Minimal Volume Entropy of Graphs: an Elementary Proof

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Submitted: Oct 1, 2025; Accepted: Apr 1, 2026; Published: Apr 24, 2026

Abstract

It is known that the minimal volume entropy of a connected finite graph of a given cyclomatic number is attained by a trivalent graph endowed with its combinatorial length. The purpose of this short note is to present a simple geometric proof of this result based solely on elementary combinatorial arguments.

Mathematics Subject Classifications: 05C38, 05C81, 53C23

1 Introduction

The volume entropy of a closed Riemannian manifold M, or more generally of a simplicial complex with a metric, is a much-studied invariant related to the length spectrum describing the asymptotic geometry of the universal cover of M. The minimal volume entropy is a topological invariant obtained by minimizing the volume entropy over all metrics of unit volume. It is closely related to the simplicial volume; see [6, 1]. Its exact value is known only in a few cases, notably on closed hyperbolic manifolds; see [8, 3, 5, 12]. In the case of finite graphs, the minimal volume entropy has been independently computed in [10] and [7] (it can also be deduced from the earlier work [15]; see also [16] for the case of graphs with a highly transitive automorphism group). We refer to [13] for yet another approach. In its simplest form, this result states that the minimal volume entropy of a connected finite graph of a given cyclomatic number is attained by a trivalent graph endowed with the combinatorial length normalized to have unit total length. Remarkably, all existing proofs of this result rely on computing the spectral radius of a transfer operator and invoke the Perron–Frobenius theorem. The purpose of this short note is to present a simple geometric proof based solely on elementary combinatorial arguments, avoiding the use of transfer operators and the Perron–Frobenius theorem.

We begin by recalling some basic topological and combinatorial notions about graphs.

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Definition 1. Let X be a finite connected graph, where loops and multiple edges are allowed. Denote by $V(X)$ and $E(X)$ the number of vertices and edges of X.

The *cyclomatic number* of X is defined as

$$c(X) = E(X) - V(X) + 1.$$

It represents the number of independent cycles of the graph. More precisely, it is equal to the rank of the first (integral) homology group of the graph. Note that $c(X) = 1 - \chi(X)$, where $\chi(X)$ is the Euler characteristic of X.

A graph X is *d-regular* if every vertex of X is of degree d.

Let us introduce the main geometric invariant of this paper.

Definition 2. The *volume entropy* of a finite connected graph X with a length metric w is defined as the exponential growth rate of the volume (*i.e.*, length) of balls in the universal cover of X, that is,

$$\mathrm{ent}(X, w) = \lim_{R \to \infty} \frac{1}{R} \log[\mathrm{vol}\, B(R)]$$

where $B(R)$ is a ball of radius R in the universal cover $\tilde{X}$ of X. Since X is a compact length space, the limit exists and does not depend on the chosen balls $B(R)$; see [11].

Example 3. Let X be a finite connected d-regular graph with $d \geqslant 3$ endowed with the combinatorial metric w_0 where every edge has unit length. Since $2\,E(X) = d\,V(X)$, we have

$$\mathrm{length}(X, w_0) = E(X) = \frac{d}{d-2}\,(c(X) - 1).$$

Furthermore, since every ball of integer radius $k \geqslant 1$ centered at a vertex in the universal cover of X has volume $d(d-1)^{k-1}$, we deduce that

$$\mathrm{ent}(X, w_0) = \log(d - 1).$$

We present a simple proof of the following result which appears in various forms in [15, 10, 7, 13].

Theorem 4. *Let X be a finite connected graph with cyclomatic number $c(X) \geqslant 2$, equipped with a length metric w. Then*

1. $$\mathrm{ent}(X, w) \cdot \mathrm{length}(X, w) \geqslant 3\,(c(X) - 1)\log(2) \tag{1}$$

   and equality holds when X is a trivalent graph equipped with the combinatorial metric.

2. *If X is d-regular with $d \geqslant 3$, then*

   $$\mathrm{ent}(X, w) \cdot \mathrm{length}(X, w) \geqslant \frac{d}{d-2}\,(c(X) - 1)\log(d - 1) \tag{2}$$

   and equality holds when w is the combinatorial metric.

The uniqueness of the minimizer in these inequalities does not follow from our argument. For a proof of uniqueness, we refer to the aforementioned references.

Other articles related to the minimal volume entropy of graphs include [2, 9].

2 Proof of the main theorem

Let us introduce some definitions and notations.

Definition 5. Let X be a finite connected graph with a length metric w. An oriented *edge-path* of X is an oriented path $\bar{e}_1 \cdots \bar{e}_k$ formed of adjacent oriented edges $\bar{e}_i$ with $\bar{e}_{i+1} \neq -\bar{e}_i$ for every $i = 1, \dots, k-1$. In particular, edge-paths do not backtrack.

Let $T > 0$. Denote by $\mathcal{P}_w^{\leqslant T}$ the set of oriented edge-paths of X of w-length at most T. For a vertex x, denote by $\mathcal{P}_w^{\leqslant T}(x)$ be the set of oriented edge-paths of X of w-length at most T starting and ending at x.

The following lemma is a variation on a classical interpretation of the volume entropy.

Lemma 6. *Let X be a finite connected graph with a length metric w. Then, for every vertex $x \in X$,*

$$\operatorname{ent}(X, w) = \lim_{T \to \infty} \frac{1}{T} \log |\mathcal{P}_w^{\leqslant T}(x)| = \lim_{T \to \infty} \frac{1}{T} \log |\mathcal{P}_w^{\leqslant T}|.$$

Proof. Given an oriented edge-path γ of X of w-length at most T, consider a lift $\tilde{\gamma}$ of γ in the universal cover $\tilde{X}$ of X. Observe that $\tilde{\gamma} = [y_1, y_2]$ is the unique segment of $\tilde{X}$ between its endpoints y_1 and y_2. The universal cover $\tilde{X}$ is tiled by translates of a fundamental domain Δ of diameter $D \leqslant 2\operatorname{diam}(X, w)$ under the deck transformation group Γ. Each endpoint y_i lies in a translate Δ_i of Δ and is at distance at most D from the lift x_i of x in Δ_i. Since $\tilde{X}$ is a tree, there exists a unique $z_i \in [x_i, y_i]$ such that $[x_i, y_i] \cap \tilde{\gamma} = [z_i, y_i]$. The curve $[x_1, x_2] = [x_1, z_1] \cup [z_1, z_2] \cup [z_2, x_2]$ is an oriented edge-path of $\tilde{X}$ of length at most $T + 2D$, which projects to an oriented edge-path γ' of X, of the same length, starting and ending at x. This defines a map $\mathcal{P}_w^{\leqslant T} \to \mathcal{P}_w^{\leqslant T+2D}(x)$ which takes γ to γ'.

The number of preimages of an edge-path γ' in the target space can be estimated as follows. Given a lift $[x_1, x_2]$ of γ' in $\tilde{X}$, the edge-paths γ which are sent to γ' arise from segments $[y_1, y_2]$ with $y_i \in \Delta_i$. Since Δ_i contains $V(X)$ vertices, this yields at most $V(X)^2$ possibilities. Thus,

$$|\mathcal{P}_w^{\leqslant T}(x)| \leqslant |\mathcal{P}_w^{\leqslant T}| \leqslant |\mathcal{P}_w^{\leqslant T+2D}(x)| \cdot V(X)^2$$

where the first inequality is obvious. Hence, $|\mathcal{P}_w^{\leqslant T}|$ and $|\mathcal{P}_w^{\leqslant T}(x)|$ have the same exponential growth rate as $T \to \infty$.

Let $\tilde{x} \in \tilde{X}$ be a lift of x. Since $\tilde{X}$ is a tree, the cardinality of $\Gamma.\tilde{x} \cap B(\tilde{x}, R)$ is equal to $|\mathcal{P}_w^{\leqslant R}(x)|$. Furthermore, it is bounded from above by the number of translates of the fundamental domain Δ contained in $B(\tilde{x}, R + D)$, and bounded from below by the number of translates of the fundamental domain Δ contained in $B(\tilde{x}, R)$, whose union covers $B(\tilde{x}, R - D)$. Thus,

$$\frac{\operatorname{vol} B(\tilde{x}, R - D)}{\operatorname{length}(X, w)} \leqslant |\mathcal{P}_w^{\leqslant R}(x)| \leqslant \frac{\operatorname{vol} B(\tilde{x}, R + D)}{\operatorname{length}(X, w)}.$$

Hence, the exponential growth rate of $|\mathcal{P}_w^{\leqslant R}|$, which agrees with that of $|\mathcal{P}_w^{\leqslant R}(x)|$, is equal to $\operatorname{ent}(X, w)$, as required. □

The following lemma relating metric and combinatorial lengths is key.

Lemma 7. *Let X be a finite connected d-regular graph with a length metric w normalized so that* $\mathrm{length}(X, w) = E(X)$*. Fix an integer $k \geqslant 1$. Then, the average w-length of the oriented edge-paths of X of combinatorial length k is equal to k. That is,*

$$\frac{1}{|\mathcal{P}_k|} \sum_{\gamma \in \mathcal{P}_k} \ell_w(\gamma) = k$$

where $\mathcal{P}_k$ denotes the set of oriented edge-paths of X of combinatorial length k.

Example 8. For a finite connected d-regular graph X with $d \geqslant 3$, the number of oriented edge-paths of combinatorial length k starting at any given vertex is equal to $d(d-1)^{k-1}$. Thus, $|\mathcal{P}_k| = d(d-1)^{k-1}\, V(X)$. In particular, the exponential growth rate of $|\mathcal{P}_k|$ as $k \to \infty$ is equal to $\log(d-1)$.

Proof of Lemma 7. Let e_i be the unoriented edges of X, with $1 \leqslant i \leqslant E(X)$. Every unoriented edge of X lies in exactly $2k\,(d-1)^{k-1}$ oriented edge-paths of combinatorial length k, counting occurence with multiplicity (the factor 2 accounts for the orientation of the edge-paths and the factor k for the position of the edge along the path). Therefore, the average w-length $\ell_w(\gamma)$ of oriented edge-paths $\gamma \in \mathcal{P}_k$ of combinatorial length k is equal to

$$\frac{1}{|\mathcal{P}_k|} \sum_{\gamma \in \mathcal{P}_k} \ell_w(\gamma) = \frac{2k\,(d-1)^{k-1}}{d(d-1)^{k-1}\, V(X)} \sum_{i=1}^{E(X)} \ell_w(e_i) = \frac{2k\, E(X)}{d\, V(X)} = k.$$

To derive these equalities, recall that

$$|\mathcal{P}_k| = d(d-1)^{k-1}\, V(X),$$
$$\sum_{i=1}^{E(X)} \ell_w(e_i) = \mathrm{length}(X, w) = E(X),$$

and $2\, E(X) = d\, V(X)$. □

We now proceed to the proof of Theorem 4. We first estalish Part 2, from which Part 1 follows.

Proof of Theorem 4. Let X be a finite connected d-regular graph with a length metric w. By scale invariance of the product $\mathrm{ent}(X, w) \cdot \mathrm{length}(X, w)$, we may assume that $\mathrm{length}(X, w) = E(X)$.

Fix $\alpha \in (0, 1)$. By Chebyshev's inequality and Lemma 7,

$$\left| \left\{ \gamma \in \mathcal{P}_k \mid \ell_w(\gamma) \geqslant \frac{k}{\alpha} \right\} \right| \leqslant \frac{\alpha}{k} \sum_{\gamma \in \mathcal{P}_k} \ell_w(\gamma) = \alpha\, |\mathcal{P}_k|.$$

Hence,

$$|\mathcal{P}_w^{\leqslant k/\alpha}| \geqslant |\{\gamma \in \mathcal{P}_k \mid \ell_w(\gamma) < \tfrac{k}{\alpha}\}| \geqslant (1-\alpha)\, |\mathcal{P}_k|.$$

Taking the exponential growth rate in this inequality as $k \to \infty$ and using Lemma 6 together with Example 8, we obtain

$$\mathrm{ent}(X, w) \geqslant \alpha \log(d-1)$$

for every $\alpha \in (0,1)$. Letting $\alpha \to 1$, it follows that $\mathrm{ent}(X, w) \geqslant \log(d-1)$.

Since the graph is d-regular, the metric normalization can be written as $\mathrm{length}(X, w) = E(X) = \frac{d}{d-2}(c(X)-1)$. This yields inequality (2).

To prove the inequality (1), consider a finite connected graph X with a length metric. Pruning the graph, removing one by one every edge with a degree one vertex (which does not affect the volume entropy but decreases the total length of the graph), we may assume that the vertices of X are of degree at least three. We then deform the graph X into a homotopy equivalent trivalent graph X' by slightly moving the edges of X introducing new edges of arbitrarily small length. Observe that X and X' have arbitrarily close lengths and volume entropies; see [14, Proposition 3.8]. (Alternatively, the natural homotopy equivalence map $X' \to X$ sends oriented edge-paths of w'-length at most T starting and ending at x' to oriented edge-paths of w-length at most λT starting and ending at x, where $\lambda > 1$ can be chosen arbitrarily close to 1. The same holds with the inverse homotopy equivalence map $X \to X'$. The continuity of the volume entropy then follows from Lemma 6.) Applying (2) to the trivalent graph X' yields (1). □

Acknowledgements

The author would like to thank the Lebanese Association for Scientific Research for the invitation to give a series of lectures in its graduate program in 2019. This article originated from the desire to present a simple geometric proof of the computation of the minimal volume entropy of graphs during one of these lectures. The author is also grateful to Ivan Babenko and Bram Petri for their useful comments on a first version of this article, and to the referees for their valuable suggestions.

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