

The Limit of Repetition Thresholds of Rich Sequences

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Abstract

The repetition threshold of a class of sequences is the smallest number r such that a sequence from the class contains no repetition with exponent $> r$. We focus on the class \mathcal{C}_d of d -ary sequences rich in palindromes. In 2020, Currie, Mol, and Rampersad determined the repetition threshold for \mathcal{C}_2 . In 2024, Currie, Mol and Peltomäki found the repetition threshold for \mathcal{C}_3 and conjectured that the repetition threshold for \mathcal{C}_d tends to 2 with d growing to infinity. Here we verify their conjecture.

Mathematics Subject Classifications: 68R15

1 Introduction

A sequence is called *rich (in palindromes)* if each of its factors contains the maximum number of distinct palindromes. The definition was motivated by an observation by Justin et al. [10] that the number of distinct non-empty palindromic factors contained in a word of length n is smaller than or equal to n . The terminology was introduced by Glen et al. in 2009 [17]. Rich sequences may be characterized in many different ways: using complete return words [17] or using a relation between factor and palindromic complexity [5] or using extensions of bispecial factors [2].

In this paper, we focus on repetitions in rich sequences. Any prefix z of length p of $u^\omega = uuu\cdots$ can be written as u^e , where $e = \frac{p}{q}$ and q is the length of u . If u is of minimum length, e is called the *exponent* of z . For example, the word *magma* can be written as $mag^{\frac{5}{3}}$.

The *critical exponent* $E(\mathbf{u})$ of a sequence \mathbf{u} is defined as

$$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : e \text{ is the exponent of a non-empty factor of } \mathbf{u}\}.$$

In addition, the *asymptotic critical exponent* $E^*(\mathbf{u})$, introduced by Cassaigne [6] under the name asymptotic index, is equal to $+\infty$, if $E(\mathbf{u}) = +\infty$, otherwise

$$E^*(\mathbf{u}) = \limsup_{n \rightarrow \infty} \{e \in \mathbb{Q} : e \text{ is the exponent of a factor of } \mathbf{u} \text{ of length } \geq n\}.$$

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It is obvious that $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

An important characteristic of a class of sequences is the infimum of critical exponents, resp. of asymptotic critical exponents, of sequences from this class. We call this parameter *repetition threshold* for C

$$\text{RT}(C) = \inf\{E(\mathbf{u}) : \mathbf{u} \text{ belongs to the class } C\},$$

resp. *asymptotic repetition threshold* for C

$$\text{RT}^*(C) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ belongs to the class } C\}.$$

Obviously, $\text{RT}^*(C) \leq \text{RT}(C)$.

In this paper, we focus on the class \mathcal{C}_d of d -ary sequences rich in palindromes. For any rich sequence \mathbf{u} , the inequality $E(\mathbf{u}) \geq E^*(\mathbf{u}) \geq 2$ holds; see [23].

Consequently, we have a lower bound

$$\text{RT}(\mathcal{C}_d) \geq 2.$$

The repetition threshold of rich sequences has so far been determined only for binary and ternary sequences. The value of the repetition threshold for the binary alphabet $\text{RT}(\mathcal{C}_2) = 2 + \frac{\sqrt{2}}{2} \approx 2.707$ was conjectured in [4], and proved by Currie, Mol, and Ramersad [7]. Recently, Currie, Mol, and Peltomäki [8] have shown that $\text{RT}(\mathcal{C}_3) = 1 + \frac{1}{3-\mu} \approx 2.259$, where μ is the unique real root of the polynomial $x^3 - 2x^2 - 1$.

Currie, Mol, and Peltomäki formulated in [8]:

Conjecture 1. $\lim_{d \rightarrow \infty} \text{RT}(\mathcal{C}_d) = 2$.

Since $\text{RT}(\mathcal{C}_{d+1}) \leq \text{RT}(\mathcal{C}_d)$, it is clear that the limit $\lim_{d \rightarrow \infty} \text{RT}(\mathcal{C}_d)$ exists. In this paper, we confirm the validity of their conjecture.

In collaboration with Klouda [12], we defined a d -ary sequence \mathbf{u}_d for every $d \in \mathbb{N}$, $d \geq 3$ as a fixed point of the morphism

$$\varphi_d : \begin{cases} 0 & \rightarrow & 01 \\ 1 & \rightarrow & 02 \\ 2 & \rightarrow & 03 \\ & \vdots & \vdots \\ d-2 & \rightarrow & 0(d-1) \\ d-1 & \rightarrow & 0(d-1)(d-1). \end{cases} \quad (1)$$

We used these sequences to show that the asymptotic repetition threshold for the class of rich sequences does not depend on the size of the alphabet. In particular, we proved that

$$\text{RT}^*(\mathcal{C}_d) = 2 = \lim_{n \rightarrow \infty} E^*(\mathbf{u}_n) \quad \text{for every } d \in \mathbb{N}, d \geq 2.$$

In this paper, we prove the following theorem.

Theorem 2. For every $d \in \mathbb{N}, d \geq 3$, there exists a morphism $\pi : \{0, 1, \dots, 2d\}^* \rightarrow \mathcal{B}^*$ with size of the alphabet $\#\mathcal{B} = d + 2^d$ such that the sequence $\pi(\mathbf{u}_{2d+1})$ is rich and

$$E(\pi(\mathbf{u}_{2d+1})) = E^*(\mathbf{u}_{2d+1}). \quad (2)$$

The above theorem implies that $\text{RT}(\mathcal{C}_{d+2^d}) \leq E^*(\mathbf{u}_{2d+1})$, thus the validity of Conjecture 1 is a straightforward consequence.

Let us emphasize an interesting feature of the sequences \mathbf{u}_3 and \mathbf{u}_5 , namely

$$\text{RT}(\mathcal{C}_2) = E^*(\mathbf{u}_3) \quad \text{and} \quad \text{RT}(\mathcal{C}_3) = E^*(\mathbf{u}_5). \quad (3)$$

In addition, the estimate $2.117 < \text{RT}(\mathcal{C}_4) < 2.12$ announced in [8] and the value $E^*(\mathbf{u}_7) \doteq 2.11971968$ together with (3) suggest that

Conjecture 3. $\text{RT}(\mathcal{C}_d) = E^*(\mathbf{u}_{2d-1})$ for every $d \in \mathbb{N}, d \geq 2$.

Determining the value $\text{RT}(\mathcal{C}_d)$ for the parameter $d = 3$ required intensive computer support. Therefore, it seems that a proof of Conjecture 3 for a general d is currently beyond our capabilities.

The paper is organized as follows. In Preliminaries, we recall the notions and theorems we use. In Sections 3 and 4, we derive properties of palindromic and non-palindromic bispecial factors of \mathbf{u}_{2d+1} . The morphism π which plays an important role in Theorem 2 is introduced in Section 5, where we also demonstrate that the image of \mathbf{u}_{2d+1} under this morphism is rich. The proof of Theorem 2 itself is the content of Section 6. In Section 7, we discuss properties of other morphisms that could lead to a proof of Conjecture 3. The appendix contains simple facts on the eigenvalues of the matrix of the morphism fixing \mathbf{u}_{2d+1} .

2 Preliminaries

An *alphabet* \mathcal{A} is a finite set of symbols, called *letters*. A *word* u over \mathcal{A} is a finite string of letters from \mathcal{A} . The length of u is denoted $|u|$ and $|u|_i$ denotes the number of occurrences of the letter i in the word u . The *Parikh vector* \vec{u} is the vector whose i -th component equals $|u|_i$. For two words u, w we write $\vec{w} \geq \vec{u}$ if $|w|_i \geq |u|_i$ for all letters i . The set of all finite words over \mathcal{A} is denoted \mathcal{A}^* . The set \mathcal{A}^* equipped with concatenation as the operation forms a monoid with the *empty word* ε as the neutral element. Consider $u, p, s, v \in \mathcal{A}^*$ such that $u = pvs$, then the word p is called a *prefix*, the word s a *suffix* and the word v a *factor* of u . For $u, v, w \in \mathcal{A}^*$, if $u = vw$, then $uw^{-1} = v$.

A *sequence* \mathbf{u} over \mathcal{A} is an infinite string $\mathbf{u} = u_0u_1u_2 \cdots$ of letters $u_j \in \mathcal{A}$ for all $j \in \mathbb{N}$. A *word* v over \mathcal{A} is called a *factor* of the sequence $\mathbf{u} = u_0u_1u_2 \cdots$ if there exists $j \in \mathbb{N}$ such that $v = u_ju_{j+1}u_{j+2} \cdots u_{j+|v|-1}$. The integer j is called an *occurrence* of the factor v in the sequence \mathbf{u} . If $j = 0$, then v is a *prefix* of \mathbf{u} .

The set of factors occurring in a sequence \mathbf{u} forms the *language* $\mathcal{L}(\mathbf{u})$ of \mathbf{u} . The language $\mathcal{L}(\mathbf{u})$ is said to be *closed under reversal* if for each factor $v = v_0v_1 \cdots v_{n-1}$, its

mirror image $\bar{v} = v_{n-1} \cdots v_1 v_0$ is also a factor of \mathbf{u} . A factor w of a sequence \mathbf{u} is *left special* if $iw, jw \in \mathcal{L}(\mathbf{u})$ for at least two distinct letters $i, j \in \mathcal{A}$. A *right special* factor is defined analogously. A factor is called *bispecial* if it is both left and right special. The set of left extensions is denoted $\text{Lext}(w)$, i.e., $\text{Lext}(w) = \{iw \in \mathcal{L}(\mathbf{u}) : i \in \mathcal{A}\}$. Similarly, $\text{Rext}(w) = \{wi \in \mathcal{L}(\mathbf{u}) : i \in \mathcal{A}\}$. The set of both-sided extensions is denoted $\text{Bext}(w) = \{iwj \in \mathcal{L}(\mathbf{u}) : i, j \in \mathcal{A}\}$. The *bilateral order* $b(w)$ of $w \in \mathcal{L}(\mathbf{u})$ is defined

$$b(w) = \#\text{Bext}(w) - \#\text{Lext}(w) - \#\text{Rext}(w) + 1.$$

A sequence \mathbf{u} is *recurrent* if each factor of \mathbf{u} has at least two occurrences in \mathbf{u} . Moreover, a recurrent sequence \mathbf{u} is *uniformly recurrent* if the distances between consecutive occurrences of each factor in \mathbf{u} are bounded. A sequence \mathbf{u} is *eventually periodic* if there exist words $x \in \mathcal{A}^*$ and $v \in \mathcal{A}^* \setminus \{\varepsilon\}$ such that \mathbf{u} can be written as $\mathbf{u} = xv v v \cdots = xv^\omega$. If \mathbf{u} is not eventually periodic, \mathbf{u} is called *aperiodic*.

Consider a factor v of a recurrent sequence $\mathbf{u} = u_0 u_1 u_2 \cdots$. Let $j < \ell$ be two consecutive occurrences of v in \mathbf{u} . Then the word $u = u_j u_{j+1} \cdots u_{\ell-1}$ is a *return word* to v in \mathbf{u} and uv is a *complete return word* to v in \mathbf{u} .

A word p is a *palindrome* if p is equal to its mirror image, i.e., $p = \bar{p}$. If p is a palindromic factor of a sequence \mathbf{u} over \mathcal{A} , then the set of palindromic extensions of p is $\text{Pext}(p) = \{ipi \in \mathcal{L}(\mathbf{u}) : i \in \mathcal{A}\}$. If a uniformly recurrent sequence \mathbf{u} contains infinitely many palindromic factors, then its language $\mathcal{L}(\mathbf{u})$ is closed under reversal. Every word u contains at most $|u| + 1$ palindromic factors (including the empty word). If u contains $|u| + 1$ palindromic factors, u is said to be *rich*. A sequence \mathbf{u} is rich if each of its factors is rich.

We use the following equivalent characterizations of richness [17, 2].

Theorem 4. *Let \mathbf{u} be a sequence with language closed under reversal. The following statements are equivalent.*

1. \mathbf{u} is rich;
2. every complete return word to a palindromic factor of \mathbf{u} is a palindrome;
3. every bispecial factor w of \mathbf{u} satisfies
 - $b(w) = 0$ if w is not a palindrome;
 - $b(w) = \#\text{Pext}(w) - 1$ if w is a palindrome.

A *morphism* is a map $\psi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ such that $\psi(uv) = \psi(u)\psi(v)$ for all words $u, v \in \mathcal{A}^*$. The morphism ψ can be naturally extended to a sequence $\mathbf{u} = u_0 u_1 u_2 \cdots$ over \mathcal{A} by setting $\psi(\mathbf{u}) = \psi(u_0)\psi(u_1)\psi(u_2) \cdots$. Consider a factor w of $\psi(\mathbf{u})$. We say that (w_1, w_2) is a *synchronization point* of w with respect to ψ if $w = w_1 w_2$ and for all $p, s \in \mathcal{L}(\psi(\mathbf{u}))$ and $v \in \mathcal{L}(\mathbf{u})$ such that $\psi(v) = p w s$ there exists a factorization $v = v_1 v_2$ of v with $\psi(v_1) = p w_1$ and $\psi(v_2) = w_2 s$.

A *fixed point* of a morphism $\psi : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is a sequence \mathbf{u} such that $\psi(\mathbf{u}) = \mathbf{u}$. We associate to a morphism $\psi : \mathcal{A}^* \rightarrow \mathcal{A}^*$ the *incidence matrix* M_ψ defined for each $i, j \in \mathcal{A}$

as $(M_\psi)_{ij} = |\psi(j)|_i$. A morphism ψ is *primitive* if the matrix M_ψ is primitive, i.e., there exists $k \in \mathbb{N}$ such that M_ψ^k is a positive matrix.

Let \mathbf{u} be a sequence over \mathcal{A} . Then the *uniform frequency* of the letter $i \in \mathcal{A}$ is equal to f_i if for any sequence (v_n) of factors of \mathbf{u} with increasing lengths

$$f_i = \lim_{n \rightarrow \infty} \frac{|v_n|_i}{|v_n|}.$$

It is known that fixed points of primitive morphisms have uniform letter frequencies [24]. In order to compute the (asymptotic) critical exponent of sequences, we use the following theorems.¹

Theorem 5 ([9], Theorem 3). *Let \mathbf{u} be a uniformly recurrent aperiodic sequence. Let $(w_n)_{n \in \mathbb{N}}$ be the sequence of all bispecial factors in \mathbf{u} ordered by length. For every $n \in \mathbb{N}$, let r_n be the shortest return word to the bispecial factor w_n in \mathbf{u} . Then*

$$E(\mathbf{u}) = 1 + \sup \left\{ \frac{|w_n|}{|r_n|} : n \in \mathbb{N} \right\}.$$

Theorem 6 ([14], Theorem 9). *Let \mathbf{v} be a sequence over an alphabet \mathcal{A} such that the uniform letter frequencies in \mathbf{v} exist. Let $\psi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ be an injective morphism. Assume there exists $L \in \mathbb{N}$ such that every factor x of $\psi(\mathbf{v})$, $|x| \geq L$, has a synchronization point. Then $E^*(\mathbf{v}) = E^*(\psi(\mathbf{v}))$.*

The focus of our attention in this article is the fixed point $\mathbf{u}_d = \mathbf{u}$ of the morphism $\varphi_d = \varphi$ defined in (1). Even though we focus only on odd indices d later, the properties derived in [12] and listed below apply to all indices $d \geq 3$.

1. The morphism φ is primitive and injective.
2. If $w \in \mathcal{A}^*$ is a palindrome, then $\varphi(w)0$ is a palindrome. Hence, \mathbf{u} contains infinitely many palindromic factors.
3. The sequence \mathbf{u} is uniformly recurrent.
4. The language of \mathbf{u} is closed under reversal.
5. The sequence \mathbf{u} is rich.
6. Every factor of \mathbf{u} has exactly d return words.
7. The asymptotic critical exponent of \mathbf{u} satisfies

$$E^*(\mathbf{u}) = 1 + \frac{1}{3 - \Lambda}, \text{ where } \Lambda \in (2, 3) \text{ is a zero of the polynomial } t^{d-1}(t-2)^2 - 1. \tag{4}$$

¹Foster et al. [15] have recently introduced a modified version of Theorem 6.

3 Palindromic bispecial factors in \mathbf{u}_{2d+1} and their return words

As follows from Theorem 5, the key objects necessary to determine the critical exponent of a sequence are bispecial factors and their return words. An efficient method for describing bispecial factors of a fixed point of a very general morphism was provided and developed in [21, 22, 18]. This method was applied in [12]. Here we summarize properties of bispecial factors of \mathbf{u}_{2d+1} . We draw from Proposition 15 and Proposition 19 in [12] and use the notation

$$F_0 = \varepsilon \quad \text{and} \quad F_k = \varphi(F_{k-1})0 \quad \text{for} \quad k = 1, 2, \dots, 2d. \quad (5)$$

Note that every F_k is a palindrome and $F_k = \varphi^{k-1}(0)\varphi^{k-2}(0)\cdots\varphi(0)0$. In the sequel, M denotes the incidence matrix of the morphism φ defined in (16). We distinguish three types of palindromic bispecial factors:

Type 0 F_k is a bispecial factor for every $k \in \{0, 1, \dots, 2d - 1\}$.

Type I An infinite series of palindromic bispecial factors $(w_n)_{n \in \mathbb{N}}$, where $w_0 = (2d)$ has the shortest return word satisfying $\vec{r}_0 = \vec{w}_0 = \vec{e}_{2d}$.

Type II An infinite series of palindromic bispecial factors $(w_n)_{n \in \mathbb{N}}$, where $w_0 = (2d)F_{2d}(2d)$ has the shortest return word satisfying $\vec{r}_0 = M^{2d}\vec{e}_0$.

Example 7. Let $d = 2$, i.e., we consider \mathbf{u}_5 , the fixed point of

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 03, 3 \mapsto 04, 4 \mapsto 044,$$

$$\mathbf{u}_5 = 0102010301020104010201030102010440102010301020104 \dots$$

The bispecial factors of Type 0 are $\varepsilon, 0, 010, 0102010$. The first five bispecial factors of Type I together with their shortest return words are given in Table 1.

n	w_n	r_n
0	4	4
1	$\varphi(4)0$	$\varphi(4)$
2	$\varphi^2(4)\varphi(0)0$	$\varphi^2(4)$
3	$\varphi^3(4)\varphi^2(0)\varphi(0)0$	$\varphi^3(4)$
4	$4\varphi^4(4)\varphi^3(0)\varphi^2(0)\varphi(0)04$	$4\varphi^4(4)4^{-1}$

Table 1: First bispecial factors of Type I

The first five bispecial factors of Type II together with their shortest return words are given in Table 2.

For the Parikh vectors of palindromic bispecial factors and their shortest return words in both infinite series (w_n) holds

$$\vec{r}_n = M\vec{r}_{n-1} \quad \text{and} \quad \vec{w}_n = M\vec{w}_{n-1} + \begin{cases} \vec{e}_0 + 2\vec{e}_{2d} & \text{if } n = 0 \pmod{2d}; \\ \vec{e}_0 & \text{otherwise.} \end{cases} \quad (6)$$

n	w_n	r_n
0	$4\varphi^3(0)\varphi^2(0)\varphi(0)04 = 4\varphi^4(0)$	$4\varphi^4(0)4^{-1}$
1	$\varphi(4)\varphi^5(0)0$	$\varphi(4)\varphi^5(0)(\varphi(4))^{-1}$
2	$\varphi^2(4)\varphi^6(0)\varphi(0)0$	$\varphi^2(4)\varphi^6(0)(\varphi^2(4))^{-1}$
3	$\varphi^3(4)\varphi^7(0)\varphi^2(0)\varphi(0)0$	$\varphi^3(4)\varphi^7(0)(\varphi^3(4))^{-1}$
4	$4\varphi^4(4)\varphi^8(0)\varphi^3(0)\varphi^2(0)\varphi(0)04 = 4\varphi^4(4)\varphi^8(0)\varphi^4(0)$	$4\varphi^4(4)\varphi^8(0)(\varphi^4(4))^{-1}4^{-1}$

Table 2: First bispecial factors of Type II

Remark 8. Let us stress one important fact we deduced in [12]: if w_n is a bispecial factor of Type I or II and r_n the shortest return word to w_n , then each return word r to w_n satisfies $\bar{r} \geq \bar{r}_n$. This property guarantees that $|\eta(r)| \geq |\eta(r_n)|$ for every morphism η .

The following lemmas help us find an explicit form of the Parikh vectors of palindromic bispecial factors.

Lemma 9. Consider a sequence $(\vec{f}_n)_{n \in \mathbb{N}}$ of vectors from \mathbb{R}^{2d+1} defined recursively:

$$\vec{f}_0 = \vec{0} \quad \text{and} \quad \vec{f}_n = M \vec{f}_{n-1} + \begin{cases} \vec{e}_0 + 2\vec{e}_{2d} & \text{if } n = 0 \pmod{2d}; \\ \vec{e}_0 & \text{otherwise.} \end{cases}$$

Then $\vec{f}_n = A M^n \vec{e}_{2d} - B_i \vec{e}_0$, where $i \in \{0, 1, \dots, 2d-1\}$ satisfies $i = n \pmod{2d}$; $A = (M - 2I)(3I - M)^{-1}$ and $B_i = (3I - M)^{-1} + 2(M - 2I)(M - I)^{-1}(M^i - I)(3I - M)^{-1}$.

Proof. Let $n \in \mathbb{N}$ be of the form $n = 2dN + i$, where $i \in \{0, 1, \dots, 2d-1\}$ and $N \in \mathbb{N}$. The recurrence relation implies

$$\vec{f}_n = \sum_{j=0}^{n-1} M^j \vec{e}_0 + 2M^i \sum_{j=0}^{N-1} M^{2dj} \vec{e}_{2d}.$$

Our goal is to express \vec{f}_n in the form $A M^n \vec{e}_{2d} - B_i \vec{e}_0$, where the matrix B_i depends only on i , not on N , and the matrix A depends neither on N nor on i . To reach this goal, we make use of the following equalities:

- $\vec{e}_0 = (M - 2I)\vec{e}_{2d}$;
- $\sum_{j=0}^{n-1} M^j = (M - I)^{-1}(M^n - I)$;
- $\sum_{j=0}^{N-1} M^{2dj+i} = (M^{2d} - I)^{-1}(M^n - M^i)$.

Consequently, it suffices to put

$$A = (M - I)^{-1}(M - 2I) + 2(M^{2d} - I)^{-1} \quad \text{and} \quad B_i = (M - I)^{-1} + 2(M^{2d} - I)^{-1}(M - 2I)^{-1}M^i.$$

By Cayley-Hamilton theorem $M^{2d}(M - 2I)^2 = I$, hence

$$(M^{2d} - I)(M - 2I)^2 = I - (M - 2I)^2 = (M - I)(3I - M).$$

Using the above relation, we get A and B_i in the declared form. \square

Lemma 10. *Let $(\vec{f}_n)_{n \in \mathbb{N}}$ be a sequence of vectors defined in Lemma 9. Then*

1. \vec{f}_k is the Parikh vector of the bispecial factor F_k for $k = 0, 1, \dots, 2d - 1$. Moreover, $\vec{f}_k \leq \vec{r}$ for every return word r to the factor F_k .
2. $\vec{f}_n + M^n \vec{e}_{2d}$ is the Parikh vector of the bispecial factor w_n of Type I for every $n \in \mathbb{N}$. The Parikh vector of the shortest return word r_n to w_n is $M^n \vec{e}_{2d}$.
3. \vec{f}_{n+2d} is the Parikh vector of the bispecial factor w_n of Type II for every $n \in \mathbb{N}$. The Parikh vector of the shortest return word r_n to w_n is $M^{n+2d} \vec{e}_0$.

Proof. The Parikh vectors of the bispecial factors F_k defined by (5) satisfy the recurrence relation $\vec{F}_0 = \vec{0}$ and $\vec{F}_k = M\vec{F}_{k-1} + \vec{e}_0$ with $\vec{F}_0 = \vec{f}_0$. Hence $\vec{F}_k = \vec{f}_k$ for $k = 0, 1, \dots, 2d - 1$.

As $F_0 = \varepsilon$, every letter $j \in \{0, 1, \dots, 2d\}$ is a return word to F_0 and has the shortest length, namely 1. Obviously, $\vec{f}_0 = \vec{0} \leq \vec{e}_j$.

$F_1 = 0$ and $\varphi(j)$ for $j = 0, \dots, 2d$ are return words to F_1 . As $\varphi(2d - 1) = 0(2d)$ is a proper prefix of $\varphi(2d) = 0(2d)(2d)$, the return word $\varphi(2d)$ can be excluded from our further considerations. The Parikh vector $M\vec{e}_j$ of the remaining return words $\varphi(j)$ satisfies $M\vec{e}_j = \vec{f}_1 + \vec{e}_{j+1} \geq \vec{f}_1$.

It follows by induction that, for $j = 0, 1, \dots, 2d - k$, the shortest return words to F_k are $\varphi^k(j)$ and the Parikh vector of $\varphi^k(j)$ is $\vec{f}_k + \vec{e}_{j+k} \geq \vec{f}_k$.

The form of recurrence relation (6) implies for both types of palindromic bispecial factors

$$\vec{w}_n = M^n \vec{w}_0 + \vec{f}_n \quad \text{and} \quad \vec{r}_n = M^n \vec{r}_0.$$

For Type I, the initial vectors are $\vec{w}_0 = \vec{r}_0 = \vec{e}_{2d}$, for Type II, the initial vectors are $\vec{w}_0 = 2\vec{e}_{2d} + \vec{F}_{2d} = 2\vec{e}_{2d} + M\vec{F}_{2d-1} + \vec{e}_0 = 2\vec{e}_{2d} + M\vec{f}_{2d-1} + \vec{e}_0 = \vec{f}_{2d}$ and $\vec{r}_0 = M^{2d} \vec{e}_0$. Using Lemma 9, we get that $M^n \vec{f}_{2d} + \vec{f}_n = \vec{f}_{n+2d}$. \square

4 Non-palindromic bispecial factors in \mathbf{u}_{2d+1} and their return words

By Proposition 19 in [12], there is an infinite series of non-palindromic bispecial factors $(w_n)_{n \in \mathbb{N}}$ associated with each $k \in \{1, 2, \dots, 2d - 1\}$, where the initial bispecial factor equals $w_0 = F_k(2d)$ and the Parikh vector of the shortest return word r_0 to w_0 satisfies $\vec{r}_0 \geq M^k \vec{e}_{2d}$ or $\vec{r}_0 \geq M^{2d} \vec{e}_0$. As the language of \mathbf{u}_{2d+1} is closed under reversal, the mirror image of every w_n is a non-palindromic bispecial factor, too. No other non-palindromic bispecial factors occur in \mathbf{u}_{2d+1} .

²We write B_i in the two-part form $(3I - M)^{-1} + 2(M - 2I)(M - I)^{-1}(M^i - I)(3I - M)^{-1}$ because it is then obvious for $i = 0$ that the second part disappears and $B_0 = (3I - M)^{-1}$.

For the Parikh vectors of w_n and their shortest return words r_n in all non-palindromic series, the following recurrence relation holds:

$$\vec{r}_n = M\vec{r}_{n-1} \quad \text{and} \quad \vec{w}_n = M\vec{w}_{n-1} + \begin{cases} \vec{e}_0 + \vec{e}_{2d} & \text{if } n = -k \text{ or } 0 \pmod{2d}; \\ \vec{e}_0 & \text{otherwise.} \end{cases} \quad (7)$$

Lemma 11. *Let w be a non-palindromic bispecial factor and r a return word to w in \mathbf{u}_{2d+1} , then*

$$\vec{r} \geq \vec{w} + \vec{e}_{2d} \quad \text{or} \quad \vec{r} \geq \vec{w} + \vec{e}_{2d-1} + \vec{e}_{2d-2} + \cdots + \vec{e}_1 + \vec{e}_0.$$

Proof. Let $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, 2d-1\}$ such that $w = w_n$ and w_n belongs to a series of non-palindromic bispecial factors starting with $w_0 = F_k(2d)$.

Case $n = 0$. By the above description of the shortest return word $r = r_0$ to the starting non-palindromic bispecial factor we know that $\vec{r} \geq M^k \vec{e}_{2d}$ or $\vec{r} \geq M^{2d} \vec{e}_0$.

Note that $w_0 = F_k(2d)$ is a proper prefix of $\varphi^k(2d)$ and the last letter of $\varphi^k(2d)$ is $(2d)$. Thus $\vec{w} + \vec{e}_{2d} \leq M^k \vec{e}_{2d} =$ the Parikh vector of $\varphi^k(2d)$.

By Lemma 18, Item 6 in [12], $\varphi^{2d}(0) = F_{2d}(2d) = \varphi^{2d-1}(0)\varphi^{2d-2}(0) \cdots \varphi^k(0)F_k(2d)$. We deduce that the Parikh vector of $\varphi^{2d}(0) \geq M^{2d-1} \vec{e}_0 + \vec{w} \geq \vec{e}_{2d-1} + \vec{e}_{2d-2} + \cdots + \vec{e}_1 + \vec{e}_0 + \vec{w}$, where the last inequality follows from the fact that $\varphi^{2d-1}(0)$ contains all letters in $\{0, 1, \dots, 2d-1\}$.

Thus the statement is valid for $n = 0$.

If for some index $n \in \mathbb{N}, n \geq 1$, the inequality $\vec{r}_{n-1} \geq \vec{w}_{n-1} + \vec{e}_{2d}$ takes place, then $\vec{r}_n = M\vec{r}_{n-1} \geq M\vec{w}_{n-1} + M\vec{e}_{2d} = M\vec{w}_{n-1} + \vec{e}_0 + 2\vec{e}_{2d} \geq \vec{w}_n + \vec{e}_{2d}$.

If for some index $n \in \mathbb{N}, n \geq 1$, the inequality $\vec{r}_{n-1} \geq \vec{w}_{n-1} + \vec{e}_{2d-1} + \vec{e}_{2d-2} + \cdots + \vec{e}_1 + \vec{e}_0$ holds, then

$$\begin{aligned} \vec{r}_n &\geq M\vec{w}_{n-1} + M\vec{e}_{2d-1} + M\vec{e}_{2d-2} + \cdots + M\vec{e}_1 + M\vec{e}_0 \\ &= M\vec{w}_{n-1} + 2d\vec{e}_0 + \vec{e}_{2d} + \vec{e}_{2d-1} + \vec{e}_{2d-2} + \cdots + \vec{e}_1 \\ &\geq \vec{w}_n + \vec{e}_{2d-1} + \vec{e}_{2d-2} + \cdots + \vec{e}_1 + \vec{e}_0 \end{aligned}$$

and the result follows. □

5 Morphism π and richness of $\pi(\mathbf{u}_{2d+1})$

Definition 12. Let $\mathcal{A} = \{0, 1, \dots, 2d\}$. Put

$$\vec{h} = (h_0, h_1, \dots, h_{2d}) = \underbrace{(1, 1, \dots, 1)}_{d\text{-items}}, \underbrace{(2^0, 2^1, \dots, 2^d)}_{(d+1)\text{-items}}.$$

A morphism $\pi : \mathcal{A}^* \mapsto \mathcal{B}^*$ satisfying for every $i \in \mathcal{A}$

1. $\pi(i)$ is a palindrome of length h_i ;
2. the number of distinct letters occurring in $\pi(i)$ is maximum;

3. any letter which occurs in $\pi(i)$ does not occur in $\pi(j)$ for $j \neq i$

is called a *weighted morphism*.

Let us remark that the cardinality of the alphabet \mathcal{B} equals $d + 2^d$.

Example 13. Let $d = 2$. Then $\vec{h} = (1, 1, 1, 2, 4)$ and $\pi(0) = 0$, $\pi(1) = 1$, $\pi(2) = 2$, $\pi(3) = 33$, $\pi(4) = 4554$. The prefix of $\pi(\mathbf{u}_5)$ equals

$$\pi(\mathbf{u}_5) = 0102010330102010455401020103301020104554455401020103301020104554 \dots$$

The factor $\pi(4) = 4554$ of $\pi(\mathbf{u}_5)$ is bispecial and $\pi(4) = 4554$ and $\pi(4010201030102010) = 45540102010330102010$ are its return words occurring in the displayed prefix of $\pi(\mathbf{u}_5)$. The reader can check in Example 7 that the word 4 is a bispecial factor of \mathbf{u}_5 and 4 and 4010201030102010 are its return words visible in the displayed prefix of \mathbf{u}_5 .

Proposition 14. Let π be a weighted morphism over $\mathcal{A} = \{0, 1, \dots, 2d\}$ and $\mathbf{u} = \mathbf{u}_{2d+1}$.

- A factor p of \mathbf{u} is a palindrome if and only if $\pi(p)$ is a palindrome.
- The sequence $\pi(\mathbf{u})$ is rich.
- If w' is a bispecial factor of $\pi(\mathbf{u})$ of length > 1 and r' a return word to w' , then $w' = \pi(w)$ and $r' = \pi(r)$, where w is a bispecial factor of \mathbf{u} and r is a return word to w in \mathbf{u} .

Proof. • The first item follows immediately from the form of π .

- By Theorem 4, Item 2, it suffices to check that any complete return word to a non-empty palindrome in $\pi(\mathbf{u})$ is a palindrome.
 - Firstly, consider p a letter. Then p is contained in the image $\pi(a)$ of a unique letter $a \in \mathcal{A}$. Then a complete return word z to p may take on one of the possible forms: either z is entirely contained in $\pi(a)$ and z is a palindrome by the definition of π , or z is contained in the factor $\pi(aya)$, where aya is a complete return word to a , consequently aya is a palindrome by richness of \mathbf{u} . Then z is a palindrome, too – it is a central factor of $\pi(aya)$, which is a palindrome by the first item.
 - Secondly, consider a palindrome p of length > 1 in $\pi(\mathbf{u})$, which is contained in $\pi(a)$ for some letter $a \in \mathcal{A}$. Then by the form of π , the letter a is uniquely given and p occurs once in $\pi(a)$ – as a central factor. Then any complete return word z to p is contained in the factor $\pi(aya)$, where aya is a complete return word to a in \mathbf{u} , hence aya is a palindrome by richness of \mathbf{u} . Consequently, z is a palindrome, too.

- Thirdly, assume p is a palindrome in $\pi(\mathbf{u})$, which is not contained in the letter images by π . By the definition of π , it can be uniquely written as $p = x\pi(q)\bar{x}$, where x is a proper suffix of $\pi(a)$ for a letter $a \in \mathcal{A}$ and q is a factor of \mathbf{u} . The words x and q may be empty, but not at the same time. By the first item, q is a palindrome. If z is a complete return word to p in $\pi(\mathbf{u})$ and q is non-empty, then again it can be uniquely written as $z = x\pi(y)\bar{x}$, where y is a complete return word to q . Hence, y is a palindrome due to richness of \mathbf{u} . Consequently, z is a palindrome, too, by the first item. If q is empty, then $p = x\bar{x}$ occurs as the central factor of $\pi(aa)$. By similar arguments as above, any complete return word z to p is a central factor of $\pi(y)$, where y is a complete return word to aa in \mathbf{u} , hence y is a palindrome and so is z .
- If w' is a bispecial factor of $\pi(\mathbf{u})$ of length > 1 , then the form of π implies that $w' = \pi(w)$ and w is a bispecial factor in \mathbf{u} . If r' is a return word to w' , then $r'w'$ starts in w' and ends in w' , therefore $r'w' = \pi(rw)$, where rw contains w as prefix and suffix and nowhere else. Thus, r is a return word to w in \mathbf{u} . \square

6 Proof of Theorem 2

If w' is a bispecial factor and r' is a return word to w' in $\pi(\mathbf{u}_{2d+1})$, then by Proposition 14, we obtain for length the following formula $|w'| = \vec{h}\vec{w}$ and $|r'| = \vec{h}\vec{r}$, where w and r are a bispecial factor and its return word in \mathbf{u}_{2d+1} . Given the relation (4) and Theorem 5, our goal is to prove that for every bispecial factor w of \mathbf{u}_{2d+1} , where $d \geq 3$, and a return word r to w , the following inequality holds:

$$\frac{\vec{h}\vec{w}}{\vec{h}\vec{r}} \leq \frac{1}{3-\Lambda}, \quad \text{equivalently,} \quad \vec{h}\vec{w} \leq \frac{1}{3-\Lambda} \vec{h}\vec{r}. \quad (8)$$

The proof of this inequality is divided into three lemmas depending on the type of bispecial factor.

Bispecial factors of Type 0 and non-palindromic bispecial factors

Lemma 15. *Let w be a bispecial factor in \mathbf{u}_{2d+1} and r a return word to w . If w equals F_k for some $k = 0, 1, \dots, 2d-1$ or w is non-palindromic, then the inequality (8) holds true.*

Proof. Due to Lemma 11 and Lemma 10, Item 1, $\vec{w} \leq \vec{r}$. Hence $\vec{h}\vec{w} \leq \vec{h}\vec{r}$. Since $1 < \frac{1}{3-\Lambda}$, the inequality (8) holds. \square

Bispecial factors of Type I

Lemma 16. *The inequality (8) holds for all bispecial factors of Type I and their return words.*

Proof. Using Lemma 10, the inequality (8) can be rewritten as $\vec{h} \vec{f}_n + \vec{h} M^n \vec{e}_{2d} \leq \frac{1}{3-\Lambda} \vec{h} M^n \vec{e}_{2d}$ for $n \in \mathbb{N}$. Applying Lemma 9, we get

$$\vec{h} \left(A - \frac{\Lambda-2}{3-\Lambda} I \right) M^n \vec{e}_{2d} \leq \vec{h} B_i \vec{e}_0 \quad \text{for } n \in \mathbb{N}. \quad (9)$$

We will show that the left side of the inequality (9) is < 1 , while the right side is > 1 . This will prove the statement.

The left side of (9) Using Lemma 22, we have

$$\vec{h} \left(A - \frac{\Lambda-2}{3-\Lambda} I \right) M^n \vec{e}_{2d} = \sum_{\lambda \in S_\Lambda \setminus \{\Lambda\}} \frac{1}{(d+1)\lambda-2d} \left(\frac{\lambda-2}{3-\lambda} - \frac{\Lambda-2}{3-\Lambda} \right) \lambda^{n+d+1}, \quad (10)$$

where S_Λ is the set of algebraic conjugates of Λ . Let us estimate the modulus of the left side. Each summand may be estimated in the same way. We use the fact that for $\lambda \neq \Lambda$ we have $|\lambda| < 1$, and moreover

$$|(d+1)\lambda - 2d| \geq 2d - |(d+1)\lambda| \geq d - 1;$$

$$\left| \frac{(\lambda-2)\lambda^d}{3-\lambda} \right| = \left| \frac{1}{3-\lambda} \right| \leq \frac{1}{2};$$

$$\frac{\Lambda-2}{3-\Lambda} = \frac{1}{\Lambda^{d-1}} \leq \frac{1}{2^{d-1}}.$$

Since the sum has d summands, the modulus of the left side is $\leq \frac{d}{d-1} \left(\frac{1}{2} + \frac{1}{2^{d-1}} \right)$. This estimate decreases with increasing value d . As we consider $d \geq 3$, we get $\frac{d}{d-1} \left(\frac{1}{2} + \frac{1}{2^{d-1}} \right) \leq \frac{27}{28} < 1$.

The right side of (9) It is easy to verify that the first column of the matrix $(3I - M)^{-1}$, i.e., the vector $(3I - M)^{-1} \vec{e}_0$, equals

$$\frac{2}{3^{2d-1}} (3^{2d-1}, 3^{2d-2}, \dots, 3^1, 3^0, 1)^T,$$

$$\text{and } \vec{h} (M - 2I) = \underbrace{(0, 0, \dots, 0)}_{d\text{-times}}, \underbrace{(1, 1, \dots, 1)}_{(d+1)\text{-times}}.$$

Let us write B in the form $B = B' + B''$, where

$$B' = (3I - M)^{-1} \quad \text{and} \quad B'' = 2(M - 2I)(M - I)^{-1}(M^i - I)(3I - M)^{-1}.$$

A straightforward calculation gives $\vec{h} B' \vec{e}_0 = \vec{h} (3I - M)^{-1} \vec{e}_0 = \frac{3^d}{3^{d-1}}$.

The number $\vec{h} B'' \vec{e}_0$ is non-negative because it is product of a non-negative vector $2\vec{h}(M - 2I)$, of a non-negative matrix $(M - I)^{-1}(M^i - I) = M^{i-1} + M^{i-2} + \dots + I$ and of a non-negative vector $(3I - M)^{-1} \vec{e}_0$.

We can conclude that the right side of (9) is $\geq \frac{3^d}{3^{d-1}} > 1$. □

Bispecial factors of Type II

Lemma 17. *The inequality (8) holds for all bispecial factors of Type II and their return words.*

Proof. By Lemma 10, the inequality (8) can be rewritten as $\vec{h}\vec{f}_{n+2d} \leq \frac{1}{3-\Lambda} \vec{h} M^{n+2d} \vec{e}_0$ for $n \in \mathbb{N}$. Equivalently, $\vec{h}\vec{f}_n \leq \frac{1}{3-\Lambda} \vec{h} M^n \vec{e}_0$ for $n \in \mathbb{N}$, $n \geq 2d$. We again use the explicit form of \vec{f}_n from Lemma 9 and the equality $\vec{e}_0 = (M - 2I)\vec{e}_{2d}$ to obtain

$$\vec{h} \left(A - \frac{1}{3-\Lambda} (M - 2I) \right) M^n \vec{e}_{2d} \leq \vec{h} B_i \vec{e}_0 \quad \text{for } n \in \mathbb{N}, n \geq 2d. \quad (11)$$

Using Lemma 22, we obtain

$$\vec{h} \left(A - \frac{1}{3-\Lambda} (M - 2I) \right) M^n \vec{e}_{2d} = \sum_{\lambda \in S_\Lambda \setminus \{\Lambda\}} \frac{1}{(d+1)\lambda - 2d} \left(\frac{\lambda-2}{3-\lambda} - \frac{\lambda-2}{3-\Lambda} \right) \lambda^{n+d+1}.$$

Let us rewrite the expression

$$\left(\frac{\lambda-2}{3-\lambda} - \frac{\lambda-2}{3-\Lambda} \right) \lambda^{n+d+1} = (\lambda - 2) \frac{\lambda - \Lambda}{(3-\lambda)(3-\Lambda)} \lambda^{n+d+1} = \frac{\lambda - \Lambda}{(3-\lambda)(3-\Lambda)} \lambda^{n+1}. \quad (12)$$

Let us emphasize that we used the equality $(\lambda-2)\lambda^d = 1$ for each algebraic conjugate of Λ , i.e., for $\lambda \in S_\Lambda$.

Let us compare the expression in (12) with a rewritten expression from (10).

$$\left(\frac{\lambda-2}{3-\lambda} - \frac{\Lambda-2}{3-\Lambda} \right) \lambda^{n+d+1} = \frac{\lambda - \Lambda}{(3-\lambda)(3-\Lambda)} \lambda^{n+1+d}. \quad (13)$$

Therefore from the validity of inequality (9) for all $n \in \mathbb{N}$, the inequality (11) follows for all $n \geq d$. \square

Proof of Theorem 2. On the one hand, Lemmas 15, 16, and 17 show that for every bispecial factor w in \mathbf{u}_{2d+1} and any of its return words r , the inequality (8) holds, i.e., $E(\pi(\mathbf{u}_{2d+1})) \leq 1 + \frac{1}{3-\Lambda} = E^*(\pi(\mathbf{u}_{2d+1}))$.

On the other hand, by Theorem 6, we have $E^*(\pi(\mathbf{u}_{2d+1})) = E^*(\mathbf{u}_{2d+1}) = 1 + \frac{1}{3-\Lambda}$. The assumptions of Theorem 6 are met since \mathbf{u}_{2d+1} is a fixed point of a primitive morphism and consequently has uniform letter frequencies and all factors of $\pi(\mathbf{u}_{2d+1})$ of length $\geq 2^d$ have a synchronization point. Since $E(\pi(\mathbf{u}_{2d+1})) \geq E^*(\pi(\mathbf{u}_{2d+1}))$, the proof of Theorem 2 is complete. \square

7 Comments on images of \mathbf{u}_{2d+1} under morphisms to smaller alphabets

The morphism π we work with in this paper maps the sequence \mathbf{u}_{2d+1} over the alphabet of size $2d+1$ onto an alphabet of size $d+2^d$. However, as follows from the papers [7, 8, 13] for $d = 2$ and $d = 3$, there exists a morphism η onto the alphabet of size d such that $E^*(\mathbf{u}_{2d-1}) = E(\eta(\mathbf{u}_{2d-1}))$. To prove at least the easier part of Conjecture 3, namely $\text{RT}(\mathcal{C}_d) \leq E^*(\mathbf{u}_{2d-1})$, it would be sufficient to find such a morphism for each d . The following propositions could provide a hint on how to find such morphisms.

Proposition 18. Let $d \in \mathbb{N}$ and $\eta : \{0, 1, \dots, 2d\}^* \mapsto \mathcal{B}^*$ be a morphism satisfying

1. there exists a finite set $\mathcal{F} \subset \mathcal{B}^*$ such that every sufficiently long bispecial factor in $\eta(\mathbf{u}_{2d+1})$ and its return word have the form $p\eta(w)s$ and $p\eta(r)p^{-1}$, respectively, with $p, s \in \mathcal{F}$ and w a bispecial factor of \mathbf{u}_{2d+1} and r a return word to w in \mathbf{u}_{2d+1} .
2. $E(\eta(\mathbf{u}_{2d+1})) = E^*(\mathbf{u}_{2d+1})$.

Then the integer row vector $(|\eta(0)|, |\eta(1)|, \dots, |\eta(2d)|)$ is orthogonal to an eigenvector of the incidence matrix M of the morphism φ_{2d+1} corresponding to the eigenvalue $\beta \in (1, 2)$.

Proof. Let \vec{h} denote the row vector $(|\eta(0)|, |\eta(1)|, \dots, |\eta(2d)|)$.

By Item 1 of the assumptions, the lengths of a sufficiently long bispecial factor in $\eta(\mathbf{u}_{2d+1})$ and any of its return words is $\vec{h}\vec{w} + |s| + |p|$ and $\vec{h}\vec{r}$, respectively, where $s, p \in \mathcal{F}$.

By Item 2 of the assumptions, $\vec{h}\vec{w} + |s| + |p| \leq \frac{1}{3-\Lambda} \vec{h}\vec{r}$. Let us specify this inequality for the sufficiently long palindromic bispecial factors of \mathbf{u}_{2d+1} .

For a sufficiently long bispecial factor w_n of Type I, the inequality reads (cf. (9))

$$\vec{h} \left(A - \frac{\Lambda-2}{3-\Lambda} I \right) M^n \vec{e}_{2d} + |s_n| + |p_n| \leq \vec{h} B_i \vec{e}_0, \quad (14)$$

and for a bispecial factor w_n of Type II (cf. (11))

$$\vec{h} \left(A - \frac{1}{3-\Lambda} (M - 2I) \right) M^n \vec{e}_{2d} + |s_n| + |p_n| \leq \vec{h} B_i \vec{e}_0. \quad (15)$$

Let S_M be the set of all eigenvalues of the incidence matrix M of the morphism φ_{2d+1} . Recall (see (17) in Appendix) that the characteristic polynomial of M has just two eigenvalues outside the unit circle, $\Lambda \in (2, 3)$ and $\beta \in (1, 2)$.

By Lemma 21, $\vec{h} M^n \vec{e}_{2d}$ equals $\sum_{\lambda \in S_M} d_\lambda \lambda^n$ for some constants d_λ 's. Moreover, the constant d_λ is zero if and only if \vec{h} is orthogonal to an eigenvector corresponding to the eigenvalue $\lambda \in S_M$.

Let us stress that $\vec{h} M^n \vec{e}_{2d}$ is a sum of $2d + 1$ expressions of the type $d_\lambda \lambda^n$. All but two of them (corresponding to $\lambda = \Lambda$ and $\lambda = \beta$) are bounded sequences.

As $A = (M - 2I)(3I - M)^{-1}$ and s_n, p_n belong to a finite set $\mathcal{F} \subset \mathcal{B}^*$, the inequalities (14) and (15) have the form

$$d_\beta \left(\frac{\beta-2}{3-\beta} - \frac{\Lambda-2}{3-\Lambda} \right) \beta^n \leq a_n, \quad \text{where } (a_n) \text{ is a bounded sequence}$$

and

$$d_\beta \left(\frac{\beta-2}{3-\beta} - \frac{\beta-2}{3-\Lambda} \right) \beta^n \leq b_n, \quad \text{where } (b_n) \text{ is a bounded sequence.}$$

For the last step of the proof, we take advantage of the facts that $\lim_{n \rightarrow \infty} \beta^n = +\infty$ and $\beta^d(\beta - 2) = -1$, see (17). Let us adjust the expressions on the left sides of the inequalities.

$$\frac{\beta-2}{3-\beta} - \frac{\Lambda-2}{3-\Lambda} = \frac{\beta-\Lambda}{(3-\beta)(3-\Lambda)} \quad \text{and} \quad \frac{\beta-2}{3-\beta} - \frac{\beta-2}{3-\Lambda} = (\beta-2) \frac{\beta-\Lambda}{(3-\beta)(3-\Lambda)} = -\frac{1}{\beta^d} \frac{\beta-\Lambda}{(3-\beta)(3-\Lambda)}.$$

Obviously, if the coefficient d_β is non-zero, the left-hand sides of the previous inequalities have opposite infinite limits. This means that at least one inequality does not hold, which is impossible. Hence, $d_\beta = 0$, equivalently, the vector \vec{h} is orthogonal to the eigenvector of M corresponding to the eigenvalue β . \square

Last but not least, let us explain that it seems reasonable to search for a suitable morphism η in the class P_{ret} defined in [3].

Definition 19. A morphism $\eta : \mathcal{A}^* \rightarrow \mathcal{B}^*$ is of class P_{ret} if η is injective and there exists a palindrome $p \in \mathcal{B}^*$ such that for all $i \in \mathcal{A}$

$$\eta(i) = pq_i,$$

where q_i is a palindrome and $\eta(i)p$ is a complete return word to p .

Durand [11] introduced the notion of derived sequence: Let \mathbf{v} be a uniformly recurrent sequence over \mathcal{A} and let x be a prefix of \mathbf{v} . There exist only finitely many return words to x in \mathbf{v} , denote them r_0, r_1, \dots, r_{k-1} . The sequence \mathbf{v} can be written as a concatenation of the return words, say $\mathbf{v} = r_{i_0}r_{i_1}r_{i_2}\dots$. The sequence $i_0i_1i_2\dots$ over the alphabet $\{0, 1, \dots, k-1\}$ is called the *derived sequence* of \mathbf{v} to the prefix x .

On the one hand, as proven in [3], if p is a palindromic factor of a uniformly recurrent rich sequence \mathbf{v} , then a suffix \mathbf{v}' of \mathbf{v} is an image of a rich sequence \mathbf{u} under a morphism from the class P_{ret} and \mathbf{u} is the derived sequence of \mathbf{v}' to the prefix p . On the other hand, the image of a rich sequence \mathbf{u} by a morphism $\eta \in P_{\text{ret}}$ may not be rich.

For morphisms η of class P_{ret} , to show richness of $\eta(\mathbf{u}_d)$, we do not have to check the bilateral orders of all bispecial factors, as required by Theorem 4, Item 3. It suffices to check the bilateral orders of the bispecial factors that are contained in the letter images by π followed by the common palindrome p . See the next proposition.

Proposition 20. Let $\eta : \{0, 1, \dots, d-1\}^* \rightarrow \mathcal{B}^*$ be a morphism of class P_{ret} from Definition 19. If every bispecial factor w in $\eta(\mathbf{u}_d)$, which does not contain the palindrome p , satisfies

- $b(w) = 0$ if w is not a palindrome;
- $b(w) = \#\text{Pext}(w) - 1$ if w is a palindrome,

then $\eta(\mathbf{u}_d)$ is rich.

Proof. In order to prove the statement, it suffices to check the condition on bilateral orders from Theorem 4, Item 3, for bispecial factors of $\eta(\mathbf{u}_d)$ containing the palindrome p . Every bispecial factor v in $\eta(\mathbf{u}_d)$ containing p arises from a bispecial factor z in \mathbf{u}_d in the following way: If $v = xp \cdots py$, where we highlight the first and the last occurrence of p in v , then $v = x\eta(w)py$, where w is a bispecial factor (possibly empty) in \mathbf{u}_d and $x = \text{lcs}\{\eta(a), \eta(b)\}$ and $py = \text{lcp}\{\eta(c)p, \eta(d)p\}$ for some a, b left extensions of w and c, d right extensions of w . In the sequel, we use the knowledge of both-sided extensions of bispecial factors w in \mathbf{u}_d from [12].

1. Assume w has two left and two right extensions.

- If w is a palindrome, then $\text{Bext}(w) = \{aw(d-1), (d-1)wa, (d-1)w(d-1)\}$ or $\text{Bext}(w) = \{aw(d-1), (d-1)wa, awa\}$. Consequently, $v = x\eta(w)p\bar{x}$, where $x = \text{lcs}\{\eta(a), \eta(d-1)\}$. Since $\eta \in P_{\text{ret}}$, the factor v is a palindrome and $b(v) = 0 = \#\text{Pext}(v) - 1$.
- If w is not a palindrome, then $\eta(w)p$ is not a palindrome and $v = x\eta(w)py$ is not a palindrome, either. Since $b(w) = 0$, clearly $b(v) = 0$.

2. Assume w has more left or right extensions. Then $w = F_i = \varphi^{i-1}(0)\varphi^{i-2}(0) \cdots \varphi(0)0$ for some $0 \leq i \leq d-3$ (for $i = 0$ we set $w = \varepsilon$) and

$$\text{Bext}(w) = \{iwj, jwi, (d-1)w(d-1) : i < j \leq d-1\}.$$

(a) Consider $j \leq d-1$, then iwj and jwi give rise to the palindromic bispecial factor $v = x\eta(w)p\bar{x}$, where $x = \text{lcs}\{\eta(i), \eta(j)\}$. If ax is a suffix of $\eta(i)$ and bx is a suffix of $\eta(j)$, then iwj, jwi , resp. $(d-1)w(d-1)$ give rise to the both-sided extensions of v : avb, bva , resp. bvb in case $j = d-1$.

If cx is a suffix of $\eta(k)$ for some $k \neq i, j$, then iwk and kwi give rise to the both-sided extensions avc, cva of v . If $k = d-1$, then cvc is also a both-sided extension. If $c = b$, the extensions avc, cva are not new. If $c = a$, we get the new extension awa . If $c \neq a, b$, we have new extensions avc, cva , eventually cvc for $k = d-1$.

To summarize, the set $\text{Bext}(v)$ of both-sided extensions of v is the union of the following sets (some of them may be empty):

- $\{avb, bva\}$;
- $\bigcup_{i < k \leq d-1, k \neq j} \{c_kva, avc_k : c_kx \text{ is suffix of } \eta(k) \text{ and } c_k \neq a, b\}$;
- $\{ava : ax \text{ is suffix of } \eta(k) \text{ for some } i < k \leq d-1\}$;
- $\{c_{d-1}vc_{d-1} : c_{d-1}x \text{ is suffix of } \eta(d-1) \text{ and } c_{d-1} \neq a\}$.

It is now straightforward to check that $b(v) = \#\text{Pext}(v) - 1$.

(b) Consider $j < d-1$, then iwj and $(d-1)w(d-1)$ give rise to the bispecial factor $v = x\eta(w)py$, where $x = \text{lcs}\{\eta(i), \eta(d-1)\}$ and $py = \text{lcp}\{\eta(j)p, \eta(d-1)p\}$. If $y = \bar{x}$, we come back to the case (a). Assume thus $y \neq \bar{x}$, i.e., v is not a palindrome. Let us treat the case where $|y| > |x|$, i.e., y has the prefix \bar{x} . The opposite case may be treated analogously.

If ax is a suffix of $\eta(i)$ and bx is a suffix of $\eta(d-1)$ and pyc is a prefix of $\eta(j)p$ and pyd is a prefix of $\eta(d-1)p$, then $iwj, (d-1)w(d-1), iw(d-1)$ give rise to the both-sided extensions of v : avc, bvd, avd .

If for some letter $f \neq c, d$ the word pyf is a prefix of $\eta(\ell)p$ for some $\ell \neq j$, then iwl gives rise to the both-sided extension avf of v .

It follows that $b(v) = 0$.

- (c) Consider $j < d - 1$, then jwi and $(d - 1)w(d - 1)$ give rise to the bispecial factor \bar{v} , where v was defined in the previous item. By closedness of $\eta(\mathbf{u}_d)$ under reversal it holds $b(\bar{v}) = 0$, too.
- (d) Consider $j, k \leq d - 1, j \neq k$, then iwj and kwi give rise to the bispecial factor $v = x\eta(w)py$, where $x = \text{lcs}\{\eta(i), \eta(k)\}$ and $py = \text{lcp}\{\eta(i)p, \eta(j)p\}$. If $y = \bar{x}$, we come back to the case (a). Assume thus $y \neq \bar{x}$, i.e., v is not a palindrome. Let us treat the case where $|y| > |x|$, i.e., y has the prefix \bar{x} . The opposite case may be treated analogously.

If ax is a suffix of $\eta(i)$ and bx is a suffix of $\eta(k)$ and pyc is a prefix of $\eta(j)p$ and pyd is a prefix of $\eta(i)p$, then iwj, kwi, jwi give rise to the both-sided extensions of v : avc, bvd, avd .

If for some letter $e \neq a, b$ the word ex is a suffix of $\eta(\ell)$ for some $\ell > i$, then ℓwi give rise to the both-sided extensions evd of v .

If for some letter $f \neq c, d$ the word pyf is a prefix of $\eta(\ell)p$ for some $\ell > i$, then iwl gives rise to the both-sided extension avf of v . By the way, the word ℓwi provides the both-sided extension avd of v , which is not new.

It follows that $b(v) = 0$. □

Based on the known repetition threshold of binary and ternary rich sequences and on some observations for quaternary rich sequences, we expect that suitable candidates for rich sequences with minimum critical exponent will show a large degree of symmetry [13].

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8 Appendix

First, let us recall a simple property of square matrices.

Lemma 21. *Let $M \in \mathbb{R}^{k \times k}$ have k distinct eigenvalues. For every eigenvalue λ , a row vector \vec{u}_λ and a column vector \vec{v}_λ denote a left and a right eigenvectors of the matrix M corresponding to λ . Then for every k -dimensional row vector \vec{h} , every k -dimensional column vector \vec{g} and every $n \in \mathbb{N}$*

$$\vec{h} M^n \vec{g} = \sum_{\lambda \in S} a_\lambda \lambda^n,$$

where S denotes the set of eigenvalues of M and $a_\lambda = \frac{1}{\vec{u}_\lambda \vec{v}_\lambda} (\vec{h} \vec{v}_\lambda) (\vec{u}_\lambda \vec{g})$.

Proof. Let $\lambda, \mu \in S$. As $\lambda \vec{u}_\lambda \vec{v}_\mu = \vec{u}_\lambda M \vec{v}_\mu = \mu \vec{u}_\lambda \vec{v}_\mu$, we deduce that if $\lambda \neq \mu$, then $\vec{u}_\lambda \vec{v}_\mu = 0$. In other words, if columns of a matrix $R \in \mathbb{R}^{k \times k}$ are formed by eigenvectors of M corresponding to distinct eigenvalues, then the rows of the inverse matrix R^{-1} are formed by the row vectors \vec{u}_λ normalized by $\frac{1}{\vec{u}_\lambda \vec{v}_\lambda}$. Obviously, $M = R D R^{-1}$, where D is a diagonal matrix with eigenvalues $\lambda \in S$ on its diagonal. Substituting M^n by $R D^n R^{-1}$ we get the required equality. \square

Henceforth, M denotes the incidence matrix of the morphism $\varphi = \varphi_{2d+1}$, i.e.,

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix} \in \mathbb{R}^{(2d+1) \times (2d+1)}. \quad (16)$$

We make use of the following properties of M that are easy to check.

P1 The characteristic polynomial of M equals

$$\chi(t) = \underbrace{(t^{d+1} - 2t^d - 1)}_{=:f(t)} \underbrace{(t^d - t^{d-1} - t^{d-2} - \cdots - t - 1)}_{=:g(t)} = \frac{1}{t-1} (t^{2d}(t-2)^2 - 1).$$

Note that the trinomial f is irreducible by [19] and satisfies the Hollander condition [20] (a polynomial $h(t) = t^n - a_{n-1}t^{n-1} - a_{n-2}t^{n-2} - \cdots - a_0$ with $a_i \in \mathbb{N}, a_0 \neq 0$, satisfies the Hollander condition if $a_{n-1} > a_{n-2} + a_{n-3} + \cdots + a_0$) and thus the dominant root of f , denote it Λ , is a Pisot number, i.e., $\Lambda > 1$ and all other roots of f are in modulus strictly smaller than 1. Similarly, the polynomial g satisfies the Frougny-Solomyak condition [16] (a polynomial $h(t) = t^n - a_{n-1}t^{n-1} - a_{n-2}t^{n-2} - \cdots - a_0$ with $a_i \in \mathbb{N}, a_0 \neq 0$, satisfies the Frougny-Solomyak condition if $a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0$). Hence g is irreducible and the dominant root, denoted β , is also Pisot. In summary, all eigenvalues in the spectrum of M are distinct and the spectrum contains just two

eigenvalues outside the closure of the unit circle, namely $\Lambda \in (2, 3)$ and $\beta \in (1, 2)$. They satisfy

$$\Lambda^{d+1} - 2\Lambda^d = 1 \quad \text{and} \quad \beta^{d+1} - 2\beta^d = -1. \quad (17)$$

Denote S_Λ the set of the algebraic conjugates of Λ and by S_β the set of the algebraic conjugates of β . The spectrum of M equals $S_\Lambda \cup S_\beta$ and the elements of the spectrum are mutually distinct.

P2 The right eigenvector of M corresponding to an eigenvalue $\lambda \in S_\Lambda \cup S_\beta$ is

$$\vec{v}_\lambda = (\lambda^{2d-1}(\lambda - 2), \lambda^{2d-2}(\lambda - 2), \dots, \lambda(\lambda - 2), \lambda - 2, 1)^T;$$

the left row eigenvector to λ is

$$\vec{u}_\lambda = (1, \lambda - 1, \lambda^2 - \lambda - 1, \dots, \lambda^{2d} - \lambda^{2d-1} - \dots - \lambda - 1),$$

equivalently, the i^{th} component of \vec{u}_λ is $\frac{1}{\lambda-1}(\lambda^{i+1} - 2\lambda^i + 1)$ for $i = 0, 1, \dots, 2d$.

In particular, the last entry of \vec{u}_λ is

$$\frac{1}{\lambda-1}(\lambda^{2d+1} - 2\lambda^{2d} + 1) = \begin{cases} \lambda^d & \text{if } \lambda \in S_\Lambda; \\ \frac{-\lambda^{d+1}}{\lambda-1} & \text{if } \lambda \in S_\beta. \end{cases}$$

P3 If $\lambda \in S_\Lambda$, then

$$\vec{u}_\lambda \vec{v}_\lambda = \frac{2\lambda^{d-1}}{\lambda-1} (d(\lambda-2) + \lambda).$$

Lemma 22. Let $n \in \mathbb{N}$ and $\vec{h} = (\underbrace{1, 1, \dots, 1}_{d\text{-items}}, \underbrace{2^0, 2^1, \dots, 2^d}_{(d+1)\text{-items}})$. Then

$$\vec{h} M^n \vec{e}_{2d} = \sum_{\lambda \in S_\Lambda} c_\lambda \lambda^n, \quad \text{where} \quad c_\lambda = \frac{1}{(d+1)\lambda - 2d} \lambda^{d+1}.$$

Proof. By P1, we can apply Lemma 21. Using P2 and (17) we derive

$$\vec{h} \vec{v}_\lambda = \begin{cases} \frac{2\lambda^d}{\lambda-1} & \text{if } \lambda \in S_\Lambda; \\ 0 & \text{if } \lambda \in S_\beta, \end{cases}$$

and $\vec{u}_\lambda \vec{e}_{2d} = \lambda^d$ for $\lambda \in S_\Lambda$. Exploiting P3, we get

$$\vec{h} M^n \vec{e}_{2d} = \sum_{\lambda \in S_\Lambda} \frac{1}{\vec{u}_\lambda \vec{v}_\lambda} \frac{2\lambda^d}{\lambda-1} \lambda^d \lambda^n = \sum_{\lambda \in S_\Lambda} \frac{\lambda^{d+1}}{(d+1)\lambda - 2d} \lambda^n. \quad \square$$