Tensor Product Formulas for the Bollobás–Riordan and Krushkal Polynomials

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Abstract

Brylawski's tensor product formula expresses the Tutte polynomial of the tensor product of two graphs in terms of Tutte polynomials arising from the tensor factors. Analogous tensor product formulas are known for the ribbon graph polynomial and transition polynomials of graphs embedded in surfaces, as well as for the Bollobás–Riordan polynomial in some special cases. We define the tensor product of graphs embedded in pseudo-surfaces and use this to generalize and unify all of the above results, providing Brylawski-style formulas for both the Bollobás–Riordan and Krushkal polynomials.

Mathematics Subject Classifications: 05C31, 05C10

1 Introduction

Let G and H be graphs, G loopless, e a non-loop edge of H, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f\in E(G)}$ a family of bijections where each φ_f sends the ends of an edge f in G to the ends of the edge e in H. The *tensor product* $G \otimes_{\boldsymbol{\varphi}} H$ is the graph obtained by identifying each edge in G with the edge e in its own copy of H in the way consistent with the φ_f, then deleting each of the identified edges. In other words, the tensor product is obtained by, for every edge in G, taking a 2-sum of G and a copy of H. In [5], Brylawski expressed the Tutte polynomial of a tensor product $G \otimes_{\boldsymbol{\varphi}} H$ in terms of polynomials arising from G and H:

$$T(G \otimes_{\boldsymbol{\varphi}} H; x, y) = \zeta^{n(G)}\xi^{r(G)} T\Big(G; \frac{T(H \setminus e; x, y)}{\xi}, \frac{T(H/e; x, y)}{\zeta}\Big), \tag{1}$$

where ζ and ξ are the unique solutions to the system of equations

$$\begin{aligned}(x-1)\zeta + \xi &= T(H \setminus e; x, y),\\ \zeta + (y-1)\xi &= T(H \,/\, e; x, y).\end{aligned}$$

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This result is known as *Brylawski's tensor product formula.* It is a useful computational aid (see the survey [20]) and is significant in the theory of the Tutte polynomial, for example playing a crucial role in Jaeger, Vertigan, and Welsh's seminal work [17] on the computational complexity of the Tutte polynomial. Brylawski proved the tensor product formula for the Tutte polynomial in the more general setting of matroids, but for our purposes we just require its formulation for graphs which we will then generalize in a different direction.

We are interested in extending Brylawski's tensor product formula to analogues of the Tutte polynomial for embedded graphs, or equivalently ribbon graphs. This problem was first considered by Huggett and Moffatt in [14]. Their paper offered three results for computing the Bollobás–Riordan polynomial [3, 4] of a tensor product of two ribbon graphs **G** and **H**. However, these results are either partial results, or require compromises that move them away from the form of (1). For example, Theorem 4.3 and Corollary 4.4 of [14] require that **H** be plane; while Theorems 5.6 and 5.7 of [14] require the use of a multivariate extension of the Bollobás–Riordan polynomial and, for the latter, **G** is replaced by a more complicated ribbon graph. Similarly, in [11] Ellis–Monaghan and Moffatt gave a tensor product formula for a restriction of the Bollobás–Riordan polynomial to two variables, but not for the full 3-variable polynomial. Thus this earlier work does not satisfactorily extend Brylawski's tensor product formula to the Bollobás–Riordan polynomial. Furthermore, we are not aware of any prior work on extending Brylawski's tensor product formula to the Krushkal polynomial [19] of a graph embedded in a surface. (Tensor products of ribbon graphs, expressed in terms of arrow presentations, are defined in Section 2, the Bollobás–Riordan polynomial in Subsection 4.1, and the Krushkal polynomial in Subsection 3.1.)

The tensor product formulas in [11] and [14] required these compromises because the Bollobás–Riordan polynomial has no known deletion-contraction relations that apply to every possible type of edge in a ribbon graph. Here we resolve this difficulty by taking a different approach. Following the work of [15, 18], we consider ribbon graphs whose vertices and boundary components are coloured (or, equivalently, with graphs that are embedded in pseudo-surfaces). We realize these as "packaged arrow presentations" below.

This language allows us to complete the definition of the 2-sum and tensor product for graphs embedded in surfaces. (Previously, the 2-sum along two loops had not been defined.) Furthermore, this more general setting allows us to make use of deletion-contraction constructions of the Bollobás–Riordan polynomial and Krushkal polynomial, to obtain both 2-sum and tensor product formula for these polynomials. Such formulas were first considered in the second-named author's PhD Thesis [23], and the results presented here develops and refines that work.

This paper is structured as follows. In Section 2 we review arrow presentations (which describe graphs cellularly embedded in surfaces) and packaged arrow presentations (which describe graphs embedded in pseudo-surfaces), and define 2-sums and tensor products for them. In Section 3 we introduce a transition polynomial for packaged arrow presentations, provide 2-sum and tensor product formulas for this polynomial, and discuss how this relates to the Krushkal polynomial. In Section 4 we show how our general result specialises to give a tensor product formula for the Bollobás–Riordan polynomial, and also give the known tensor product formulas for the transition polynomial, ribbon graph polynomial and Tutte polynomial.

2 Arrow presentations and packaged arrow presentations

2.1 Arrow presentations

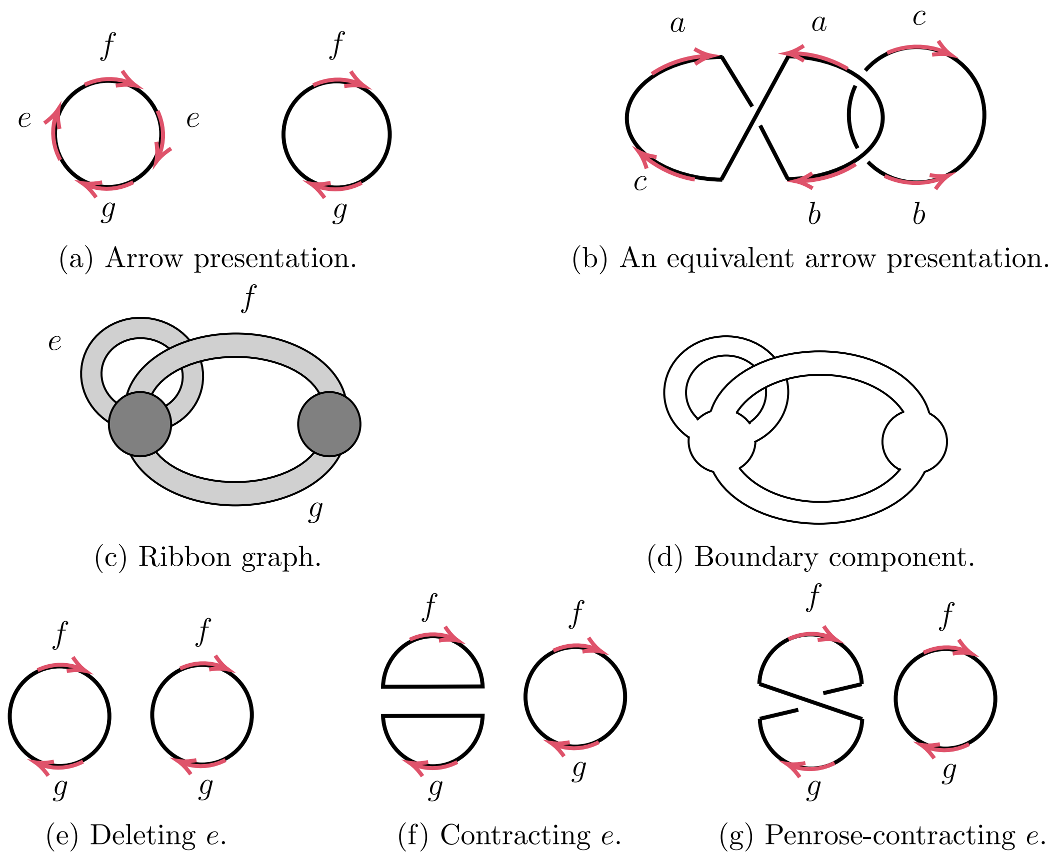
Our interest is in *topological Tutte polynomials*, which are analogues of the Tutte polynomial for graphs embedded in surfaces. In the context of topological Tutte polynomials, graphs cellularly embedded in closed surfaces are usually described using ribbon graphs. Here, however, we find it most natural to describe them using Chmutov's arrow presentations [7], which we review in this section. Additional background on arrow presentations, ribbon graphs and embedded graphs can be found in the book [9].

An *arrow presentation* $\mathbb{G}$ consists of a set of circles (i.e., closed 1-manifolds) called *vertices*, a set of arrows lying disjointly on the circles, and a set of labels. Each label is assigned to exactly two arrows and each arrow has a label. The set of labels is called the *edge set* and its elements are *edges*. Thus an edge is identified with a pair of arrows, and we say that this pair of arrows *constitutes* the edge. We shall use $V(\mathbb{G})$ to denote the vertex set of $\mathbb{G}$, and $E(\mathbb{G})$ its edge set. A vertex is *incident* with an edge e if there is an e-labelled arrow lying on it, and two vertices are *adjacent* if they are both incident to the same edge. (These terms correspond with the standard graph terms.) A vertex is *isolated* if it is not incident to any edge. At times we will express individual arrows in an arrow presentation in the form $\overrightarrow{pq}$ meaning that the arrow lies on the directed arc of a vertex (i.e., a circle) from a point p to a point q. See Figure 1a for an example of an arrow presentation. It has three edges and two vertices. We say two arrow presentations are *distinct* if the share no vertices or edges, and that a collection of arrow presentations is distinct if the arrow presentations in it are all pairwise distinct.

Two arrow presentations are *equivalent* if one can be obtained from the other by a diffeomorphism of the vertices that sends arrows to arrows, reversing the directions of both e-labelled arrows for some (potentially empty) subset of edges e, and a bijection between the edge sets (where the labelling of the arrows respects the bijection). Here we consider arrow presentations up to equivalence. For example, the arrow presentations shown in Figures 1a and 1b are equivalent.

Arrow presentations describe ribbon graphs, which in turn describe graphs cellularly embedded in closed surfaces. We briefly review the connection here. The reader may safely skip this discussion of ribbon graphs, which is included only for context. A *ribbon graph* $\mathbf{G} = (V, E)$ is a surface with boundary represented by the union of a set of discs V, called the *vertices*, and another set of discs E, called the *edges*, satisfying the following conditions: (i) the vertices and edges intersect in disjoint arcs; (ii) each such arc lies on the boundary of precisely one vertex and precisely one edge; (iii) every edge contains precisely two such arcs.

Ribbon graphs can be represented by arrow presentations. Given a ribbon graph,



(a) Arrow presentation. (b) An equivalent arrow presentation. (c) Ribbon graph. (d) Boundary component. (e) Deleting e. (f) Contracting e. (g) Penrose-contracting e.

Figure 1: An arrow presentation, its corresponding ribbon graph, and the results of ribbon graph operations.

obtain its arrow presentation as follows: for each edge e, arbitrarily orient its boundary, place an arrow labelled e in the direction of the orientation along each arc where the edge intersects a vertex. Taking the boundaries of the vertices together with the labelled arrows results in an arrow presentation. Conversely, every arrow presentation gives rise to a ribbon graph. Given an arrow presentation, obtain a ribbon graph by identifying each circle with the boundary of a vertex disc, then for each label e take a disc, arbitrarily orient its boundary, and identify disjoint arcs on its boundary to the two arrows labelled e such that the direction of the arrows agree with that of the boundary. Figures 1a and 1c provide an example of a ribbon graph and its representation as an arrow presentation. These processes set up a 1-1 correspondence between (equivalence classes of) arrow presentations and (equivalence classes of) ribbon graphs. Further details can be found in [7, 9].

While boundary components are immediately defined for ribbon graphs, being the boundary components of the underlying surface, they are a little awkward to define and work with in arrow presentations. (This is the one drawback here of working with arrow presentations, but the benefits outweigh the costs). Let $\mathbb{G}$ be an arrow presentation. Construct a set of closed curves as follows. Let $\overrightarrow{p_{e,1}p_{e,2}}$ and $\overrightarrow{q_{e,1}q_{e,2}}$ be the two arrows constituting an edge e. For each edge e, delete the arrows together with the interiors of

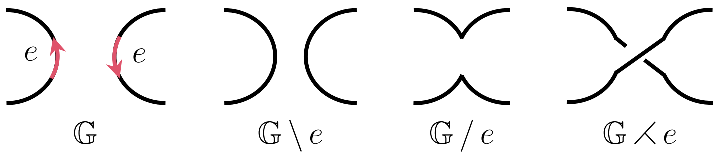


Table 1: Operations on an edge e of an arrow presentation $\mathbb{G}$.

the vertex arcs they lie on. Next add a curve joining $q_{e,2}$ and $p_{e,1}$, and another one joining $p_{e,2}$ and $q_{e,1}$. This results in a set of closed curves. Each curve is a *boundary component* of $\mathbb{G}$. We use $B(\mathbb{G})$ to denote the set of all boundary components of $\mathbb{G}$. We say a boundary component containing any of the points $p_{e,1}$, $p_{e,2}$, $q_{e,1}$, or $q_{e,2}$ is *incident* to the edge e, and two boundary components are *adjacent* if they are incident to a common edge. (These terms correspond exactly to their standard usage for ribbon graphs.) See Figure 1d, which shows the single boundary component of the arrow presentation of Figure 1a.

We recall the operations of deletion, contraction and Penrose-contraction. These are illustrated in Table 1. Let $\mathbb{G}$ be an arrow presentation with edge e. Let $\overrightarrow{p_1p_2}$ and $\overrightarrow{q_1q_2}$ be the two arrows constituting the edge e. Then $\mathbb{G}$ *delete* e, denoted by $\mathbb{G} \setminus e$, is the arrow presentation obtained by removing the pair of arrows labelled e (but not the arcs of the vertices they lie on) from $\mathbb{G}$, and removing e from the edge set. Next, $\mathbb{G}$ *contract* e, denoted by $\mathbb{G} / e$, is the arrow presentation obtained by deleting the arrows $\overrightarrow{p_1p_2}$ and $\overrightarrow{q_1q_2}$ together with the interiors of the vertex arcs they lie on. Then identifying p_1 and q_2, and identifying p_2 and q_1. Then, finally, removing e from the edge set. Similarly, $\mathbb{G}$ *Penrose-contract* e, denoted by $\mathbb{G} \curlywedge e$, is the arrow presentation obtained by deleting the arrows $\overrightarrow{p_1p_2}$ and $\overrightarrow{q_1q_2}$ together with the interiors of the vertex arcs they lie on. Then identifying p_1 and q_1, and identifying p_2 and q_2. Then, finally, removing e from the edge set. As an example, Figures 1e–1g show the results of applying these operations to the edge e of the arrow presentation shown in Figure 1a.

Note that the operations of deletion, contraction and Penrose-contraction all commute with each other and themselves when applied to distinct edges. This follows immediately from the observation that the operations act locally at edges. They also correspond to the usual ribbon graph operations of the same name.

2.2 2-sums and tensor products for arrow presentations

Next we consider 2-sums and tensor products. Although somewhat awkward in terms of ribbon graphs, these operations are straightforward to work with when expressed in the language of arrow presentations. This is the main reason why we work with arrow presentations here rather than ribbon graphs.

Let f be an edge of an arrow presentation $\mathbb{G}$, and e be an edge of an arrow presentation $\mathbb{H}$. A *coupling* of f and e is a bijection φ from the pair of arrows in $\mathbb{G}$ that constitute the edge f to the pair of arrows in $\mathbb{H}$ that constitute the edge e.

The reader may find it helpful to consult Figure 2 when reading the following definition.

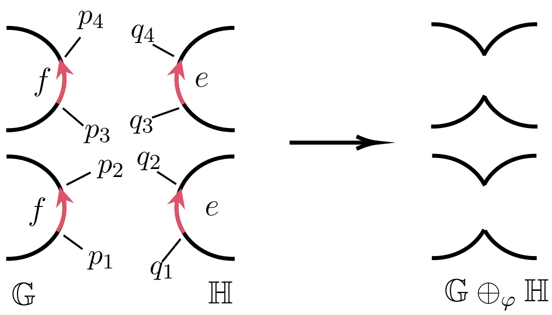


Figure 2: The 2-sum of $\mathbb{G}$ and $\mathbb{H}$ with respect to the coupling φ.

Definition 1. Let $\mathbb{G}$ and $\mathbb{H}$ be arrow presentations, f be an edge of $\mathbb{G}$, e be an edge of $\mathbb{H}$ and φ be a coupling of f and e. Further suppose φ sends the arrow $\overrightarrow{p_1p_2}$ to the arrow $\overrightarrow{q_1q_2}$, and sends the arrow $\overrightarrow{p_3p_4}$ to the arrow $\overrightarrow{q_3q_4}$. Then the *2-sum* of $\mathbb{G}$ and $\mathbb{H}$ with respect to φ, denoted by $\mathbb{G} \oplus_\varphi \mathbb{H}$ is the arrow presentation obtained by deleting the four arrows together with the interiors of the arcs of the vertices that they lie on, and then, for $i = 1, \dots, 4$, identifying p_i and q_i.

As an example, Figure 3 shows a 2-sum of a one-vertex arrow presentation with a three-vertex arrow presentation resulting in a two-vertex arrow presentation. Due to the symmetry of the given example, forming the 2-sum with respect to the other possible coupling of f and e would result in an equivalent arrow presentation, however in general the 2-sum does depend upon φ.

Note that $|V(\mathbb{G}\oplus_\varphi\mathbb{H})|$ equals $|V(\mathbb{G})|+|V(\mathbb{H})|-i$ for some $i \in \{0, 1, 2\}$. The possibilities $i = 0, 1$ occur when each of f and e are incident to only one vertex (i.e., when both edges are loops). Otherwise $i = 2$. This can be verified by examining Figure 2 and considering the ways the arcs shown may be connected to form circles in $\mathbb{G}$ and $\mathbb{H}$, and how this carries through to give the vertices in $\mathbb{G} \oplus_\varphi \mathbb{H}$.

Remark 2. As with graphs, the 2-sum may also be defined using edge operations. For example, using the notation of Definition 1, swap the names of the arrows $\overrightarrow{p_1p_2}$ and $\overrightarrow{q_3q_4}$ then Penrose-contract both e and f in this new arrow presentation. Or alternatively, delete both e and f, then add arrows $\overrightarrow{p_1p_2}$ and $\overrightarrow{q_2q_1}$ constituting a new edge a, and arrows and $\overrightarrow{p_3p_4}$ and $\overrightarrow{q_4q_3}$ constituting a new edge b. Then contract a and b.

The next two results are obvious, following immediately upon drawing how the operations change an arrow presentation.

Lemma 3. *Let $\mathbb{G}$ be an arrow presentation with an edge f, let $\mathbb{H}$ be an arrow presentation with distinct edges e and g, and let $\mathbb{K}$ an arrow presentation with an edge h. In addition let $\varphi_{f,e}$ be a coupling of f and e, and let $\varphi_{g,h}$ be a coupling of g and h. Then*

$$\mathbb{G} \oplus_{\varphi_{f,e}} \mathbb{H} = \mathbb{H} \oplus_{\varphi_{f,e}^{-1}} \mathbb{G} \qquad \textit{and} \qquad \mathbb{G} \oplus_{\varphi_{f,e}} (\mathbb{H} \oplus_{\varphi_{g,h}} \mathbb{K}) = (\mathbb{G} \oplus_{\varphi_{f,e}} \mathbb{H}) \oplus_{\varphi_{g,h}} \mathbb{K}.$$

Lemma 4. *Let $\mathbb{G}$ and $\mathbb{H}$ be arrow presentations, f be an edge of $\mathbb{G}$, and e and g be distinct edges of $\mathbb{H}$. Additionally let φ be a coupling of f and e. Then*

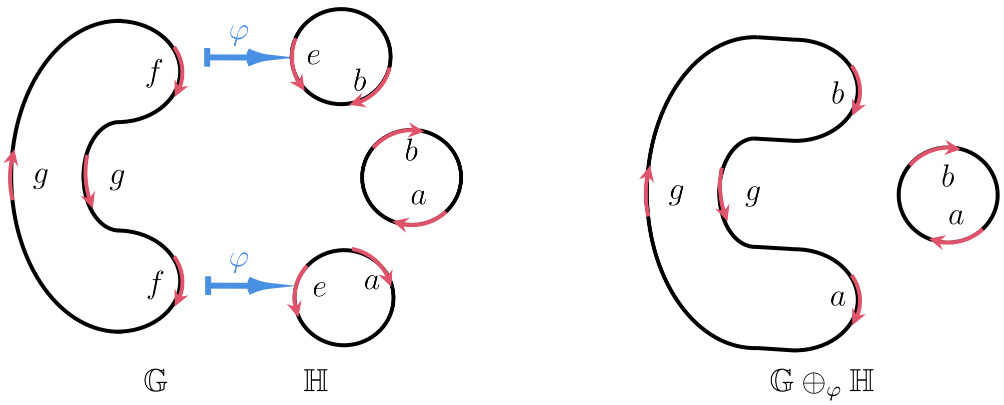


Figure 3: Two arrow presentations, $\mathbb{G}$ and $\mathbb{H}$, with a coupling φ, and the 2-sum $\mathbb{G} \oplus_\varphi \mathbb{H}$.

1. $(\mathbb{G} \oplus_\varphi \mathbb{H}) \setminus g = \mathbb{G} \oplus_\varphi (\mathbb{H} \setminus g)$,

2. $(\mathbb{G} \oplus_\varphi \mathbb{H}) \,/\, g = \mathbb{G} \oplus_\varphi (\mathbb{H} \,/\, g)$, *and*

3. $(\mathbb{G} \oplus_\varphi \mathbb{H}) \curlywedge g = \mathbb{G} \oplus_\varphi (\mathbb{H} \curlywedge g)$.

We now make use of Lemma 3 to define tensor products. The idea is to form a 2-sum at each edge f of $\mathbb{G}$ with some arrow presentation $\mathbb{H}^{(f)}$.

Definition 5. Let $\mathbb{G}$ be an arrow presentation and $\{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})}$ be a family of arrow presentations indexed by the edges of $\mathbb{G}$. All the arrow presentations here are distinct. Further suppose that for each edge f of $\mathbb{G}$ there is a coupling φ_f of f with an edge $e^{(f)}$ of $\mathbb{H}^{(f)}$. Let $\boldsymbol{\varphi} = \{\varphi_f\}_{f\in E(\mathbb{G})}$ denote the indexed set of such couplings. Then the *tensor product* $\mathbb{G} \otimes_{\boldsymbol{\varphi}} \{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})}$ is the arrow presentation obtained by starting with $\mathbb{G}$ and then, for each edge f in $\mathbb{G}$, taking the 2-sum with $\mathbb{H}^{(f)}$ with respect to φ_f:

$$\mathbb{G} \otimes_{\boldsymbol{\varphi}} \{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})} = \mathbb{G} \bigoplus_{\varphi_f \in \boldsymbol{\varphi}} \mathbb{H}^{(f)}.$$

The following special case of a tensor product which repeatedly 2-sums a copy of the same arrow presentation is the analogue of the tensor product appearing in Brylawski's tensor product formula (1).

Definition 6. Let $\mathbb{G}$ be and $\mathbb{H}$ be arrow presentations. Let e be a fixed edge of $\mathbb{H}$ and for each edge f of $\mathbb{G}$ let φ_f be a coupling of f and e, and let $\boldsymbol{\varphi} = \{\varphi_f\}_{f\in E(\mathbb{G})}$. Then

$$\mathbb{G} \otimes_{\boldsymbol{\varphi}} \mathbb{H} = \mathbb{G} \otimes_{\boldsymbol{\psi}} \{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})},$$

where each $\mathbb{H}^{(f)}$ is an arrow presentation equivalent to $\mathbb{H}$ (with all copies distinct); $e^{(f)}$ is the edge in $\mathbb{H}^{(f)}$ corresponding to e; ψ_f is the coupling of f and $e^{(f)}$ induced from φ_f under the equivalence; and $\boldsymbol{\psi} = \{\psi_f\}_{f\in E(\mathbb{G})}$.

2.3 Packaged arrow presentations

Let $\mathbb{G}$ be an arrow presentation. A *vertex partition* of $\mathbb{G}$ is a partition $\mathcal{V}$ of its vertex set $V(\mathbb{G})$. We shall call the blocks of the partition *vertex classes* and use $[v]_{\mathcal{V}}$ to denote the vertex class containing v. Similarly, a *boundary partition* $\mathcal{B}$ of $\mathbb{G}$ is a partition of its set of boundary components $B(\mathbb{G})$. We shall call the blocks of this partition *boundary classes* and use $[b]_{\mathcal{B}}$ to denote the boundary class containing b. At times, when there is no potential for confusion, we omit subscripts and write $[v]$ for $[v]_{\mathcal{V}}$, and $[b]$ for $[b]_{\mathcal{B}}$.

Definition 7. A *packaged arrow presentation* $\underline{\mathbb{G}}$ is a triple $(\mathbb{G}, \mathcal{V}, \mathcal{B})$ where $\mathbb{G}$ is an arrow presentation, $\mathcal{V}$ is a vertex partition of $\mathbb{G}$, and $\mathcal{B}$ is a boundary partition of $\mathbb{G}$.

Terms (and the corresponding notation) such as edges, vertices, boundary components, couplings, etc. of a packaged arrow presentation refer to the edges, vertices, boundary components, couplings, etc., of its arrow presentation.

Convention 8. When depicting packaged arrow presentations in figures we shall draw the vertices (i.e., circles) of the arrow presentation using solid lines, and the boundary components as dashed lines sitting a little away from the solid lines of the arrow presentation. (Which side of these lines they are drawn on is not significant.) We shall indicate to which vertex or boundary class a curve belongs with annotations of the form $[u]_{\mathcal{V}}$, $[b]_{\mathcal{B}}$, etc. Unless otherwise indicated, it is possible that the classes with different labels, $[u]_{\mathcal{V}}$, $[v]_{\mathcal{V}}$ etc., are equal.

As an illustration of these conventions, Figure 4 shows some packaged arrow presentations. Additionally, Figure 8 shows all the packaged arrow presentations on one edge that have no isolated vertices. In that figure $[u]_{\mathcal{V}} \neq [v]_{\mathcal{V}}$ and $[a]_{\mathcal{B}} \neq [b]_{\mathcal{B}}$.

Remark 9. As noted above, arrow presentations correspond to ribbon graphs. Packaged arrow presentations correspond to Huggett and Moffatt's coloured ribbon graphs [15]. (The correspondence is the obvious one: vertex and boundary components in the same block of the partition are given the same colour.) Coloured ribbon graphs arose as a model for graphs embedded in pseudo-surfaces (see [15, Section 2]), and the results in this paper may be expressed in that language too. We shall shortly give definitions of deletion and contraction for packaged arrow presentations. These correspond to the operations on coloured ribbon graphs from [15].

We consider five operations acting on the edges of packaged arrow presentations: deletion, contraction, Penrose-contraction, merge-deletion and merge-contraction. The reader may find it helpful to refer to Table 2 when reading the definitions. Examples of contraction and merge-deletion are given in Figure 4.

For defining the five operations, let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ be a packaged arrow presentation containing an edge e. Let u and v be the vertices incident to e (u may equal v), $[u]$ and $[v]$ be their vertex classes ($[u]$ may equal $[v]$); and let a and b be the boundary components incident to e (a may equal b), and $[a]$ and $[b]$ be their boundary classes ($[a]$ may equal $[b]$). The vertices of $\mathbb{G}$ and $\mathbb{G} \setminus e$ are naturally identified. For $\mathbb{G}/e$ (respectively,

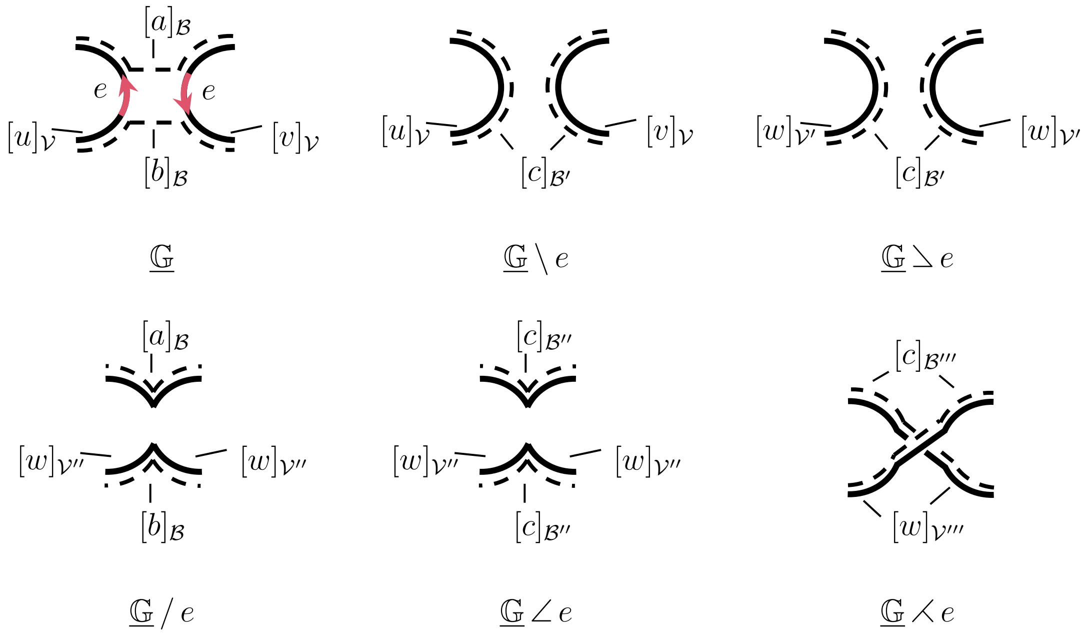


Table 2: Operations on an edge e of a packaged arrow presentation $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$. Here $[u]_{\mathcal{V}}$ and $[v]_{\mathcal{V}}$ may be equal, as may $[a]_{\mathcal{B}}$ and $[b]_{\mathcal{B}}$.

$\mathbb{G} \leftthreetimes e$), each vertex of $\mathbb{G}$ other than u and v is naturally identified with a vertex in $\mathbb{G}/e$ (respectively, $\mathbb{G} \leftthreetimes e$), as these vertices are not altered. We say the remaining vertices in $\mathbb{G}/e$ (respectively, $\mathbb{G} \leftthreetimes e$) are *created by the contraction* (respectively, *created by the Penrose-contraction*). Similarly, the boundary components of $\mathbb{G}$ and $\mathbb{G}/e$ are naturally identified. For $\mathbb{G} \setminus e$ (respectively, $\mathbb{G} \leftthreetimes e$) each boundary component of $\mathbb{G}$ other than a and b is naturally identified with a boundary component in $\mathbb{G} \setminus e$ (respectively, $\mathbb{G} \leftthreetimes e$), as these boundary components are not altered by the operation. We say the remaining boundary components in $\mathbb{G} \setminus e$ (respectively, $\mathbb{G} \leftthreetimes e$) are *created by the deletion* (respectively, *created by the Penrose-contraction*).

- Let S be the set of boundary components of $\mathbb{G} \setminus e$ created by the deletion, and let

  $$\mathcal{V}' = (\mathcal{V} - \{[u], [v]\}) \cup \{[u] \cup [v]\},$$

  and

  $$\mathcal{B}' = (\mathcal{B} - \{[a], [b]\}) \cup \{[a] \cup [b] \cup S - \{a, b\}\}.$$

  Then

  - $\underline{\mathbb{G}}$ *delete* e, denoted by $\underline{\mathbb{G}} \setminus e$, is $(\mathbb{G} \setminus e, \mathcal{V}, \mathcal{B}')$; and
  - $\underline{\mathbb{G}}$ *merge-delete* e, denoted by $\underline{\mathbb{G}} \underline{\setminus} e$, is $(\mathbb{G} \setminus e, \mathcal{V}', \mathcal{B}')$.

- Let T be the set of vertices of $\mathbb{G}/e$ created by the contraction, and let

  $$\mathcal{V}'' = (\mathcal{V} - \{[u], [v]\}) \cup \{[u] \cup [v] \cup T - \{u, v\}\},$$

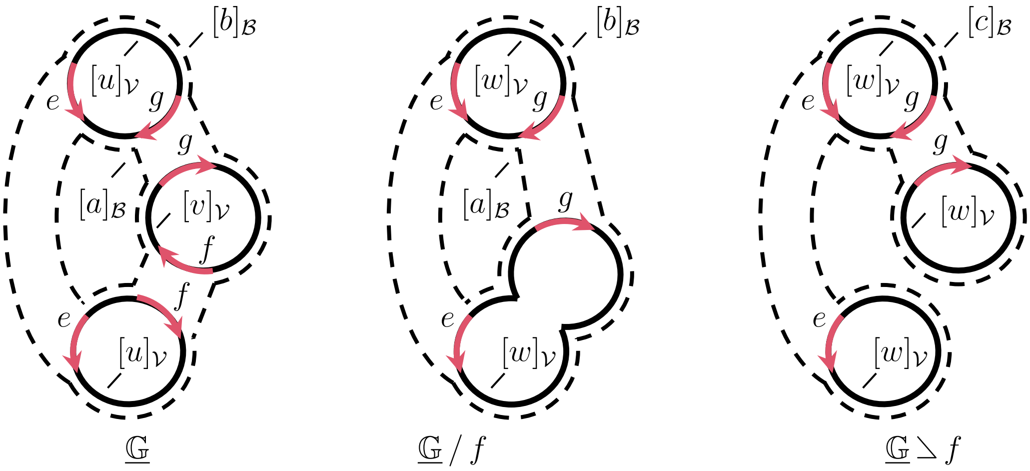


Figure 4: Contracting and merge-deleting the edge f.

and

$$\mathcal{B}'' = (\mathcal{B} - \{[a],[b]\}) \cup \{[a] \cup [b]\}.$$

Then

– $\underline{\mathbb{G}}$ *contract* e, denoted by $\underline{\mathbb{G}}/e$, is $(\mathbb{G}/e, \mathcal{V}'', \mathcal{B})$; and
– $\underline{\mathbb{G}}$ *merge-contract* e, denoted by $\underline{\mathbb{G}} \angle e$, is $(\mathbb{G}/e, \mathcal{V}'', \mathcal{B}'')$.

- $\underline{\mathbb{G}}$ *Penrose-contract* e, denoted by $\underline{\mathbb{G}} \curlywedge e$, is $(\mathbb{G} \curlywedge e, \mathcal{V}''', \mathcal{B}''')$ where

$$\mathcal{V}''' = (\mathcal{V} - \{[u],[v]\}) \cup \{[u] \cup [v] \cup T - \{u,v\}\},$$

where T is the set of vertices of $\mathbb{G} \curlywedge e$ created by the Penrose-contraction; and

$$\mathcal{B}''' = (\mathcal{B} - \{[a],[b]\}) \cup \{[a] \cup [b] \cup S - \{a,b\}\},$$

where S is the set of boundary components of $\mathbb{G} \curlywedge e$ created by the Penrose-contraction.

Lemma 10. *The operations of deletion, contraction, Penrose-contraction, merge-deletion and merge-contraction commute with each other and with one another when acting on different edges.*

Proof. As noted above, deletion, contraction and Penrose-contraction commute for arrow presentations. All that remains is to check that the resulting partitions are the same. Checking this is a straightforward but lengthy calculation and we omit the details. □

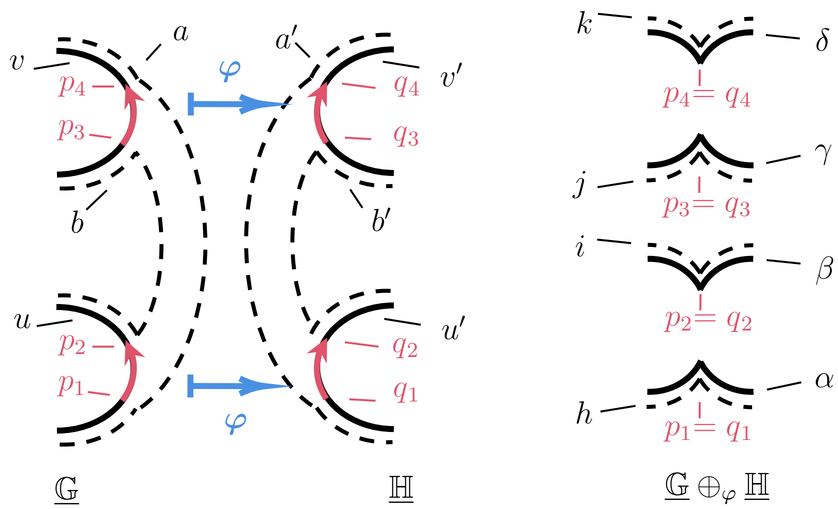


Figure 5: Naming conventions used in Definition 11.

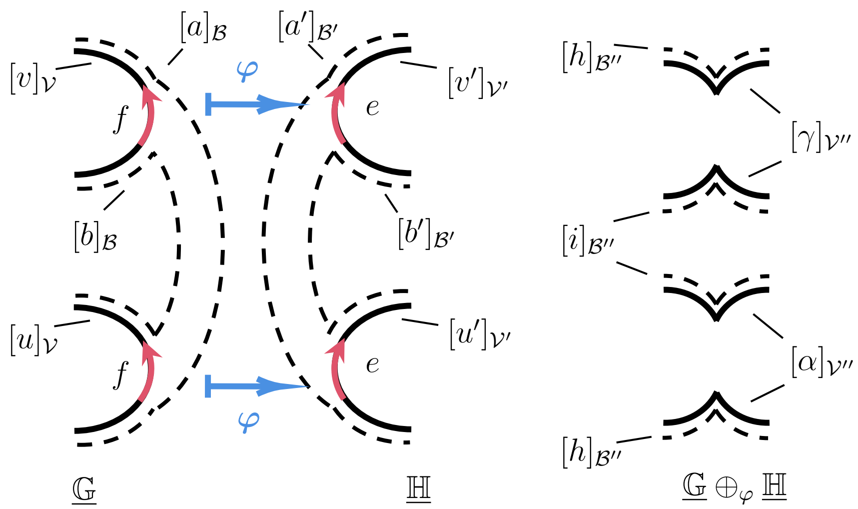


Figure 6: Forming the 2-sum of packaged arrow presentations.

2.4 2-sums and tensor products for packaged arrow presentations

We shall now extend the definition of 2-sums to packaged arrow presentations. Although this 2-sum is in practice straightforward the formal definition is rather cumbersome. This is because we need to keep track of the vertices and boundary components affected by the 2-sum. To aid the reader the naming of the vertex and boundary components used in the definition is indicated in Figure 5, while Figure 6 indicates how the vertex and boundary partitions are determined. Note that in the definition named vertices and boundary components need not be distinct (for example, $[u]_\mathcal{V}$ may equal $[v]_\mathcal{V}$, etc.). An example of a 2-sum of two packaged arrow presentations is given in Figure 7.

Definition 11. Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ and $\underline{\mathbb{H}} = (\mathbb{H}, \mathcal{V}', \mathcal{B}')$ be packaged arrow presentations, f be an edge of $\mathbb{G}$, e be an edge of $\mathbb{H}$ and φ be a coupling of f and e. Further suppose φ

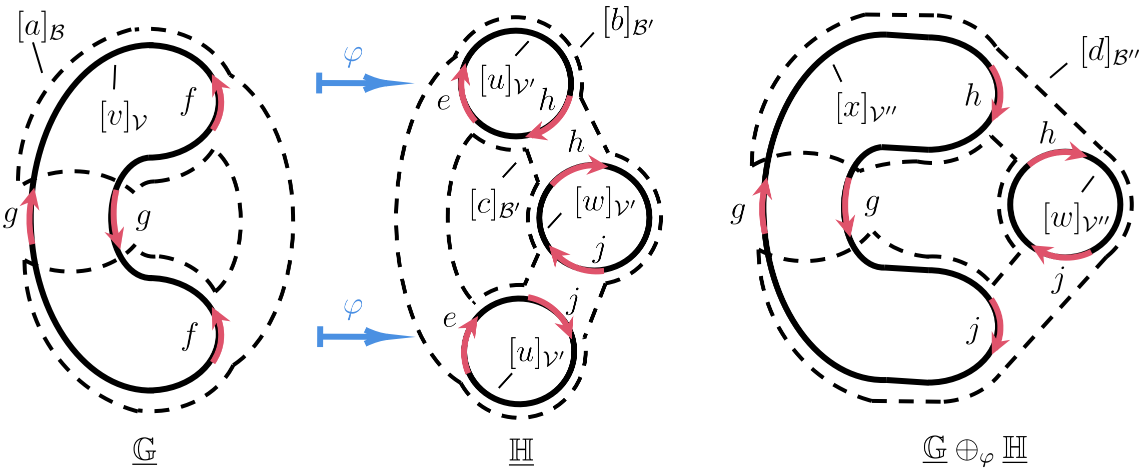


Figure 7: Two packaged arrow presentations, $\underline{\mathbb{G}}$ and $\underline{\mathbb{H}}$, with a coupling φ, and the 2-sum $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{H}}$.

sends the arrow $\overrightarrow{p_1p_2}$ to the arrow $\overrightarrow{q_1q_2}$, and sends the arrow $\overrightarrow{p_3p_4}$ to the arrow $\overrightarrow{q_3q_4}$.

- In $\mathbb{G}$ let u denote the vertex containing $\overrightarrow{p_1p_2}$; v denote the vertex in containing $\overrightarrow{p_3p_4}$; a denote the boundary component containing p_1 and p_4; and b denote the boundary component containing p_2 and p_3.

- In $\mathbb{H}$ let u' denote the vertex containing $\overrightarrow{q_1q_2}$; v' denote the vertex in containing $\overrightarrow{q_3q_4}$; a' denote the boundary component containing q_1 and q_4; and b' denote the boundary component containing q_2 and q_3.

- In $\mathbb{G} \oplus_\varphi \mathbb{H}$ let α denote the vertex, and h the boundary component, containing $p_1 = q_1$; β denote the vertex, and i the boundary component, containing $p_2 = q_2$; γ denote the vertex, and j the boundary component, containing $p_3 = q_3$; and δ denote the vertex, and k the boundary component, containing $p_4 = q_4$.

Then the *2-sum* of $\underline{\mathbb{G}}$ and $\underline{\mathbb{H}}$ with respect to φ, denoted by $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{H}}$ is the packaged arrow presentation $(\mathbb{G} \oplus_\varphi \mathbb{H}, \mathcal{V}'', \mathcal{B}'')$ where

$$\mathcal{V}'' = (\mathcal{V} \cup \mathcal{V}' - \{[u]_\mathcal{V}, [v]_\mathcal{V}, [u']_{\mathcal{V}'}, [v']_{\mathcal{V}'}\}) \cup \{[u]_\mathcal{V} \cup [u']_{\mathcal{V}'} \cup \{\alpha, \beta\} - \{u, u'\}\} \cup \{[v]_\mathcal{V} \cup [v']_{\mathcal{V}'} \cup \{\gamma, \delta\} - \{v, v'\}\},$$

and

$$\mathcal{B}'' = (\mathcal{B} \cup \mathcal{B}' - \{[a]_\mathcal{B}, [b]_\mathcal{B}, [a']_{\mathcal{B}'}, [b']_{\mathcal{B}'}\}) \cup \{[a]_\mathcal{B} \cup [a']_{\mathcal{B}'} \cup \{h, k\} - \{a, a'\}\} \cup \{[b]_\mathcal{B} \cup [b']_{\mathcal{B}'} \cup \{i, j\} - \{b, b'\}\}.$$

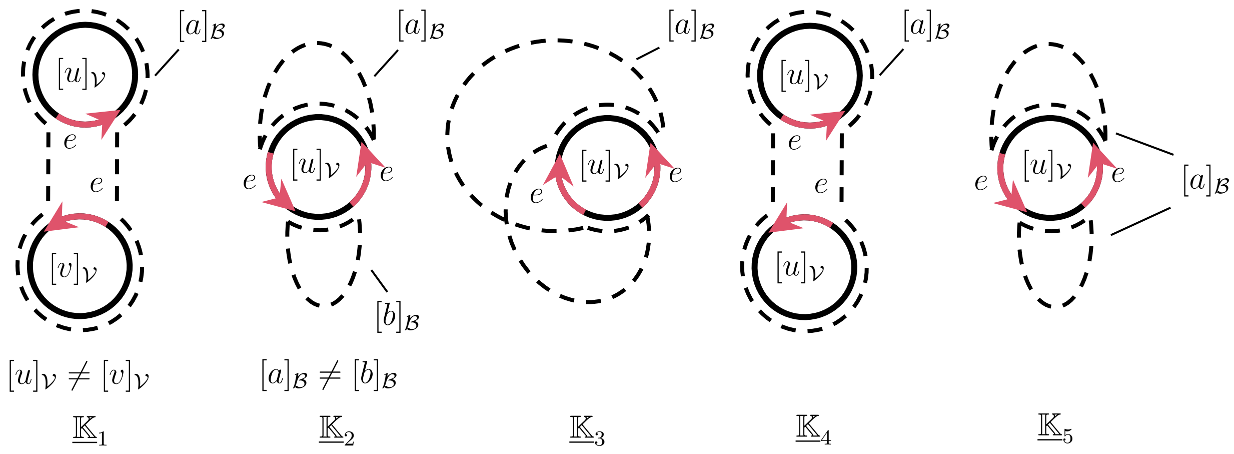


Figure 8: The five packaged arrow presentations with one edge and no isolated vertices. Here $[u]_{\mathcal{V}} \neq [v]_{\mathcal{V}}$ and $[a]_{\mathcal{B}} \neq [b]_{\mathcal{B}}$.

The following two results extend Lemmas 3 and 4. As verifying the results is straightforward but lengthy, we omit their proofs.

Lemma 12. *Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$, $\underline{\mathbb{H}} = (\mathbb{H}, \mathcal{V}', \mathcal{B}')$ and $\underline{\mathbb{K}} = (\mathbb{K}, \mathcal{V}'', \mathcal{B}'')$ be packaged arrow presentations, f be an edge of $\mathbb{G}$, e and g be distinct edges of $\mathbb{H}$, and h be an edge of $\mathbb{K}$. In addition let $\varphi_{f,e}$ be a coupling of f and e, and $\varphi_{g,h}$ be a coupling of g and h. Then*

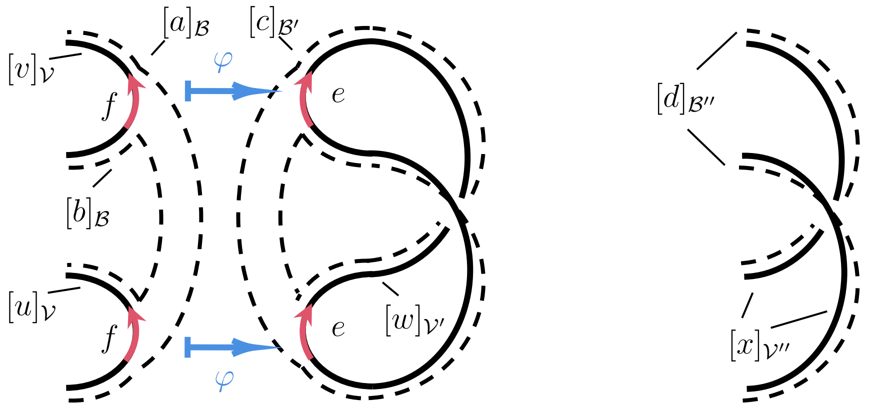
$$\underline{\mathbb{G}} \oplus_{\varphi_{f,e}} \underline{\mathbb{H}} = \underline{\mathbb{H}} \oplus_{\varphi_{f,e}^{-1}} \underline{\mathbb{G}} \qquad \text{and} \qquad \underline{\mathbb{G}} \oplus_{\varphi_{f,e}} (\underline{\mathbb{H}} \oplus_{\varphi_{g,h}} \underline{\mathbb{K}}) = (\underline{\mathbb{G}} \oplus_{\varphi_{f,e}} \underline{\mathbb{H}}) \oplus_{\varphi_{g,h}} \underline{\mathbb{K}}.$$

Lemma 13. *Let $\underline{\mathbb{G}}$ and $\underline{\mathbb{H}}$ be packaged arrow presentations, f be an edge of $\underline{\mathbb{G}}$, and e and g be distinct edges of $\underline{\mathbb{H}}$. Additionally let φ be a coupling of f and e. Then the following hold.*

1. $(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}) \setminus g = \underline{\mathbb{G}} \oplus_{\varphi} (\underline{\mathbb{H}} \setminus g)$.
2. $(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}) \,/\, g = \underline{\mathbb{G}} \oplus_{\varphi} (\underline{\mathbb{H}} \,/\, g)$.
3. $(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}) \curlywedge g = \underline{\mathbb{G}} \oplus_{\varphi} (\underline{\mathbb{H}} \curlywedge g)$.
4. $(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}) \searrow g = \underline{\mathbb{G}} \oplus_{\varphi} (\underline{\mathbb{H}} \searrow g)$.
5. $(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}) \angle g = \underline{\mathbb{G}} \oplus_{\varphi} (\underline{\mathbb{H}} \angle g)$.

Each of the five operations on edges in a packaged arrow presentation (deletion, contraction, Penrose-contraction, merge-deletion, and merge-contraction) can be realized by forming the 2-sum with one of the five distinct packaged arrow presentations on one edge.

Lemma 14. *Let $\underline{\mathbb{G}}$ be a packaged arrow presentation with an edge f, and let $\underline{\mathbb{K}}_1$–$\underline{\mathbb{K}}_5$ be the packaged arrow presentations given in Figure 8. Then*



$\underline{\mathbb{G}}$, $\underline{\mathbb{K}}_3$ and φ $\qquad$ $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_3$ or $\underline{\mathbb{G}} \leftthreetimes f$

Figure 9: Showing $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_3 = \underline{\mathbb{G}} \leftthreetimes f$.

1. $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_1 = \underline{\mathbb{G}} \setminus f$,
2. $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_2 = \underline{\mathbb{G}} \,/\, f$,
3. $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_3 = \underline{\mathbb{G}} \leftthreetimes f$,
4. $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_4 = \underline{\mathbb{G}} \underline{\setminus} f$,
5. $\underline{\mathbb{G}} \oplus_\varphi \underline{\mathbb{K}}_5 = \underline{\mathbb{G}} \angle f$,

here φ is any coupling of f and the singleton edge e in the appropriate $\underline{\mathbb{K}}_i$.

Proof. The result can be verified by comparing the resulting packaged arrow presentations on the left- and right-hand sides of each equation. Figure 9 illustrates this for Item 3. The remaining cases are verified similarly and we omit the details. □

Definition 15. Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ be a packaged arrow presentation and $\{\underline{\mathbb{H}}^{(f)}\}_{f \in E(\mathbb{G})}$ be a family of packaged arrow presentations $\underline{\mathbb{H}}^{(f)} = (\mathbb{H}^{(f)}, \mathcal{V}^{(f)}, \mathcal{B}^{(f)})$ indexed by the edges of $\mathbb{G}$. All the arrow presentations here are distinct. Further suppose that for each edge f of $\mathbb{G}$ there is a coupling φ_f of f with an edge $e^{(f)}$ of $\mathbb{H}^{(f)}$. Let $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\mathbb{G})}$. Then the *tensor product* is the packaged arrow presentation

$$\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \{\underline{\mathbb{H}}^{(f)}\}_{f \in E(\mathbb{G})} = \underline{\mathbb{G}} \bigoplus_{\varphi_f \in \boldsymbol{\varphi}} \{\underline{\mathbb{H}}^{(f)}\}_{f \in E(\mathbb{G})}.$$

Definition 16. Let $\underline{\mathbb{G}}$ be and $\underline{\mathbb{H}}$ be packaged arrow presentations. Let e be a fixed edge of $\underline{\mathbb{H}}$ and for each edge f of $\underline{\mathbb{G}}$ let φ_f be a coupling of f and e, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\mathbb{G})}$. Then

$$\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}} = \mathbb{G} \otimes_{\boldsymbol{\psi}} \{\underline{\mathbb{H}}^{(f)}\}_{f \in E(\mathbb{G})},$$

where each $\underline{\mathbb{H}}^{(f)}$ is a packaged arrow presentation equivalent to $\underline{\mathbb{H}}$ (all the copies are distinct); $e^{(f)}$ is the edge in $\underline{\mathbb{H}}^{(f)}$ corresponding to e; ψ_f is the coupling of f and $e^{(f)}$ induced from φ_f under the equivalence; and $\boldsymbol{\psi} = \{\psi_f\}_{f \in E(\mathbb{G})}$.

2.5 Vertex-partitioned arrow presentations

In Section 4.1 we consider the Bollobás–Riordan polynomial. For this we need to make use of vertex-partitioned arrow presentations. A *vertex-partitioned arrow presentation* $\widehat{\mathbb{G}}$ is a pair $(\mathbb{G}, \mathcal{V})$ where $\mathbb{G}$ is an arrow presentation and $\mathcal{V}$ a partition of its vertex set. We note that as arrow presentations correspond to ribbon graphs, vertex-partitioned arrow presentations correspond to Krajewski, Moffatt, and Tanasa's vertex-partitioned ribbon graphs from [18, Section 4.7] and therefore to graphs embedded in pseudo-surfaces by [15, Corollary 16].

A vertex-partitioned arrow presentation $(\mathbb{G}, \mathcal{V})$ can be regarded as a packaged arrow presentation $(\mathbb{G}, \mathcal{V}, \{B(G)\})$ in which all boundary components are in the same block and vice versa. With this we define versions of all the above packaged arrow presentation terms and operations (such as, deletion, contraction, 2-sums etc.) as those induced from packaged arrow presentations. Alternatively, just remove all mention of boundary partitions in the previous sections.

3 Tensor product formulas for topological Tutte polynomials

3.1 The Krushkal polynomial

3.1.1 A review of the Krushkal polynomial

The *Krushkal polynomial* was introduced by Krushkal in [19] for graphs in orientable surfaces, and extended to graphs in non-orientable surfaces by Butler in [6]. It is a four-variable polynomial that we denote here by $K(G \subset \Sigma; x, y, a, b)$ where $G \subset \Sigma$ is a graph G embedded in a closed surface Σ (but not necessarily cellularly embedded).

Our aim is to extend Brylawski's tensor product formula from the Tutte polynomial to the Krushkal polynomial. A difficulty with the Krushkal polynomial, as originally defined, is that it has no known deletion-contraction relations that apply to an arbitrary edge. (A deletion-contraction relation here is a linear relation between the value of the polynomial $G \subset \Sigma$ and its two values after deleting and contraction one of its edges. It provides a recursive way to compute the polynomial.)

By extending the Krushkal polynomial to vertex partitioned embedded graphs, Krajewski, Moffatt and Tanasa, in [18], found deletion-contraction relations that apply to any edge. This resulted in a way to compute the Krushkal polynomial recursively with the base case consisting of its value on edgeless graphs. A more combinatorial approach to this work was taken by Huggett and Moffatt, in [15], who framed the polynomial as an invariant, $T_{ps}(\underline{\mathbb{G}}; w, x, y, z)$, of partitioned ribbon graphs. As we do not need the details, we omit the definition of T_{ps}, which can be found in [15, Definition 28]. Via Remark 9, both partitioned ribbon graphs and graphs embedded in pseudo-surfaces can be expressed in terms of packaged arrow presentations.

We use an alternative form of T_{ps} that has a simple deletion-contraction relation. For a packaged arrow presentation $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ we let $Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma)$ be the polynomial uniquely defined by

$$Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma) = \begin{cases} a\, Z(\underline{\mathbb{G}} \setminus e; a, b, \alpha, \beta, \gamma) + b\, Z(\underline{\mathbb{G}} \,/\, e; a, b, \alpha, \beta, \gamma) & \text{for any edge } e, \\ \alpha^{|V(\mathbb{G})|}\beta^{|\mathcal{V}|}\gamma^{|\mathcal{B}|} \quad \text{if } \mathbb{G} \text{ is edgeless.} \end{cases} \tag{2}$$

It follows from [15, Theorem 24] that $Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma)$ is well-defined. (As taking $a_i = a$, $b_i = b$, $\alpha = \beta$, $\beta = \gamma$, and $\gamma = \alpha$ in [15, Theorem 24] gives Equation (2).) The polynomial $Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma)$ can be written in terms of T_{ps} by [15, Theorem 29]. An application of [15, Corollary 42] then shows that the Krushkal polynomial can be recovered from $Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma)$.

Remark 17. For reference we include the specific relations between the above polynomials, although we do not use the details here. There are various normalizations of the Krushkal polynomial in the literature. We use the following one. For an embedded graph $G \subset \Sigma$,

$$K(G \subset \Sigma; x, y, a, b) = \sum_{A \subseteq E} x^{r(G)-r(A)} y^{\kappa(A)} a^{\frac{1}{2}s(A)} b^{\frac{1}{2}s^{\perp}(A)}.$$

Using $N(X)$ to denote the surface with boundary defined by neighbourhood of a given subset X of Σ, and $\gamma(N(X))$ its Euler genus, here $s(A) = \gamma(N(V \cup A))$, $s^{\perp}(A) = \gamma(\Sigma \backslash N(V \cup A))$, and $\kappa(A) = \#\text{components}(\Sigma \backslash N(V \cup A)) - \#\text{components}(\Sigma)$. Note that we use here the form of the exponent of y from the proof of Lemma 4.1 of [2] rather than the homological definition given in [19].

For reference, $T_{ps}\,(\underline{\mathbb{G}}; w, x, y, z)$ here is exactly that stated in [15, Definition 28]. By Theorem [15, Theorem 29],

$$Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma) = \beta^{k(\mathbb{G}/\mathcal{V})} \gamma^{k(\mathbb{G}^*/\mathcal{B})} \alpha^{|V(\mathbb{G})|-\rho(\mathbb{G})} b^{\rho(\mathbb{G})} a^{|E(\mathbb{G})|-\rho(\mathbb{G})} \\ T_{ps}\,(\underline{\mathbb{G}}; (\alpha\beta a)/b, (\alpha a)/b, (\alpha\gamma b)/a, (\alpha b)/a)\,. \tag{3}$$

Translating from [15], $k(\mathbb{G}/\mathcal{V})$ is the number of components of the graph with vertex set $\mathcal{V}$ and for edges, if two vertices of $\mathbb{G}$ are adjacent to a common edge, then add an edge between the blocks they lie in. $k(\mathbb{G}^*/\mathcal{B})$ is the number of components of the graph with vertex set $\mathcal{B}$ and for edges, if two boundary components of $\mathbb{G}$ are adjacent to a common edge, then add an edge between the blocks they lie in. Finally $\rho(\mathbb{G}) = \frac{1}{2}(|E(\mathbb{G})| + |V(\mathbb{G})| - |B(\mathbb{G})|)$ (for readers familiar with the terms, we note this equals the rank of $\mathbb{G}$ plus one-half of its Euler genus).

On the other hand T_{ps} can be obtained from Z as follows.

$$T_{ps}\,(\underline{\mathbb{G}}; w, x, y, z) = (x/w)^{k(\mathbb{G}/\mathcal{V})} \, (z/y)^{k(\mathbb{G}^*/\mathcal{B})} \left(\sqrt{xz}\right)^{\rho(\mathbb{G})-|V(\mathbb{G})|} \sqrt{x}^{\,\rho(\mathbb{G})} \sqrt{z}^{\,|E(\mathbb{G})|-\rho(\mathbb{G})} \\ Z\left(\underline{\mathbb{G}}; 1/\sqrt{z}, 1/\sqrt{x}, \sqrt{xz}, w/x, y/z\right). \tag{4}$$

An application of [15, Corollary 42] then relates $Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, \gamma)$ to the Krushkal polynomial. If $\underline{\mathbb{G}}$ is a packaged arrow presentation in which every vertex partition contains a unique element, and $G \subset \Sigma$ is its corresponding graph embedded in a surface constructed as in [15, Section 2.3] then

$$K\left(G \subset \Sigma; x, y, a, b\right) = \sqrt{a}^{-|V(\mathbb{G})|} \sqrt{b}^{\gamma(\Sigma)-|E(\mathbb{G})|+|V(\mathbb{G})|-2k(\mathbb{G}^*/\mathcal{B})} (1/x)^{k(\mathbb{G})} (1/y)^{k(\mathbb{G}^*/\mathcal{B})} \\ Z(\underline{\mathbb{G}}; \sqrt{b}, \sqrt{a}, 1/\sqrt{ab}, ax, by). \quad (5)$$

3.1.2 The tensor product formula

Our aim is to extend Brylawski's tensor product formula, stated here in Equation (1), to $Z(\underline{\mathbb{G}})$. To do so we need to consider a generalisation $Q(\underline{\mathbb{G}})$ of $Z(\underline{\mathbb{G}})$. The reason for this is detailed in Remark 26 below. The generalisation is defined via the following proposition.

Proposition 18. *Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ be a packaged arrow presentation. Then there is a unique map $Q(\underline{\mathbb{G}}) = Q(\underline{\mathbb{G}}; a, b, c, x, y, \alpha, \beta, \gamma)$ from packaged arrow presentations to $\mathbb{Z}[a, b, c, x, y, \alpha, \beta, \gamma]$ such that*

$$Q(\underline{\mathbb{G}}) = \begin{cases} aQ(\underline{\mathbb{G}} \setminus e) + bQ(\underline{\mathbb{G}} \,/\, e) + cQ(\underline{\mathbb{G}} \curlywedge e) + xQ(\underline{\mathbb{G}} \succeq e) + yQ(\underline{\mathbb{G}} \angle e), & \text{for any edge } e; \\ \alpha^{|V(\mathbb{G})|}\beta^{|\mathcal{V}|}\gamma^{|\mathcal{B}|}, & \text{if } \mathbb{G} \text{ is edgeless.} \end{cases}$$

We delay the routine proof of this proposition and instead deduce it from a more general one below.

We shall next state our tensor product formulas for $Q(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}})$ and $Z(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}})$ before considering multivariate versions of them. These two formulas follow immediately from the multivariate extensions, but we state the simpler versions of the results first to help the reader digest the notation.

Theorem 19. *Let $\underline{\mathbb{G}}$ be and $\underline{\mathbb{H}}$ be packaged arrow presentations. Let e be a fixed edge of $\underline{\mathbb{H}}$ and for each edge f of $\underline{\mathbb{G}}$ let φ_f be a coupling of f and e, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\underline{\mathbb{G}})}$. Then*

$$Q(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}}; a, b, c, x, y, \alpha, \beta, \gamma) = Q(\underline{\mathbb{G}}; \phi_{(=,2)}, \phi_{(\|,2)}, \phi_{\times}, \phi_{(=,1)}, \phi_{(\|,1)}, \alpha, \beta, \gamma),$$

where the ϕ's are the unique solutions to

$$\alpha\beta\gamma \begin{bmatrix} \alpha\beta & 1 & 1 & \alpha & 1 \\ 1 & \alpha\gamma & 1 & 1 & \alpha \\ 1 & 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \phi_{(=,2)} \\ \phi_{(\|,2)} \\ \phi_{\times} \\ \phi_{(=,1)} \\ \phi_{(\|,1)} \end{bmatrix} = \begin{bmatrix} Q(\underline{\mathbb{H}} \setminus e; a, b, c, x, y, \alpha, \beta, \gamma) \\ Q(\underline{\mathbb{H}} \,/\, e; a, b, c, x, y, \alpha, \beta, \gamma) \\ Q(\underline{\mathbb{H}} \curlywedge e; a, b, c, x, y, \alpha, \beta, \gamma) \\ Q(\underline{\mathbb{H}} \succeq e; a, b, c, x, y, \alpha, \beta, \gamma) \\ Q(\underline{\mathbb{H}} \angle e; a, b, c, x, y, \alpha, \beta, \gamma) \end{bmatrix}. \quad (6)$$

This result will follow immediately from Theorem 23 below.

Care needs to be taken when applying our results. Here we treat the $a, b, c, x, y, \alpha, \beta, \gamma$ in $Q(\underline{\mathbb{G}}; a, b, c, x, y, \alpha, \beta, \gamma)$ as formal variables. When we write expressions such as, say, $Q(\underline{\mathbb{G}}; a, b, 0, 0, 0, \alpha, \beta, \gamma)$ we mean that we evaluate the polynomial at $c = x = y = 0$. An analogous comment holds for all the polynomials we consider here. In what follows, care must be taken when evaluating the formal variables α, β, γ as it is possible that the matrices specified in our results become singular. There are no such issues when evaluating the other formal variables, so evaluation for a, b, c, x, y may occur at any stage in the application of the theorem. Thus we may treat $a, b, c, x, y, \alpha, \beta, \gamma$ as parameters provided the matrices in the theorem statements remain non-singular and α, β, γ are nonzero.

Taking $c = x = y = 0$ in Theorem 19 gives our tensor product formula for $Z(\underline{\mathbb{G}})$, and hence for $T_{ps}(\underline{\mathbb{G}})$ and $K(G \subset \Sigma)$ via Equations (4) and (5), respectively. Remark 26 below details why in general Q cannot be replaced by Z in the theorem.

Corollary 20. *Let $\underline{\mathbb{G}}$ and $\underline{\mathbb{H}}$ be packaged arrow presentations and e be a fixed edge of $\underline{\mathbb{H}}$. For each edge f of $\underline{\mathbb{G}}$, let φ_f be a coupling of f and e, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\mathbb{G})}$. Then*

$$Z(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}}; a, b, \alpha, \beta, \gamma) = Q(\underline{\mathbb{G}}; \phi_{(=,2)}, \phi_{(\|,2)}, \phi_{\times}, \phi_{(=,1)}, \phi_{(\|,1)}, \alpha, \beta, \gamma),$$

where the ϕ's are the unique solutions to

$$\alpha\beta\gamma \begin{bmatrix} \alpha\beta & 1 & 1 & \alpha & 1 \\ 1 & \alpha\gamma & 1 & 1 & \alpha \\ 1 & 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \phi_{(=,2)} \\ \phi_{(\|,2)} \\ \phi_{\times} \\ \phi_{(=,1)} \\ \phi_{(\|,1)} \end{bmatrix} = \begin{bmatrix} Z(\underline{\mathbb{H}} \setminus e; a, b, \alpha, \beta, \gamma) \\ Z(\underline{\mathbb{H}} / e; a, b, \alpha, \beta, \gamma) \\ Z(\underline{\mathbb{H}} \curlywedge e; a, b, \alpha, \beta, \gamma) \\ Z(\underline{\mathbb{H}} \succeq e; a, b, \alpha, \beta, \gamma) \\ Z(\underline{\mathbb{H}} \angle e; a, b, \alpha, \beta, \gamma) \end{bmatrix}.$$

We turn our attention to a multivariate version of Theorem 19.

Proposition 21. *Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ be a packaged arrow presentation. For each edge e in $E(\mathbb{G})$ let $(a_e, b_e, c_e, x_e, y_e)$ be an ordered tuple of formal variables. Then there is a unique map $\mathbf{Q}(\underline{\mathbb{G}})$ from packaged arrow presentations to $\mathbb{Z}[a_e, b_e, c_e, x_e, y_e, \alpha, \beta, \gamma : e \in E(\mathbb{G})]$ such that*

$$\mathbf{Q}(\underline{\mathbb{G}}) = \begin{cases} a_e \mathbf{Q}(\underline{\mathbb{G}} \setminus e) + b_e \mathbf{Q}(\underline{\mathbb{G}} / e) + c_e \mathbf{Q}(\underline{\mathbb{G}} \curlywedge e) + x_e \mathbf{Q}(\underline{\mathbb{G}} \succeq e) + y_e \mathbf{Q}(\underline{\mathbb{G}} \angle e), & \text{for any edge } e; \\ \alpha^{|V(\mathbb{G})|} \beta^{|\mathcal{V}|} \gamma^{|\mathcal{B}|}, & \text{if } \mathbb{G} \text{ is edgeless.} \end{cases} \tag{7}$$

Proof. We use a standard argument. Assign an arbitrary linear order to the edges of $\underline{\mathbb{G}}$ and compute $\mathbf{Q}(\underline{\mathbb{G}})$ by applying Equation (7) to the edges with respect to this order. The resulting summands of $\mathbf{Q}(\underline{\mathbb{G}})$ (before collecting terms) are in bijection with ordered tuples $(A_1, A_2, A_3, A_4, A_5)$ where each $A_i \subseteq E(\mathbb{G})$, $\bigcup_{i=1}^{5} A_i = E(\mathbb{G})$ and the A_i's are pairwise disjoint. Here A_1 is the set of edges that are deleted in forming the summand, A_2 those that are contracted, A_3 those that are Penrose-contracted, A_4 those that are merge-deleted, and A_5 those that are merge-contracted. By Lemma 10 we may carry these operations out in any order without changing the end result. Thus the summand corresponding to $(A_1, A_2, A_3, A_4, A_5)$ contributes

$$\Big(\prod_{e \in A_1} a_e\Big)\Big(\prod_{e \in A_2} b_e\Big)\Big(\prod_{e \in A_3} c_e\Big)\Big(\prod_{e \in A_4} x_e\Big)\Big(\prod_{e \in A_5} y_e\Big) \alpha^{|V(\mathbb{G}')|} \beta^{|\mathcal{V}'|} \gamma^{|\mathcal{B}'|}, \tag{8}$$

where $(\mathbb{G}', \mathcal{V}', \mathcal{B}') = \underline{\mathbb{G}} \setminus A_1 \,/\, A_2 \curlywedge A_3 \veebar A_4 \angle A_5$. The polynomial $Q(\underline{\mathbb{G}})$ can then be obtained by summing these contributions over all possible tuples $(A_1, A_2, A_3, A_4, A_5)$. As this sum is clearly independent of the choice of linear order of the edges, the result follows. □

Observe that Proposition 18 follows immediately from Proposition 21. We also note that Equation (8) gives rise to a state sum expression for $\mathbf{Q}(\underline{\mathbb{G}})$.

We use $\mathbf{Q}(\underline{\mathbb{G}}; \mathbf{w}, \alpha, \beta, \gamma)$, where $\mathbf{w} = \{(a_e, b_e, c_e, x_e, y_e) : e \in E(\mathbb{G})\}$ to denote the polynomial defined by Equation (7). Of course the result still holds if we take the formal variables to be parameters instead.

The following result is immediate.

Proposition 22. *Let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ be a packaged arrow presentation and let $\underline{\mathbb{H}} = (\mathbb{H}, \{\{v\}\}, \{\{b\}\})$ be a packaged arrow presentation consisting of a single vertex v, a single boundary component b and no edges (i.e., one circle with no arrows). Denote the disjoint union of their arrow presentations by $\mathbb{G} \sqcup \mathbb{H}$. Let $\underline{\mathbb{G}} \sqcup \underline{\mathbb{H}}$ be a packaged arrow presentation of the form $(\mathbb{G} \sqcup \mathbb{H}, \mathcal{V}', \mathcal{B}')$ where $\mathcal{V}'$ and $\mathcal{B}'$ restrict to the partitions $\mathcal{V}$ and $\mathcal{B}$, respectively. Then*

$$\mathbf{Q}(\underline{\mathbb{G}} \sqcup \underline{\mathbb{H}}; \mathbf{w}, \alpha, \beta, \gamma) = \alpha \beta^p \gamma^q \, \mathbf{Q}(\underline{\mathbb{G}}; \mathbf{w}, \alpha, \beta, \gamma),$$

where $p = 1$ if v is in its own block of the partition $\mathcal{V}'$, and $p = 0$ otherwise; and $q = 1$ if b is in its own block of the partition $\mathcal{B}'$, and $q = 0$ otherwise.

The following theorem is the main result of this paper. It provides the tensor product formula from which formulas for the Krushkal, Bollobás–Riordan, ribbon graph, topological transition and Tutte polynomials will follow. In particular, it satisfactorily completes the work initiated by Huggett and Moffatt in [14]. We note that Corollary 24 offers a cleaner restatement of the result.

Theorem 23. *Let $\underline{\mathbb{G}}$ and $\underline{\mathbb{H}}^{(f_i)}$, for $i = 1, \ldots, k$, be packaged arrow presentations. Suppose that each $\underline{\mathbb{H}}^{(f_i)}$ has an edge $e^{(i)}$, and that $f_1, \ldots, f_k$ are distinct edges of $\underline{\mathbb{G}}$. Suppose also that for each i there is coupling φ_i of f_i and $e^{(i)}$. Let*

$$\underline{\mathbb{G}}[k] = \underline{\mathbb{G}} \oplus_{\varphi_1} \underline{\mathbb{H}}^{(f_1)} \oplus_{\varphi_2} \cdots \oplus_{\varphi_k} \underline{\mathbb{H}}^{(f_k)}.$$

Then

$$\mathbf{Q}(\underline{\mathbb{G}}[k]; \mathbf{w}, \alpha, \beta, \gamma) = \mathbf{Q}(\underline{\mathbb{G}}; \mathbf{w}', \alpha, \beta, \gamma), \tag{9}$$

where $\mathbf{w} = \{(a_f, b_f, c_f, x_f, y_f) : f \in E(\underline{\mathbb{G}}[k])\}$ and

$$\mathbf{w}' = \big\{ \big(\phi^{(f_i)}_{(=,2)}, \phi^{(f_i)}_{(\|,2)}, \phi^{(f_i)}_{\times}, \phi^{(f_i)}_{(=,1)}, \phi^{(f_i)}_{(\|,1)} \big) : i = 1, \ldots, k \big\} \\ \cup \big\{ (a_f, b_f, c_f, x_f, y_f) : f \in E(\underline{\mathbb{G}}) - \{f_1, \ldots, f_k\} \big\}.$$

and where, for each f_i, each of $\phi^{(f_i)}_{(=,2)}$, $\phi^{(f_i)}_{(\|,2)}, \phi^{(f_i)}_{\times}$, $\phi^{(f_i)}_{(=,1)}$, and $\phi^{(f_i)}_{(\|,1)}$ arise as the unique solution to

$$\alpha\beta\gamma \begin{bmatrix} \alpha\beta & 1 & 1 & \alpha & 1 \\ 1 & \alpha\gamma & 1 & 1 & \alpha \\ 1 & 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \phi^{(f_i)}_{(=,2)} \\ \phi^{(f_i)}_{(\|,2)} \\ \phi^{(f_i)}_{\times} \\ \phi^{(f_i)}_{(=,1)} \\ \phi^{(f_i)}_{(\|,1)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}(\underline{\mathbb{H}}^{(f_i)} \setminus e^{(i)}; \mathbf{w}, \alpha, \beta, \gamma) \\ \mathbf{Q}(\underline{\mathbb{H}}^{(f_i)} \,/\, e^{(i)}; \mathbf{w}, \alpha, \beta, \gamma) \\ \mathbf{Q}(\underline{\mathbb{H}}^{(f_i)} \curlywedge e^{(i)}; \mathbf{w}, \alpha, \beta, \gamma) \\ \mathbf{Q}(\underline{\mathbb{H}}^{(f_i)} \succeq e^{(i)}; \mathbf{w}, \alpha, \beta, \gamma) \\ \mathbf{Q}(\underline{\mathbb{H}}^{(f_i)} \angle\, e^{(i)}; \mathbf{w}, \alpha, \beta, \gamma) \end{bmatrix}. \tag{10}$$

Proof. We prove the result by induction on k. If $k = 0$ there is nothing to prove. Next suppose that Equation (9) is true in the $\underline{\mathbb{G}}[k-1]$ case.

Apply Equation (7) to each of the edges in $\underline{\mathbb{H}}^{(f_k)}$ other than $e^{(k)}$. Then use Proposition 22 to remove any vertices that were created in this application of Equation (7) as well as any isolated vertices that were part of $\underline{\mathbb{H}}^{(f_k)}$. This results in an expression of $\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)})$ as a linear combination of values of $\mathbf{Q}$ on the five one-edge arrow presentations with the coefficients of this linear combination lying in $\mathbb{Z}[a_e, b_e, c_e, x_e, y_e, \alpha, \beta, \gamma : e \in E(\mathbb{G})]$. Using $\underline{\mathbb{K}}_1, \dots, \underline{\mathbb{K}}_5$ to denote the one-edge arrow presentations as specified in Figure 8, we may therefore write

$$\begin{aligned} \mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}; \mathbf{w}, \alpha, \beta, \gamma) = {} & \phi^{(f_k)}_{(=,2)} \mathbf{Q}(\underline{\mathbb{K}}_1; \mathbf{w}, \alpha, \beta, \gamma) + \phi^{(f_k)}_{(\|,2)} \mathbf{Q}(\underline{\mathbb{K}}_2; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(\times)} \mathbf{Q}(\underline{\mathbb{K}}_3; \mathbf{w}, \alpha, \beta, \gamma) + \phi^{(f_k)}_{(=,1)} \mathbf{Q}(\underline{\mathbb{K}}_4; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(\|,1)} \mathbf{Q}(\underline{\mathbb{K}}_5; \mathbf{w}, \alpha, \beta, \gamma). \end{aligned} \tag{11}$$

Next, applying Equation (7) and Proposition 22 to the copy of $\underline{\mathbb{H}}^{(f_k)}$ in $\underline{\mathbb{G}}[k]$ and making use of Lemma 10 we can write

$$\begin{aligned} \mathbf{Q}(\underline{\mathbb{G}}[k]; \mathbf{w}, \alpha, \beta, \gamma) = {} & \phi^{(f_k)}_{(=,2)} \mathbf{Q}(\underline{\mathbb{G}}[k-1] \oplus_{\varphi_k} \underline{\mathbb{K}}_1; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(\|,2)} \mathbf{Q}(\underline{\mathbb{G}}[k-1] \oplus_{\varphi_k} \underline{\mathbb{K}}_2; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(\times)} \mathbf{Q}(\underline{\mathbb{G}}[k-1] \oplus_{\varphi_k} \underline{\mathbb{K}}_3; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(=,1)} \mathbf{Q}(\underline{\mathbb{G}}[k-1] \oplus_{\varphi_k} \underline{\mathbb{K}}_4; \mathbf{w}, \alpha, \beta, \gamma) \\ & + \phi^{(f_k)}_{(\|,1)} \mathbf{Q}(\underline{\mathbb{G}}[k-1] \oplus_{\varphi_k} \underline{\mathbb{K}}_5; \mathbf{w}, \alpha, \beta, \gamma). \end{aligned} \tag{12}$$

By Lemma 14, we may rewrite this as

$$\begin{aligned}\mathbf{Q}(\underline{\mathbb{G}}[k];\mathbf{w},\alpha,\beta,\gamma) &= \phi^{(f_k)}_{(=,2)}\mathbf{Q}(\underline{\mathbb{G}}[k-1]\setminus f_k;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad + \phi^{(f_k)}_{(\|,2)}\mathbf{Q}(\underline{\mathbb{G}}[k-1]\,/\,f_k;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad + \phi^{(f_k)}_{(\times)}\mathbf{Q}(\underline{\mathbb{G}}[k-1]\diagup f_k;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad + \phi^{(f_k)}_{(=,1)}\mathbf{Q}(\underline{\mathbb{G}}[k-1]\succeq f_k;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad + \phi^{(f_k)}_{(\|,1)}\mathbf{Q}(\underline{\mathbb{G}}[k-1]\angle f_k;\mathbf{w},\alpha,\beta,\gamma)\\ &= \mathbf{Q}(\underline{\mathbb{G}}[k-1];\mathbf{w}'',\alpha,\beta,\gamma),\end{aligned} \tag{13}$$

where

$$\mathbf{w}'' = \{(\phi^{(f_k)}_{(=,2)},\phi^{(f_k)}_{(\|,2)},\phi^{(f_k)}_{\times},\phi^{(f_k)}_{(=,1)},\phi^{(f_k)}_{(\|,1)})\}\cup\{(a_f,b_f,c_f,x_f,y_f): f\in E(\underline{\mathbb{G}}[k-1])-\{f_k\}\}.$$

The last equality follows from (7) (treating the ϕ's as formal variables). By the inductive hypothesis it follows that $\mathbf{Q}(\underline{\mathbb{G}}[k];\mathbf{w},\alpha,\beta,\gamma)=\mathbf{Q}(\underline{\mathbb{G}};\mathbf{w}';\alpha,\beta,\gamma)$.

It remains to show that the ϕ's are the solutions to the given systems of equations. Proceeding as for Equation (11), and with exactly the same ϕ's, we can write

$$\begin{aligned}\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}\setminus e;\mathbf{w},\alpha,\beta,\gamma) &= \phi^{(f_k)}_{(=,2)}\mathbf{Q}(\underline{\mathbb{K}}_1\setminus e;\mathbf{w},\alpha,\beta,\gamma)+\phi^{(f_k)}_{(\|,2)}\mathbf{Q}(\underline{\mathbb{K}}_2\setminus e;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad+\phi^{(f_k)}_{(\times)}\mathbf{Q}(\underline{\mathbb{K}}_3\setminus e;\mathbf{w},\alpha,\beta,\gamma)+\phi^{(f_k)}_{(=,1)}\mathbf{Q}(\underline{\mathbb{K}}_4\setminus e;\mathbf{w},\alpha,\beta,\gamma)\\ &\quad+\phi^{(f_k)}_{(\|,1)}\mathbf{Q}(\underline{\mathbb{K}}_5\setminus e;\mathbf{w},\alpha,\beta,\gamma),\\ &= \phi^{(f_k)}_{(=,2)}\alpha^2\beta^2\gamma+\phi^{(f_k)}_{(\|,2)}\alpha\beta\gamma+\phi^{(f_k)}_{(\times)}\alpha\beta\gamma+\phi^{(f_k)}_{(=,1)}\alpha^2\beta\gamma+\phi^{(f_k)}_{(\|,1)}\alpha\beta\gamma,\end{aligned}$$

and similarly for $\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}\,/\,e;\mathbf{w},\alpha,\beta,\gamma)$, $\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}\diagup e;\mathbf{w},\alpha,\beta,\gamma)$, $\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}\succeq e;\mathbf{w},\alpha,\beta,\gamma)$ and $\mathbf{Q}(\underline{\mathbb{H}}^{(f_k)}\angle e;\mathbf{w},\alpha,\beta,\gamma)$. The five resulting equations can be rewritten as the matrix equation (10). As the matrix of formal variables is non-singular, it uniquely determines the ϕ's. □

The following corollary is immediate from the proof of Theorem 23 and offers a cleaner statement of it and a form similar to Brylawski's original tensor product formula for the Tutte polynomial.

Corollary 24. *Let $\underline{\mathbb{G}}$, $\{\underline{\mathbb{H}}^{(f)}\}_{f\in E(\underline{\mathbb{G}})}$, $e^{(f)}$ and $\boldsymbol{\varphi}$ be as in Definition 15. Then*

$$\mathbf{Q}(\underline{\mathbb{G}}\otimes_{\boldsymbol{\varphi}}\{\underline{\mathbb{H}}^{(f)}\}_{f\in E(\underline{\mathbb{G}})};\mathbf{w},\alpha,\beta,\gamma)=\mathbf{Q}(\underline{\mathbb{G}};\mathbf{w}',\alpha,\beta,\gamma),$$

where $\mathbf{w}=\{(a_g,b_g,c_g,x_g,y_g): g\in E(\underline{\mathbb{G}}\otimes_{\boldsymbol{\varphi}}\{\underline{\mathbb{H}}^{(f)}\}_{f\in E(\underline{\mathbb{G}})})\}$ and

$$\mathbf{w}'=\big\{\big(\phi^{(f)}_{(=,2)},\phi^{(f)}_{(\|,2)},\phi^{(f)}_{\times},\phi^{(f)}_{(=,1)},\phi^{(f)}_{(\|,1)}\big): f\in E(\mathbb{G})\big\},$$

and where, for each f, $\phi^{(f)}_{(=,2)}$, $\phi^{(f)}_{(\|,2)}$, $\phi^{(f)}_{\times}$, $\phi^{(f)}_{(=,1)}$ and $\phi^{(f)}_{(\|,1)}$ arise from the unique solution to the system of equations of the form (10)*, replacing the f_i with f and $e^{(i)}$ with $e^{(f)}$ in the displayed matrices.*

Proof of Theorem 19. The result follows from Corollary 24. □

We highlight the following 2-sum formula as a special case.

Corollary 25. *Let $\underline{\mathbb{G}}$, $\underline{\mathbb{H}}$, e and φ be as in Definition 11. Then*

$$Q(\underline{\mathbb{G}} \oplus_{\varphi} \underline{\mathbb{H}}; a,b,c,x,y,\alpha,\beta,\gamma) = \begin{bmatrix} Q(\underline{\mathbb{G}} \setminus e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{G}} / e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{G}} \curlywedge e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{G}} \searrow e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{G}} \angle e; a,b,c,x,y,\alpha,\beta,\gamma) \end{bmatrix}^{T} \cdot \begin{bmatrix} \phi_{(=,2)} \\ \phi_{(\|,2)} \\ \phi_{\times} \\ \phi_{(=,1)} \\ \phi_{(\|,1)} \end{bmatrix},$$

where the ϕ's arise as the unique solution to

$$\alpha\beta\gamma \begin{bmatrix} \alpha\beta & 1 & 1 & \alpha & 1 \\ 1 & \alpha\gamma & 1 & 1 & \alpha \\ 1 & 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \phi_{(=,2)} \\ \phi_{(\|,2)} \\ \phi_{\times} \\ \phi_{(=,1)} \\ \phi_{(\|,1)} \end{bmatrix} = \begin{bmatrix} Q(\underline{\mathbb{H}} \setminus e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{H}} / e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{H}} \curlywedge e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{H}} \searrow e; a,b,c,x,y,\alpha,\beta,\gamma) \\ Q(\underline{\mathbb{H}} \angle e; a,b,c,x,y,\alpha,\beta,\gamma) \end{bmatrix}.$$

Remark 26. Our aim was to extend Brylawski's tensor product formula to $Z(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}})$. However, we needed to consider the more general polynomial $Q(\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \underline{\mathbb{H}})$. The reason for this is as follows. If we followed the proof of Theorem 23 using the polynomial Z instead of Q, then the argument goes through except the second equality in Equation (13) does not hold. Continuing to adapt the proof ends up with what is effectively Corollary 30. An analogous situation occurs in [11, Section 6] where a tensor product formula is given for the transition polynomial rather than the ribbon graph polynomial. Avoiding the use of Q altogether, and having a tensor product formula completely in terms of Z can be done via Technique 27 below if we insist all vertex and boundary partitions have only one block and all the arrow presentations represent orientable ribbon graphs (which ensures the $\phi_{\times}^{(f_i)}$ terms are zero, as in the proof of [11, Corollary 6.3]). As we note in Remark 33, doing so results in [14, Theorem 2].

The following observation, which we present as a technique, allows us to recover known results from Theorem 23.

Technique 27. If we can deduce from the properties of $\underline{\mathbb{H}}^{(f_i)}$ that specific $\underline{\mathbb{K}}_i$ cannot arise after applying Equation (7) to each of its edges except $e^{(i)}$, then we can simplify the matrix Equation (10).

For example, if $e^{(i)}$ is incident to two vertices that are in the same block of the vertex partition (or it is only incident to one vertex), then $\underline{\mathbb{K}}_1$ cannot arise and $\phi_{(=,2)}^{(f_i)} = 0$. Similarly, we can deduce that if the boundary components adjacent to $e^{(i)}$ are in the same boundary partition (or if $e^{(i)}$ is only adjacent to one boundary component), then $\underline{\mathbb{K}}_2$ cannot arise and $\phi_{(\|,2)}^{(f_i)} = 0$. If each $\underline{\mathbb{H}}^{(f_i)}$ is orientable (i.e., corresponds to an orientable ribbon graph) and we set $c_f = 0$ for every $f \in E(\underline{\mathbb{G}})$ then, since $\underline{\mathbb{K}}_5$ cannot arise, we have $\phi_{\times}^{(f_i)} = 0$. In all of the above cases, we can simplify the matrix Equation (9) by substituting in $\phi_{(=,2)}^{(f_i)} = 0$, $\phi_{(\|,2)}^{(f_i)} = 0$ or $\phi_{\times}^{(f_i)} = 0$ accordingly and removing the redundant rows and columns.

A similar argument can be applied to Theorem 19 to simplify the matrix equation (6) under the same conditions. In general, this technique can be used whenever one can deduce that a specific $\underline{\mathbb{K}}_i$ cannot arise after resolving the other edges in $\underline{\mathbb{H}}^{(f_i)}$.

Theorem 23 does not directly apply if we set $\alpha = -2, 0, 1$, $\beta = 0, 1$ or $\gamma = 0, 1$ since the determinant of the left-hand matrix in Equation (10) is $\alpha^2 (\alpha + 2) (\alpha - 1)^2 (\beta - 1) (\gamma - 1)$. However, Technique 27 provides a way to indirectly apply the theorem: if we can eliminate suitable rows and columns using the technique we need only consider a non-singular submatrix of the original. (For example, if we eliminate the need for the row and column containing γ, we may set $\gamma = 1$ in the polynomials. This is key in the next section.)

4 Some special cases

4.1 The Bollobás–Riordan polynomial

The Bollobaś–Riordan polynomial [3, 4] is probably the best-known embedded graph polynomial. For a ribbon graph $\mathbf{G}$, the *Bollobás–Riordan polynomial*, $R(\mathbf{G}; x, y, z) \in \mathbb{Z}[x, y, z^{1/2}]$, is

$$R(\mathbf{G}; x, y, z) = \sum_{A \subseteq E(\mathbf{G})} (x-1)^{r(E)-r(A)} y^{|A|-r(A)} z^{\gamma(A)},$$

where $r(A)$ is the rank of the ribbon graph $\mathbf{G} \setminus (E - A)$ and $\gamma(A)$ its Euler genus. (Again we do not need the specifics of this definition, but include it for reference.) As shown in [19, Lemma 4.1] and [6, Theorem 5.1], the Bollobás–Riordan polynomial can be recovered from the Krushkal polynomial

$$R(\mathbf{G}; x+1, y, z) = y^{\frac{1}{2}\gamma(\mathbf{G})} K(G \subset \Sigma; x, y, yz^2, 1/y), \tag{14}$$

where $\mathbf{G}$ is a ribbon graph and $G \subset \Sigma$ its corresponding cellularly embeddded graph.

As with the Krushkal polynomial, there are no known deletion-contraction relations for the Bollobás–Riordan polynomial that apply to an arbitrary edge. It was shown in [18] that extending the Bollobás–Riordan polynomial to vertex-partitioned arrow presentations results in a polynomial that does have deletion-contraction relations that apply to any edge. That polynomial was expressed in terms of a three-variable polynomial T_{cps} of graphs embedded in pseudo-surfaces, or equivalently vertex-partitioned arrow presentations, in [15, Definition 38]. We shall consider a polynomial equivalent to T_{cps}.

Let $\widehat{\mathbb{G}}$ be a vertex-partitioned arrow presentation and $\underline{\mathbb{G}}$ be any packaged arrow presentation obtained by partitioning the boundary components of $\widehat{\mathbb{G}}$. Set

$$\widehat{Z}(\widehat{\mathbb{G}}; a, b, \alpha, \beta) = Z(\underline{\mathbb{G}}; a, b, \alpha, \beta, 1).$$

Note that the value of $\widehat{Z}$ is independent of the choice of boundary partition in forming $\underline{\mathbb{G}}$.

It follows from Equation (2) that

$$\widehat{Z}(\widehat{\mathbb{G}};a,b,\alpha,\beta)=\begin{cases} a\,\widehat{Z}(\widehat{\mathbb{G}}\setminus e;a,b,\alpha,\beta)+b\,\widehat{Z}(\widehat{\mathbb{G}}\,/\,e;a,b,\alpha,\beta), & \text{for any edge } e;\\ \alpha^{|V(\mathbb{G})|}\beta^{|\mathcal{V}|}, \quad \text{if } \widehat{\mathbb{G}}=(\mathbb{G},\mathcal{V}) \text{ is edgeless.} \end{cases}$$

By [15, Theorem 39], it follows that T_{cps} is equivalent to $\widehat{Z}$. (Alternatively use the fact that $T_{cps}(\underline{\mathbb{G}},w,x,y)=T_{ps}(\widehat{\mathbb{G}},w,x,y,y)$ together with Equations (3) and (4).) Thus by giving a tensor product formula for $\widehat{Z}$ we give one for $R(\mathbf{G};x,y,z)$.

Remark 28. Although we do not use it here, for reference, and using the notation in [15], we will specify how $\widehat{Z}$ and R are related. If $\mathbf{G}$ is a ribbon graph with corresponding arrow presentation $\mathbb{G}$, $\widehat{\mathbb{G}}=(\mathbb{G},\mathcal{V})$ where each vertex is in its own block in $\mathcal{V}$, and $\underline{\mathbb{G}}=(\mathbb{G},\mathcal{V},\mathcal{B})$ where $\mathcal{B}$ is any boundary partition, Equations (5) and (14) give that

$$\begin{aligned} R(\mathbf{G};x+1,y,z) &= y^{\frac{1}{2}\gamma(\widehat{\mathbb{G}})}K(\underline{\mathbb{G}};x,y,yz^2,1/y)\\ &= (1/xyz^2)^{k(\mathbb{G}/\mathcal{V})}\sqrt{y}^{|E|-2r(\mathbb{G}/\mathcal{V})}z^{v(\mathbb{G})-2r(\mathbb{G}/\mathcal{V})}\\ &\qquad Z(\widehat{\mathbb{G}};1/\sqrt{y},\sqrt{yz^2},1/z,xyz^2,1). \end{aligned}$$

As it is a specialisation of the Krushkal polynomial we can recover a tensor product formula for the extension $\widehat{Z}$ of the Bollobás–Riordan polynomial from Theorem 19. To do so, for $\widehat{\mathbb{G}}$ a vertex-partitioned arrow presentation, let $\underline{\mathbb{G}}$ be any packaged arrow presentation obtained by partitioning the boundary components of $\widehat{\mathbb{G}}$ and set

$$\widehat{Q}(\widehat{\mathbb{G}};a,b,c,x,\alpha,\beta)=Q(\underline{\mathbb{G}};a,b,c,x,0,\alpha,\beta,1). \tag{15}$$

Note that the value of $\widehat{Q}$ is independent of how the boundary components are partitioned.

Theorem 29. *Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be vertex-partitioned arrow presentations. Let e be a fixed edge of $\widehat{\mathbb{H}}$ and for each edge f in $\widehat{\mathbb{G}}$ let φ_f be a coupling of f and e, and $\boldsymbol{\varphi}=\{\varphi_f\}_{f\in E(\widehat{\mathbb{G}})}$. Then*

$$\widehat{Q}(\widehat{\mathbb{G}}\otimes_{\boldsymbol{\varphi}}\widehat{\mathbb{H}};a,b,c,x,\alpha,\beta)=\widehat{Q}(\widehat{\mathbb{G}};\phi_{(=,2)},\phi_{\|},\phi_{\times},\phi_{(=,1)},\alpha,\beta),$$

where the ϕ's arise as the unique solution to

$$\alpha\beta\begin{bmatrix}\alpha\beta & 1 & 1 & \alpha\\ 1 & \alpha & 1 & 1\\ 1 & 1 & \alpha & 1\\ \alpha & 1 & 1 & \alpha\end{bmatrix}\cdot\begin{bmatrix}\phi_{(=,2)}\\ \phi_{\|}\\ \phi_{\times}\\ \phi_{(=,1)}\end{bmatrix}=\begin{bmatrix}\widehat{Q}(\widehat{\mathbb{H}}\setminus e;a,b,c,x,\alpha,\beta)\\ \widehat{Q}(\widehat{\mathbb{H}}\,/\,e;a,b,c,x,\alpha,\beta)\\ \widehat{Q}(\widehat{\mathbb{H}}\nearrow e;a,b,c,x,\alpha,\beta)\\ \widehat{Q}(\widehat{\mathbb{H}}\searrow e;a,b,c,x,\alpha,\beta)\end{bmatrix}.$$

Proof. Let $\underline{\mathbb{G}}$ be a packaged arrow presentation obtained from $\widehat{\mathbb{G}}$ by partitioning the boundary components, and $\underline{\mathbb{H}}$ be the packaged arrow presentation obtained from $\widehat{\mathbb{H}}$ such that its boundary partition contains precisely one block. Using Equation (15) and Theorem 19, we can obtain an expression for $\widehat{Q}(\widehat{\mathbb{G}}\otimes_{\boldsymbol{\varphi}}\widehat{\mathbb{H}})$ in terms of $Q(\underline{\mathbb{G}})$ with formal variables arising from $Q(\underline{\mathbb{H}} \setminus e)$, $Q(\underline{\mathbb{H}} / e)$, $Q(\underline{\mathbb{H}} \curlywedge e)$, $Q(\underline{\mathbb{H}} \angle e)$ and $Q(\underline{\mathbb{H}} \angle e)$. Due to our choice of $\underline{\mathbb{H}}$, $Q(\underline{\mathbb{H}} / e) = Q(\underline{\mathbb{H}} \angle e)$ and as observed in Technique 27, $\phi_{(\|,2)} = 0$. So we simplify the matrix equation (6) accordingly, relabel $\phi_{(\|,1)}$ as $\phi_{\|}$, reorder the rows where appropriate. As setting $\gamma = 1$ no longer results in a singular matrix we can use Equation (15) to obtain the result. □

The following is immediate upon setting $c = x = 0$ in Theorem 29.

Corollary 30. *Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be the vertex-partitioned arrow presentation obtained from $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$, respectively, by placing each vertex in its own block of the partition. Let e be a fixed edge of $\widehat{\mathbb{H}}$ and for each edge f in $\widehat{\mathbb{G}}$ let φ_f be a coupling of f and e, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\widehat{\mathbb{G}})}$. Then*

$$\widehat{Z}(\widehat{\mathbb{G}} \otimes_{\boldsymbol{\varphi}} \widehat{\mathbb{H}}; a, b, \alpha, \beta) = \widehat{Q}(\widehat{\mathbb{G}}; \phi_{(=,2)}, \phi_{\|}, \phi_{\times}, \phi_{(=,1)}, \alpha, \beta),$$

where the ϕ's arise as the unique solution to

$$\alpha\beta \begin{bmatrix} \alpha\beta & 1 & 1 & \alpha \\ 1 & \alpha\gamma & 1 & 1 \\ 1 & 1 & \alpha & 1 \\ \alpha & 1 & 1 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \phi_{(=,2)} \\ \phi_{\|} \\ \phi_{\times} \\ \phi_{(=,1)} \end{bmatrix} = \begin{bmatrix} \widehat{Z}(\widehat{\mathbb{H}} \setminus e; a, b, \alpha, \beta) \\ \widehat{Z}(\widehat{\mathbb{H}} / e; a, b, \alpha, \beta) \\ \widehat{Z}(\widehat{\mathbb{H}} \curlywedge e; a, b, \alpha, \beta) \\ \widehat{Z}(\widehat{\mathbb{H}} \angle e; a, b, \alpha, \beta) \end{bmatrix}.$$

We will now indicate how to recover Huggett and Moffatt's tensor product formula [14, Theorem 4.3], which is stated below as Corollary 31. Let $\mathbb{G}$ be an arrow presentation, and $\mathbf{b} = \{b_e : e \in E(\mathbb{G})\}$ be a set of indeterminates indexed by the edges of $\mathbb{G}$. The *multivariate Bollobás–Riordan polynomial* [21] is defined by

$$\widetilde{\mathbf{Z}}(\mathbb{G}; a, \mathbf{b}, c) = \sum_{A \subseteq E(\mathbb{G})} a^{k(A)} \Big(\prod_{e \in A} b_e \Big) c^{b(A)},$$

where $k(A)$ is the number of connected components of (the ribbon graph corresponding to) $\mathbb{G} \setminus (E(\mathbb{G}) - A)$ and $b(A)$ its number of boundary components.

For convenience write $\mathbf{W}(\underline{\mathbb{G}})$ for $Q(\underline{\mathbb{G}}; \{(1,0,0,0,b_e)\}_{e \in E(\mathbb{G})}, c, a, 1)$, where $\underline{\mathbb{G}}$ is a packaged arrow presentation. Proposition 21 gives that

$$\mathbf{W}(\underline{\mathbb{G}}) = \begin{cases} \mathbf{W}(\underline{\mathbb{G}} \setminus e) + b_e \mathbf{W}(\underline{\mathbb{G}} \angle e) & \text{for any edge } e, \\ a^{|\mathcal{V}|} c^{|V(\mathbb{G})|} & \text{if } \mathbb{G} \text{ is edgeless.} \end{cases}$$

By making use of Equation (8),

$$\mathbf{W}(\underline{\mathbb{G}}) = \sum_{A \subseteq E(\underline{\mathbb{G}})} \big(\prod_{e \in A} b_e \big) c^{|V(\mathbb{G} \angle A \setminus (E(\underline{\mathbb{G}}) - A))|} a^{|\mathcal{V}(\underline{\mathbb{G}} \angle A \setminus (E(\underline{\mathbb{G}}) - A))|}. \tag{16}$$

Thus if $\mathbb{G}$ is an arrow presentation and $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ places each vertex in its own block, and $\mathcal{B}$ is any boundary partition, we see that $|V(\mathbb{G} \angle A \setminus (E(\underline{\mathbb{G}}) - A))| =$

$b(\mathbb{G} \setminus (E(\underline{\mathbb{G}}) - A)))$ since contraction does not change the number of boundary components, and it is readily seen that $|\mathcal{V}(\underline{\mathbb{G}} \angle A \setminus (E(\underline{\mathbb{G}}) - A))| = k(\mathbb{G} \setminus (E(\underline{\mathbb{G}}) - A))$. Thus Equation (16) and the definition of $\mathbf{W}(\underline{\mathbb{G}})$ gives that

$$\widetilde{\mathbf{Z}}(\mathbb{G}; a, \mathbf{b}, c) = \mathbf{Q}(\underline{\mathbb{G}}; \{(1, 0, 0, 0, b_e)\}_{e \in E(\mathbb{G})}, c, a, 1). \tag{17}$$

With this observation we can now recover [14, Theorem 4.3] as a corollary of Theorem 23 as follows.

Corollary 31. *Let $\mathbb{G}$ be an arrow presentation that describes an orientable ribbon graph, and let $\mathbb{H}$ be an arrow presentation that describes a plane ribbon graph. Let e be a fixed edge of $\mathbb{H}$ and for each edge f in $\mathbb{G}$ let φ_f be a coupling of f and e, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(\mathbb{G})}$. Then*

$$\widetilde{\mathbf{Z}}(\mathbb{G} \otimes_{\boldsymbol{\varphi}} \mathbb{H}; a, b, c) = (ac)^{-e(\mathbb{G})} \Big(\prod_{e \in E(\mathbb{G})} g_e \Big) \widetilde{\mathbf{Z}}(\mathbb{G}; a, \{f_e/g_e\}_{e \in E(\mathbb{G})}, c),$$

where, for each e, both f_e and g_e arise as the unique solution to

$$\begin{aligned} acg_e + f_e &= \widetilde{\mathbf{Z}}(\mathbb{H} \setminus e; a, b, c), \\ g_e + cf_e &= \widetilde{\mathbf{Z}}(\mathbb{H} \setminus e; a, b, c). \end{aligned}$$

Proof. Let $\underline{\mathbb{G}}$ be a packaged arrow presentation obtained from $\mathbb{G}$, and $\underline{\mathbb{H}}$ from $\mathbb{H}$, in which each vertex and each boundary component is in a block of size one. Using Theorem 23 and Equation (17), we can obtain an expression for $\widetilde{\mathbf{Z}}(\mathbb{G} \otimes_{\boldsymbol{\varphi}} \mathbb{H})$ in terms of $\mathbf{Q}(\underline{\mathbb{G}})$ with formal variables arising from $\mathbf{Q}(\underline{\mathbb{H}} \setminus e)$, $\mathbf{Q}(\underline{\mathbb{H}} / e)$, $\mathbf{Q}(\underline{\mathbb{H}} \times e)$, $\mathbf{Q}(\underline{\mathbb{H}} \searrow e)$ and $\mathbf{Q}(\underline{\mathbb{H}} \angle e)$. As $\mathbb{H}$ is plane and the partitions of $\underline{\mathbb{H}}$ only contain blocks of size one, applying the methods in Technique 27 we can deduce that $\phi_\times = 0$, $\phi_{(=,1)} = 0$ and $\phi_{(\|,1)} = 0$. So we simplify matrix equation (6) accordingly, and rename $\phi_{(=,2)}$ as g_e and $\phi_{(\|,2)}$ as f_e. Since setting $\gamma = 1$ no longer results in a singular matrix, we can use Equation (17) to obtain the result. □

We highlight, in particular that [14, Corollary 4.3] applies the above result to find a tensor product formula for the Bollobás–Riordan polynomial (with the same orientability and planarity conditions). Thus Theorem 29 is strictly stronger than Huggett and Moffatt's result for the Bollobás–Riordan polynomial.

4.2 The topological transition and ribbon graph polynomials

The *topological transition polynomial*, introduced in [8, 13], is a multivariate polynomial of ribbon graphs,or equivalently arrow presentations. It contains both the 2-variable version of Bollobás and Riordan's ribbon graph polynomial (described below) and the Penrose polynomial [1, 10] as specializations, and is intimately related to Jaeger's transition polynomial [16] and the generalized transition polynomial of [12].

Let $\mathbb{G}$ be an arrow presentation, let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \{(a_e, b_e, c_e)\}_{e \in E(\mathbb{G})}$ be an indexed family of triples of indeterminates, and t be another indeterminate. Then, as shown in [8] the *topological transition polynomial*, $\widetilde{\mathbf{Q}}(\mathbb{G};(\mathbf{a},\mathbf{b},\mathbf{c}),t)$, is defined by the recursion relation

$$\widetilde{\mathbf{Q}}(\mathbb{G}) = \begin{cases} a_e\,\widetilde{\mathbf{Q}}(\mathbb{G}\,/\,e) + b_e\,\widetilde{\mathbf{Q}}(\mathbb{G}\setminus e) + c_e\,\widetilde{\mathbf{Q}}(\mathbb{G}\,\overline{\times}\,e), & \text{for any edge } e;\\ t^p & \text{if } \mathbb{G} \text{ is edgeless with } p \text{ vertices;}\end{cases} \tag{18}$$

where we have written $\widetilde{\mathbf{Q}}(\mathbb{G})$ for $\widetilde{\mathbf{Q}}(\mathbb{G};(\mathbf{a},\mathbf{b},\mathbf{c}),t)$ in the recursion relation. (Note the different order of the deletion and contraction terms in Equations (7) and (18).)

Observe that if $\underline{\mathbb{G}}$ is any packaged arrow presentation obtained by partitioning the vertices and boundary components of $\mathbb{G}$ then

$$\widetilde{\mathbf{Q}}(\mathbb{G};(\mathbf{a},\mathbf{b},\mathbf{c}),t) = \mathbf{Q}(\underline{\mathbb{G}};\mathbf{w},t,1,1),$$

where $\mathbf{w} = \{(b_e,a_e,c_e,0,0) : e \in E(\underline{\mathbb{G}})\}$.

The following result extends [22, Theorem 28] and [11, Theorem 6.1].

Theorem 32. *Let $\mathbb{G}$, $\{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})}$, $e^{(f)}$ and $\boldsymbol{\varphi}$ be as in Definition 5. Then*

$$\widetilde{\mathbf{Q}}(\mathbb{G}\otimes_{\boldsymbol{\varphi}}\{\mathbb{H}^{(f)}\}_{f\in E(\mathbb{G})};(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),t) = \widetilde{\mathbf{Q}}(\mathbb{G};(\boldsymbol{\phi}_{\|},\boldsymbol{\phi}_{=},\boldsymbol{\phi}_{\times}),t),$$

where $(\boldsymbol{\phi}_{\|},\boldsymbol{\phi}_{=},\boldsymbol{\phi}_{\times}) = \{(\phi_{\|}^{(f)},\phi_{=}^{(f)},\phi_{\times}^{(f)}) : f \in E(\mathbb{G}))\}$, and, for each f, we have that $\phi_{=}^{(f)}$, $\phi_{\|}^{(f)}$ and $\phi_{\times}^{(f)}$ arise as the unique solution to

$$\begin{bmatrix} t^2 & t & t\\ t & t^2 & t\\ t & t & t^2\end{bmatrix}\cdot\begin{bmatrix}\phi_{\|}^{(f)}\\ \phi_{=}^{(f)}\\ \phi_{\times}^{(f)}\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbf{Q}}(\mathbb{H}^{(f)}\,/\,e;(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),t)\\ \widetilde{\mathbf{Q}}(\mathbb{H}^{(f)}\setminus e;(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),t)\\ \widetilde{\mathbf{Q}}(\mathbb{H}^{(f)}\,\overline{\times}\,e;(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),t)\end{bmatrix}.$$

Proof. Let $\underline{\mathbb{G}}$ and each $\underline{\mathbb{H}}^{(f)}$ be the packaged arrow presentations consisting of $\mathbb{G}$ and the $\mathbb{H}^{(f)}$ with all the partitions containing precisely one block. Then the result follows from Corollary 24 upon noting that the $\phi_{(=,2)}^{(f)}$ and $\phi_{(\|,2)}^{(f)}$ are zero, and that $\mathbb{H}^{(f)}\setminus e = \mathbb{H}^{(f)} \,\underline{\angle}\, e$ and $\mathbb{H}^{(f)}\,/\,e = \mathbb{H}^{(f)}\angle\, e$. (Again we have used the technique of removing rows and columns to ensure that setting $\beta=\gamma=1$ no longer results in a singular matrix, circumventing the issue that just setting $\beta=\gamma=1$ in Corollary 24 results in a singular matrix.) ☐

Remark 33. We note that as was detailed for the proof of [11, Corollary 6.3], if $\mathbb{G}$ and $\mathbb{H}$ represent orientable ribbon graphs and if we set each $c_e = 0$ throughout Theorem 32, then $\phi_{\times} = 0$ and so the $\widehat{Q}(\mathbb{H}\,\overline{\times}\,e)$ need not be considered. This results in [14, Theorem 4.3], restated here as Corollary 31.

4.3 The Tutte polynomial

We conclude by indicating how to recover Brylawski's original formula (1) from Theorem 19. Let $G=(V,E)$ be a graph, Then set

$$\dot{Z}(G;a,b,c) = \sum_{A\subseteq E} a^{k(A)} b^{|A|} c^{|E-A|},$$

where $k(A)$ is the number of connected components of the spanning subgraphs (V, A) of G. This polynomial satisfies the deletion-contraction formula

$$\dot{Z}(G) = \begin{cases} c\dot{Z}(G \setminus e) + b\dot{Z}(G / e) & \text{for any edge } e \text{ of } G, \\ a^{|V|} & \text{when } G \text{ is edgeless with } |V| \text{ vertices.} \end{cases}$$

Here $G \setminus e$ and G / e are the usual graph deletion and contraction operations, and $\dot{Z}(G)$ denotes $\dot{Z}(G; a, b, c)$.

Let G and H be graphs, G loopless, e a non-loop edge of H, and $\boldsymbol{\varphi} = \{\varphi_f\}_{f \in E(G)}$ a family of bijections where each φ_f sends the ends of the edge f in G to the ends of the edge e in H. Let $\mathbb{G}$ be any arrow presentation that represents any embedding of G. (So the vertices and edges of G correspond to the vertices and edges of $\mathbb{G}$, and for each edge in G place e-labeled arrows on circles of the arrow presentation corresponding to the ends of that edge.) Let $\mathcal{V}$ be the vertex partition of $\mathbb{G}$ that places each vertex in its own block, and let $\mathcal{B}$ be any boundary partition of $\mathbb{G}$. Finally let $\underline{\mathbb{G}} = (\mathbb{G}, \mathcal{V}, \mathcal{B})$. For the graph H, construct some $\underline{\mathbb{H}}$ similarly. It is easily seen that $\underline{\mathbb{G}} \otimes_{\boldsymbol{\varphi}'} \underline{\mathbb{H}}$ represents a tensor product of graphs $G \otimes_{\boldsymbol{\varphi}} H$ analogously where $\boldsymbol{\varphi}'$ is induced by $\boldsymbol{\varphi}$. (This uses the assumptions that G is loopless and e is not a loop.) With G and $\mathbb{G}$ as above, we have

$$\dot{Z}(G; a, b, c) = Q(\underline{\mathbb{G}}; c, 0, 0, 0, b, 1, a, 1).$$

This follows from the observation that the vertex partition of $\underline{\mathbb{G}} \angle e$ records when the two vertices of G are merged under contraction and the vertex partitions are unchanged by deletion. Clearly $\dot{Z}(H \setminus e; a, b, 1) = Q(\underline{\mathbb{H}} \setminus e; c, 0, 0, 0, b, 1, a, 1)$ as e is not a loop, and so it follows by the recursion formulas that $\dot{Z}(H / e; a, b, 1) = Q(\underline{\mathbb{H}} \angle e; c, 0, 0, 0, b, 1, a, 1)$.

An application of Theorem 19, after some rewriting of the system of equations, again using eliminating rows and columns from the matrices just as above to enable us to apply it to $Q(\underline{\mathbb{G}}; c, 0, 0, 0, b, 1, a, 1)$, gives that

$$\dot{Z}(G \otimes_{\boldsymbol{\varphi}} H; a, b, 1) = \dot{Z}(G; a, f, g),$$

where f and g arise as the unique solution to

$$\begin{aligned} af + a^2 g &= \dot{Z}(H \setminus e; a, b, 1), \\ af + ag &= \dot{Z}(H / e; a, b, 1). \end{aligned}$$

This is exactly the result displayed in [14, Corollary 3.7]. The proof of Corollary 3.9 of that reference then shows how to rewrite the above formula to obtain Brylawski's tensor product formula (1).

Declarations

 No underlying data is associated with this article.

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