Nearly Cospectral Graphs and the Polynomial Reconstruction Problem

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Abstract

The polynomial reconstruction problem, proposed by Cvetković in 1973, asks whether the characteristic polynomial $\phi(G;x)$ of a graph G with at least 3 vertices can be reconstructed from the polynomial deck $\mathcal{P}(G) = \{\phi(G - v_i; x)\}_{v_i \in V(G)}$. In 2000, Hagos proposed a question of whether $\phi(G;x)$ is reconstructible from the two decks $\mathcal{P}(G)$ and $\mathcal{P}(\bar{G})$. Recently, strengthening a theorem of Ji et al. (2024), Spier (2025) proved that the characteristic polynomials of two graphs are congruent modulo 4 if and only if the characteristic polynomials of their complements are congruent modulo 4. Motivated by the above, we prove that for any two graphs G and H, if $\phi(G;x) - \phi(H;x) \equiv c \pmod 4$ and $\phi(\bar{G};x) - \phi(\bar{H};x) \equiv d \pmod 4$ for two constants c and d, respectively, then $c \equiv d \pmod 4$. In particular, if n is even, then $c \equiv d \equiv 0 \pmod 4$. This strengthens Spier's results, and provides some non-trivial information about the constant coefficients of $\phi(G;x)$ and $\phi(H;x)$ for any potential counterexample pair (G, H) to the polynomial reconstruction problem. We also obtain a similar result for any potential counterexample pair (G, H) to the problem proposed by Hagos in 2000.

Mathematics Subject Classifications: 05C50, 05C60

1 Introduction

Let G be a *simple* graph of order n with vertex set $V(G)$ and edge set $E(G)$. Let $A = [a_{ij}]_{n\times n}$ be the adjacency matrix of G, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The *characteristic polynomial* $\phi(G;x)$ of a graph G is defined as

$$\phi(G;x) = \det(xI - A) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n.$$

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A subgraph of G obtained by deleting vertex v_i and all its incident edges is called a *vertex-deleted subgraph* of G and is denoted by $G_i = G - v_i$. The multisets $\mathcal{D}(G) = \{G_1, G_2, \ldots, G_n\}$ is referred to as the *deck* of G. The collection of characteristic polynomials of the vertex-deleted subgraphs is called the *polynomial deck* of G, denoted by $\mathcal{P}(G) = \{\phi(G_1; x), \phi(G_2; x), \ldots, \phi(G_n; x)\}$. The *complement* $\bar{G}$ of G is the graph with the same vertex set as G, where two distinct vertices are adjacent whenever they are non-adjacent in G. The polynomial deck of $\bar{G}$, denoted by $\mathcal{P}(\bar{G})$, is just the multiset $\{\phi(\bar{G}_1; x), \phi(\bar{G}_2; x), \ldots, \phi(\bar{G}_n; x)\}$.

The famous reconstruction conjecture, also known as the Ulam-Kelly's conjecture, states that every simple graph with at least three vertices can be reconstructed up to isomorphism from $\mathcal{D}(G)$ [7]. As a spectral variant of Ulam-Kelly's reconstruction conjecture, in 1973, Cvetković [3] proposed the following polynomial reconstruction problem.

Problem 1 (Polynomial reconstruction problem [3])**.** Given two graphs G and H of order at least 3, is it true that

$$\mathcal{P}(G) = \mathcal{P}(H) \implies \phi(G; x) = \phi(H; x)?$$

The same problem can also be found in the work of Schwenk [9], who expressed his suspicion that $\phi(G; x)$ may not be reconstructible from $\mathcal{P}(G)$ but that counterexamples will be difficult to find. Up to now, this problem has received considerable attention. Despite many results have been obtained, the problem remains widely open; see [10] for a recent survey.

We call the collection of the characteristic polynomials of both the vertex-deleted subgraphs and their complements the *generalized polynomial deck* of G and denote it by

$$\mathcal{GP}(G) = \{(\phi(G_i; x), \phi(\bar{G}_i; x)) \mid v_i \in V(G)\}.$$

In 2000, Hagos proved that the characteristic polynomial of a graph is reconstructible from the generalized polynomial deck of the graph [5].

Theorem 2 ([5])**.** *The characteristic polynomial of a graph G is reconstructible from the generalized polynomial deck $\big\{\big(\phi(G_i; x), \phi(\bar{G}_i; x)\big) \mid v_i \in V(G)\big\}$ of G.*

Hagos also proposed a different problem of whether $\phi(G; x)$ is reconstructible from the two decks $\mathcal{P}(G)$ and $\mathcal{P}(\bar{G})$.

Problem 3 ([5])**.** Given two graphs G and H, is it true that

$$\mathcal{P}(G) = \mathcal{P}(H) \text{ and } \mathcal{P}(\bar{G}) = \mathcal{P}(\bar{H}) \implies \phi(G; x) = \phi(H; x) \text{ or } \phi(\bar{G}; x) = \phi(\bar{H}; x)?$$

Remark 4. It should be noted that, in contrast to the generalized polynomial deck given in Theorem 2, the condition in Problem 3 lacks a priori knowledge about which characteristic polynomials in the two decks originate from a vertex-deleted subgraph and its complement. This distinction is critical to the proof strategy employed in Theorem 2.

Note that the derivative of the characteristic polynomial of a graph G equals the sum of the characteristic polynomials of its vertex-deleted subgraphs [4], that is,

$$\phi'(G;x) = \sum_{i=1}^{n} \phi(G_i;x). \tag{1}$$

We can readily determine the characteristic polynomial of G except for the constant term from its polynomial deck $\mathcal{P}(G)$.

Both Problem 1 and Problem 3 appear highly challenging to solve directly. In light of Eq. (1), for any potential counterexample pair (G, H) to Problem 1, there exists a nonzero constant $c \in \mathbb{Z}$ such that

$$\phi(G;x) - \phi(H;x) = c. \tag{2}$$

For any potential counterexample pair (G, H) to Problem 3, meanwhile, there exist two nonzero constants $c, d \in \mathbb{Z}$ satisfying

$$\phi(G;x) - \phi(H;x) = c \text{ and } \phi(\bar{G};x) - \phi(\bar{H};x) = d. \tag{3}$$

Recall that two graphs are called *cospectral* if their characteristic polynomials coincide, i.e., the difference of their characteristic polynomials is zero. This motivates the following definition.

Definition 5. A pair of graphs G and H are called *nearly-cospectral*, if the difference between their characteristic polynomials is a constant, i.e., if Eq. (2) holds for some constant c (not necessarily zero). They are further termed *generalized nearly-cospectral* if they are nearly-cospectral, and so are their complements, i.e., if Eq. (3) holds for some constants c and d (not necessarily zero).

So a natural question arises :

Problem 6. What can be said about the (generalized) nearly-cospectral graphs?

Even Problem 6, however, has proven difficult to resolve. Fortunately, progress can be made when considering it modulo 4: more recently, Ji et al. [6] introduced a new invariant for cospectral graphs, which allows $\phi(\bar{G};x)$ modulo 4 to be determined from $\phi(G;x)$.

Theorem 7 ([6])**.** *Let G and H be two cospectral graphs with complements $\bar{G}$ and $\bar{H}$, respectively. Then we have $\phi(\bar{G};x) \equiv \phi(\bar{H};x) \pmod 4$.*

Subsequently, Spier [11] found that, without the full knowledge of polynomial $\phi(G;x)$, $\phi(G;x)$ modulo 4 suffices to derive $\phi(\bar{G};x)$ modulo 4. Consequently, Theorem 7 can be strengthened as follows.

Theorem 8 ([11])**.** *For any graph G, given $\phi(G;x) \pmod 4$, we can compute $\phi(\bar{G};x) \pmod 4$.*

Apparently Theorem 8 can also be restated as follows:

$$\phi(G;x) - \phi(H;x) \equiv 0 \pmod 4 \Leftrightarrow \phi(\bar{G};x) - \phi(\bar{H};x) \equiv 0 \pmod 4.$$

Building upon the above discussions, we restrict ourselves to the following more modest questions:

- If $\phi(G;x) \pmod 4$ and $\phi(H;x) \pmod 4$ differ by a constant, what's the relationship between $\phi(\bar{G};x) \pmod 4$ and $\phi(\bar{H};x) \pmod 4$?
- Moreover, if $\phi(G;x) \pmod 4$ and $\phi(H;x) \pmod 4$ differ by a constant, and $\phi(\bar{G};x) \pmod 4$ and $\phi(\bar{H};x) \pmod 4$ also differ by a constant, then what is the relationship between these two constants?

Remark 9. It is tempting to consider the above problem modulo an arbitrary integer $m > 1$, instead of modulo 4. However, results as in Theorems 7 and 8 are only available when $m = 4$.

In this paper, we shall give some partial answers to the above questions. The main results are as follows.

Theorem 10. *Let G and H be two graphs of order $n \geqslant 3$. If $\phi(G;x) - \phi(H;x) \equiv c \pmod 4$ and $\phi(\bar{G};x) - \phi(\bar{H};x) \equiv d \pmod 4$, where c and d are two constants, respectively. Then $c \equiv d \pmod 4$. In particular, if n is even, then $c \equiv d \equiv 0 \pmod 4$; and if n is odd, then $c \equiv d \equiv 0$ or $2 \pmod 4$.*

The proof of Theorem 10 is based on Theorem 24 and Theorem 25, which are of independent interest. For the first question mentioned above, if $\phi(G;x) - \phi(H;x) \pmod 4$ is a non-zero constant, the form of the polynomial $\phi(\bar{G};x) - \phi(\bar{H};x) \pmod 4$ depends on the parity of the number of vertices n of graphs G and H (see Section 3).

As an application to the polynomial reconstruction problem, we provide some non-trivial information about the constant coefficients of $\phi(G;x)$ and $\phi(H;x)$ for any potential counterexample pair (G,H) to Problem 3.

The following lemmas provide some properties that can be inferred from the polynomial deck.

Lemma 11 ([11])**.** *Let G be a graph with at least 3 vertices. Then the constant coefficients of $\phi(G;x) \pmod 2$ and $\phi(\bar{G};x) \pmod 2$, as well as the top n coefficients of $\phi(\bar{G};x) \pmod 4$, can be reconstructed by the polynomial deck $\mathcal{P}(G)$.*

According to Theorem 10 and Lemma 11, one obtains that $\phi(G;x) - \phi(\bar{G};x) \pmod 4$ can be reconstructed from the polynomial deck.

Corollary 12. *Let G be a graph with at least 3 vertices. Then the polynomial $\phi(G;x) - \phi(\bar{G};x) \pmod 4$ is reconstructible from $\mathcal{P}(G)$.*

Proof. Suppose that $\mathcal{P}(G) = \mathcal{P}(H)$. By Lemma 11, both $\phi(G; x) - \phi(H; x)$ and $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ are constants. According to Theorem 10, we have $\phi(G; x) - \phi(H; x) \equiv \phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$. Then we obtain $\phi(G; x) - \phi(\bar{G}; x) \equiv \phi(H; x) - \phi(\bar{H}; x) \pmod 4$. Therefore, $\phi(G; x) - \phi(\bar{G}; x) \pmod 4$ is reconstructible from $\mathcal{P}(G)$. □

If the number of vertices n is even, Spier obtained a result which is stronger than Lemma 11. We can provide a simpler proof of Lemma 13 in our context, compared to the approach in [11].

Lemma 13 ([11])**.** *Let G be a graph with at least 3 vertices. If n is even, then the constant coefficients of $\phi(G; x) \pmod 4$ and $\phi(\bar{G}; x) \pmod 4$ can be reconstructed from $\mathcal{P}(G)$.*

Proof. Suppose $\mathcal{P}(G) = \mathcal{P}(H)$. Then we can get $\phi(G; x) - \phi(H; x) \equiv c \pmod 4$ and $\phi(\bar{G}; x) - \phi(\bar{H}; x) \equiv d \pmod 4$. According to Theorem 10, there must be $c \equiv d \equiv 0 \pmod 4$. That is, $\phi(G; x) \equiv \phi(H; x) \pmod 4$ and $\phi(\bar{G}; x) \equiv \phi(\bar{H}; x) \pmod 4$. □

Combining Lemma 11 and Lemma 13, Spier established the following result.

Corollary 14 (cf. [11])**.** *If G and H are graphs on n vertices that form a counterexample pair to the polynomial reconstruction problem, then $\phi(G; x) = \phi(H; x) + 2k$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $\phi(\bar{G}; x) \equiv \phi(\bar{H}; x) + 2k' \pmod 4$. Moreover, if n is even, then k and k' are also even.*

Furthermore, it follows from Theorem 10 that we can strengthen Corollary 14 to the following Corollary 15.

Corollary 15. *If G and H are graphs on n vertices that form a counterexample pair to the polynomial reconstruction problem, then $\phi(G; x) = \phi(H; x) + 2k$ and $\phi(\bar{G}; x) \equiv \phi(\bar{H}; x) + 2k \pmod 4$ for some $k \in \mathbb{Z} \setminus \{0\}$. Moreover, if n is even, then k is also even.*

Moreover, if G and H are graphs on n vertices that form a counterexample pair to Problem 3, Corollary 14 can be strengthened as Corollary 16.

Corollary 16. *If G and H are graphs on n vertices that form a counterexample pair to Problem 3, then $\phi(G; x) = \phi(H; x) + 2k$, $\phi(\bar{G}; x) = \phi(\bar{H}; x) + 2k'$ and $2k \equiv 2k' \pmod 4$ for some $k, k' \in \mathbb{Z} \setminus \{0\}$. Moreover, if n is even, then k and k' are also even.*

The rest of the paper is organized as follows. Section 2 gives some preliminary results that are needed later in the paper. In Section 3, we provide the proof of Theorem 10.

2 Preliminaries

In this section, we present some necessary preliminary results that will be used to prove the main results of the paper.

The following theorem is known as Sachs' Coefficient Theorem, which establishes the relationship between the coefficients of a graph's characteristic polynomial and its subgraph structures. Here, an *elementary graph* is a graph in which each component is K_2 or a cycle.

Theorem 17 (Sachs' Coefficient Theorem [8]). *Let $\phi(G;x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$, and let $\mathcal{H}_i$ be the set of elementary subgraphs of G with i vertices. Then*

$$b_i = \sum_{H \in \mathcal{H}_i} (-1)^{p(H)} 2^{c(H)} \, (i = 1, \ldots, n), \tag{4}$$

where $p(H)$ and $c(H)$ denote the number of components and cycles in H, respectively.

The following corollary is a direct consequence of Theorem 17.

Corollary 18. *For any graph G, b_i is even for every odd i.*

Proof. Let $\phi(G;x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be the characteristic polynomial of a graph G. It follows from Theorem 17 that $b_1 = 0$. For any odd $i > 1$, there must be a cycle in every $H \in \mathcal{H}_i$, that is, $c(H) \geqslant 1$ for every $H \in \mathcal{H}_i$. Otherwise, H is the disjoint union of copies of K_2. This implies that H has an even number of vertices, which contradicts the fact that H is an elementary graph with i vertices. Therefore, it follows from Eq. (4) that b_i is even. □

Let $w^G(x) = \sum_{k \geqslant 0} \frac{e^{\mathrm{T}} A^k e}{x^{k+1}}$ be the generating function that counts the total number of walks in the graph G, where e is the all-one vector of dimension n, and a *walk* of length k in a graph is a sequence of vertices and edges $v_0e_1v_1e_2 \ldots v_{k-1}e_kv_k$ such that $e_i = (v_{i-1}, v_i) \in E(G)$ for $1 \leqslant i \leqslant k$ and v_i's are not necessarily distinct. Then we can express the walk generating function in terms of $\phi(G;x)$ and $\phi(\bar{G};x)$ [11]:

$$w^G(x) = \sum_{k \geqslant 0} \frac{e^{\mathrm{T}} A^k e}{x^{k+1}} = \frac{(-1)^n \phi(\bar{G}; -x-1) - \phi(G;x)}{\phi(G;x)}. \tag{5}$$

The following Theorem 19 establishes the specific relationship among the coefficients of $w^G(x)$, $\phi(G;x)$, and $(-1)^n\phi(\bar{G};-x-1) - \phi(G;x)$.

Theorem 19 ([11]). *Let $p(x) = a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}$ and $q(x) = x^n + b_1x^{n-1} + \cdots + b_n$ be polynomials in $\mathbb{C}[x]$. Then, $\frac{p(x)}{q(x)} = \sum_{k \geqslant 0} \frac{s_k}{x^{k+1}}$ with*

$$\begin{bmatrix} s_0 & 0 & \cdots & \cdots & 0 \\ s_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ s_{n-1} & \cdots & s_1 & s_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

For a prime number p and integer m, let $v_p(m)$ denote the largest non-negative integer k such that p^k divides m. Let $v_p(0) := \infty$. The following theorem shows the equivalence between the top k coefficients of $\phi(G;x) \pmod{p^l}$ and $\mathrm{Tr} A^m \pmod{p^{v_p(m)+l}}$ for $m \in \{0, 1, \ldots, k\}$.

Theorem 20 ([2, 11]). *Suppose that G and H are graphs with the adjacency matrices A and B, respectively. Let $\phi(G;x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ and $\phi(H;x) = b'_0x^n + b'_1x^{n-1} + \cdots + b'_{n-1}x + b'_n$ and $b_0 = b'_0 = 1$. Let p^l be a prime power and $k \geqslant 0$. Then, $b_i \equiv b'_i \pmod{p^l}$ for $i \in \{0, 1, \dots, k\}$ if and only if $\mathrm{Tr}A^m \equiv \mathrm{Tr}B^m \pmod{p^{v_p(m)+l}}$ for $m \in \{0, 1, \dots, k\}$.*

The following result established by Ji et al. [6] gives an explicit formula for $e^{\mathrm{T}}A^me$ (mod 4) in terms of $\mathrm{Tr}A^{2s+1}, \mathrm{Tr}A^{2(2s+1)}, \dots, \mathrm{Tr}A^{2^t(2s+1)}, \mathrm{Tr}A^{2^{t+1}(2s+1)}$ for $m = 2^t(2s+1)$.

Lemma 21 ([6]). *Let A be the adjacency matrix of a simple graph G, $m = 2^t(2s+1)$ be an integer with $t, s \in \mathbb{Z}_{\geqslant 0}$. Then we have*

$$e^{\mathrm{T}}A^me \equiv \frac{\mathrm{Tr}\,A^{2m}}{2^{t+1}} + \sum_{\ell=0}^{t+1} \frac{\mathrm{Tr}\,A^{2^\ell(2s+1)}}{2^\ell} \pmod 4. \tag{6}$$

Lemma 22 gives the determinant of the adjacency matrix of a graph G (i.e., the constant term of the characteristic polynomial of G) modulo 4 for graphs with even order, which is the key to proving Theorem 25.

Lemma 22 ([1]). *Let A be an n by n integral symmetric matrix with even entries on the main diagonal. Then*

1. *if $n \equiv 0 \pmod 4$, then $\det(A) \equiv 0$ or $1 \pmod 4$;*
2. *if $n \equiv 2 \pmod 4$, then $\det(A) \equiv 0$ or $-1 \pmod 4$.*

To describe the subsequent results, we introduce some concepts and notations. Let λ be a partition of an integer $m > 0$, denoted $\lambda \vdash m$. A partition λ of m is a finite ordered tuple $(\lambda_1, \lambda_2, \dots, \lambda_{k(\lambda)})$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{k(\lambda)} \geqslant 1$ and $\sum_{j=1}^{k(\lambda)} \lambda_j = m$. Here, $k(\lambda)$ denotes the number of parts of the partition λ. We define $r_j(\lambda)$ as the number of parts of size j in the partition λ, that is, $r_j(\lambda) = |\{l \in [k(\lambda)] \mid \lambda_l = j\}|$, where $[k(\lambda)] = \{1, 2, \dots, k(\lambda)\}$. Note that $\sum_{j\geqslant 1} r_j(\lambda) = k(\lambda)$, and this sum is finite. Let $(a_1, a_2, \dots, a_t)$ be a sequence of nonnegative integers summing to n, and suppose that we have m categories $C_1, \dots, C_m$. Let $\binom{n}{a_1, a_2, \dots, a_t}$ denote the number of ways of assigning each element of an n-set S to one of the categories $C_1, \dots, C_t$ so that exactly a_i elements are assigned to C_i. The number $\binom{n}{a_1, a_2, \dots, a_m}$ is called a *multinomial coefficient.*

The following lemma provides an expression for $\mathrm{Tr}A^m$ in terms of $b_1, b_2, \dots, b_m$ by using the partitions of m.

Lemma 23 ([2]). *For every $m \geqslant 1$,*

$$\mathrm{Tr}\,A^m = (-1)^m m \sum_{\lambda \vdash m} \frac{(-1)^{k(\lambda)}}{k(\lambda)} \binom{k(\lambda)}{r_1(\lambda), r_2(\lambda), \dots} \prod_{j\geqslant 1} \left((-1)^j b_j\right)^{r_j(\lambda)}.$$

3 Proofs of Theorem 10

In this section, we present the proof of Theorem 10 which relies on the following two theorems.

Theorem 24. *Let G and H be two graphs of odd order. Then* $\phi(G;x) - \phi(H;x) \equiv c \pmod 4$ *if and only if* $\phi(\bar{G};x) - \phi(\bar{H};x) \equiv c \pmod 4$*, where is a constant. Moreover, we have* $c \equiv 0$ or $2 \pmod 4$.

Proof. Let G and H be graphs of odd order n. Suppose that $\phi(G;x) - \phi(H;x) \equiv c \pmod 4$ is a constant. If $c \equiv 0 \pmod 4$, then we must have $\phi(\bar{G};x) - \phi(\bar{H};x) \equiv c \pmod 4$ by Theorem 8. Next, assume that $c \not\equiv 0 \pmod 4$. Then we have $c \equiv 2 \pmod 4$ by Corollary 18. In particular, let $\phi(G;x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ and $\phi(H;x) = b'_0x^n + b'_1x^{n-1} + \cdots + b'_{n-1}x + b'_n$ and $b_0 = b'_0 = 1$. Then $b_n - b'_n \equiv 2 \pmod 4$ and

$$b_i \equiv b'_i \pmod 4, \quad i = 0, 1, \ldots, n-1. \tag{7}$$

Let A and B be the adjacency matrices of graphs G and H, respectively. According to Lemma 20, we have that

$$\frac{\mathrm{Tr} A^m}{2^{v_2(m)}} \equiv \frac{\mathrm{Tr} B^m}{2^{v_2(m)}} \pmod 4, \quad m = 0, 1, \ldots, n-1. \tag{8}$$

It follows from Lemma 21 that

$$e^{\mathrm{T}} A^m e \equiv e^{\mathrm{T}} B^m e \pmod 4, \quad m = 0, 1, \ldots, \frac{n-1}{2}. \tag{9}$$

For $m = n+1, n+3, \ldots, 2(n-1)$, if λ is a partition of m with $\lambda_1 = n$, then we must have $\lambda_2 < n$, hence $r_n(\lambda) = 1$. By Lemma 23, we get an expression for $\frac{\mathrm{Tr}\, A^m}{2^{v_2(m)}}$ involving $b_1, b_2, \ldots, b_n$. That is,

$$\begin{aligned}
\frac{\mathrm{Tr}\, A^m}{2^{v_2(m)}} &= \frac{(-1)^m m}{2^{v_2(m)}} \sum_{\lambda \vdash m,\, \lambda_1 = n} \frac{(-1)^{k(\lambda)}}{k(\lambda)} \binom{k(\lambda)}{r_1(\lambda), r_2(\lambda), \ldots} \prod_{j \geqslant 1} \left((-1)^j b_j\right)^{r_j(\lambda)} \\
&\quad + \frac{(-1)^m m}{2^{v_2(m)}} \sum_{\lambda \vdash m,\, \lambda_1 < n} \frac{(-1)^{k(\lambda)}}{k(\lambda)} \binom{k(\lambda)}{r_1(\lambda), r_2(\lambda), \ldots} \prod_{j \geqslant 1} \left((-1)^j b_j\right)^{r_j(\lambda)} \\
&:= \frac{(-1)^m m}{2^{v_2(m)}} \sum_{\lambda \vdash m,\, \lambda_1 = n} \frac{(-1)^{k(\lambda)}}{k(\lambda)} \binom{k(\lambda)}{r_1(\lambda), r_2(\lambda), \ldots} (-1)^n b_n f_\lambda(b_1, b_2, \ldots, b_{n-1}) \\
&\quad + \frac{(-1)^m m}{2^{v_2(m)}} \sum_{\lambda \vdash m,\, \lambda_1 < n} \frac{(-1)^{k(\lambda)}}{k(\lambda)} \binom{k(\lambda)}{r_1(\lambda), r_2(\lambda), \ldots} f_\lambda(b_1, b_2, \ldots, b_{n-1}),
\end{aligned} \tag{10}$$

where $f_\lambda(b_1, b_2, \ldots, b_{n-1})$ is a function with integer coefficients in $b_1, b_2, \ldots, b_{n-1}$ corresponding to the partition λ.

Let $m = 2^t(2s+1)$. Since $\frac{m}{2} \in \{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1\}$, it follows from Lemma 21, Eq. (8) and Eq. (10) that

$$\begin{aligned} e^{\mathrm{T}} A^{\frac{m}{2}} e - e^{\mathrm{T}} B^{\frac{m}{2}} e &\equiv \left(2\frac{\operatorname{Tr} A^m}{2^{v_2(m)}} + \frac{\operatorname{Tr} A^{\frac{m}{2}}}{2^{v_2(\frac{m}{2})}} + \cdots + \frac{\operatorname{Tr} A^{2(2s+1)}}{2} + \operatorname{Tr} A^{2s+1} \right) \\ &\quad - \left(2\frac{\operatorname{Tr} B^m}{2^{v_2(m)}} + \frac{\operatorname{Tr} B^{\frac{m}{2}}}{2^{v_2(\frac{m}{2})}} + \cdots + \frac{\operatorname{Tr} B^{2(2s+1)}}{2} + \operatorname{Tr} B^{2s+1} \right) \\ &\equiv 2\frac{\operatorname{Tr} A^m}{2^{v_2(m)}} - 2\frac{\operatorname{Tr} B^m}{2^{v_2(m)}} \\ &:\equiv 2(b_n - b'_n) g(b_1, b_2, \ldots, b_{n-1}) \\ &\equiv 0 \pmod 4, \end{aligned} \tag{11}$$

where $g(b_1, b_2, \ldots, b_{n-1})$ is the coefficient of b_n in Eq. (10), and is a function of $b_1, b_2, \ldots,$ b_{n-1} with integer coefficients.

By combining Eq. (9) and Eq. (11), we obtain that

$$e^{\mathrm{T}} A^m e \equiv e^{\mathrm{T}} B^m e \pmod 4, \quad m \in \{0, 1, \ldots, n-1\}. \tag{12}$$

Note that $w^G(x) = \sum_{k \geqslant 0} \frac{e^{\mathrm{T}} A^k e}{x^{k+1}} = \frac{p(x)}{q(x)}$, where $p(x) = (-1)^n \phi(\bar{G}; -x-1) - \phi(G, x)$ is a polynomial of degree $n-1$ and $q(x) = \phi(G; x)$ is a monic polynomial of degree n. According to Theorem 19, it follows from Eq. (7) and Eq. (12) that we can get

$$a_i \equiv a'_i \pmod 4, \quad i = 0, 1, \ldots, n-1, \tag{13}$$

where a_i and a'_i ($i = 0, 1, \ldots, n-1$) represent the coefficients of $(-1)^n \phi(\bar{G}; -x-1) - \phi(G; x)$ and $(-1)^n \phi(\bar{H}; -x-1) - \phi(H; x)$, respectively. Let $\phi(\bar{G}; x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n$ and $\phi(\bar{H}; x) = c'_0 x^n + c'_1 x^{n-1} + \cdots + c'_{n-1} x + c'_n$ and $c_0 = c'_0 = 1$. Then we have

$$c_i \equiv c'_i \pmod 4, \quad i = 0, 1, \ldots, n-1. \tag{14}$$

Therefore, $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ must be a constant, denoted by d. By Corollary 18 and Theorem 8, we have $d \equiv 2 \pmod 4$. Hence $\phi(\bar{G}; x) - \phi(\bar{H}; x) \equiv 2 \pmod 4$. This completes the proof. □

Next, we consider the case of even order.

Theorem 25. *Let G and H be two graphs of even order. Suppose that $\phi(G; x) - \phi(H; x) \equiv c \pmod 4$, where c is a constant and $c \not\equiv 0 \pmod 4$. Then $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ must be a polynomial of degree $\frac{n}{2} - 1$.*

Proof. Let $\phi(G; x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n$ and $\phi(H; x) = b'_0 x^n + b'_1 x^{n-1} + \cdots + b'_{n-1} x + b'_n$ and $b_0 = b'_0 = 1$. Note that

$$b_i \equiv b'_i \pmod 4, \quad i = 0, 1, \ldots, n-1. \tag{15}$$

Let A and B be the adjacency matrices of graphs G and H, respectively. It follows from Theorem 20 that

$$\mathrm{Tr}A^m \equiv \mathrm{Tr}B^m \pmod{2^{v_2(m)+2}}, \quad m = 0, 1, \dots, n-1. \tag{16}$$

It follows from Lemma 21 that

$$e^{\mathrm{T}}A^m e \equiv e^{\mathrm{T}}B^m e \pmod 4, \quad m = 0, 1, \dots, \frac{n}{2} - 1. \tag{17}$$

Note that $w^G(x) = \sum_{k\geqslant 0} \frac{e^{\mathrm{T}}A^k e}{x^{k+1}} = \frac{p}{q}(x)$, where $p(x) = (-1)^n\phi(\bar{G}; -x-1) - \phi(G, x)$ is a polynomial of degree $n-1$ and $q(x) = \phi(G; x)$ is a monic polynomial of degree n. According to Theorem 19, it follows from Eq. (15) and Eq. (17) that we can get

$$a_i \equiv a'_i \pmod 4, \quad i = 0, 1, \dots, \frac{n}{2} - 1, \tag{18}$$

where a_i and a'_i $(i = 0, 1, \dots, n-1)$ represent the coefficients of $(-1)^n\phi(\bar{G}; -x-1) - \phi(G; x)$ and $(-1)^n\phi(\bar{H}; -x-1) - \phi(H; x)$, respectively. Let $\phi(\bar{G}; x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ and $\phi(\bar{H}; x) = c'_0x^n + c'_1x^{n-1} + \cdots + c'_{n-1}x + c'_n$ and $c_0 = c'_0 = 1$. Then we have

$$c_i \equiv c'_i \pmod 4, \quad i = 0, 1, \dots, \frac{n}{2}. \tag{19}$$

If we also have $c_{\frac{n}{2}+1} \equiv c'_{\frac{n}{2}+1} \pmod 4$, then

$$a_i \equiv a'_i \pmod 4, \quad i = 0, 1, \dots, \frac{n}{2}. \tag{20}$$

Reverse the above derivation process. According to Theorem 19, it follows from Eq. (15) and Eq. (20) that we have

$$e^{\mathrm{T}}A^m e \equiv e^{\mathrm{T}}B^m e \pmod 4, \quad m = 0, 1, \dots, \frac{n}{2}. \tag{21}$$

Using Lemma 21 again, it follows from Eq. (16) and Eq. (21) that we get $2\frac{\mathrm{Tr}A^n}{2^{t+1}} \equiv 2\frac{\mathrm{Tr}B^n}{2^{t+1}}$ (mod 4) for $\frac{n}{2} = 2^t(2s+1)$. That is,

$$\mathrm{Tr}A^n \equiv \mathrm{Tr}B^n \pmod{2^{v_2(n)+1}}. \tag{22}$$

Moreover, we also have $\mathrm{Tr}A^m \equiv \mathrm{Tr}B^m \pmod{2^{v_2(m)+1}}$ for $m \in \{0, 1, \dots, n-1\}$ from Eq. (16). According to Theorem 20, we can get $b_n \equiv b'_n \pmod 2$. Note that $b_n = \det A$ and $b'_n = \det B$. Then it follows from Lemma 22 that $b_n \equiv b'_n \pmod 4$, which contradicts the assumption $c \not\equiv 0 \pmod 4$. Therefore, $c_{\frac{n}{2}+1} \not\equiv c'_{\frac{n}{2}+1} \pmod 4$. This means that $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ must be a polynomial of degree $\frac{n}{2} - 1$. □

Now we are ready to present the proof of Theorem 10.

Proof of Theorem 10. From Theorem 24, the conclusion clearly holds when n is odd.

Suppose that n is even. If $c \equiv 0 \pmod 4$, it follows from Theorem 8 that $d \equiv 0 \pmod 4$. According to Theorem 25, if $c \not\equiv 0 \pmod 4$, then $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ must be a polynomial of degree $\frac{n}{2} - 1$. Moreover, when $n \geqslant 4$, $\phi(\bar{G}; x) - \phi(\bar{H}; x) \pmod 4$ is a non-constant polynomial. Therefore, when $n \geqslant 4$, if $\phi(G; x) - \phi(H; x) \equiv c \pmod 4$ and $\phi(\bar{G}; x) - \phi(\bar{H}; x) \equiv d \pmod 4$, then we must have $c \equiv d \equiv 0 \pmod 4$. This completes the proof. □

4 Concluding Remarks

Motivated by the polynomial reconstruction problem and ongoing efforts to address it, we introduce the notions of nearly-cospectral graphs and generalized nearly-cospectral graphs, and propose investigating problems related to these graph classes. However, even these related problems remain challenging to resolve. Fortunately, building on the recent work of Ji et al. [6] and Spier [11], we consider these notions modulo 4 - an approach that not only strengthens existing results but also yields applications to the polynomial reconstruction problem, either reinforcing or simplifying Spier's findings.

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