\begin{aligned}
A(u,x) &= \sum_{t\geqslant 0} \left((t+1)A_0 - tC\right)(uxC)^t \\
&= \frac{A_0 - uxC^2}{(1-uxC)^2} \\
&= \frac{x^2C^3}{(1-uxC)^2(1-x)\sqrt{1-4x}} + \frac{C}{1-uxC}.
\end{aligned}$$

We now turn to the generating function $B(u,x)$, which enumerates paths in $\mathcal{B}_{n,t}$. For a nonnegative integer t, let $\widetilde{\mathcal{B}}_{n,t}$ denote the subset of $\mathcal{B}_{n,t}$ consisting of paths whose unique down step appears at the end and is a complex down step. Define

$$\begin{aligned}
\widetilde{B}_t = \widetilde{B}_t(x) &:= \sum_{n\geqslant 0} \left|\widetilde{\mathcal{B}}_{n,t}\right| x^n, \\
\widetilde{B}(u,x) &:= \sum_{n,t\geqslant 0} \left|\widetilde{\mathcal{B}}_{n,t}\right| u^t x^n = \sum_{t\geqslant 0} \widetilde{B}_t(x)\, u^t.
\end{aligned}$$

By decomposing the labeled F-paths in $\cup_{n\geqslant 1}\widetilde{\mathcal{B}}_{n,t}$, we obtain the relation

$$\widetilde{B}_t = (xC)^t \widetilde{B}_0, \tag{13}$$

where $C = C(x)$ denotes the generating function for the Catalan numbers C_n, which counts the number of labeled F-paths in $\mathcal{A}_{n,0}$ of semilength n, consisting only of north and up steps.

To find the generating function $\widetilde{B}_0$, let us examine the last step (a, b) of a labeled F-path $Q \in \widetilde{\mathcal{B}}_{n,0}$. For each value $s = a - b$, the last step is a complex down step $(a, a - s)$ labeled as $(a; b_1, \ldots, b_k)$, where

$$1 \leqslant a \leqslant s-1, \quad k \geqslant 2, \quad b_1, \ldots, b_k \leqslant 0, \quad b_1 + \cdots + b_k = a - s, \quad (b_2, \ldots, b_k) \neq (0, \ldots, 0).$$

Thus, the contribution corresponding to a fixed value of s is

$$\sum_{k \geqslant 2} \sum_{a=1}^{s-1} \left(\binom{s+k-a-1}{k-1} - 1 \right) x^k \cdot x^s C^{s+1} = \left(\frac{1}{(1-x)^s} - \frac{sx}{1-x} - 1 \right) x^s C^{s+1}.$$

Therefore, we get the following equation for $\widetilde{B}_0$:

$$\begin{aligned}
\widetilde{B}_0 &= \sum_{s \geqslant 2} \left(\frac{1}{(1-x)^s} - \frac{sx}{1-x} - 1 \right) x^s C^{s+1}. \\
&= \frac{x^2 C^5}{1-x} - \frac{x^2 C^2}{1-x} \left(C^2 - 1 \right) - x^2 C^4 \\
&= \frac{x^4 C^7}{1-x}.
\end{aligned}$$

The simplification can be carried out as in the computation of A_0, using similar generating function manipulations. Due to (13), we have

$$\widetilde{B}(u, x) = \sum_{t \geqslant 0} \widetilde{B}_t u^t = \frac{\widetilde{B}_0}{1 - uxC}.$$

Since all steps of $Q \in \mathcal{B}_{n,t}$ following the complex down step (if any) are north steps, each such sequence contributes an additional factor of $\frac{1}{1-ux}$. Therefore, we obtain

$$\begin{aligned}
B(u, x) = \widetilde{B}(u, x) \cdot \frac{1}{1-ux} &= \frac{x^4 C^7}{(1-x)(1-uxC)(1-ux)} \\
&= \frac{x^3 C^5}{1-x} \left(\frac{C}{1-uxC} - \frac{1}{1-ux} \right).
\end{aligned}$$

Combining the two types of labeled F-paths, we define the total generating function

$$G(u, x) := \sum_{n,t \geqslant 0} \left| \mathcal{LF}_{n,t}(210) \right| u^t x^n.$$

Then,

$$G(u, x) = A(u, x) + B(u, x),$$

and by substituting the expressions derived above, we obtain:

$$G(u,x) = \frac{x^2C^3/(1-x)}{\sqrt{1-4x}}\frac{1}{(1-xCu)^2} + \frac{C}{1-xCu} + \frac{x^3C^6/(1-x)}{1-xCu} - \frac{x^3C^5/(1-x)}{1-xu}.$$

Thus, we obtain

$$\left[u^t\right] G(u,x) = \frac{(t+1)x^{t+2}C^{t+3}}{(1-x)\sqrt{1-4x}} + x^tC^{t+1} + \frac{x^{t+3}C^{t+6}}{1-x} - \frac{x^{t+3}C^5}{1-x}.$$

Since

$$\left[x^{n-1}\right]\left(\left[u^t\right] G(u,x)\right) = |\mathcal{LF}_{n-1,t}(210)| = |\mathcal{IS}_{n,t}(102,210)|$$

for $n \geqslant 1$, we arrive at the following enumerative result:

Proposition 18. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-1$, the number of $(102,210)$-avoiding inversion sequences e of length n with $\operatorname{rank}(e) = t$ is given by*

$$c(n-t-1,t+1) + (t+1)\sum_{i=0}^{n-t-3}\binom{2i+t+3}{i} + \sum_{i=0}^{n-t-4}\left(c(i,t+6) - c(i,5)\right),$$

where $c(j,k) := \frac{k}{2j+k}\binom{2j+k}{j} = [x^j]C^k$.

5.8 (102, 110)-avoiding inversion sequences

In this subsection, we aim to enumerate $(102,110)$-avoiding inversion sequences by employing a similar generating function approach as in the previous case of $(102,210)$-avoidance.

Let $\mathcal{LF}_{n,t}(110)$ be the set of $Q \in \mathcal{LF}_{n,t}$ satisfying the following conditions:

- Q contains no complex down steps;
- every 0-tailed down step in Q is preceded only by north or up steps;
- no north step occurs after any down step.

Lemma 19. *For nonnegative integers n and t, we have*

$$\phi(\mathcal{LF}_{n,t}(110)) = \mathcal{IS}_{n+1,t}(102,110).$$

Proof. Let $Q \in \mathcal{LF}_{n,t} \backslash \mathcal{LF}_{n,t}(110)$. Suppose $Q = s_1 \dots s_\ell$ contains a down step s_i, labeled $(a; b_1, \dots, b_k)$ with $k \geqslant 2$. Assume that $\phi(s_1 \dots s_{i-1})$ has already been constructed, and let r be the maximum value of $\phi(s_1 \dots s_{i-1})$. The step s_i inserts a new value $m = r + a$.

- If s_i is complex, that is, if there exists $j \in \{2, \dots, k\}$ such that $b_j < 0$, then m is inserted to the left of r at least twice. It follows that the pattern 110 appears in $\phi(s_1 \dots s_i)$ and consequently in $\phi(Q)$.

- If s_i is 0-tailed, then m is inserted exactly k times into the sequence. More precisely, one occurrence of m is inserted to the left of r, and the remaining m's are placed consecutively to the right of r. If there exists at least one down step among $s_1, \ldots, s_{i-1}$, that is, if $\phi(s_1 \ldots s_{i-1})$ already contains a descent, then the insertion of m's gives rise to the pattern 110 in $\phi(s_1 \ldots s_i)$ and consequently in $\phi(Q)$.

Finally, if a north step appears after a down step in Q, then this necessarily produces the pattern 110 in $\phi(Q)$. Combining the above, $Q \notin \mathcal{LF}_{n,t}(110)$ implies $\phi(Q) \notin \mathcal{IS}_{n+1,t}(102, 110)$.

For the converse direction, suppose there exists a path $Q = s_1 \ldots s_\ell \in \mathcal{LF}_{n,t}(110)$ such that $\phi(Q)$ has a 110 pattern. Then there is the first step s_i which makes a 110 pattern in $\phi(Q)$. Since $Q \in \mathcal{LF}_{n,t}(110)$, s_i cannot be a complex down step. To make a 110 pattern in $\phi(Q)$ using s_i, the step s_i cannot be an up step or a pure down step. Also, if s_i is a north step or a 0-tailed down step, then one of $s_1, \ldots, s_{i-1}$ should be a down step. These contradict $Q \in \mathcal{LF}_{n,t}(110)$. Thus, $Q \in \mathcal{LF}_{n,t}(110)$ implies $\phi(Q) \in \mathcal{IS}_{n+1,t}(102, 110)$.

Thus, we conclude that

$$\phi(\mathcal{LF}_{n,t}(110)) = \mathcal{IS}_{n+1,t}(102, 110). \qquad \square$$

We now turn to the enumeration of such paths using generating functions. Let

$$H_t = H_t(x) := \sum_{n \geqslant 0} |\mathcal{LF}_{n,t}(110)|\, x^n,$$
$$H(u, x) := \sum_{n,t \geqslant 0} |\mathcal{LF}_{n,t}(110)|\, u^t x^n = \sum_{t \geqslant 0} H_t\, u^t.$$

Recall $C = C(x)$ denotes the generating function for the Catalan numbers C_n, and in this context, C_n counts the number of labeled F-paths in $\mathcal{LF}_{n,0}(110)$ of semilength n, consisting only of north and up steps. Since no north step is followed by a down step, decomposing labeled F-paths in $\cup_{n \geqslant 1} \mathcal{LF}_{n,t}(110)$ induce that

$$\begin{aligned} H_t &= (xC)^t H_0, \\ H(u, x) &= \frac{H_0}{1 - uxC}. \end{aligned} \tag{14}$$

To find the generating function H_0, let us consider the last step of $Q \in \mathcal{LF}_{n,0}(110)$ with $n \geqslant 1$.

- If the last step is an up step of the form $(t + 1, 1)$, it contributes $x \cdot H_t$.
- If the last step is a pure down step $(a, a - t)$ for $1 \leqslant a \leqslant t$, it contributes $tx \cdot H_t$. As before, the factor t accounts for the number of choices for a.

- If the last step is a 0-tailed down step $(a, a-t)$ for $1 \leqslant a \leqslant t$, labeled as $(a; a-t, 0, \ldots, 0)$, it contributes

$$tx \cdot (x + x^2 + x^3 + \cdots) \cdot x^t C^{t+1},$$

since the last step is the only down step.

Thus, we get

$$\begin{aligned}
H_0 - 1 &= \sum_{t\geqslant 0} xH_t + \sum_{t\geqslant 1} txH_t + \sum_{t\geqslant 1} t\,(x^2 + x^3 + x^4 + \cdots)x^t C^{t+1} \\
&= xH_0 \sum_{t\geqslant 0}(t+1)(xC)^t + \frac{x^2C}{1-x}\sum_{t\geqslant 1} t(xC)^t && (\because (14)) \\
&= \frac{xH_0}{(1-xC)^2} + \frac{x^3C^2}{(1-x)(1-xC)^2} \\
&= xC^2H_0 + \frac{x^3C^4}{1-x}, && (\because C = \tfrac{1}{1-xC})
\end{aligned}$$

which yields that

$$\begin{aligned}
H_0 &= \frac{1}{1-xC^2}\left(1 + \frac{x^3C^4}{1-x}\right) \\
&= \frac{1 - x + x(C-1)^2}{(1-xC^2)(1-x)} && (\because xC^2 = C - 1) \\
&= \frac{1-2x}{(1-x)\sqrt{1-4x}}. && (\because \tfrac{C}{1-xC^2} = \tfrac{1}{\sqrt{1-4x}})
\end{aligned}$$

From $H(u,x) = \frac{H_0}{1-xCu}$, we have

$$\begin{aligned}
[u^t]H(u,x) &= H_0\,(xC)^t \\
&= \frac{1-2x}{1-x}\frac{(xC)^t}{\sqrt{1-4x}} \\
&= \left(1 - x - x^2 - x^3 - \cdots\right)\sum_{n\geqslant t}\binom{2n-t}{n-t}x^n.
\end{aligned}$$

Since

$$\left[x^{n-1}\right]\left(\left[u^t\right]H(u,x)\right) = |\mathcal{LF}_{n-1,t}(110)| = |\mathcal{IS}_{n,t}(102,110)|,$$

we have the following proposition.

Proposition 20. *For integers $n \geqslant 2$ and $0 \leqslant t \leqslant n-1$, the number of $(102, 110)$-avoiding inversion sequences e of length n with $\mathrm{rank}(e) = t$ is given by*

$$\binom{2n-t-2}{n-t-1} - \sum_{i=2}^{n-t}\binom{2n-t-2i}{n-t-i}.$$

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