# Lattices of Cyclic Flats of Matroids: from Finite Rank to Finitary and Beyond

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**Abstract**

In the finite case, there are close connections between lattices and matroids. More notably, every lattice can be realised as the lattice of cyclic flats of some matroid. The main concern of this paper is finding analogous connections in the infinite case. In particular, we introduce a class of set systems (which forms a superclass of matroids) whose lattices of cyclic flats model all (possibly infinite) complete lattices.

**Mathematics Subject Classifications:** 05C63, 05B35, 06B05

## 1 Introduction

There are strong and well-established connections between the theory of finite matroids and that of finite lattices. To start with, it is easily seen that the flats of a matroid, ordered by inclusion, form a lattice. However, not every lattice is the lattice of flats of some matroid. For example, a chain of three elements is not atomistic and so cannot be realised as the lattice of flats of any matroid. We shall focus ourselves to the cyclic flats, i.e. flats that are union of circuits, which enjoy two advantages over the flats. Firstly, while an individual lattice of cyclic flats is less structured than the corresponding lattice of flats, interestingly, when going through all the finite(-rank) matroids, the corresponding lattices of cyclic flats produce all the finite (length) lattices, as proved by Sims [16] (see also Bonin and de Mier [3]). The second point to look at cyclic flats is duality, which is an essential aspect of matroid theory. More precisely, the complement of a cyclic flat is a cyclic flat of the dual matroid. But this statement does not always hold if we drop the word 'cyclic'.

While cyclic flats enjoy these properties, they contain less information about the matroid than the set of flats. In fact, a matroid can be described by its flats but not by its

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cyclic flats alone. On the other hand, the pair of cyclic flats and their ranks determines (up to a set of coloops) a finite-rank matroid [8]. Then the question arises: which pairs of a collection of sets and an integer-valued function on this collection come from a matroid? Attractively, Sims [17] (see also [3]) proved a characterisation of pairs which do.

The main objective of this work is extending these results to infinity with the toolbox opened by a landmark paper on infinite matroids by Bruhn et al [6]. After reviewing the basics of matroids and lattices in Section 2 (we tried to make the paper self-contained), in Section 3, we extend the aforementioned characterisation in [17] to finitary matroids, a well-studied class of infinite matroids. By doing so we get necessary and sufficient conditions on a collection of sets together with an integer-valued function to be the collection of cyclic flats and the rank function of a finitary matroid (Theorem 7). For the direction of the proof where we have to show that a suitable finitary matroid exists, we apply the finite-rank version to subsets at which the discussed integer-valued function is finite and stick the resulting finite-rank matroids together to get the desired finitary matroid. We also review a result of Skublics [18] and a few examples as well as give some thoughts on the existence of submodular functions on possibly infinite lattices. In the finite case, such functions always exist by a classical result of Dilworth [10].

Going beyond finitary matroids, we note that the cyclic flats of any matroid form a complete lattice. Whether every complete lattice takes this form or not remains open. In [14], Mao claimed an equivalence between the class of bounded lattices and that of pre-independence spaces in which the closure operator is closed and every dependent set contains a minimal one. However as we shall see, the statement has a minor error and the proof has more serious deficiencies. In Section 4, after reviewing Mao's claim and pointing out the gaps in the proof, we shall prove a variant of that claim, where we show that, instead of a general bounded lattice, every complete lattice is the lattice of cyclic flats of some set system drawn from a superclass of matroids (Theorem 15). This answers a weaker form of the above open question. To do so, we will make use of the notion of thin-sums representability, which was introduced by Bruhn and Diestel in [5] and studied by Afzali and Bowler in [1] to extend the concept of vector matroids to non-finitary representable matroids.

## 2 Preliminaries

We recall some definitions and basic results of matroid theory and lattice theory. Our matroid terminology follows [15] and [6] and for lattices we refer to [12]. Given a family $\mathcal{F}$ of subsets of a set $E$, we denote $\bigcup_{F\in\mathcal{F}} F$ by $\bigcup\mathcal{F}$ and $\bigcap_{F\in\mathcal{F}} F$ by $\bigcap\mathcal{F}$. Given a set $I$ and a singleton $\{x\}$, we write $I+x$ for $I\cup\{x\}$ and $I-x$ for $I\setminus\{x\}$.

For a set $E$ and $\mathcal{I}\subseteq 2^E$, let $\mathcal{I}^{max}$ be the set of maximal elements of $\mathcal{I}$. Then $M=(E,\mathcal{I})$ is called a *matroid* if

(I1) $\emptyset\in\mathcal{I}$;

(I2) if $I\subseteq I'$ and $I'\in\mathcal{I}$, then $I\in\mathcal{I}$;

(I3) for all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$; and

(IM) whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

The sets in $\mathcal{I}$ are the *independent sets* of $M$. The subsets of $E$ not in $\mathcal{I}$ are called *dependent*. Maximal independent sets are *bases* and minimal dependent sets are *circuits* of the matroid. We denote the set of circuits of $M$ by $\mathcal{C}(M)$. A subset of $E$ is *cyclic* [3, 14] (aka fully dependent set [16]) if it is a union of circuits. The *span* of $X \subseteq E$ is

$$\mathrm{Span}(X) := X \cup \{x \in E : \exists I \in \mathcal{I}, I \subseteq X \text{ such that } I + x \text{ is dependent}\}.$$

If $X = \mathrm{Span}(X)$, then $X$ is a *flat*. If there exists a finite largest independent subset $I$ of $X$ then the *rank* of $X$ is defined to be $|I|$, otherwise the rank is $\infty$. The *rank function* of $M$ is a map $r_M : 2^E \to \mathbb{N} \cup \{0, \infty\}$ which to each $X \subseteq E$ assigns its rank.

For a set system $M = (E, \mathcal{I})$ (which is not necessarily a matroid) we still use the same terminology. For example, the sets in $\mathcal{I}$ and the sets not in $\mathcal{I}$ are the independent sets and the dependent sets of $M$, respectively. Moreover, maximal independent sets and minimal dependent sets (which might not exist) are respectively bases and circuits of $M$. Furthermore, a cyclic set, the span of a set, as well as a flat are defined for set systems in analogy to matroids.

A set system, in which a set is independent as soon as all its finite subset are, is called *finitary*. Finitary matroids form an important subclass of infinite matroids, and can be axiomatised in terms of independent sets [6, Corollary 4.4] or circuits. Clearly a matroid is finitary if and only if all its circuits are finite. In the first part of this article, we live in the finitary framework.

For a set $E$ and $\mathcal{I} \subseteq 2^E$, the set system $M = (E, \mathcal{I})$ is a finitary matroid if and only if (I1), (I2), and the following conditions hold:

(I3)$_{\mathbf{fin}}$ for any pair of finite sets $I'$ and $I$ in $\mathcal{I}$ with $|I'| > |I|$ there is $x \in I' \setminus I$ such that $I + x$ is independent; and

(I4) an infinite set is independent if and only if each of its finite subsets is independent.

Alternatively, a finitary matroid can be defined in terms of its circuits. Let $\mathcal{C}$ be a collection of finite subsets of $E$ and $\mathcal{I} := \{I \subseteq E : C \not\subseteq I \text{ for all } C \in \mathcal{C}\}$. Then $(E, \mathcal{I})$ is a finitary matroid if and only if

(C1) $\emptyset \notin \mathcal{C}$;

(C2) no element in $\mathcal{C}$ contains another; and

(C3)$_{\mathbf{fin}}$ for distinct $C_1$ and $C_2$ and $x \in C_1 \cap C_2$ there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - x$.

Clearly $\mathcal{C}$ is the set of circuits of $M$. The proof of the equivalence between these two sets of axioms is very similar to the finite case [15, Theorem 1.1.4]. Although we

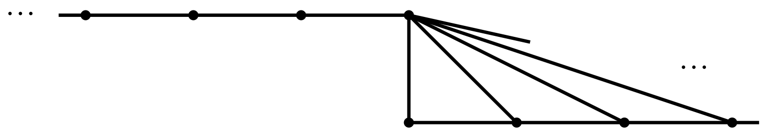

Figure 1: The Bean graph

do not prove it here, contrary to general infinite matroids, finitary matroids could be characterised in terms of their rank function.

A set system satisfying only (I1), (I2) and (I3) is called a *pre-independence space*. A pre-independence space in which the sizes of independent sets are bounded is called a *finite-rank matroid*. A pre-independence space does not need to be a finitary matroid but every finite-rank matroid is clearly a finitary matroid.

Almost all classes of finite matroids, e.g., vector matroids and graphic matroids, naturally extend to finitary matroids. Many of these classes have also non-finitary extensions. For example, given a graph $G$, the finite cycles and the double-rays (infinite paths with no initial and no terminal vertices) form the circuits of the *algebraic cycle matroid* of $G$ on $E(G)$ if and only if $G$ does not contain any subdivision of the *bean graph* (Figure 1) [13] (see also [1]).

Let $(\mathcal{L}, \leqslant)$ be a poset and $S \subseteq \mathcal{L}$. An *upper* (resp. *lower*) bound of $S$ is an element $x$ in $\mathcal{L}$ such that $s \leqslant x$ (resp. $s \geqslant x$) for any $s \in S$. If this bound $x$ happens to be in $S$, then it is unique and is called the *greatest (resp. the least) element* of $S$. The poset is *bounded* if the least element $0_\mathcal{L}$ and the greatest element $1_\mathcal{L}$ of $\mathcal{L}$ exist (the subscripts are often omitted). If the least element of the set of upper bounds of $S$ exists then this element is called the *join* of $S$, denoted by $\bigvee S = \bigvee_{s \in S} s$. Dually, the *meet* of $S$ is the greatest lower bound, if it exists, of $S$ and is denoted by $\bigwedge S = \bigwedge_{s \in S} s$. Denote by $\mathcal{L}_j$ the set of non-zero elements $x \in \mathcal{L}$ that are *join-irreducible*, i.e., $x = y \vee z$ for elements $y, z \in \mathcal{L}$ implies that $x$ equals $y$ or $z$. A chain from $x$ to $y$ is *maximal* if it is not properly contained in any chain from $x$ to $y$. For $x \leqslant y$ the *interval* $[x, y]$ is the set of elements $z$ such that $x \leqslant z \leqslant y$.

Let $\mathcal{L}$ be a *lattice*, i.e., a poset in which every two-element set has a join and a meet. The lattice $\mathcal{L}$ is *complete* if the join and the meet of every subset exist. Clearly every complete lattice is bounded. An element $x$ of a complete lattice $\mathcal{L}$ is *compact* if for any $J \subseteq \mathcal{L}$ with $x \leqslant \bigvee J$, there is a finite subset $J_0$ of $J$ such that $x \leqslant \bigvee J_0$. A lattice $\mathcal{L}' \subseteq \mathcal{L}$ is a *sublattice* of $\mathcal{L}$ if the join and the meet of any two elements in $\mathcal{L}'$ equals to respectively the join and the meet of these two elements in $\mathcal{L}$. A bijective map $f$ from a lattice to another is a *lattice isomorphism* if the join and meet operation interchange with $f$. For $x, y \in \mathcal{L}$, the element $x$ *covers* $y$, denoted by $x \succ y$, if $x > y$ and $\{z : x > z > y\} = \emptyset$. An *atom* is an element covering 0. The *length* of any chain of $k$ elements ($k \geqslant 1$) is $k - 1$. The *height* of an element $x$ is the maximum length of a chain from 0 to $x$ and if there is no finite bound on the length of a chain from 0 to $x$ then the height of $x$ is infinite. We denote the finite height elements of $\mathcal{L}$ by $\mathcal{L}_0$. If $1_\mathcal{L}$ is of finite height, then $\mathcal{L}$ is called of *finite length*.

For a set system $M$, denote by $\mathcal{L}(M)$ the poset of cyclic flats of $M$ (ordered by inclusion). Similar to the finite case [3], we can prove the following.

**Proposition 1.** *The poset of cyclic flats of any matroid $M$ is a complete lattice.*

*Proof.* Let $\mathcal{F}$ be a collection of cyclic flats of $M$. Then

$$\bigvee \mathcal{F} = \mathrm{Span}(\bigcup \mathcal{F})$$

and

$$\bigwedge \mathcal{F} = \bigcup \{C \subseteq \bigcap \mathcal{F} : C \text{ is a circuit}\}. \qquad \square$$

In this paper, a *finitary lattice* $\mathcal{L}$ is a lattice for which there is a finitary matroid $M$ such that $\mathcal{L} \cong \mathcal{L}(M)$. If $M$ is a finite-rank matroid, then $\mathcal{L}(M)$ is a complete lattice of finite length. It can be easily checked that the finite-rank cyclic flats of a matroid form a sublattice of the lattice of cyclic flats.

**Lemma 2.** *Given a finitary matroid $M$, the elements of $\mathcal{L}(M)_0$ are exactly the finite-rank cyclic flats of $M$.*

*Proof.* Consider $F \in \mathcal{L}(M)_0$ and let $0 = F_1, \dots, F_n = F$ be a longest chain from 0 to $F$ in $\mathcal{L}(M)$. Let $C$ be a circuit contained in $F_{i+1}$ but not in $F_i$. Then the span of $C \cup F_i$ must be $F_{i+1}$. As $M$ is finitary, $C$ is finite. Therefore, we can obtain a spanning set of $F_{i+1}$ by adding finitely many elements to a spanning set of $F_i$. So we can inductively see $F$ has a finite spanning set, which means that $F$ has finite rank.

The converse is trivial. $\square$

**Corollary 3.** *For any finitary matroid $M$, any element $F$ of finite height in $\mathcal{L}(M)$ is compact.*

*Proof.* Let $\mathcal{F}$ be a collection of cyclic flats such that $F \leqslant \bigvee \mathcal{F}$. By Lemma 2, $F$ has a finite base $B$. For each $b \in B \cap (\bigcup \mathcal{F})$ pick some $F_b \in \mathcal{F}$ with $b \in F_b$. For each $b \in B \setminus \bigcup \mathcal{F}$ fix a circuit $C_b$ with $b \in C_b$ and $C_b - b \subseteq \bigcup \mathcal{F}$. Pick a finite collection of the elements of $\mathcal{F}$, say $\mathcal{F}_b$, that covers $C_b - b$. Clearly the join of all $F_b$, where $b \in B \cap (\bigcup \mathcal{F})$ and all the elements of $\bigcup_b \mathcal{F}_b$ (which is a finite subcollection of $\mathcal{F}$ as $b$ runs over $B \setminus \bigcup \mathcal{F}$) contains $F$, which means $F$ is compact. $\square$

Let $k$ be a field, $E$ and $A$ be two sets and $f : E \to k^A$ be a function. A *thin dependence* of a set $I \subseteq E$ is a function $c \in k^E$ such that $\mathrm{supp}(c) \subseteq I$ and for any $a \in A$, the set $\{e : c(e)f(e)(a) \neq 0\}$ is finite and

$$\sum_{e \in E} c(e)f(e)(a) = 0.$$

The set $I$ is *thinly independent* iff the only thin dependence of $I$ is the zero function. Let $M_{ts}(f) := (E, \mathcal{I})$, where $\mathcal{I}$ consists of the thinly independent sets defined by $f$. It is not the case that thin-sums systems are always matroids but when they are, they are called *thin-sums matroids.*

## 3 The lattice of cyclic flats of a finitary matroid

Given a finitary matroid and its rank function $r$, a set $I$ is independent if and only if for every finite-rank cyclic flat $F$ the inequality $|F \cap I| \leqslant r(F)$ holds. So a finitary matroid (and thus also a finite-rank matroid) is uniquely determined (up to a set of coloops) by its set of cyclic flats and their ranks. In this section, we extend the following characterisation of collections of subsets of a ground set together with an integer-valued function that form the set of cyclic flats of a finite-rank matroid and their ranks.

**Theorem 4.** *([3],[17])*[1] *Given a set $E$, a subset $\mathcal{L}$ of $2^E$, and a map $r : \mathcal{L} \to \mathbb{N} \cup \{0\}$, there is a finite-rank matroid on $E$ for which $\mathcal{L}$ is the collection of cyclic flats and $r$ the restriction of its rank function to $\mathcal{L}$ if and only if*

*(Z0) $\mathcal{L}$ is a bounded lattice (under inclusion);*

*(Z1) $r(0) = 0$;*

*(Z2) $0 < r(G) - r(F) < |G \backslash F|$ for all sets $F, G$ in $\mathcal{L}$ with $F \subsetneq G$; and*

*(Z3) for any $F$ and $G$ in $\mathcal{L}$,*

$$r(F) + r(G) \geqslant r(F \vee G) + r(F \wedge G) + |(F \cap G) \setminus (F \wedge G)|.$$

Clearly, to extend this theorem to finitary matroids (of possibly infinite rank), we do not only have to allow $r$ to attain the value $\infty$ and find a well-defined formulation of (Z2), but we also have to find the connection between cyclic flats of infinite rank and those of finite rank. Lemma 5 illustrates this connection in finitary matroids and inspires (Z4) of Theorem 7 as an extended version of the above theorem.

**Lemma 5.** *Let $M = (E, \mathcal{I})$ be a finitary matroid. Then a set $F \subseteq E$ is a cyclic flat of $M$ if and only if*

$$F = \bigcup \{F_1 \vee \cdots \vee F_n : n \in \mathbb{N}, F_i \in \mathcal{L}(M)_0, F_i \subseteq F\}.$$

*Proof.* Clearly every cyclic flat of $M$ has this form. So assume that $F = \bigcup\{F_1 \vee \cdots \vee F_n : n \in \mathbb{N}, F_i \in \mathcal{L}(M)_0, F_i \subseteq F\}$. In order to show that $F$ is a flat, let $e \in \mathrm{Span}(F)$. Then there is a circuit $C$ such that $e \in C \subseteq F + e$. The finite set $C - e$ is covered by finitely many sets of the form $F_1 \vee \cdots \vee F_n$ such that each $F_i$ is a finite-rank cyclic flat and a subset of $F$. So their join is a cyclic flat inside $F$ that witnesses that $e \in F$. As the join and the union of cyclic flats is again cyclic, $F$ is a cyclic flat of $M$. □

[1]This is taken from [3], where the authors state that it is a re-discovery of a result from [17]. In fact in [3] Theorem 4 is stated for finite matroids but the proof verbatim works for finite-rank matroids: In the matroid $M_{\mathcal{L}}$ that is constructed in the proof, 1 is a cyclic flat of $M_{\mathcal{L}}$ such that every circuit is contained in it and the length of every chain of $\mathcal{L}$ is bounded by $r(1)$. This makes it possible to show 3.2.7 via induction on the rank of a flat.

**Corollary 6.** *The lattice of finite-rank cyclic flats of $M$ determine the lattice of cyclic flats.*

*Proof.* The two lattices differ only possibly in the join of arbitrary collection of cyclic flats of finite rank. For any collection $\mathcal{F}$ of cyclic flats of finite rank,

$$\bigvee \mathcal{F} = \{F : F \text{ is contained in the join of finitely many elements in } \mathcal{F}\},$$

by viewing an element $F$ in $\mathcal{L}(M)$ as the set of elements less than or equal to $F$. □

We can now formulate our theorem.

**Theorem 7.** *Let $E$ be a set, $\mathcal{L}$ a set of subsets of $E$, and $r$ a map from $\mathcal{L}$ to $\mathbb{N} \cup \{0, \infty\}$. Denote the set $\{F \in \mathcal{L} \colon r(F) < \infty\}$ by $\mathcal{L}_f$. Then there is a finitary matroid on ground set $E$ for which $\mathcal{L}$ is the collection of cyclic flats and $r$ the restriction of its rank function to $\mathcal{L}$ if and only if*

*(Z0) $\mathcal{L}$ is a complete lattice (under inclusion);*

*(Z1) $r(0) = 0$;*

*(Z2) $r(F) \leqslant r(G)$ for any $F$ and $G$ in $\mathcal{L}$ with $F \subsetneq G$, and if further $F$ and $G$ are in $\mathcal{L}_f$ then*

$$0 < r(G) - r(F) < |G \backslash F|;$$

*(Z3) for any $F$ and $G$ in $\mathcal{L}$,*

$$r(F) + r(G) \geqslant r(F \vee G) + r(F \wedge G) + |(F \cap G) \backslash (F \wedge G)|; \text{ and}$$

*(Z4) a set $F \subseteq E$ is in $\mathcal{L}$ if and only if*

$$F = \bigcup \{G_1 \vee \cdots \vee G_n : n \in \mathbb{N}, G_i \in \mathcal{L}_f, G_i \subseteq F\}.$$

*Proof.* If there is such a finitary matroid $M$, then (Z0)-(Z2) clearly hold and (Z4) is true by Lemma 5 and the fact that $\mathcal{L}_f = \mathcal{L}(M)_0$ by Lemma 2. For the proof of (Z3) consider $F, G \in \mathcal{L}(M)$. Either $r(F \vee G)$ is infinite and (Z3) holds as $r(F)$ or $r(G)$ has to be infinite; or $r(F \vee G)$ is finite and (Z3) holds by applying Theorem 4 to $\mathcal{L}(M \upharpoonright F \vee G)$.

Now we assume that (Z0)-(Z4) hold and show that there is such a finitary matroid $M$. By (Z0), $\mathcal{L}$ is a lattice and by (Z3), $\mathcal{L}_f$ is a sublattice of $\mathcal{L}$. For each $F \in \mathcal{L}_f$, applying Theorem 4 to the interval $[0, F]$ in $\mathcal{L}$ yields a finite-rank matroid $M_F$ on ground set $F$ such that its cyclic flats are the elements of $[0, F]$ and the restriction of the rank function of $M_F$ to $[0, F]$ equals the restriction of $r$ to this set. Because a finite-rank matroid is determined by its set of cyclic flats and their ranks, we have that $M_F$ is the restriction of $M_G$ to $F$ whenever $F \leqslant G \in \mathcal{L}_f$.

Let $\mathcal{C} = \bigcup_{F \in \mathcal{L}_f} \mathcal{C}(M_F)$. Now we show that it is the set of circuits of a finitary matroid. Property (C1) is clear. For (C2) and $(\mathrm{C3})_{\mathrm{fin}}$ note that for any two circuits $C_1, C_2$ in $\mathcal{C}$

there are $F_i \in \mathcal{L}_f$ such that $C_i$ is a circuit of $M_{F_i}$. Because both $C_i$ are circuits of $M_{F_1 \vee F_2}$ they satisfy (C2) and $(\mathrm{C3})_{\mathrm{fin}}$ and so $\mathcal{C}$ is the set of circuits of a finitary matroid $M$.

Let $F \in \mathcal{L}_f$ and $C$ a circuit of $M$ that is not a circuit of $M_F$. Then $C$ is a circuit of some matroid $M_G$ with $G \in \mathcal{L}_f$, so it also is a circuit of $M_{F \vee G}$. Because $M_F$ is the restriction of $M_{F \vee G}$ to $F$, the circuit $C$ uses at least one element of $E$ not in $F$. Because $F$ is a flat of $M_{F \vee G}$, the circuit $C$ uses at least two elements of $E$ not in $F$. Hence $M_F$ is the restriction of $M$ to $F$ and $F$ is a finite-rank cyclic flat of $M$.

Let $F$ be a finite-rank cyclic flat of $M$. Then $F$ is the span of the union of finitely many circuits $C_1, \ldots, C_n$ in $M$. Each $C_i$ is a circuit of some $M_{G_i}$, so $C_1 \cup \cdots \cup C_n$ is a subset of $G = G_1 \vee \cdots \vee G_n$. Hence also their span is contained in the cyclic flat $G$ and thus $F \subseteq G$. So $F$ is a cyclic flat of $M_G$ and thus contained in $\mathcal{L}_f$.

Thus $\mathcal{L}_f = \mathcal{L}(M)_0$. By Lemma 5 condition (Z4) translates to $\mathcal{L}(M) = \mathcal{L}$. A cyclic flat in $\mathcal{L}$ but not in $\mathcal{L}_f$ is contained in $\mathcal{L}(M)$ but not in $\mathcal{L}(M)_0$ and thus has infinite rank in $M$. A cyclic flat $F \in \mathcal{L}_f$ has rank $r(F)$ in $M_F$. Because $M_F$ is the restriction of $M$ to $F$, the rank of $F$ in $M_F$ equals the rank of $F$ in $M$. Hence $r$ is the restriction of the rank function of $M$ to $\mathcal{L}$. □

The key point in the proof of Theorem 7 was applying the finite-rank version to sublattices of $\mathcal{L}$ of the form $[0, F]$ with $r(F) < \infty$, and pasting the resulting finite-rank matroids together to a finitary matroid. This approach is also helpful when trying to extend a theorem by Sims that given a lattice $\mathcal{L}$ of finite length, there is a matroid whose lattice of cyclic flats is isomorphic to $\mathcal{L}$ [16][2]. Such an extension is proved by Skublics in [18, Theorem 1][3] without the help of matroid theory, constructing the geometric lattice explicitly by hand. This theorem has one extra requirement on the lattice that does not refer directly to the structure of the lattice itself: the existence of a pseudorank function $p$ on the lattice, i.e., a submodular function mapping the least element to 0, every other element of finite height to a natural number and every element of infinite height to $\infty$ such that $F < G$ implies $p(F) < p(G)$ for all $F$ and $G$ of finite height.

It would be good to have a characterisation of when a pseudorank function exists that is more directly related to the structure of the lattice. As a lattice $\mathcal{L}$ admits a pseudorank function if and only if $\mathcal{L}_0$ is a sublattice that admits a pseudorank function, it is sufficient to characterise which lattices $\mathcal{L}$ with $\mathcal{L} = \mathcal{L}_0$ admit a pseudorank function. We give two examples of lattices not having pseudorank functions in which every element has finite height.

**Example 8.** Let $\mathcal{L}_1$ be the lattice shown on the left in Figure 2. Assume for a contradiction that $\mathcal{L}_1$ admits a pseudorank function $p$, that is, a strictly increasing submodular function mapping the least element 0 to 0 and all other elements to a natural number.

[2] A rank-finite independence space there is what we call a finite-rank matroid here.

[3] What is called a geometric lattice there can easily be seen to be the lattice of flats of a finitary matroid $M$. Adding, for each element of $M$, a parallel one turns all flats of $M$ into cyclic flats while the resulting matroid is still finitary, so $\mathcal{L}$ is also isomorphic to the lattice of cyclic flats of a matroid. The extra assumptions on $\mathcal{L}$ are all necessarily true for a lattice of cyclic flats of a finitary matroid.

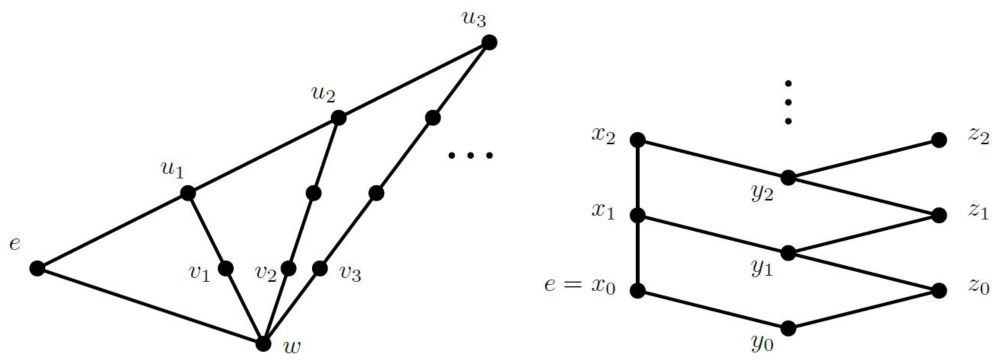


Figure 2: Two lattices in which every element has finite height and on which there is no strictly increasing submodular integer-valued function

Then for each natural number $i$

$$p(e) - p(w) = p(e) - p(e \wedge v_i) \geqslant p(e \vee v_i) - p(v_i) = p(u_i) - p(v_i) \geqslant i$$

where the first and the second inequalities follow from the submodularity and the strict monotonicity of $p$ respectively. This is the desired contradiction because $p(e)$ and $p(w)$ are finite numbers.

Now consider the lattice $\mathcal{L}_2$ shown on the right in Figure 2. Assume again for a contradiction that there is a suitable function $p$. Then for each natural number $i$ by the submodularity and strict monotonicity of $p$

$$p(x_i) - p(y_i) = p(x_i) - p(x_i \wedge z_i) \geqslant p(x_{i+1}) - p(z_i) \geqslant p(x_{i+1}) - p(y_{i+1}) + 1.$$

So by induction

$$p(x_0) - p(y_0) \geqslant p(x_{i+1}) - p(y_{i+1}) + i \geqslant i + 1$$

for all natural numbers $i$, which is again a contradiction.

The authors of [6] successfully extend matroid rank function to infinity by introducing relative rank functions. So, analogously, one may try to reintroduce submodular functions in a way more adaptable to infinity. Although we do not have any strong evidence, it might be worth to try the following candidate. For a lattice $\mathcal{L}$ and $x \leqslant y \in \mathcal{L}$, let $f_0(x, y)$ be the maximum length of a chain from $x$ to $y$. Then inductively define $f_i$ as

$$f_{i+1}(x, y) = \sup\left( \{\sum_{j=0}^{n-1} f_i(z_j, z_{j+1}) \colon y = z_0 < \cdots < z_n = y \vee z : x = y \wedge z\} \right)$$

and finally let $f$ be $\lim_{i \to \infty} f_i(x, y)$. What are the properties of lattice $\mathcal{L}$ that entail finiteness of $f(x, y)$ for each $(x, y)$? What is the connection between $f$ and Dilworth's famous submodular function [10]? Can $f$ be used to prove connections between infinite matroids and lattices? They already do not seem to be trivial questions!

## 4 Complete lattice and cyclic flats

In the finite case, the cyclic flats of a matroid form a lattice under set inclusion, and on the other hand, each lattice is the lattice of cyclic flats of some matroid [16]. While the cyclic flats of any infinite matroid still form a complete lattice, it is not known if any complete lattice is the lattice of cyclic flats of some matroid! A paper by Mao, [14], contains a theorem about an equivalence between bounded lattices and certain pre-independence spaces, which would settle a variant of the problem. However there is an immediate problem with the claim, namely no incomplete lattice could arise as the lattice of cyclic flats of these set systems. In fact, there are more serious gaps in the proof. After demonstrating these gaps, we shall prove a new theorem that is very close to the one claimed by Mao.

Consider the following properties of a set system:

(C) any dependent set contains a minimal one; and

(Cl') the span of any subset of $E$ is a flat.

In [14, Theorem 3.1], the following is claimed:

> *Every bounded lattice is isomorphic to the lattice of cyclic flats of a set system satisfying (I1), (I2), (I3)$_{fin}$, (C) and (Cl').*

First of all, the span of any union of cyclic flats in a set system satisfying the requirements of this statement is a cyclic flat. Therefore, any lattice of cyclic flats must be complete. Hence, boundedness is insufficient. Moreover, the proof breaks down in two places. Firstly, limit ordinals were not treated when arguing for (IM) by transfinite induction. Secondly, the construction uses linear dependence, which does not allow infinite circuits. In other words, the resulting pre-independence spaces (of which every complete lattice was going to be the lattice of cyclic flats) are finitary. However, as we shall see, there is a lattice that cannot be the lattice of cyclic flats of any finitary pre-independence space. Recall that finitary pre-independence spaces are exactly finitary matroids and automatically satisfy (C) and (Cl') [6].

**Example 9.** There is a complete lattice that cannot be the lattice of cyclic flats of any finitary matroid.

*Proof.* Let $G$ be the graph such that the vertex set $V(G) = \{v_i, u_i : i \in \mathbb{Z}\}$ and $E(G) = \{v_i v_{i+1},\ v_i u_i, u_i v_{i+1} : i \in \mathbb{Z}\}$ (Figure 3). Consider the algebraic cycle matroid $M$ of $G$ and let $\mathcal{L} := \mathcal{L}(M)$. The atoms of $\mathcal{L}$ are the triangles $T_i = \{v_i v_{i+1}, v_i u_i, u_i v_{i+1}\}$ and the double-ray $R = \{v_i v_{i+1} : i \in \mathbb{Z}\}$.

As $R$ is contained in the union of the triangles $T_i$, it is below $\bigvee_{i \in \mathbb{Z}} T_i$ in $\mathcal{L}$. Moreover, as $R$ is an atom (and so of finite height), it is compact by Corollary 3. However, this contradicts the fact that $R \not\leq \bigvee_{i \in I} T_i$ for any finite set $I \subseteq \mathbb{Z}$. □

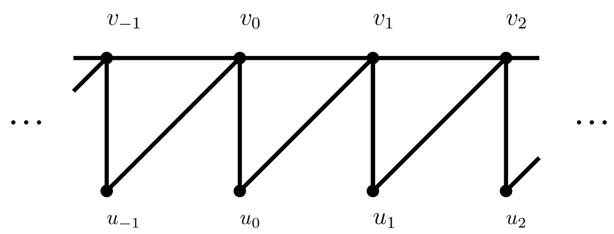


Figure 3: The graph for Example 9

Next, we introduce a superclass of matroids that also contains the spaces considered in [14]. For a set $E$ and $\mathcal{I} \subseteq 2^E$, the set system $M = (E, \mathcal{I})$ is an *almost-closed space* if it satisfies (I1), (I2), (C) and the following condition:

(Cl) the span of any union of cyclic flats is a flat.

Matroids are examples of almost-closed spaces. On the other hand, the algebraic cycle system of the Bean graph does not satisfy (I3) (e.g., [1]), and therefore is not a matroid, but it is an almost-closed space. For the latter, as proved in [6] and [1], the algebraic cycle system of the Bean graph satisfies (C) and a simple case analysis argument gives a proof of (Cl). Later in Proposition 17 we will see an almost-closed space that does not satisfy (IM).

**Lemma 10.** *Given any almost-closed space $M = (E, \mathcal{I})$ and $X \subseteq E$, for any $x \in$ Span$(X) \setminus X$ there is a circuit $C$ with $x \in C \subseteq X + x$.*

*Proof.* If $x \in \mathrm{Span}(X) \setminus X$, then by definition there is $I \in \mathcal{I}$ with $I \subseteq X$ and $I + x$ dependent. So, by (C), $I + x$ contains a minimal dependent set $C$. On the other hand, $C \nsubseteq I$ by (I2), and so $x \in C \subseteq I + x \subseteq X + x$. □

Applying this lemma, similar to Proposition 1 we can prove the following.

**Proposition 11.** *Given any almost-closed space $M$, the poset $\mathcal{L}(M)$ is a complete lattice.*

With the aid of thin-sums representability, we prove the converse of Proposition 11. We recall a few lemmas and definitions.

**Lemma 12.** *Let $V$ be a vector space, $H$ be a hyperplane and $W$ be a subspace of $V$ such that dim$(W \cap H) = 0$. Then dim$(W) \leqslant 1$.*

*Proof.* Suppose $w_1, w_2 \in W$, where $w_1$ is non-zero, so that $w_1 \notin H$. As $H$ is a hyperplane, the union of a base and $w_1$ is a base of $V$. Hence, $w_2 = w + h$ where $w \in \mathrm{Span}(w_1)$ and $h \in H$. It follows that $w_2 - w \in W \cap H = \{0\}$ and $w_2 \in \mathrm{Span}(w_1)$. □

Given a field $k$, sets $E$ and $A$, a map $f : E \to k^A$ and $X \subseteq E$, we are interested in the set of thin dependences of a circuit of $M_{ts}(f)$. Note that $k^E$ is a vector space over $k$ with the usual addition and scalar multiplication. Moreover, the set of thin dependences of a fixed circuit form a subspace of $k^E$. Let $N(X)$ denote the set $\{a \in A : f(x)(a) \neq 0 \text{ for some } x \in X\}$. The following lemma was proved in [1].

**Lemma 13.** *The support of any thin dependence that is not the zero function is a union of circuits in $M_{ts}(f)$.*

Recall that an affine equation over a set $X$ with coefficients in $k$ is a pair of some $a \in k$ and a collection $(\alpha_x \in k : x \in X)$ such that only finitely many $\alpha_x$ are non-zero. A family $(a_x \in k : x \in X)$ is a solution of this equation if $\sum_{x\in X} \alpha_x a_x = a$. The following was proved in [9], and then again in [1] using ideas from [7].

**Lemma 14.** *Given a system $\mathcal{A}$ of affine equations over a fixed set $X$ and with coefficients in $k$, there is a solution of $\mathcal{A}$ if and only if there is a solution for every finite subset of $\mathcal{A}$.*

Suppose that $(\mathcal{L}, \leqslant)$ is a complete lattice. Recall that we can replace the elements of $\mathcal{L}$ with sets and $\leqslant$ with set inclusion as in [3]. To be precise, for each $z \in \mathcal{L}$, let $V_z := \{y \in \mathcal{L} : z \nleqslant y\}$. Then $\mathcal{L}$ is isomorphic to $(\{V_z : z \in \mathcal{L}\}, \subseteq)$ via the bijection $z \mapsto V_z$. Indeed, if $z \leqslant z'$, then any element $y$ in $V_z$ is an element of $V_{z'}$ as $y \ngeqslant z$ implies that $y \ngeqslant z'$. On the other hand, if $V_z \subseteq V_{z'}$, then $z' \in \mathcal{L} \setminus V_{z'} \subseteq \mathcal{L} \setminus V_z$, so $z' \geqslant z$.

Let $B := \mathcal{L} \setminus \{1\}$[4]. Let $A := \{b' : b \in B\}$ be a disjoint copy of $B$, and $S := \{s_z : z \in \mathcal{L}\}$ be a set disjoint from $B \cup A$. Let $E := B \cup S$. Let

$$T := \{bb' : b \in B\} \cup \{s_z b' : z \in \mathcal{L}, b \in V_z\}$$

be a set of indeterminates algebraically independent over $\mathbb{Q}$ and $k = \mathbb{Q}(T)$. Note that each of $bb'$ and $s_z b'$ is treated as a single indeterminate. For any $e \in E$ and $b' \in A$, let $f : E \to k^A$ be

$$f(e)(b') = \begin{cases} bb' & e = b, \\ s_z b' & e = s_z,\ b \in V_z, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 15.** *Given any complete lattice $(\mathcal{L}, \leqslant)$, there is an almost-closed space $M = (E, \mathcal{I})$ such that $\mathcal{L} \cong \mathcal{L}(M)$.*

*Proof.* We claim that for $f$ constructed as above, $M := M_{ts}(f)$ is a desired set system. By the definition of thin independence, $M$ satisfies (I1) and (I2). By definition, any thin dependent set has a thin dependence with non-trivial support. By Lemma 13, this support is a union of circuits. As this support is a subset of the given thin dependent set, (C) is satisfied. We will verify (Cl) and exhibit an isomorphism after characterising the cyclic flats of $M$. First, note that $V_x \cup \{s_x\}$ is a circuit.

**Claim 1:** For any $z \in \mathcal{L}$, the set $F := V_z \cup \{s_x : x \leqslant z\}$ is a cyclic flat.

*Proof.* Note that $F = \bigcup_{x \leqslant z} (V_x + s_x)$, since $V_x \subseteq V_z$ if and only if $x \leqslant z$. Hence, $F$ is cyclic. For any $e \in E \setminus F$, there is some $a \in A \setminus N(F)$ such that $f(e)(a) \neq 0$. Indeed, if $e \in B \setminus F$, we take $a = e'$. If $e \in E \setminus B$, i.e., $e = s_x$ for some $x \nleqslant z$, then $z \in V_x \setminus V_z$ and

[4] Defining $B$ as $\mathcal{L}$ works too, because $M_{ts}(f)$ would have $\{1\}$ as a coloop, exclusion of which ensures that the ground set is a cyclic flat.

we take $a = z'$. As $f(x)(a) = 0$ for any element $x \in F$ and $f(e)(a) \neq 0$, by the definition of thin dependence, $e$ is not in any dependent set contained in $F + e$. In other words, $F = \text{Span}(F)$, which means $F$ is a flat. $\square$

**Claim 2:** Every cyclic flat $F$ of $M$ is of the form $V_z \cup \{s_x : x \leqslant z\}$ where $z = \bigvee\{x : s_x \in F\}$.

*Proof.* We first prove that $N(F) = N(F \cap B)$. Suppose this is not true. Then there is $b \in B \setminus F$ such that $b' \in N(u)$ for some $u \in F$. As $F$ is a cyclic flat, there exists a circuit $C \subseteq F$ containing $u$. Since $b \notin F = \text{Span}(F)$, the set $C - u + b$ is independent in $M$ by the minimality of $C$ and the definition of span.

Note that $f(b)(a) = 0$ unless $a = b'$. Let

$$S := \{d \in k^E : \text{supp}(d) \subseteq C, \sum_{e \in C} d(e)f(e)(a) = 0 \ \forall a \in N(C) - b'\}.$$

It follows that $S \cap \{d \in k^E : d(u) = 0\}$ contains only the zero function, as any non-zero function $d$ in the set can be extended to a thin dependence $d'$ of $C - u + b$ by setting

$$d'(e) = \begin{cases} -\sum_{x \in C-u} d(x)f(x)(b')/bb' & e = b \\ d(e) & e \neq b \end{cases}.$$

Hence, by Lemma 12, $\dim S \leqslant 1$. As the non-zero subspace $D$ of thin dependences of $C$ is a subspace of $S$, the spaces $S$ and $D$ are equal. Consider a non-zero map $d$ in $S$. As $d$ solves all the equations at $N(C) - b'$, it follows that every finite subset of these equations has a solution that does not involve $ub'$. Hence by Lemma 14 the system of equations at $N(C) - b'$ has a solution $d'$ that does not involve $ub'$. But as $S = D$, the map $d'$ must also solve the equation at $b'$ that involves $ub'$. This contradiction shows that $N(F) = N(F \cap B)$.

By completeness, there exists a join $z$ of $\{x : s_x \in F\}$. As $F$ is cyclic, for any $b \in B \cap F$, there is some $s_x \in F$ such that $b' \in N(s_x)$. Hence, $V_z \cup \{s_x : x \leqslant z\} \supseteq F$. On the other hand, let $b \in V_z$. Since $z \not\leqslant b$, by the definition of join, $b \not\geqslant x$ for some $x$ with $s_x \in F$, which means that $b \in V_x$. Since $s_x \in F$ and $N(V_x) = N(s_x) \subseteq N(F) = N(F \cap B)$, the set $V_x$ is a subset of $F \cap B$. This shows that $V_z \subseteq F$. Since $F$ is a flat,

$$\begin{aligned} F &\supseteq \text{Span}(V_z) \\ &= V_z \cup \{s_x : x \leqslant z\} \\ &\supseteq F. \end{aligned}$$

Hence, $F = V_z \cup \{s_x : x \leqslant z\}$. $\square$

To verify (Cl), let $U := \bigcup_{u \in J} \text{Span}(V_u)$ for some $J \subseteq \mathcal{L}$ be a union of cyclic flats. Let $z = \bigvee J$. Then $U = \bigcup_{u \in J} \text{Span}(V_u) \subseteq \bigcup \text{Span}(V_z) = \text{Span}(V_z)$. Hence, $\text{Span}(U) \subseteq \text{Span}(V_z)$. On the other hand, if $b \in V_z$, then $b \not\geqslant u$ for some $u \in J$. So, $b \in V_u$, and so

$V_z \subseteq \bigcup_{u \in J} V_u \subseteq U$. Hence, $\mathrm{Span}(V_z) \subseteq \mathrm{Span}(U)$. Therefore, $\mathrm{Span}(U) = \mathrm{Span}(V_z) = F$ is a flat, and (Cl) is verified.

Define a map $\phi : \mathcal{L} \to \mathcal{L}(M)$ by $\phi(z) = V_z \cup \{s_x : x \leqslant z\} = \mathrm{Span}(V_z)$. We check that $\phi$ is order preserving. Suppose that $z \leqslant z'$. Then $V_z \subseteq V_{z'}$, hence, $\mathrm{Span}(V_z) \subseteq \mathrm{Span}(V_{z'})$. If $z \nleqslant z'$, then $z' \in \mathrm{Span}(V_z) \setminus \mathrm{Span}(V_{z'})$, hence, $\mathrm{Span}(V_z) \nsubseteq \mathrm{Span}(V_{z'})$. Injectivity of $\phi$ is implied by the antisymmetry of the lattice. Moreover, it follows from the claims that $\phi$ is surjective. Hence, $\phi$ is a lattice isomorphism showing that $\mathcal{L} \cong \mathcal{L}(M)$. □

It would be great if almost-closed spaces in the above theorem could be replaced by matroids.

**Question 16.** Given any complete lattice $(\mathcal{L}, \leqslant)$, is there a matroid $M$ such that $\mathcal{L}$ is the lattice of cyclic flats of $M$?

Nathan Bowler [4] conjectured that the closed interval $[0, 1]$ (which is a complete lattice with its natural ordering) is not the lattice of cyclic flats of any matroid. We next show that the almost-closed space in Theorem 15 does not give a counterexample to that conjecture.

**Proposition 17.** *Let $\mathcal{L}$ be a complete lattice that contains an element $t$ such that both $V_t$ and $\mathcal{L} \setminus V_t$ are infinite (e.g., the lattice $[0, 1]$ with any $t \in (0, 1)$). Then the corresponding almost-closed space $M := M_{ts}(f)$ defined by $f$ constructed before Theorem 15 is not a matroid.*

*Proof.* Recall that the ground set of $M$ is $E = B \cup S$. We will show that $M$ does not satisfy (IM) so that $M$ is not a matroid. For $J \subseteq V_t \subseteq B$ define $I(J) := (B \setminus J) \cup \{s_z : z \in \mathcal{L} \setminus V_t\}$. We will argue that $I(J)$ is dependent if and only if $J$ is finite.

Suppose $J = \{b_1, \ldots, b_n\}$ for some $n \in \mathbb{N}$. Let $Z := \{z_1, \ldots, z_n, z_{n+1}\} \subseteq \mathcal{L} \setminus V_t$. If we write $f(s_{z_i})(b'_j)$ as a matrix, there are more columns than rows. Hence there exist

$$c(s_{z_1}), c(s_{z_2}), \ldots, c(s_{z_{n+1}}) \in k,$$

not all zero, such that for each $j \in [n]$,

$$\sum_{i=1}^{n+1} c(s_{z_i}) f(s_{z_i})(b'_j) = 0.$$

Then $c$ can be extended to a non-zero thin dependence of $I(J)$ by defining

$$c(e) := \begin{cases} -\sum_{i=1}^{n+1} c(s_{z_i}) f(s_{z_i})(e')/ee' & \text{for } e \in B \setminus J \\ 0 & \text{for } e \in J \cup \{s_z : z \in \mathcal{L} \setminus (V_t \cup Z)\}. \end{cases}$$

Hence, $I(J)$ is a dependent set.

On the other hand, suppose $J$ is infinite. Let $c$ be a thin dependence of $I(J)$. As $f(s_z)(0') \neq 0$ for all $z \in \mathcal{L} - 0$, by the definition of thin dependence there are only finitely many $z \in \mathcal{L}$ with $c(s_z) \neq 0$, say $z_1, \ldots, z_n$. Let $\{b_1, \ldots, b_n\} \subseteq J$. Then for $j \in [n]$,

$$\sum_{i=1}^{n} c(s_{z_i}) f(s_{z_i})(b'_j) \quad = \quad \sum_{e \in E} c(e) f(e)(b'_j) = 0.$$

By construction, the entries in the $n$ by $n$ matrix defined by $f(s_{z_i})(b'_j)$, as $i, j$ run from 1 to $n$, are non-zero and algebraically independent, so that the matrix is non-singular. Therefore, $c(s_{z_i}) = 0$ for all $i \in [n]$ and hence $c$ is zero on $S$. Thus, for any $b' \in A$,

$$0 = \sum_{e \in E} c(e) f(e)(b') = \sum_{e \in B} c(e) f(e)(b').$$

Since $f(e)(b') \neq 0$ if and only if $e = b$, we conclude that $c(b) = 0$ for all $b \in B$. Hence $c$ is zero everywhere, so $I(J)$ is independent.

Let $J_0 := V_t$. As the independent set $I(J_0)$ cannot be extended to a maximal one in $I(J_0) \cup J_0$, (IM) is violated. Hence, $M$ is not a matroid. □


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