Counterexamples to conjectures on strong maximality and minimality

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Abstract

We provide counterexamples to several conjectures concerning strongly maximal and strongly minimal structures in infinite graphs and hypergraphs. In particular, we construct 3-uniform hypergraphs without strongly maximal matchings and without strongly minimal covers, and from our construction for covers we build a graph with no strongly minimal colouring. We also consider several refinements of these problems.

Our results resolve conjectures and questions of Aharoni; Aharoni and Berger; Aharoni, Berger, Georgakopoulos, and Sprüssel; Aharoni and Korman; and Tardos.

Mathematics Subject Classifications: 05C63, 05C65, 05C69, 05C70

1 Introduction

Many problems in graph theory ask when one may find a certain object of the greatest possible size. For example, Hall's theorem gives a condition for the existence of a matching which covers every vertex, and König's theorem tells us that the largest matching and the smallest vertex-cover (that is, a set of vertices of the graph which meet every edge) have the same size.

When moving to the infinite case, versions of these results concerning only cardinality tend to be weak, and easy to prove. For example, given Hall's condition in an infinite bipartite graph, one can simply apply a greedy algorithm to construct a matching with the same cardinality as the vertex set of the graph. However, the fact that the object—a matching, in this case—attains the maximum possible size tells us very little about its structure: in contrast with the finite case, it may not even be maximal. However, although in fact we may show a maximal matching exists, even that may not capture adequately the notion of being a 'largest' object.

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One way around these issues is to strengthen our desired notion of 'maximality' to 'strong maximality', which informally means that we can make no local improvement to our object. More precisely, a *strongly maximal matching* of a graph G is a matching M such that, for every matching M' of G, we have $|M \setminus M'| \geqslant |M' \setminus M|$. Similarly, a *strongly minimal vertex-cover* of G is a vertex-cover C such that, for every vertex-cover C' of G, we have $|C \setminus C'| \leqslant |C' \setminus C|$.

The notion of strong maximality more accurately captures our intuition of a 'largest' object. Returning to König's theorem, it was proved by Aharoni [1] in 1984, confirming a conjecture of Erdős (see e.g. [11]) that in any bipartite graph G one may find a strongly maximal matching M and a strongly minimal vertex-cover C, and furthermore M and C are orthogonal—that is, one may find a bijection between C and M such that each vertex of C is contained in the corresponding edge of M. In fact, strong maximality and minimality of these objects follows from orthogonality alone, much like in the finite case.

Following on from this result, it is natural to then ask whether hypergraphs must also have strongly maximal matchings and strongly minimal covers. Here a *hypergraph* H comprises a vertex set V together with a set E of edges, each of which is a subset of V. We say that H is *k-uniform* if every edge has size k. Then a *matching* is a subset of E whose edges are pairwise disjoint; an *edge-cover* is a subset of E whose union is V; and a *vertex-cover* is a set of vertices which has non-empty intersection with every edge of E.

Indeed, Aharoni [2, Problem 5.5] asked in 1991 whether every hypergraph with finite edges contains a strongly maximal matching, and in 1992, Aharoni and Korman [6, Conjecture 5.4] conjectured that, in the absence of isolated vertices, there should also be a strongly minimal edge-cover. Erdős [9] expected the conjecture to be false and, indeed, counterexamples were found for both matchings and edge-covers by Ahlswede and Khachatrian [8], and van der Zypen [12] respectively, in results which we shall return to later. Nevertheless, these conjectures were later refined by Aharoni, Berger, Georgakopoulos, and Sprüssel [7, Conjecture 1.4] to the following.

Conjecture 1. In any hypergraph with finitely bounded size of edges and no isolated vertices there exist a strongly maximal matching and a strongly minimal edge-cover.

Here the requirement that there be no isolated vertices is simply to ensure that there is any edge-cover at all. Our first two results show that both parts of Theorem 1 are false.

Theorem 2. *There is a* 3*-uniform hypergraph* H_1 *with no strongly maximal matching.*

Theorem 3. *There is a* 3*-uniform hypergraph* H_2 *with no isolated vertices and no strongly minimal edge-cover.*

We remark here that, by taking a disjoint union of H_1 and H_2, we obtain a 3-uniform hypergraph which has neither a strongly maximal matching not a strongly minimal edge-cover.

Covers and matchings are not the only objects for which one might consider strong minimality and maximality. Indeed, Aharoni, Berger, Georgakopoulos, and Sprüssel [7] also highlighted the following problem.

Conjecture 4. In every graph there exists a strongly minimal cover of the vertex set by independent sets.

The structure posited to exist by Theorem 4 is referred to by Aharoni and Berger in [5] as a *strongly minimal colouring*, in that such a strongly minimal cover would be a proper vertex-colouring of the graph satisfying a strongly minimal condition on the colour classes.

Problems of this form were also considered in Aharoni and Berger's seminal paper [4] proving Menger's theorem for infinite graphs. Indeed, alongside Theorem 1, they also conjectured that every *flag complex*—a hypergraph which is down-closed (every subset of an edge is an edge) and 2-determined (if all 2-element subsets of a set are edges, then so is that set)—has a strongly minimal edge-cover. We show that the above problem of flag complexes is equivalent to Theorem 4, and thus deduce the following result.

Theorem 5. *There is a graph G with no strongly minimal colouring.*

Aharoni and Berger also stated the following problem, attributed to Tardos (see [4, Problem 10.4]).

Question 6. Is it true that, in every hypergraph with finite edges, there exists a matching M such that no matching M' exists for which $|M \setminus M'| = 1$ and $|M' \setminus M| = 2$?

Our last result resolves Theorem 6 in the negative.

Theorem 7. *There is a hypergraph H_3 such that, for every matching M of H_3, there is a matching M' of H_3 with $|M \setminus M'| = 1$ and $|M' \setminus M| = 2$.*

Finally, Theorem 1 was also repeated in [3], where it was also conjectured that every hypergraph has a strongly minimal vertex-cover, but this is easily seen to be false, as shown in Section 3.4.

The constructions for Theorems 2 and 3 both begin with the counterexamples found for the case in which edges may have unbounded size, which are presented in Section 1.1. Both constructions work by replacing the edges in the known counterexamples with a 'gadget', which simulates the large edges with only 3-edges and 2-edges, before the 2-edges are later replaced with 3-edges. This gadget is described in Section 2.

1.1 Known counterexamples for unbounded edge size

As discussed in the introduction, Theorem 1 was initially stated for strongly maximal matchings and strongly minimal covers in hypergraphs without the bound on the size of the edges, but these were shown to be false by Ahlswede and Khachatrian and by van der Zypen respectively. For completeness, we include these counterexamples. We first consider the case of strongly maximal matchings, due to Ahlswede and Khachatrian [8].

Theorem 8. *Let H_1^* be the hypergraph on vertex set $\mathbb{N}$ with edge set consisting of those finite subsets $E \subseteq \mathbb{N}$ for which $|E| = \min E$. Then H_1^* has no strongly maximal matching.*

To see that H_1^* has no strongly maximal matching, take some infinite matching M, and take $E_1, E_2 \in M$ such that, letting $E_1 = \{v_1, v_2, \dots, v_t\}$ with $v_1 < v_2 < \dots < v_t$, we have $\min E_2 = |E_2| \geqslant v_2 + v_3$. Then $|E_1 \cup E_2| = |E_1| + |E_2| \geqslant v_1 + v_2 + v_3$, and so we can find pairwise disjoint $F_1, F_2, F_3 \subseteq E_1 \cup E_2$ with $|F_i| = \min F_i = v_i$ for all $i = 1, 2, 3$.

We may also note that H_1^* also has no strongly minimal vertex-cover, as any vertex-cover $A \subseteq \mathbb{N}$ is cofinite, and if $\mathbb{N} \setminus A = \{v_1, \dots, v_n\}$, then it is not hard to see that $(A \cup \{v_1\}) \setminus \{v_n + 1, v_n + 2\}$ is also a vertex-cover.

We next give van der Zypen's construction for strongly minimal edge-covers [12].

Theorem 9. *Let H_2^* be the hypergraph on vertex set $\mathbb{N}$ with edge set consisting of all finite subsets of $\mathbb{N}$. Then H_2^* has no strongly minimal edge-cover.*

To see that H_2^* has no strongly minimal edge-cover, take some edge cover C of H_2^*, and let $E_1, E_2 \in C$ be two arbitrary edges. Then $(C \setminus \{E_1, E_2\}) \cup \{E_1 \cup E_2\}$ is also an edge-cover of H_2^*.

2 The gadget

We construct the hypergraphs of Theorems 2 and 3 from the counterexamples in Section 1.1, by 'simulating' edges of size larger than 3 using a certain gadget with smaller edges. Given an edge e, we now define the graph G_e, the *gadget on* e. First, fix an arbitrary labelling $\{v_1, \dots, v_k\}$ of e. We add $2k-2$ new vertices in total, which will be used only in the gadget on e: $v_i^+ = v_i^+(e)$ for $1 \leqslant i \leqslant k-1$, and $v_i^- = v_i^-(e)$ for $2 \leqslant i \leqslant k$. The gadget G_e will have vertex set

$$V(G_e) = e \cup \{v_i^+ : 1 \leqslant i \leqslant k-1\} \cup \{v_i^- : 2 \leqslant i \leqslant k\},$$

and edge set

$$E(G_e) = \{v_i^+ v_{i+1}^- : 1 \leqslant i \leqslant k-1\} \cup \{v_i v_i^- v_i^+ : 2 \leqslant i \leqslant k-1\} \cup \{v_1 v_1^+, v_k v_k^-\}.$$

We call the k edges incident to vertices of e the *outer edges*, and call the remaining $k-1$ edges the *inner edges*, drawn in red in Figure 1.

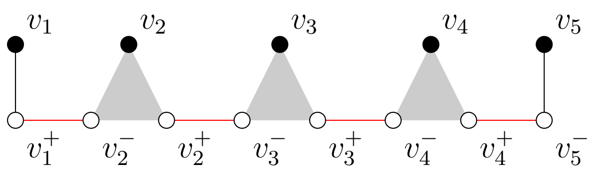

Figure 1: The gadget G_e on the edge $e = \{v_1, \dots, v_5\}$, with inner edges marked in red, and outer edges in grey and black.

A key observation is that the outer edges are a matching of G_e of size k, and that this is the unique largest matching on G_e. However, notice also that the inner edges form a matching of size $k-1$ which does not meet any of the points of e.

The gadget will be used by replacing every edge e of a hypergraph H^* with the gadget G_e, to form a hypergraph in which every edge has size at most 3, no matter how large the edges were in H^*.

3 Proofs

We now produce our counterexamples in turn, proving Theorems 2 and 3 in Sections 3.1 and 3.2 respectively. Then, in Section 3.3 we prove Theorem 5 by means of constructing a flag complex with no strongly minimal edge-cover. In Section 3.4, we give a short construction of a graph with no strongly minimal vertex-cover. Finally, in Section 3.5, we prove Theorem 7.

3.1 Strongly maximal matchings

We first construct a hypergraph H_1 with no strongly maximal matching in which all edges have size at most 3. We do this by replacing each edge e of H_1^* with a gadget G_e on the vertices of that edge. So

$$H_1 = \bigcup \{G_e : e \in H_1^*\}.$$

Recall that the sets of added vertices $v_i^{\pm}(e)$ for each gadget are disjoint: so each new vertex is only used in one edge. This graph will essentially 'simulate' the graph H_1^*.

We first make the following observation about the structure of strongly maximal matchings of H_1.

Lemma 10. *Suppose that M is a strongly maximal matching of H_1. Then for any edge $e \in H_1^*$ of size $k = |e|$, the intersection $M \cap G_e$ either has size $k-1$, or is exactly the k outer edges.*

Proof. First note that the outer edges are the unique matching of size at least k. So it suffices to show the intersection has size at least $k-1$.

Suppose some e is such that $|M \cap G_e| < k-1$. Then replacing these edges by the inner edges of G_e removed fewer than $k-1$ edges, and adds $k-1$. Certainly the inner edges of G_e don't intersect any other edges of M, as no edges outside G_e meet these vertices. So M is not a strongly maximal matching, giving a contradiction and thus the desired result. □

We next note that in order to prove Theorem 2, it is enough to show that H_1, which is not uniform, has no strongly maximal matching.

Lemma 11. *There is a 3-uniform hypergraph H_1' and a bijection $\phi_1 : E(H_1) \to E(H_1')$ such that, for all $M \subseteq E(H_1)$, the image of M under ϕ is a matching of H_1' if and only if M is a matching of H_1.*

In particular, H_1 has a strongly maximal matching if and only if H_1' does.

Proof. We construct the 3-uniform hypergraph H_1' from H_1 by adding to each edge e of size 2 a new vertex v_e. For such edges, we set $\phi_1(e) = e \cup \{v_e\}$, and for edges that already had size 3 we take $\phi_1(e) = e$. But each v_e is in exactly one edge, so for any two distinct edges e, e' of H_1, we must have $\phi_1(e) \cap \phi_1(e') = e \cap e'$. In particular, e and e' intersect if and only if $\phi_1(e)$ and $\phi_1(e')$ do.

But now ϕ_1 and its inverse both preserve matchings. So if a matching M of H_1 is not strongly maximal, this is witnessed by some other matching N with $|N \setminus M| > |M \setminus N|$. But now

$$|\phi_1[N] \setminus \phi_1[M]| = |N \setminus M| > |M \setminus N| = |\phi_1[M] \setminus \phi_1[N]|,$$

so $\phi_1[M]$ also cannot be strongly maximal. Hence we have the desired result. □

We now prove Theorem 2.

Proof of Theorem 2. By Theorem 11, it is enough to prove that H_1 has no strongly maximal matching. Suppose for contradiction that M is a strongly maximal matching of H_1. Now we define M^* to be the set of those edges e of H_1^* such that $M \cap G_e$ is exactly the outer edges of G_e. But notice that for each edge e of M^*, the vertices of e are all covered in M by edges of G_e. In particular, no vertex can be in two edges of M^*, so M^* is a matching. Notice that by Theorem 10, M contains $|e|$ edges of G_e for each $e \in M^*$, and $|e| - 1$ edges of G_e otherwise.

Then by Theorem 8, M^* cannot be strongly maximal, and so there is a matching N^* of H_1^* such that $|N^* \setminus M^*| > |M^* \setminus N^*|$ and $|N^* \setminus M^*|$ is finite. We now define a matching N of H_1 as follows:

- For $e \in M^* \setminus N^*$, we take $N \cap G_e$ to be exactly the inner edges of G_e. Thus N uses one fewer edge of G_e than M.
- For any other edge e of H_1^* such that $M \cap G_e$ contains an edge that meets an edge of $N^* \setminus M^*$, we again take $N \cap G_e$ to be exactly the inner edges of G_e. Notice that there are at most $|\bigcup(N^* \setminus M^*)|$ such edges, and in particular there are finitely many.

  Note also that e is not in M^*, as the only remaining such edges are also in N^*, and so do not meet any other edge of N^*, let alone an edge of $N^* \setminus M^*$. Thus N uses the same number of edges of G_e as M, namely $|e| - 1$.
- For $e \in N^* \setminus M^*$, we take $N \cap G_e$ to be exactly the *outer* edges of G_e. Now N uses one more edge of G_e than M.
- For any other edge e of H_1^*, we take $N \cap G_e$ to be exactly the same as $M \cap G_e$.

We now have to check that this is still a matching. Every set $N \cap G_e$ is a matching by definition, so we just need to check that no vertex of H_1^* is contained in outer edges of N in two different gadgets $G_e, G_{e'}$. As M is a matching, at least one of these outer edges would be in $N \setminus M$, and so we may assume $e \in N^* \setminus M^*$.

Then e, e' must meet, so $e' \notin N^*$. Hence the only way N can contain an outer edge of $G_{e'}$ is if $N \cap G_{e'} = M \cap G_{e'}$. But now this meets e, an edge of $N^* \setminus M^*$, so in fact $N \cap G_{e'}$

is exactly the inner edges of $G_{e'}$. Thus in fact no vertex can be contained in outer edges of both G_e and $G_{e'}$. Hence N is a matching.

But N differs from M on G_e for finitely many choices of e, so $N \setminus M$ and $M \setminus N$ are both finite. And N contains one more edge of G_e than M whenever $e \in N^* \setminus M^*$; one fewer whenever $e \in M^* \setminus N^*$; and the same number otherwise. Hence

$$|N \setminus M| - |M \setminus N| = |N^* \setminus M^*| - |M^* \setminus N^*| > 0,$$

and so M is not strongly maximal. Thus H_1 has no strongly maximal matching. □

3.2 Strongly minimal edge-covers

To prove Theorem 3, we use the same method as in Section 3.1, first constructing a graph H_2 in which all edges have size 2 or 3:

$$H_2 = \bigcup \{G_e : e \in H_2^*\}.$$

Once again, recall that the added vertices $v_i^{\pm}(e)$ are each used only in the edge e. We begin with the following observation about strongly minimal edge-covers of H_2.

Lemma 12. *Suppose that C is a strongly minimal edge-cover of H_2. Then for any edge $e \in H_2^*$ of size $k = |e|$, the intersection $C \cap G_e$ either has size k, or is exactly the inner edges.*

Proof. There are $2k-2$ added vertices which are only covered by edges of G_e, each of which covers at most two. So certainly there are least $k-1$ edges in $C \cap G_e$. If there are exactly $k-1$ edges, then each edge must cover exactly two added vertices, and no added vertex can be contained in two such edges. But clearly such a matching must be the $k-1$ inner edges.

Now suppose that $C \cap G_e$ contains more than k edges. Then we can replace these edges by the k outer edges, as these cover the whole vertex set of G_e. But this would contradict strong minimality, and hence we have the desired result. □

We again note that in order to prove Theorem 3, it is enough to show that H_2, which is not uniform, has no strongly minimal edge-cover.

Lemma 13. *There is a 3-uniform hypergraph H_2' and a bijection $\phi_2 : E(H_2) \to E(H_2')$ such that, for all $C \subseteq E(H_2)$, the image of C under ϕ is an edge-cover of H_2' if and only if C is an edge-cover of H_2.*

In particular, H_2 has a strongly minimal edge-cover if and only if H_2' does.

Proof. We construct the 3-uniform hypergraph H_2' from H_2 by adding one new vertex v to the graph, and adding that vertex to each edge of size 2. We take $\phi_2(e)$ to be $e \cup \{v\}$ for edges of size 2, and take $\phi_2(e)$ to be e for edges of size 3. Now $\bigcup \phi_2[C] = (\bigcup C) \cup \{v\}$ if C contains an edge of size 2, and $\bigcup \phi_2[C] = \bigcup C$ otherwise. Hence certainly C is an edge-cover if $\phi_2[C]$ is. But now the definition of a gadget implies that some vertex of H_2

is only incident to edges of size 2. Hence any edge-cover of C must contain an edge of size 2, and so must have $\cup\phi_2[C] = V(H_2) \cup \{v\}$. But then $\phi_2[C]$ is also an edge-cover, giving the desired result.

But now ϕ_2 and its inverse both preserve edge-covers. So just as in the proof of Theorem 11, if an edge-cover C of H_2 is not strongly minimal, this is witnessed by some other edge-cover D with $|D \setminus C| < |C \setminus D|$. But now

$$|\phi_2[D] \setminus \phi_2[C]| = |D \setminus C| < |C \setminus D| = |\phi_2[C] \setminus \phi_2[D]|\,,$$

so $\phi_2[C]$ also cannot be strongly maximal. Hence we have the desired result. □

We now prove Theorem 3, noting that this has many similarities to the proof of Theorem 2 in the previous section.

Proof of Theorem 3. By Theorem 13, it is enough to prove that H_2 has no strongly minimal edge-cover. Suppose for contradiction that C is a strongly minimal edge-cover of H_2. Now we define C^* to be the set those edges e of H_2^* such that $C \cap G_e$ contains at least one outer edge of G_e. Now each vertex of H_2^*, viewed as a vertex of H_2, is covered by at least one edge of C, which is necessarily an outer edge of some G_e. But then that vertex is contained in e, an edge of C^*. So C^* is an edge-cover of H_2^*. Notice that by Theorem 12, C contains $|e|$ edges of G_e for each $e \in C^*$, and $|e| - 1$ edges of G_e otherwise.

Then, by Theorem 9, C^* cannot be strongly minimal, and so there is an edge-cover D^* of H_2^* such that $|D^* \setminus C^*| < |C^* \setminus D^*|$ and these quantities are finite. We now define an edge-cover D of H_2 as follows:

- For each $e \in C^* \setminus D^*$, we take $D \cap G_e$ to be exactly the inner edges of G_e. Thus D uses one fewer edge of G_e than C.
- For each $e \in D^* \setminus C^*$, we take $D \cap G_e$ to be exactly the outer edges of G_e. Thus D uses one more edge of G_e than C.
- For each vertex v contained in an edge of $C^* \setminus D^*$, note that that vertex is covered by some edge e_v of D^* (picking one such edge arbitrarily if there are many options). For each such edge e_v which was already in C^*, take $D \cap G_e$ to be exactly the outer edges of G_e. Notice there are finitely many such edges e_v.

  Now for $e = e_v$, the intersection $D \cap G_e$ certainly covers every vertex that $C \cap G_e$ did, as well as covering v, and uses the same number of edges of G_e as C.
- For any other edge e of H_2^*, we take $D \cap G_e$ to be exactly the same as $C \cap G_e$.

We now have to check that this is indeed an edge-cover. By definition, every set $D \cap G_e$ covers the added vertices of G_e, so we just need to check that the vertices of H_2^* are covered by D. The only time a vertex v covered by $C \cap G_e$ is not also covered by $D \cap G_e$ is when $e \in C^* \setminus D^*$. But then either some edge e of $D^* \setminus C^*$ contains v, or there is an assigned edge $e = e_v$. Either way, v is covered by $D \cap G_e$, which is exactly the outer edges of that gadget. Thus D is an edge-cover of H_2.

But D differs from C on G_e for finitely many choices of e, so $D \setminus C$ and $C \setminus D$ are both finite. And D contains one more edge of G_e than C whenever $e \in D^* \setminus C^*$; one fewer whenever $e \in C^* \setminus D^*$; and the same number otherwise. Hence

$$|D \setminus C| - |C \setminus D| = |D^* \setminus C^*| - |C^* \setminus D^*| < 0,$$

and so C is not strongly minimal. Thus H_2 has no strongly minimal edge-cover. □

3.3 Strongly minimal graph colourings

We recall that a hypergraph H is a *flag complex* if every subset of an edge of H is itself an edge of H, and every set of vertices in which every possible edge of size 2 is present in H is itself an edge of H. Recall further that Aharoni and Berger [4] conjectured that every flag complex should have a strongly minimal edge-cover. However, it is easy to see that the graph $H_2^+ = H_2 \cup \{e : e \subset e' \in H_2\}$ is a flag complex, and likewise has no strongly minimal edge-cover: if C^+ were a strongly minimal edge-cover of H_2^+, then we could extend all edges of $E(H_2^+) \setminus E(H_2)$ in C^+ to be in $E(H_2)$ by simply adding vertices, producing a strongly minimal edge-cover of H_2.

This will allow us to prove Theorem 5, which we recall states that there is a graph with no strongly minimal colouring.

Proof of Theorem 5. We define the graph G on the same vertex set as H_2^+ to contain exactly those edges uv which are not edges of H_2^+. Now notice that as H_2^+ is a flag complex, the independent sets of G are exactly the edges of H_2^+.

Now suppose that $\chi\colon V(G) \to P$ is a strongly minimal colouring of G for some palette P. Then for each colour $c \in P$, the set $\chi^{-1}(c)$ of vertices with that colour is an independent set of G, and thus is an edge e_c of H_2^+. Note also that e_c and $e_{c'}$ are disjoint for any two distinct colours $c, c' \in P$.

But now certainly each vertex has some colour, so the edges $C = \{e_c : c \in P\}$ form an edge-cover of H_2^+. Then let D be an edge-cover of H_2^+ that witnesses that C is not strongly minimal—that is to say, $C \setminus D = \{e_c : c \in I\}$ is larger than $D \setminus C = \{e_c : c \in J\}$ for some sets $I, J \subseteq P$. But now we may recolour each vertex v for which $\chi(v) \in I$ with some colour c such that $v \in e_c \in J$. But then the colouring χ could not have been strongly minimal, giving the desired result. □

3.4 Strongly minimal vertex-covers

We now construct a graph with no strongly minimal vertex-cover.

Let G be the graph on vertex set $\mathbb{N}$ with edge set E defined as follows.

$$E = \big\{\{u, v\} \in \mathbb{N}^{(2)} : 2u \leqslant v\big\}.$$

Assume for contradiction that some $A \subseteq \mathbb{N}$ is a strongly minimal vertex-cover.

Firstly, note that we may assume that $A \neq \mathbb{N}$, as in this case we can simply remove an element of A to produce a smaller vertex-cover. Define $x \coloneqq \min(\mathbb{N} \setminus A)$ to be the minimal element of the complement of A. If there was a $y \in \mathbb{N} \setminus A$ with $y \geqslant 2x$, then $\{x, y\}$ would be an edge of G, contradicting the fact that A is a vertex-cover. Thus we may let $y := \max(\mathbb{N} \setminus A)$ be the largest element not in A.

We claim that $A' := (A \cup \{x\}) \setminus \{y+1, y+2\}$ is a vertex-cover of G. Indeed, we know that $y < 2x$, and so $y + 2 < 2 \min A'$, so the complement of A' induces no edge of G, and so A' is a vertex-cover of G. This contradicts the supposed strong minimality of A, as required.

3.5 The problem of Tardos

We now prove Theorem 7 by directly constructing a hypergraph H_3 such that, for every matching M of H_3, there is a matching M' of H_3 with $|M \setminus M'| = 1$ and $|M' \setminus M| = 2$.

Proof. Let H_3 be the hypergraph with vertex set $\mathbb{N} \times \mathbb{N}$ and edge set $\{e_{x,y} : x, y \geqslant 1\}$, where

$$e_{x,y} = \{(x, 1), (x, 2), \ldots, (x, y), (x, y), (x + 1, y), \ldots, (2x - 1, y), (2x, y)\}.$$

Intuitively, we may think of the x and y axes as going to the right and upward respectively, as is standard, and then $e_{x,y}$ roughly forms the shape of the Greek letter Γ of height y and width x, where it also meets the horizontal axis at x, as shown in Figure 2.

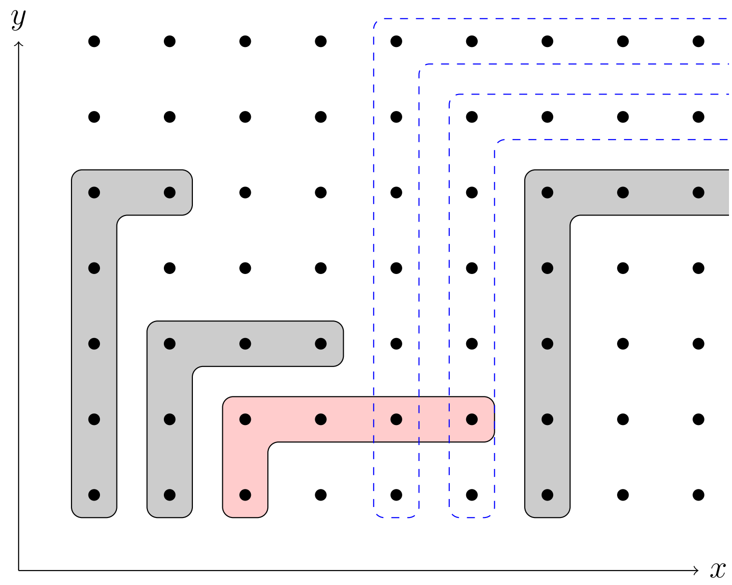

Figure 2: A matching in the hypergraph H_3. The edge filled in red can be replaced by the edges drawn in dashed blue.

Let $M \subseteq E$ be a matching. We locate an edge which can be removed, with two added in its place. Indeed, assume that $E = \{e_{x_1,y_1}, e_{x_2,y_2}, \ldots\}$, where $x_1 < x_2 < \ldots$, noting that we cannot have two edges with the same value of x, as they would intersect. We write e_i for e_{x_i,y_i} for short.

Let n be minimal such that $y_{n+1} \geqslant y_n$, noting that such an n must exist. We will remove edge e_n from M and add two edges in its place. Define

$$A = \{(x, y) : \ x \in \{2x_n - 1, 2x_n\}, y \in \mathbb{N}\}.$$

The key claim is that e_n is the only edge in M which intersects A.

Indeed, as $y_{n+1} \geqslant y_n$, we must have $x_{n+1} \geqslant 2x_n + 1$ for e_n and e_{n+1} to be disjoint from each other, and so $e_{n+1}, e_{n+2}, \dots$ are disjoint from A. Moreover, $x_{n-1} \leqslant x_n - 1$, and so e_{n-1} does not extend far enough to reach A, and so $e_{n-1}, e_{n-2}, \dots$ are also all disjoint from A. Thus e_n is the only edge in M meeting A, as claimed.

We can now define the edges which will replace e_n to form a new matching:

$$M' = (M \setminus \{e_n\}) \cup \{e_{2x_n-1,y_{n+1}+2}, e_{2x_n,y_{n+1}+1}\}.$$

Write $f_1 = e_{2x_n-1,y_{n+1}+2}$ and $f_2 = e_{2x_n,y_{n+1}+1}$ for convenience. It suffices to prove that M' is a matching, i.e. that f_1 and f_2 are disjoint from each other and from all other edges of M'. It is immediate that f_1 and f_2 are disjoint from each other, and they are disjoint from $e_1, e_2, \dots, e_{n-1}$ as none of these edges intersect A.

Let k be minimal such that $x_{n+k} \geqslant 4x_n + 1$. Then the minimal x-coordinate of all edges $e_{n+k}, e_{n+k+1}, \dots$ is at least $4x_n + 1$, which is larger than the x-coordinate of any point in f_1 or f_2.

Finally, if $k \geqslant 2$, then the minimal y-coordinate of any edge $e_{n+1}, \dots, e_{n+k-1}$ is at most y_{n+1}, as for all $2 \leqslant j \leqslant k-1$ we have $(x_{n+j}, y_{n+1}) \in e_{n+1}$, and so we must have $y_{n+j} \leqslant y_{n+1}$ in order that e_{n+1} and e_{n+j} be disjoint. Thus e_{n+j} is also disjoint from f_1 and f_2, and so M' is indeed a matching, as required, and Theorem 7 is proved. ☐

4 Concluding remarks

We now discuss some directions of further research, and state several problems which remain open.

Firstly, we may notice that our construction to prove Theorem 7 required edges of unbounded size, and so we may refine the question of Tardos to require a hypergraph with bounded edges. It appears difficult to reduce our counterexample to Theorem 6 to one with bounded edges by means of a gadget, as we did for Theorem 1. Moreover, in the setting of bounded edge-size, the corresponding question for edge-covers also remains open.

Question 14. Does every hypergraph with finitely bounded edges contain a matching M such that no matching M' has $|M \setminus M'| = 1$ and $|M' \setminus M| = 2$?

Question 15. Does every hypergraph with finitely bounded edges and no isolated vertices contain an edge-cover C such that no edge-cover C' has $|C \setminus C'| = 2$ and $|C' \setminus C| = 1$?

In a related direction, if Theorem 1 had been true, it would have implied the following refinement of a conjecture of Aharoni and Korman (see [3, Conjecture 2.2]) by a compactness argument.

Conjecture 16. Let P be a poset of finite width (i.e. there is an integer w such that all antichains of P have size bounded above by w). Then there exists a chain C in P and a partition of P into antichains such that C meets every antichain in the partition.

This conjecture, also known as the fishbone conjecture, was first stated in 1992 [6] with the condition that the antichains were merely of finite size, rather than of size at most k. That version of the conjecture was recently resolved by the first author [10]. Theorem 16 is known for the case $w = 2$ [6], but remains open even for the case of $w = 3$.

We also draw attention to the following problem of Pach (see introduction of [9]).

Question 17. Is it true that for any convex body C in $\mathbb{R}^d$, there is a packing of congruent copies of C with the property that none of its finite subfamilies can be replaced by a larger system?

Although this problem is much more geometric than those we consider, we nevertheless include it here, as it again concerns a strongly maximal object.

Finally, we conclude with a few words about the case of fractional matchings and covers. One natural direction to look to weaken the conjectures considered here would be to instead ask whether a hypergraph H necessarily contains a strongly maximal fractional matching, and similarly for covers. We recall that a fractional matching is a function $f\colon E(H) \to [0,1]$ such that, for all vertices v, we have $\sum_{e \ni v} f(e) \leqslant 1$. Then f is strongly maximal if there is no function $g\colon E(H) \to [0,1]$ which differs from f on only finitely many values and has $\sum_{e \in E(H)} (g(e) - f(e)) > 0$, i.e. has greater total weight on the values on which it differs from f. Strongly minimal fractional vertex-covers and edge-covers are defined similarly.

One could then ask whether any hypergraph H with bounded edges necessarily contains a strongly maximal fractional matching, a strongly minimal fractional edge-cover, and a strongly minimal fractional vertex-cover. However, we remark that it is not too hard to show that H_1, H_2, and H_1 respectively provide examples to show that these objects need not exist.

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