Chain Tutte Polynomials

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Abstract

The Tutte polynomial and Derksen's $\mathcal{G}$-invariant are the universal deletion-contraction and valuative matroid and polymatroid invariants, respectively. There are only a handful of well known invariants (like the matroid Kazhdan-Lusztig polynomials) between (in terms of fineness) the Tutte polynomial and Derksen's $\mathcal{G}$-invariant. The aim of this study is to define a spectrum of generalized Tutte polynomials to fill the gap between the Tutte polynomial and Derksen's $\mathcal{G}$-invariant. These polynomials are built by taking repeated convolution products of universal Tutte characters studied by Dupont, Fink, and Moci and using the framework of Ardila and Sanchez for studying valuative invariants. We develop foundational aspects of these polynomials by showing they are valuative on generalized permutahedra and present a generalized deletion-contraction formula. We apply these results on chain Tutte polynomials to obtain formulas for the Möbius polynomial, the opposite characteristic polynomial, a generalized Möbius polynomial, Ford's expected codimension of a matroid variety, and Derksen's $\mathcal{G}$-invariant.

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1 Introduction

For any set X we denote the set of subsets of X by 2^X. Also, we use $\mathbb{N} = \{0, 1, 2, \dots\}$ and for $n \in \mathbb{N}\backslash\{0\}$ we let $[n] = \{1, 2, \dots, n\}$. Notationally we will denote a sequence of objects $(n_1, \dots, n_k)$ by $(n_i)_1^k$ or (n_i) when the range of indices is clear from the context. A *polymatroid* is a pair $M = (\mathcal{A}, \mathrm{rk})$ where $\mathcal{A}$ is a finite set, called the ground set and $\mathrm{rk} : 2^{\mathcal{A}} \to \mathbb{N}$ is a function that satisfies

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1. $\mathrm{rk}(\emptyset) = 0$
2. for all $X \subseteq Y \subseteq \mathcal{A}$, $\mathrm{rk}(X) \leqslant \mathrm{rk}(Y)$
3. for all $X \subseteq Y \subseteq \mathcal{A}$, $\mathrm{rk}(X) + \mathrm{rk}(Y) \geqslant \mathrm{rk}(X \cap Y) + \mathrm{rk}(X \cup Y)$.

A *matroid* is a polymatroid that also satisfies for all $a \in \mathcal{A}$, $\mathrm{rk}(a) \in \{0, 1\}$. There are many equivalent definitions of matroids and we follow the standard conventions in [34] and [2]. The set of *independent sets* of a matroid is the set

$$\mathcal{I} = \{X \subseteq \mathcal{A} \mid \ \mathrm{rk}(X) = |X|\}$$

and one can define a matroid in terms of independent sets as in [2, Sec 7.2]. Given a matroid defined by independent sets $\mathcal{I}$ one can define the rank function $\mathrm{rk} : 2^{\mathcal{A}} \to \mathbb{N}$ by $\mathrm{rk}(X) = \max\{|I| \ \mid \ I \in \mathcal{I},\ I \subseteq X\}$. This gives a cryptomorphism between the rank function and the independent set definition (again see [2, Sec 7.2]). We will utilize both of these definitions: when we write $M = (\mathcal{A}, \mathrm{rk})$ we are considering the rank function and when we write $M = (\mathcal{A}, \mathcal{I})$ we are considering $\mathcal{I}$ as the set of independent sets. For any polymatroidal function, when the matroid is not clear from context we will include the polymatroid as a subscript, for example we may denote rk as rk_M. An element $a \in \mathcal{A}$ is a *loop* in M if $\mathrm{rk}(a) = 0$ and a *coloop* if $\mathrm{rk}(\mathcal{A} - a) = \mathrm{rk}(M) - 1$. We also say that two elements $a, b \in \mathcal{A}$ which are not loops are *parallel* if $\mathrm{rk}(\{a, b\}) = 1$. Finally we say that a matroid is *simple* if it has no loops or parallel elements.

The *Whitney rank generating function of* a polymatroid $M = (\mathcal{A}, \mathrm{rk})$ is

$$W_M(a_1; b_1) = \sum_{S \subseteq \mathcal{A}} a_1^{\mathrm{rk}(M) - \mathrm{rk}(S)} b_1^{|S| - \mathrm{rk}(S)}$$

and the *Tutte polynomial* of M is

$$T_M(x; y) = W_M(x - 1, y - 1) = \sum_{S \subseteq \mathcal{A}} (x - 1)^{\mathrm{rk}(M) - \mathrm{rk}(S)} (y - 1)^{|S| - \mathrm{rk}(S)}.$$

Tutte polynomials were originally defined on graphs by Tutte in [41] while their coefficients were first investigated by Whitney in [43] (for some history see [22]). Crapo popularized and generalized Tutte polynomials to matroids in [12] . Since then Tutte polynomials have become arguably the most studied invariants on graphs, matroids, hyperplane arrangements, polymatroids, and extended generalized permutahedra. Recently an entire CRC handbook [18] edited by Ellis-Monaghan and Moffatt is entirely dedicated to the study of Tutte polynomials. Most importantly Tutte polynomials are universal among deletion-contraction invariants. This means one can formulate any deletion-contraction invariant, like the characteristic polynomial of a matroid which agrees with the chromatic polynomial for graphs, as an evaluation of the Tutte polynomial.

Recently there has been significant developments on understanding the foundational aspects of Tutte polynomials. Speyer in [39] showed that the Tutte polynomial over matroids is valuative from the polytope perspective. The work of Dupont, Fink, and Moci in [16] presents the Tutte polynomial as a specialization of Tutte characters on set species which allows them to examine Tutte polynomials on objects like Delta matroids and prove various convolution formulas. Ardila and Sanchez in [3] conducted a study of valuative invariants and built a framework for understanding them with convolutions. This led to another proof that the Tutte polynomial is valuative but over a much larger class of objects, in particular extended generalized permutahedra. Then, Ferroni and Schröter in [24] studied in detail some foundational valuations on certain small matroid subdivisions. Further recent developments on the Tutte polynomial have brought new results on the structure of the coefficients [11], different interpretations of the Tutte polynomial [30], and the dimension of the space spanned by its coefficients [32].

In [13] Derksen defined an invariant for polymatroids which has since been studied by many authors. Now we present a definition of this invariant. A *complete chain* of a polymatroid $M = (\mathcal{A}, \mathrm{rk})$ is a sequence of subsets $\bar{X} = (X_i)_0^{|\mathcal{A}|}$ of $\mathcal{A}$ such that $X_0 = \emptyset \subset X_1 \subset \cdots \subset X_{|\mathcal{A}|} = \mathcal{A}$. Let

$$r(\bar{X}) = (\mathrm{rk}(X_1) - \mathrm{rk}(X_0), \mathrm{rk}(X_2) - \mathrm{rk}(X_1), \ldots, \mathrm{rk}(X_{|\mathcal{A}|}) - \mathrm{rk}(X_{|\mathcal{A}|-1}))$$

be the rank vector of $\bar{X}$. Next choose $\{U_s\}$ as the basis (see [13] for details of this basis) for the ring of quasi-symmetric functions Qsym where s runs over all the finite sequences of non-negative integers. The *Derksen $\mathcal{G}$-invariant* of M is

$$\mathcal{G}(M) = \sum_{\bar{X}} U_{r(\bar{X})}.$$

Derksen showed in [13] that his $\mathcal{G}$-invariant is valuative and that the Tutte polynomial can be derived from $\mathcal{G}$. Then in [14] Fink and Derksen proved that Derksen's $\mathcal{G}$-invariant is universal among valuative invariants. Hence between the Tutte polynomial and Derksen's $\mathcal{G}$-invariant are the invariants which do not satisfy a deletion-contraction formula but are valuations. Many of these non-deletion-contraction yet valuative invariants, like the matroid Kazhdan-Lusztig polynomials of [17] and the Billera-Jia-Reiner quasi-symmetric function of a matroid from [5] and the matroid volume polynomial of Eur in [19], have not been calculated for some simple families of matroids. One reason is that the known recursions for these invariants are unwieldy. An exception is the result of Ferroni and Larson in [23] where they compute the Kazhdan-Lusztig polynomials for braid matroids using series parallel matroids. A central motivation for this work is to build structures that might eventually help compute these invariants. So, one aim here is to develop a generalized deletion-contraction to help compute valuative matroid invariants.

The main focus of this note is the following generalization of Tutte polynomials.

Definition 1. Let $M = (\mathcal{A}, \mathrm{rk})$ be a polymatroid with ground set $\mathcal{A}$. For a positive integer k let

$$\mathcal{C}^k_{\mathcal{A}} = \left\{ (S_1, \dots, S_k) \in \left(2^{\mathcal{A}}\right)^k \;\mid\; S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \right\}$$

and set $\mathcal{C}^0_{\mathcal{A}} = \{()\}$ where we are considering $()$ as the empty chain. The k^{th} *chain Whitney rank generating polynomial* of M is

$$W^k_M((a_i)^k_1; (b_i)^k_1) = \sum_{(S_i)^k_1 \in \mathcal{C}^k_{\mathcal{A}}} \prod_{i=1}^{k} (a_i)^{\mathrm{rk}(M) - \mathrm{rk}(S_i)} (b_i)^{|S_i| - \mathrm{rk}(S_i)}.$$

Then the k^{th} *chain Tutte polynomial* of M is

$$T^k_M((x_i)^k_1; (y_i)^k_1) = W^k_M((x_i - 1)^k_1; (y_i - 1)^k_1).$$

For convenience we may suppress the variables and just write T^k_M. Also, we will define $W^0_M = T^0_M = 1$.

This definition shows that $T^1_M(x_1; y_1)$ is the classic (poly)matroid Tutte polynomial. In section 4 we present the definition of this polynomial for any generalized permutahedron but for the majority of the paper we restrict our study to matroids. Let $\mathrm{Mat} = \bigcup \mathrm{Mat}_{r,n}$ be the collection of all matroids where $\mathrm{Mat}_{r,n}$ is the collection of matroids of rank r on a ground set of n elements. Then both the chain Whitney W^k_M and Tutte T^k_M polynomials are matroid invariants and we view them as functions

$$W^k : \mathrm{Mat}_{r,n} \to \mathbb{Z}[a_1, \dots, a_n, b_1, \dots, b_n]$$

and

$$T^k : \mathrm{Mat}_{r,n} \to \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n].$$

Many authors have studied generalizations of Tutte polynomials but they all seem to be different from that given in Definition 1. First, we note that our chain Tutte polynomials are far from the chain polynomials studied in [35] by Read and Whitehead and [40] by Traldi. We do not consider weighted graphs or matroids like that of Zaslavsky in [45]. Farr in [21] considered a rank generating function for objects with rank functions defined over the entire real field (not just the non-negative integers). Cameron, Dinu, Michałek, and Seynnaeve in [10] defined a Tutte polynomial on a flag matroid which is a sequence of matroids. There are also generalizations to finer invariants. An important example in the multivariable Tutte polynomial (see [38]) which is a complete invariant for matroids. There is also the work of Bernardi, Kálmán, and Postnikov in [4] where they define one polynomial for each positive integer n that parametrizes the Tutte polynomials of all polymatroids with a fixed ground set size n. Another perspective was given by Krajewski, Moffatt, and Tanasa in [31] where a framework is outlined to build Tutte polynomials using Hopf algebras. Miezaki, Oura, Sakuma, and Shinohara in an announcement paper [33] define genus g Tutte polynomials which are probably the most similar to the study here. However, the polynomials of Miezaki, Oura, Sakuma, and Shinohara have many more terms and are much finer invariants than ours. Actually the main result (Theorem 3.1 in [33]) of Miezaki, Oura, Sakuma, and Shinohara is that their collection of genus g Tutte polynomials are complete invariants for matroids where ours will never be. The polynomials in [31] seem close to ours but they are not built from sequences of ground set elements. The polynomials in [31] satisfy a classical "2-term" deletion-contraction formula instead of our k-term version.

The first goal of this work is to present a generalized deletion-contraction formula for T^k_M. In order to state a kind of generalized deletion-contraction we first need the following polynomials. These polynomials have three inputs: the matroid, a ground set element, and an integer for where the split occurs. In order to define these polynomials we need to recall some more matroid operations. For a matroid $M = (\mathcal{A}, \mathcal{I})$ the *deletion* of M by $a \in \mathcal{A}$ is the matroid $M \backslash S = (\mathcal{A} \backslash a, \mathcal{I}')$ where $\mathcal{I}' = \{I \subseteq \mathcal{A} \backslash a \quad | \quad I \in \mathcal{I}\}$. The *contraction* of M by $a \in \mathcal{A}$ is the matroid $M \backslash a = (\mathcal{A} \backslash a, \mathcal{I}'')$ where

$$\mathcal{I}'' = \begin{cases} \{I \subseteq \mathcal{A} \backslash S \mid I \cup a \in \mathcal{I}\} & \text{if } a \text{ is not a loop} \\ \mathcal{I} & \text{if } a \text{ is a loop.} \end{cases} .$$

If $S \subseteq \mathcal{A}$ the deletion $M \backslash S$ and the contraction M/S are the matroids defined by repeatedly deleting and contracting by elements in S respectively. The rank functions of these successive deletions and contractions satisfy $\mathrm{rk}_{M \backslash S}(A) = \mathrm{rk}_M(A)$ and $\mathrm{rk}_{M/S}(A) = \mathrm{rk}_M(A \cup S) - \mathrm{rk}_M(S)$ for $A \subseteq \mathcal{A} \backslash S$.

Definition 2. Let M be a matroid with atoms $\mathcal{A}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Then we define the k^{th} *split chain Tutte polynomial* of M and $a \in \mathcal{A}$ *at split term* j where $0 \leqslant j \leqslant k$

by $sT^{k,0}_{M,a} = T^k_{M/a}$ for the case $j = 0$, $sT^{k,k}_{M,a} = T^k_{M\backslash a}$ for the case $j = k$, and for $j \in \{1, \dots, k-1\}$

$$sT^{k,j}_{M,a}((x_i),(y_i)) = \sum_{(S_i)_1^j \in \mathcal{C}^j_{\mathcal{A}-a}} \prod_{i=1}^{j} (x_i - 1)^{\mathrm{rk}(M\backslash a) - \mathrm{rk}_{M\backslash a}(S_i)} (y_i - 1)^{|S_i| - \mathrm{rk}_{M\backslash a}(S_i)}$$

$$\cdot \sum_{\substack{(S_i)_{j+1}^k \in \mathcal{C}^{k-j}_{\mathcal{A}-a} \\ S_j \subseteq S_{j+1}}} \prod_{i=j+1}^{k} (x_i - 1)^{\mathrm{rk}(M/a) - \mathrm{rk}_{M/a}(S_i)} (y_i - 1)^{|S_i| - \mathrm{rk}_{M/a}(S_i)}.$$

We present two other views of $sT^{k,j}_{M,a}((x_i),(y_i))$ in section 3 and note that these are all just reordering of the terms of T^k containing and not containing a ground set element a. The decomposition in (2) gives a kind of minor presentation of $sT^{k,j}_{M,a}((x_i),(y_i))$. Now we state the generalized deletion-contraction formula.

Theorem 3. *Let $M = (\mathcal{A}, \mathrm{rk})$ be a matroid and $a \in \mathcal{A}$.*

1. *If $a \in \mathcal{A}$ is a loop then*

$$T^k_M((x_i);(y_i)) = T^k_L((x_i);(y_i)) T^k_{M\backslash a}((x_i);(y_i))$$

where L is the matroid of a loop.

2. *If $a \in \mathcal{A}$ is a coloop then*

$$T^k_M((x_i);(y_i)) = T^k_C((x_i);(y_i)) T^k_{M/a}((x_i);(y_i))$$

where C is the matroid of a coloop.

3. *If $a \in \mathcal{A}$ is not a loop and not a coloop then*

$$T^k_M((x_i);(y_i)) = \sum_{j=0}^{k} sT^{k,j}_{M,a}((x_i);(y_i)).$$

Theorem 3 together with (2) gives a recursion on chain Tutte polynomials. However, this formula seems unwieldy as the recursion summation is over all subsets of the deleted ground set with terms being products of chain Tutte polynomials on deletions and contractions with coefficients consisting of large products.

The next main result is relating the chain Tutte polynomials to Derksen's $\mathcal{G}$-invariant. We show that we can "linearly" determine Derksen's $\mathcal{G}$-invariant from the $|\mathcal{A}|^{th}$ chain Tutte polynomial. In order to properly define the homomorphism for the next theorem we have to consider the chain Whitney polynomial W_M^k as an element in the infinite direct sum of the multivariable Laurent polynomial ring $\bigoplus_{r,n} \mathbb{Z}[a_i^{\pm 1}, b_i^{\pm 1}]_{i\in[n]}$.

Theorem 4. *There exists an additive homomorphism*

$$\Psi : \bigoplus_{r,n} \mathbb{Z}[a_i^{\pm 1}, b_i^{\pm 1}]_{i\in[n]} \to \mathrm{Qsym}$$

such that $\Psi(W_M^{|\mathcal{A}|}) = \mathcal{G}(M)$ *for any matroid* M*.*

Since T_M^1 is the classic Tutte polynomial and $T_M^{|\mathcal{A}|}$ is essentially equivalent to Derksen's $\mathcal{G}$-invariant, the polynomials T_M^k are a sequence of polynomials between, in terms of successive refinement, the classical Tutte polynomial and Derksen's $\mathcal{G}$-invariant.

A starting place for this study was the work of Dupont, Fink, and Moci in [16] where they define universal Tutte characters and their convolutions. For this study we use the language and perspective in [3] where Ardila and Sanchez make the setting of Tutte characters concrete. In section 4 we define generalized permutahedra and valuations on matroids. Then we define a slightly different chain Tutte polynomial in the setting of generalized permutahedra by convolutions of Tutte characters. Applying the tools of Ardila and Sanchez we prove these chain Tutte polynomials are valuations on generalized permutahedra. Then restricting this result to matroids we obtain the following (see section 4 for details).

Theorem 5. *The k^{th} chain Whitney polynomial and chain Tutte polynomial are matroid valuations.*

Since Derksen's $\mathcal{G}$-invariant is valuatively universal, which was proved by Fink and Derksen in [14], we get that the $|\mathcal{A}|^{th}$ chain Tutte polynomial is equivalent to $\mathcal{G}$. Hence we have proved that the chain Tutte polynomials make a spectrum of finer invariants from the classical Tutte polynomial to the $\mathcal{G}$-invariant.

Another aim of this paper is to study some relatively unknown matroid invariants as evaluations of chain Tutte polynomials. First we look at some constant evaluations of the chain Tutte polynomials. While the constant evaluation formulas we present are not surprising, we include them to illustrate how chain Tutte polynomials generalize classic formulas for the Tutte polynomial. Then we consider some polynomial evaluations of chain Tutte polynomials. The *Möbius polynomial* of a matroid M is

$$\bar{\chi}_M(s,t) = \sum_{X \leqslant Y \in L(M)} \mu(X,Y) s^{\mathrm{rk}(M)-\mathrm{rk}(X)} t^{\mathrm{rk}(M)-\mathrm{rk}(Y)} \tag{1}$$

where μ is the Möbius function defined in section 2. It turns out one can obtain the Möbius polynomial of a matroid from the second chain Tutte polynomial and together with Theorem 5 we get the following.

Theorem 6. *If M is a matroid then*

$$T^2_M(1-s, 1-t; 0, 0) = \bar{\chi}_M(s,t)$$

and $\bar{\chi}_M$ is a matroid valuation.

The Möbius polynomial has seen some recent activity in coding theory (see [27] and [29]) and some applications in non-commutative algebras defined by layered graphs in [36] by Retakh and Wilson which originated from work of Gelfand, Retakh, Serconek, and Wilson in [26]. Using the generalized deletion-contraction formula applied to the evaluation formulas from chain Tutte polynomials we get a different presentation for the Möbius polynomial (Corollary 35).

We conclude our study by applying the chain Tutte polynomial to a few other related polynomials. We study the opposite characteristic polynomial of the lattice of flats of a matroid. This opposite characteristic polynomial is also an easy evaluation of the second chain Tutte polynomial. The reason to study the opposite characteristic polynomial is that in [28] Johnson and the author presented a problem to prove that the generalized Möbius function defined using the J-function from [28] is a valuation. We prove this result by proving that the opposite characteristic polynomial is a valuation since it is an evaluation of the second chain Tutte polynomial. Next we examine the expected codimension of a matroid variety defined by Ford in [25]. We show that the expected codimension is determined by the second chain Tutte polynomial and then apply our inductive result to obtain a formula for the expected codimension.

We organize this paper as follows. In section 2 we collect some classic facts about matroids, generalized permutahedra, Möbius functions, Tutte polynomials and Derksen's $\mathcal{G}$-invariant. Then in section 3 we present basic results about chain Tutte polynomials, including Theorem 3, which gives a generalized deletion-contraction formula. Next in section 4 we define chain Tutte polynomials on generalized permutahedra and prove that chain Tutte polynomials are valuations. We end with section 5 where we examine various applications of chain Tutte polynomials by computing specific polynomial evaluations. There we study the Möbius polynomial, the opposite characteristic polynomial, the generalized J-Möbius polynomial, and the expected codimension of a matroid variety.

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2 Preliminaries

In this section we review some definitions and results as well as some reformulations of basic properties of well known invariants on matroids and lattices. For matroid terminology we follow [2] and [34].

2.1 Posets and polymatroids

In a finite poset L a maximal chain is a strictly ordered sequence of elements $a_1 < a_2 < \cdots < a_k$ such that there does not exist $b \in L$ and $1 \leqslant i \leqslant k-1$ with $a_i < b < a_{i+1}$. A finite poset L is a ranked lattice if all least upper bounds and greatest lower bounds exist and the lengths of all maximal chains are the same. If L is a ranked lattice then the least upper bound is called a join and denoted by $\vee$ and the greatest lower bound is called a meet and denoted by $\wedge$. If a finite lattice L is ranked then it has a rank function usually denoted by $\mathrm{rk} : L \to \mathbb{Z}_{\geqslant 0}$ where $\mathrm{rk}(X)$ is the length of a maximal chain from the least element to X. An atom in a lower bounded lattice with bottom element $\hat{0}$ is any element $a \in L$ such that $a > \hat{0}$ and there does not exist $b \in L$ such that $\hat{0} < b < a$. We say a lower bounded lattice is atomic if every element is a join of atoms. Let L be a finite ranked and atomic lattice with atoms $\mathcal{A}$ and rank function rk. If $S \subseteq \mathcal{A}$ then we define the rank of S as $\mathrm{rk}(\bigvee S)$.

A *flat* of a matroid M is a set $F \subseteq \mathcal{A}$ such that $\mathrm{rk}(F \cup \{x\}) > \mathrm{rk}(F)$ for all $x \in \mathcal{A} - F$. The *lattice of flats* of M, denoted by $L(M)$, is the set of flats of M ordered by inclusion. The lattice of flats is a geometric lattice; in particular it is a ranked atomic lattice. We often work with M through $L(M)$. When M is simple, the atoms of $L(M)$ are the elements of $\mathcal{A}$. In general, the atoms of $L(M)$ are the parallelism classes of non-loops in $\mathcal{A}$.

Note that from an undirected graph $G = (V, E)$ with vertices V and edges E one can construct a matroid $M = (E, \mathrm{rk})$ where the rank function is defined by $\mathrm{rk}(S) = |S| - c$ where c is the number of connected components of the graph (V, S). We allow graphs to have multiedges and loops. In Section 3 we will consider some examples defined by graphs.

2.2 Möbius function

The Möbius function of a locally finite poset $\mathcal{P}$ is the function $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ defined by setting $\mu(X, X) = 1$ and

$$\sum_{X \leqslant Y \leqslant Z} \mu(X, Y) = 0$$

for all $X, Z \in \mathcal{P}$. First we recall a classic result on Möbius functions on matroids, by which we mean the Möbius function on the lattice of flats $L(M)$ of the matroid M. For any subset of atoms $X \subseteq \mathcal{A}$ we write $\bigvee X$ for the join of these atoms inside the lattice of flats (a.k.a. the closure of X in $L(M)$).

Lemma 7 ([44], Proposition 7.1.4)**.** *Let M be a simple matroid with minimum flat $\hat{0} = \emptyset$ and maximum flat $\hat{1} = \mathcal{A}$. Then*

$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{X \subseteq \mathcal{A} \\ \bigvee X = \hat{1}}} (-1)^{|X|}.$$

For our purposes we need to extend this result slightly. Surely this is in the literature somewhere, but the author could not find this particular formulation so we include the proof.

Lemma 8. *Let M be a matroid with ground set $\mathcal{A}$ and $L(M)$ its lattice of flats. For all $X, Y \in L(M)$,*

$$\mu(X, Y) = \sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ \bigvee A = X \\ \bigvee B = Y}} (-1)^{|A| + |B|}.$$

Proof. For $X, Y \in L(M)$ we define a function

$$g(X,Y) = \sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ \bigvee A = X \\ \bigvee B = Y}} (-1)^{|A|+|B|}$$

and note here that $A, B \subseteq \mathcal{A}$ are not necessarily flats. Then for any fixed flats $X, Z \in L(M)$

$$\begin{aligned}
\sum_{\substack{Y \in L(M) \\ X \leqslant Y \leqslant Z}} g(X,Y) &= \sum_{\substack{Y \in L(M) \\ X \leqslant Y \leqslant Z}} \left[\sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ \bigvee A = X \\ \bigvee B = Y}} (-1)^{|A|+|B|} \right] \\
&= \sum_{\substack{Y \in L(M) \\ X \leqslant Y \leqslant Z}} \left[\sum_{\substack{A \subseteq \mathcal{A} \\ \bigvee A = X}} (-1)^{|A|} \left[\sum_{\substack{B \supseteq A \\ \bigvee B = Y}} (-1)^{|B|} \right] \right] \\
&= \sum_{\substack{A \subseteq \mathcal{A} \\ \bigvee A = X}} (-1)^{|A|} \left[\sum_{\substack{Y \in L(M) \\ X \leqslant Y \leqslant Z}} \left[\sum_{\substack{B \supseteq A \\ \bigvee B = Y}} (-1)^{|B|} \right] \right] \\
&= \sum_{\substack{A \subseteq \mathcal{A} \\ \bigvee A = X}} (-1)^{|A|} \left[\sum_{\substack{B \subseteq \mathcal{A} \\ A \subseteq B \subseteq Z}} (-1)^{|B|} \right].
\end{aligned}$$

The sum in the brackets in the last line above is zero in all cases except when $A = Z$ which implies that $X = Z$ and the entire expression is $(-1)^{2|Z|} = 1$. Hence we have shown that the function g satisfies the same recursion as the Möbius function μ. $\square$

2.3 Characteristic polynomials and deletion-contraction

The characteristic polynomial has been one of the most studied matroid invariants. This invariant will appear in various places in this work.

Definition 9. Let $M = (\mathcal{A}, \mathrm{rk})$ be a polymatroid and $L(M)$ its lattice of flats. The *characteristic polynomial* of M is

$$\chi_M(t) = \sum_{X \in L(M)} \mu(\hat{0}, X) t^{\mathrm{crk}(X)}$$

where $\operatorname{crk}(X) = \operatorname{rk}(M) - \operatorname{rk}(X)$ (we sometime use this notation to condense exponents when needed).

A key fact is that the characteristic polynomial of a matroid is an evaluation of the Tutte polynomial:

$$\chi_M(t) = (-1)^{\operatorname{rk}(M)} T_M(1-t; 0).$$

Now we recall the functions that satisfy a deletion-contraction formula.

Definition 10. Let R be a commutative ring. We say that a function $f : \mathrm{Mat} \to R$ is a *generalized Tutte-Grothendieck invariant* if for any $M \in \mathrm{Mat}$ with ground set $\mathcal{A}$ and $e \in \mathcal{A}$,

$$f(M) = \begin{cases} af(M\backslash e) + bf(M/e) & \text{if } e \text{ is neither a loop nor a coloop} \\ f(M\backslash e)f(L) & \text{if } e \text{ is a loop} \\ f(M/e)f(C) & \text{if } e \text{ is a coloop} \end{cases}$$

for some fixed non-zero constants $a, b \in R$ where C is the matroid of one coloop and L is a matroid consisting of one loop.

3 Basic properties of chain Tutte polynomials

For the majority of this note we focus primarily on the chain generalization of the classical Tutte polynomial given in Definition 1. In section 4 we also study an equivalent version (a change of coordinates) in Definition 24.

For most of this study we will focus on the setting of matroids. However, we may examine the case of a more general ranked, atomic lattice L in which case we write $T_L^k((x_i);(y_i))$ where $\operatorname{rk}(S_i)$ means the rank of $\operatorname{rk}(\bigvee S_i)$ in L. In this case the atoms are the ground set elements and for a subset of ground set elements S_i the rank $\operatorname{rk}(S_i)$ is the usual matroid rank function.

3.1 Formulas between different chain Tutte polynomials

As desired, the higher chain Tutte polynomials determine all the lower chain Tutte polynomials.

Lemma 11. *For any matroid M and any $k \geqslant 1$*

$$T_M^{k+1}(2, 2x_1 - 1, x_2, \dots, x_k; 2, 2^{-1}y_1 + 2^{-1}, y_2, \dots, y_k) = 2^{\operatorname{rk}(M)} T_M^k(x_1, \dots, x_k; y_1, \dots, y_k).$$

Proof. Computing the evaluation we get

$$T_M^{k+1}(2, 2x_1 - 1, x_2, \ldots, x_k; 2, 2^{-1}y_1 + 2^{-1}, y_2, \ldots, y_k)$$

$$= \sum_{(S_i)_1^{k+1} \in \mathcal{C}_{\mathcal{A}}^{k+1}} 2^{\mathrm{rk}(M)-|S_2|} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(M)-\mathrm{rk}(S_{i+1})} (y_i - 1)^{|S_{i+1}|-\mathrm{rk}(S_{i+1})}$$

$$= 2^{\mathrm{rk}(M)} \sum_{(S_i)_2^{k+1} \in \mathcal{C}_{\mathcal{A}}^{k}} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(M)-\mathrm{rk}(S_{i+1})} (y_i - 1)^{|S_{i+1}|-\mathrm{rk}(S_{i+1})}$$

since the number of subsets S_1 of S_2 is exactly $2^{|S_2|}$. By noting the last quantity above is just a reindexing of $2^{\mathrm{rk}(M)} T_M^k(x_1, \ldots, x_k; y_1, \ldots, y_k)$ we have completed the proof. □

Hence the classic Tutte polynomial T^1 is determined by any higher chain Tutte polynomial T^k for $k \geqslant 2$. Moreover, we have the following which gives that the $|\mathcal{A}|^{th}$ chain Tutte polynomial $T_M^{|\mathcal{A}|}$ for M determines all the other chain Tutte polynomials T_M^k.

Proposition 12. *For any $n, k \geqslant 1$ there exists is a homomorphism*

$$\sigma^k : \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathbb{Z}[x_1, \ldots, x_k, y_1, \ldots, y_k]$$

such that $\sigma^k(T_M^n) = T_M^k$ for any matroid $M = (\mathcal{A}, \mathrm{rk})$ with $|\mathcal{A}| = n$.

Proof. For $k \leqslant |\mathcal{A}|$ Lemma 11 gives the result. For $k > |\mathcal{A}|$ we build the homomorphism on the chain Whitney polynomial W_M^k instead of T_M^k in order to write a map in terms of monomials. Then we can make the easy substitution back to T_M^k. Any term of W_M^k will have at least $k - |\mathcal{A}|$ of the same consecutive exponents on both the a_i and the b_i of the form $a_i^e a_{i+1}^e$ and $b_i^f b_{i+1}^f$. Conversely, in W_M^j, for any j, whenever a term has the same exponents consecutively in both a_i, a_{i+1} and b_i, b_{i+1} matching variables in the product this means that both $\mathrm{rk}(S_i) = \mathrm{rk}(S_{i+1})$ and $|S_i| = |S_{i+1}|$. Since the (S_i) are a chain we have that $S_i = Si + 1$. This leads us to define shifted evaluations, but we have to keep track of when the exponents of the terms are the same in order to avoid getting over counted repeats in our map below.

For any monomial $m = \prod_{i=1}^{|\mathcal{A}|} a_i^{e_i} b_i^{f_i} \in \mathbb{Z}[a_1, \ldots, a_{|\mathcal{A}|}, b_1, \ldots, b_{|\mathcal{A}|}]$ we set

$$\begin{aligned} D_k(m) = \ & \{(n_1, \ldots, n_{|\mathcal{A}|}) \in \mathbb{N}^{|\mathcal{A}|} \mid \ n_1 + \cdots + n_{|\mathcal{A}|} = k - |\mathcal{A}|, \\ & \text{and if } e_i = e_{i+1} = \cdots = e_{i+\lambda} \text{ and } f_i = f_{i+1} = \cdots = f_{i+\lambda} \\ & \text{then } n_{i+1} = \cdots = n_{i+\lambda} = 0\} \end{aligned}$$

be the set of distributions for determining new terms from $W_M^{|\mathcal{A}|}$ to W_M^k. Also, define $B_1 = 0$ and for any $2 \leqslant j \leqslant |\mathcal{A}|$ set $B_j = n_1 + n_2 + \cdots + n_{j-1}$. Then for any monomial $m = \prod_{i=1}^{|\mathcal{A}|} a_i^{e_i} b_i^{f_i} \in \mathbb{Z}[a_1, \ldots, a_{|\mathcal{A}|}, b_1, \ldots, b_{|\mathcal{A}|}]$ we define

$$\rho_k : \mathbb{Z}[a_1, \ldots, a_{|\mathcal{A}|}, b_1, \ldots, b_{|\mathcal{A}|}] \to \mathbb{Z}[a_1, \ldots, a_k, b_1, \ldots, b_k]$$

by setting

$$\rho_k(m) = \sum_{(n_1, \ldots, n_{|\mathcal{A}|}) \in D_k(m)} \prod_{i=1}^{|\mathcal{A}|} \prod_{j=0}^{n_i} (a_{i+B_i+j})^{e_i} (b_{i+B_i+j})^{f_i}.$$

Extending ρ_k linearly and applying to all the terms of $W_M^{|\mathcal{A}|}$ we get $\rho_k(W_M^{|\mathcal{A}|}) = W_M^k$. Finally, as noted above σ_k is defined by $s^{-1} \circ \rho_k \circ s$ where s is the substitution $x_i \mapsto a_i + 1$ and $y_i \mapsto b_i + 1$. □

3.2 Derksen's $\mathcal{G}$-invariant

In [14] Derksen and Fink proved that the $\mathcal{G}$-invariant is universal among valuative invariants.

Theorem 13 ([14, Theorem 1.4])**.** *The coefficients of $\mathcal{G}$ span the vector space of all valuative matroid invariants with values in $\mathbb{Q}$.*

We juxtapose this with a formula for determining Derksen's $\mathcal{G}$-invariant in terms of $T^{|\mathcal{A}|}$.

Proof of Theorem 4. First we fix the rank to be $r \geqslant 0$ and the number of ground set elements to be $n = |\mathcal{A}|$. Then we consider the ring homomorphism

$$\psi_{r,n} : \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}] \to \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}]$$

defined by $\psi_{r,n}(1) = 1$,

$$a_i^{\pm 1} \mapsto \begin{cases} (a_i^{-1} a_{i+1} b_i^{-1})^{\pm 1} & \text{if } 1 \leqslant i < n \\ (a_n^{-1} b_n^{-1})^{\pm 1} & \text{if } i = n \end{cases}$$

and

$$b_i^{\pm 1} \mapsto b_i^{\pm 1}$$

for all $1 \leqslant i \leqslant n$ and extend polynomially. Also, define a module homomorphism

$$s_{r,n} : \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}] \to \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}]$$

by

$$s_{r,n}(p) = p \cdot a_1^r \prod_{i=1}^{n} b_i^r.$$

Then composing these two maps we get

$$\begin{aligned}
&(s_{r,n} \circ \psi_{r,n})(W_M^n) \\
&\quad = s_{r,n}\left(\sum_{(S_i)\in\mathcal{C}_{\mathcal{A}}^n}\left[\prod_{i=1}^{n-1}(a_i^{-1}a_{i+1}b_i^{-1})^{\mathrm{rk}(M)-\mathrm{rk}(S_i)} b_i^{|S_i|-\mathrm{rk}(S_i)}\right](a_n^{-1}b_n^{-1})^{\mathrm{rk}(M)-\mathrm{rk}(S_n)}\right) \\
&\quad = a_1^{\mathrm{rk}(M)}\prod_{i=1}^{n} b_i^{\mathrm{rk}(M)}\left(\sum_{(S_i)\in\mathcal{C}_{\mathcal{A}}^n} a_1^{\mathrm{rk}(S_1)-\mathrm{rk}(M)}\left[\prod_{i=2}^{n} a_i^{\mathrm{rk}(S_i)-\mathrm{rk}(S_{i-1})}\right]\left[\prod_{i=1}^{n} b_i^{|S_i|-\mathrm{rk}(M)}\right]\right) \\
&\quad = \sum_{(S_i)\in\mathcal{C}_{\mathcal{A}}^n}\prod_{i=1}^{n} a_i^{\mathrm{rk}(S_i)-\mathrm{rk}(S_{i-1})} b_i^{|S_i|}
\end{aligned}$$

where we consider $S_0 = \emptyset$ and note that $\mathrm{rk}(\emptyset) = 0$.

Consider the decomposition $\mathrm{QSym} = \bigoplus_{r,n} \mathrm{QSym}_{r,n}$ where $\mathrm{QSym}_{r,n}$ are the quasi-symmetric functions spanned by U_s where s has length n and the sum of the elements in s is r. With this decomposition we can define our final map

$$\theta_{r,n} : \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}] \to \mathrm{QSym}_{r,n}$$

on monomials by

$$\prod_{i=1}^{n} a_i^{e_i} b_i^{f_i} \mapsto \left\{ \begin{array}{cc} U_{(e_1,\ldots,e_n)} & \text{if } \forall i,\ f_i = i, \text{ and } \sum e_i = r \\ 0 & \text{else} \end{array} \right. .$$

By construction for any polymatroid $M \in \mathrm{Mat}_{r,n}$ we have $(\theta_{r,n} \circ s_{r,n} \circ \psi_{r,n})(W_M^n) = \mathcal{G}(M)$. To finish the proof we consider the extension

$$\Psi = \bigoplus_{r,n} (\theta_{r,n} \circ s_{r,n} \circ \psi_{r,n})$$

on all of

$$\bigoplus_{r,n} \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, b_1^{\pm 1}, \ldots b_n^{\pm 1}] \to \mathrm{QSym}$$

to get the desired result that $\Psi(W_M^n) = \mathcal{G}(M)$ for any polymatroid. □

3.3 Direct sums and duals

Next we present a few basic facts about these polynomials which generalize the properties of the Tutte polynomial T^1_M. First we show that up to a change of coordinates a matroid and its dual have the same the chain Tutte polynomial.

Proposition 14. *If M^* is the dual matroid of M then*

$$T^k_{M^*}(x_1, \ldots, x_k; y_1, \ldots, y_k) = T^k_M(y_k, \ldots, y_1; x_k, \ldots, x_1).$$

Proof. First we note that the map sending a flag $(A_1, \ldots, A_k)$ to its complement $(\mathcal{A} - A_k, \ldots, \mathcal{A} - A_1)$ is a permutation of $\mathcal{C}^k(\mathcal{A})$. Then just use Definition 1 and the fact that $\mathrm{rk}_{M^*}(X) = |X| - \mathrm{rk}_M(\mathcal{A}) + \mathrm{rk}_M(\mathcal{A} - X)$. $\square$

The next fact is that T^k is multiplicative over matroid direct sum. Recall that if $M_1 = (\mathcal{A}_1, \mathcal{I}_1)$ and $M_2 = (\mathcal{A}_2, \mathcal{I}_2)$ are matroids then $M_1 \oplus M_2 = (\mathcal{A}_1 \sqcup \mathcal{A}_2, \{I_1 \sqcup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\})$.

Proposition 15. *If $M_1 = (\mathcal{A}_1, \mathcal{I}_1)$ and $M_2 = (\mathcal{A}_2, \mathcal{I}_2)$ are matroids then*

$$T^k_{M_1 \oplus M_2}((x_i); (y_i)) = T^k_{M_1}((x_i); (y_i)) \cdot T^k_{M_2}((x_i); (y_i)).$$

Proof. The proof is done by decomposing the definition of $T^k_{M_1 \oplus M_2}((x_i); (y_i))$ in terms of the disjoint sets $\mathcal{A}_1$ and $\mathcal{A}_2$. We find that

$$\begin{aligned}
&T^k_{M_1 \oplus M_2}((x_i); (y_i)) = \\
&= \sum_{(S_i) \in \mathcal{C}^k_{\mathcal{A}}} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(\mathcal{A}_1 \sqcup \mathcal{A}_2) - \mathrm{rk}(S_i)} (y_i - 1)^{|S_i| - \mathrm{rk}(S_i)} \\
&= \sum_{(S^1_i) \in \mathcal{C}^k_{\mathcal{A}_1}} \sum_{(S^2_i) \in \mathcal{C}^k_{\mathcal{A}_2}} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(\mathcal{A}_1 \sqcup \mathcal{A}_2) - \mathrm{rk}(S^1_i \sqcup S^2_i)} (y_i - 1)^{|S^1_i \sqcup S^2_i| - \mathrm{rk}(S^1_i \sqcup S^2_i)} \\
&= \sum_{(S^1_i) \in \mathcal{C}^k_{\mathcal{A}_1}} \sum_{(S^2_i) \in \mathcal{C}^k_{\mathcal{A}_2}} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(\mathcal{A}_1) + \mathrm{rk}(\mathcal{A}_2) - \mathrm{rk}(S^1_i) - \mathrm{rk}(S^2_i)} (y_i - 1)^{|S^1_i| + |S^2_i| - \mathrm{rk}(S^1_i) - \mathrm{rk}(S^2_i)} \\
&= \sum_{(S^1_i) \in \mathcal{C}^k_{\mathcal{A}_1}} \prod_{i=1}^{k} (x_i - 1)^{\mathrm{rk}(\mathcal{A}_1) - \mathrm{rk}(S^1_i)} (y_i - 1)^{|S^1_i| - \mathrm{rk}(S^1_i)}
\end{aligned}$$

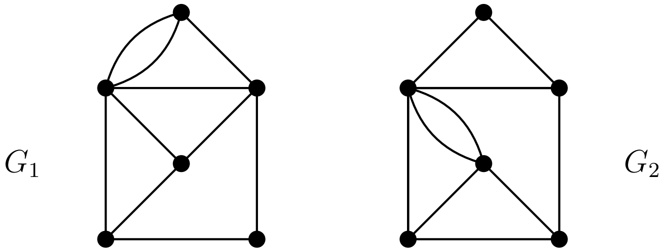

Figure 1: The Gray graphs

$$\cdot \sum_{(S_i^2)\in\mathcal{C}^k_{\mathcal{A}_2}} \prod_{i=1}^{k} (x_i-1)^{\mathrm{rk}(\mathcal{A}_2)-\mathrm{rk}(S_i^2)}(y_i-1)^{|S_i^2|-\mathrm{rk}(S_i^2)}$$

which is exactly the product $T^k_{M_1}((x_i);(y_i)) \cdot T^k_{M_2}((x_i);(y_i))$. □

3.4 Basic Examples

Through some examples we see that chain Tutte polynomials can have negative coefficients, but have more information than the classic Tutte polynomial.

Example 16. Let $\mathcal{B}_1$ be the rank 1 Boolean lattice (subset lattice of a set with 1 element) also known as the free matroid on one element or $U_{1,1}$ or the matroid of one coloop C. Then by direct computation

$$T^k_{\mathcal{B}_1}((x_i);(y_i)) = 1 + \sum_{i=1}^{k}\prod_{j=1}^{i}(x_j-1).$$

Notice that even $T^2_{\mathcal{B}_1}(x_1,x_2;y_1,y_2)$ has negative coefficients. Then by Theorem 15 we have that

$$T^k_{\mathcal{B}_n}((x_i);(y_i)) = \left(1 + \sum_{i=1}^{k}\prod_{j=1}^{i}(x_j-1)\right)^n$$

where $\mathcal{B}_n$ is the free matroid on n elements (aka Boolean subset lattice or the uniform matroid $U_{n,n}$).

Now we show that the second chain Tutte polynomial T^2 has more information than the Tutte polynomial T^1 for a few classic examples.

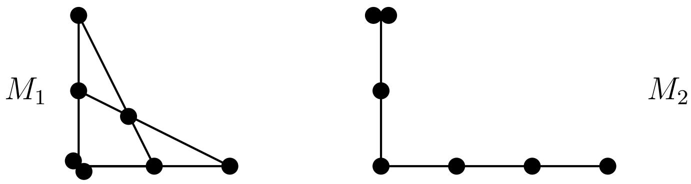

Figure 2: Two non-isomorphic matroids with same Tutte polynomial T^1 but different chain Tutte polynomial T^2

Example 17. The two Gray graphs G_1 and G_2, pictured in Figure 1 (see [13, Example 3.4] and [9]), have the same Tutte polynomial even though they are not isomorphic graphs. However, the second chain Tutte polynomials differ for G_1 and G_2. For example, the coefficient of $y_1y_2^3$ in $T^2_{G_1}$ is 1 but in $T^2_{G_1}$ it is 0. These polynomials both have 312 non-zero terms hence we do not list all coefficients (these computations were made using Sage [15]).

Example 18. Let M_1 and M_2 be the rank 3 matroids on 7 elements in Figure 2 ([8, Example 6.2.18]). The coefficient of $a_1^2a_2b_1b_2^2$ in $W^2_{M_1}$ is 2, but the coefficient of the same monomial in $W^2_{M_2}$ is 1. However, the classic Tutte polynomial of these two matroids is the same. The Derksen invariant $\mathcal{G}$ also distinguishes these two matroids ([13, Example 3.6]).

We will see below in Section 4.2 that Derksen's $\mathcal{G}$-invariant determines T^k for any $k > 0$. Hence, also the chain Tutte polynomials T^k for all k will not distinguish the examples built by Bonin in [6] and Bonin and Long in [7].

3.5 Generalized deletion-contraction for chain Tutte polynomials

First we look at a few examples to show that T^k for $k > 1$ is not a generalized Tutte-Grothendieck invariant.

Example 19. For $r, n \in \mathbb{N}$ and $r \leqslant n$ let $U_{r,n}$ be the matroid with ground set $[n]$ and independent sets are all subsets of $[n]$ of size less than or equal to r of $[n]$; $U_{r,n}$ are called uniform matroids. In order to show a deletion-contraction recursion does not exist we study the matroids $U_{2,4}$, $U_{2,3}$, and $U_{1,2}$. The deletions and contractions independently of the choice of ground set element e of these matroids are as follows: $U_{2,4}\backslash e = U_{2,3}$, $U_{2,4}/e = U_{1,3}$, $U_{2,3}\backslash e = U_{2,2}$, $U_{2,3}/e = U_{1,2}$, $U_{1,2}\backslash e = U_{1,1}$ and $U_{1,2}/e = U_{0,1}$. We can compute $T^2_{U_{0,1}} = (x_1-1)(x_2-1) + (x_1-1)(x_2-1)(y_2-1) + (x_1-1)(x_2-1)(y_1-1)(y_2-1)$, $T^2_{U_{1,1}} = 1 + (x_1-1)(x_2-1) + (x_1-1)$, and

$$T^2_{U_{1,2}} = (x_1-1)(x_2-1) + 2(x_1-1) + (x_1-1)(y_2-1) + 2(y_2-1) + (y_1-1)(y_2-1) + 2.$$

Then using the computer algebra system Sage [15] we get

$$T^2_{U_{2,3}} = x_1^2x_2^2 + x_1^2x_2 - 2x_1x_2^2 + x_1^2y_2 + x_1x_2 + x_2^2 + x_1y_2 + y_1y_2 - 2x_2 - y_1 + 1$$

and

$$\begin{aligned} T^2_{U_{2,4}} = \;& x_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2 + 2x_1^2x_2 - 2x_1x_2^2 + 2x_1^2y_2 \\ & -2y_1^2y_2 + 2x_1y_2^2 + 2y_1y_2^2 + x_2^2 + y_1^2 - 2x_2 - 2y_1 + 1. \end{aligned}$$

If there was a deletion-contraction recursion from Definition 10 then we would have the following equations $T^2_{U_{1,2}} = aT^2_{U_{1,1}} + bT^2_{U_{0,1}}$, $T^2_{U_{2,3}} = aT^2_{U_{2,2}} + bT^2_{U_{1,2}}$, and $T^2_{U_{2,4}} = aT^2_{U_{2,3}} + bT^2_{U_{1,3}}$. Hence, elements in the kernel of the matrix

$$A = \begin{bmatrix} T^2_{U_{1,1}} & T^2_{U_{0,1}} & T^2_{U_{1,2}} \\ T^2_{U_{2,2}} & T^2_{U_{1,2}} & T^2_{U_{2,3}} \\ T^2_{U_{2,3}} & T^2_{U_{1,3}} & T^2_{U_{2,4}} \end{bmatrix}$$

over the field of rational functions would be candidates for a and b. However, the determinant of A is a non-zero polynomial of degree 10 (again computed using Sage). Hence it's not possible for T^2 to be a generalized Tutte-Grothendieck invariant. Since T^2 is an evaluation of T^k for all $k > 1$ we know that these are also not generalized Tutte-Grothendieck invariants.

Next we study the split chain Tutte polynomials in Definition 2. Note that these split chain Tutte polynomials are just reorderings of the original chain Tutte polynomials separated by terms containing or not containing the element $a \in \mathcal{A}$. Gathering terms in order to present a formula for the split chain Tutte polynomials we have

$$sT^{k,j}_{M,a}((x_i),(y_i)) =$$

$$\sum_{(S_i)_1^{k-j} \in \mathcal{C}^{k-j}_{\mathcal{A}-a}} T^j_{M|S_1} \prod_{i=1}^{k-j} (x_{i+j}-1)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S_i)} (y_{i+j}-1)^{|S_i|-\mathrm{rk}_{M/a}(S_i)} \prod_{i=1}^{j} (x_i-1)^{\mathrm{rk}(M/S_1)}.$$

Condensing this by decomposing the middle terms as the shifted terms in the contraction $M/(S_1 \cup a)$ and then summing only over the first subsets in the chain, assuming that $k \geqslant 2$ and $1 \leqslant j \leqslant k-1$, we have that

$$sT^{k,j}_{M,a}((x_i),(y_i)) = \sum_{S_1 \subseteq \mathcal{A}-a} g(j,a,S_1) T^j_{M|_{S_1}}((x_i)_1^j;(y_i)_1^j) T^{k-j-1}_{M/(S_1\cup a)}((x_i)_{2+j}^k;(y_i)_{2+j}^k) \quad (2)$$

where

$$g(j,a,S_1) = \left[\prod_{i=1}^{j}(x_i-1)^{\mathrm{rk}(M/S_1)}\right](x_{j+1}-1)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S_1)}\prod_{i=1}^{k-j}(y_{i+j}-1)^{|S_1|-\mathrm{rk}_{M/a}(S_1)}.$$

Note that we can view the first chain Tutte polynomial in (2) over the matroid $M\backslash(\mathcal{A}-(S_1\cup a))$.

By construction for $a\in\mathcal{A}$ not a loop and not a coloop we can state the classical recursion for the Tutte polynomial as

$$T_M^1 = sT_{M,a}^{1,0} + sT_{M,a}^{1,1}.$$

Now Theorem 3 generalizes this for $k>1$ which we now prove.

Proof of Theorem 3. We split the terms of T_M^k up by which sets contain $a\in\mathcal{A}$. For $j\in\{1,\dots,k-1\}$ let

$$st_{M,a}^{k,0}((x_i);(y_i)) = \sum_{\substack{(S_i)_1^k\in\mathcal{C}_{\mathcal{A}}^k\\ a\in S_1}}\prod_{i=1}^{k}(x_i-1)^{\mathrm{rk}(M)-\mathrm{rk}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}(S_i)},$$

$$st_{M,a}^{k,k}((x_i);(y_i)) = \sum_{(S_i)_1^k\in\mathcal{C}_{\mathcal{A}-a}^k}\prod_{i=1}^{k}(x_i-1)^{\mathrm{rk}(M)-\mathrm{rk}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}(S_i)},$$

and

$$st_{M,a}^{k,j}((x_i);(y_i)) = \sum_{\substack{(S_i)_{j+1}^k\in\mathcal{C}_{\mathcal{A}}^{k-j}\\ a\in S_{j+1}}}\ \sum_{(S_i)_1^j\in\mathcal{C}_{S_{j+1}-a}^j}\prod_{i=1}^{k}(x_i-1)^{\mathrm{rk}(M)-\mathrm{rk}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}(S_i)}.$$

Then by construction

$$T_M^k = \sum_{j=0}^{k} st_{M,a}^{k,j}((x_i);(y_i)).$$

We know that for all $S\subseteq\mathcal{A}$ with $a\in S$ we have $\mathrm{rk}_M(S) = \mathrm{rk}_{M/a}(S-a)+\mathrm{rk}_M(a)$. Since $a\in\mathcal{A}$ is not a loop for all $S\subseteq\mathcal{A}$ with $a\in S$ we have $|S|-\mathrm{rk}(S) = |S-a|+1-(\mathrm{rk}_{M/a}(S-a)+\mathrm{rk}_M(a)) = |S-a|-\mathrm{rk}_{M/a}(S-a)$. Also, for $S\subseteq\mathcal{A}$ with

$a \in S$ we have $\mathrm{rk}(M)-\mathrm{rk}_M(S) = (\mathrm{rk}_{M/a}(\mathcal{A}-a)+\mathrm{rk}_M(a))-(\mathrm{rk}_{M/a}(S-a)+\mathrm{rk}_M(a)) = \mathrm{rk}(M/a) - \mathrm{rk}_{M/a}(S-a)$. Putting this into the $j=0$ term we have

$$st^{k,0}_{M,a}((x_i);(y_i)) = \sum_{\substack{(S_i)_1^k \in \mathcal{C}^k_{\mathcal{A}} \\ a \in S_1}} \prod_{i=1}^{k} (x_i-1)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S_i-a)}(y_i-1)^{|S_i-a|-\mathrm{rk}_{M/a}(S_i-a)}$$

$$= \sum_{(S_i)_1^k \in \mathcal{C}^k_{\mathcal{A}-a}} \prod_{i=1}^{k} (x_i-1)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}_{M/a}(S_i)} = T^k_{M/a} = sT^{k,0}_{M,a}.$$

In the $j=k$ case the argument is much simpler. Since $a \in A$ is not a coloop we know that $rk(M) = \mathrm{rk}(M\backslash a)$. So,

$$st^{k,k}_{M,a} = \sum_{(S_i)_1^k \in \mathcal{C}^k_{\mathcal{A}-a}} \prod_{i=1}^{k} (x_i-1)^{\mathrm{rk}(M\backslash a)-\mathrm{rk}_{M\backslash a}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}_{M\backslash a}(S_i)} = T^k_{M\backslash a} = sT^{k,k}_{M,a}.$$

Now for any $j \in \{1,\dots,k-1\}$ we see that

$$st^{k,j}_{M,a}((x_i);(y_i)) = \sum_{\substack{(S_i)_{j+1}^k \in \mathcal{C}^{k-j}_{\mathcal{A}} \\ a \in S_{j+1}}} \prod_{i=j+1}^{k} (x_i-1)^{\mathrm{rk}(M)-\mathrm{rk}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}(S_i)}\phi_j$$

where

$$\phi_j = \sum_{(S_i)_1^j \in \mathcal{C}^j_{S_{j+1}-a}} \prod_{i=1}^{j} (x_i-1)^{\mathrm{rk}(M)-\mathrm{rk}(S_i)}(y_i-1)^{|S_i|-\mathrm{rk}(S_i)}.$$

Notice that $\mathrm{rk}(M) = \mathrm{rk}_{M\backslash[\mathcal{A}-(S_{j+1}-a)]}(S_{j+1}-a) + \mathrm{rk}_{M/(S_{j+1}-a)}(\mathcal{A}-(S_{j+1}-a))$. Using this we see that

$$\phi_j = \left[\prod_{i=1}^{j}(x_i-1)^{\mathrm{rk}(M/(S_{j+1}-a))}\right] T^j_{M\backslash[\mathcal{A}-(S_{j+1}-a)]}.$$

Then we note that for $a \in S_{j+1} \subseteq \cdots \subseteq S_k$ we know that $\mathrm{rk}(M) - \mathrm{rk}(S_i) = \mathrm{rk}(M/a) - \mathrm{rk}_{M/a}(S_i - a)$ and that $|S_i| - \mathrm{rk}_M(S_i) = |S_i - a| - \mathrm{rk}_{M/a}(S_i - a)$ since a is not a loop. Hence

$$st^{k,j}_{M,a} = \sum_{\substack{(S_i)_{j+1}^k \in \mathcal{C}^{k-j}_{\mathcal{A}} \\ a \in S_{j+1}}} \prod_{i=j+1}^{k} (x_i-1)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S_i-a)}(y_i-1)^{|S_i-a|-\mathrm{rk}_{M/a}(S_i-a)}\phi_j.$$

Now that each term has a taken out we reindex and write this as

$$st^{k,j}_{M,a} = \sum_{(S_i)_1^{k-j} \in \mathcal{C}^{k-j}_{\mathcal{A}-a}} \prod_{i=1}^{k-j} (x_{i+j} - 1)^{\mathrm{rk}(M/a) - \mathrm{rk}_{M/a}(S_i)} (y_{i+j} - 1)^{|S_i| - \mathrm{rk}_{M/a}(S_i)} \phi'_j$$

where

$$\phi'_j = \left[\prod_{i=1}^{j} (x_i - 1)^{\mathrm{rk}(M/S_1)} \right] T^j_{M \setminus [\mathcal{A} - S_1]}.$$

Since this is exactly the definition of our split Tutte polynomials we are done. The formulas for the cases where $a \in \mathcal{A}$ is a loop or coloop follow from a direct application of Proposition 15. ☐

4 Generalized permutahedra and valuations

We start with the definition of generalized permutahedra where we follow Ardila-Sanchez [3]. We will study a convolution product definition of a generalized version of chain Tutte polynomials to the setting of generalized permutahedra. Then we will restrict this new definition to the submonoid of matroids in order to show how chain Tutte polynomials can be derived from these convolution products.

4.1 Generalized permutahedra

First we note that a function $f : 2^{\mathcal{A}} \to \mathbb{R}$ is *submodular* if for all $A, B \subseteq \mathcal{A}$ the function f satisfies $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$.

Definition 20. A *generalized permutahedron* is a polytope P in $\mathbb{R}^{\mathcal{A}}$ which is of the form

$$P = \{x \in \mathbb{R}^{\mathcal{A}} \mid \sum_{i \in \mathcal{A}} x_i = z(\mathcal{A}) \text{ and } \sum_{i \in S} x_i \leqslant z(S) \text{ for all } S \subseteq \mathcal{A}\}$$

where $z : 2^{\mathcal{A}} \to \mathbb{R}$ is a submodular function. We denote the set species of generalized permutahedra by **GP** (as in [3], meaning for any finite set I, $\mathbf{GP}[I]$ is a vector space spanned by the set of all generalized permutahedra in $\mathbb{R}^I$). **GP** has coproduct $\Delta_{S_1,\ldots,S_k}$ defined by $\Delta_{S_1,\ldots,S_k}(P) = (P_1, \ldots, P_k)$ where P_i are generalized permutahedra in $\mathbb{R}^{S_i}$ such that the maximal face of P with respect to the indicator vector $e_{(S_1,\ldots,S_k)} = e_{S_1} + e_{S_1 \sqcup S_2} + \cdots + e_{S_1 \sqcup \cdots \sqcup S_k}$ is $P_{e_{(S_1,\ldots,S_k)}} = P_1 \times \cdots \times P_k$ (see [1, Proposition 1.4.4] for details).

Next we need to consider how polytopes decompose into pieces. The idea is to construct invariants on polytopes inductively by their pieces.

Definition 21. A generalized permutahedral *subdivision* of a polytope P is a set of generalized permutahedra $\{P_1, \ldots, P_s\}$ whose vertices are vertices of P,

$$P = \bigcup_{i=1}^{s} P_i$$

and for all $1 \leqslant i < j \leqslant s$ if $P_i \cap P_j \neq \emptyset$ then $P_i \cap P_j$ is a proper face of both P_i and P_j.

Now we can state the definition of a valuation using subdivisions, which is the main subject of the section.

Definition 22. A function $f : \mathbf{GP} \to R$ where R is an algebra is a *valuation* if $f(\emptyset) = 0$ and for any $P \in \mathbf{GP}$ and any subdivision $\{P_1, \ldots, P_k\}$ of P we have

$$f(P) = \sum_{\emptyset \neq \{j_1, \ldots, j_i\} \subseteq [k]} (-1)^i f(P_{j_1} \cap \cdots \cap P_{j_i}).$$

Remark 23. In the literature there are many studies on different classes of combinatorial objects which properly contain all generalized permutahedra as well as many different notions of valuative functions (see [20, Appendix A] for an excellent summary). In this paper we restrict to **GP** and the definition of a valuation in Definition 22.

Next we focus on some polynomials which are slightly different than those defined in Definition 1. To do this, again we follow Ardila and Sanchez [3]. In [3, Definition 6.2] Ardila and Sanchez present a convolution of species maps $f_i : \mathbf{GP} \to R$ for $i \in [k]$ where R is an algebra over $\mathbb{F}$ with multiplication m. The *convolution* of these functions is the species map $f_1 \star f_2 \star \cdots \star f_k : \mathbf{GP} \to R$ defined by

$$f_1 \star f_2 \star \cdots \star f_k[\mathcal{A}](P) = \sum_{S_1 \sqcup \cdots \sqcup S_k = \mathcal{A}} m^{k-1} \circ f_1[S_1] \otimes \cdots \otimes f_k[S_k] \circ \Delta_{S_1, \ldots, S_k}(P)$$

where $f_i[S_i] : \mathbf{GP}[S_i] \to R$ are restrictions.

Now we present a chain Tutte polynomial defined on **GP** using a species map which can be found in the proof of Proposition 7.4 in [3].

Definition 24. Let $N_i[\mathcal{A}](P) = u_i^{|\mathcal{A}|} v_i^{z_P(\mathcal{A})}$ where z_P is the semimodular function defining P. The k^{th} *permutahedral chain Tutte polynomial* is

$$\mathcal{T}_P^k((u_i); (v_i)) = N_1 \star N_2 \star \cdots \star N_k[\mathcal{A}](P).$$

The next fact follows from the construction in Definition 24 and results in [3].

Proposition 25. *The k^{th} permutahedral chain Tutte polynomial $\mathcal{T}^k$ is a valuation on* **GP**.

Proof. In [3, Proposition 7.4] Ardila and Sanchez show that N_i is a valuation on the species of extended generalized permutahedra. Then by [3, Corollary 6.3] we have the conclusion. □

4.2 Matroid valuations

First we recall the basis matroid polytope (using [3] as our general reference for this material). A matroid M can be defined via its set of bases $\mathcal{B}(M)$ which are all the independent sets of M whose size is the rank of M. Then the matroid polytope of M is

$$P(M) = \mathrm{Conv}\{e_B \mid B \in \mathcal{B}(M)\}$$

where $e_B = e_{i_1} + \cdots + e_{i_r}$ with $B = \{i_1, \ldots, i_r\}$. Now we need a few key definitions to state our main result. These are exactly the same as Definitions 21 and 22 restricted to the Hopf submonoid of matroids.

Definition 26. A *matroid polyhedral subdivision* of a matroid polytope $P(M)$ is a subdivision of $P(M)$ where all the pieces are matroid polytopes.

Now we want to know how invariants decompose across subdivisions which gives rise to valuations. We note again that there are many notions of valuations on matroids, but we call a function $f : \mathrm{Mat} \to R$ a *matroid valuation* if it is a valuation of matroid polytopes.

Now we note that the monoid of matroids is a submonoid of **GP** so we can restrict our permutahedral chain Tutte polynomial to the monoid of matroids. In the case of matroids the coproduct on a set decomposition $\mathcal{A} = S_1 \sqcup \cdots \sqcup S_k$ with $A_0 = \emptyset$ and $A_i = S_1 \sqcup \cdots \sqcup S_i$ is ([3, Lemma 2.7])

$$\Delta_{S_1,\ldots,S_k}(M) = M[A_0, A_1] \otimes M[A_1, A_2] \otimes \cdots \otimes M[A_{k-1}, A_k]$$

where $M[A, B] = (M \mid B)/A$. For matroid polytopes the associated semimodular function is exactly the rank function i.e. $z_{P(M)} = \mathrm{rk}_M$. Then the i^{th} term in the product of $\mathcal{T}^k$ is

$$N_i[S_i](P(M[A_{i-1}, A_i])) = u_i^{|S_i|} v_i^{\mathrm{rk}_{(M|A_i)/A_{i-1}}(S_i)}.$$

Putting this all together the k^{th} permutahedral chain Tutte polynomial of a matroid polytope $P(M)$ is

$$\mathcal{T}^k_{P(M)}((u_i);(v_i)) = \sum_{S_1 \sqcup \cdots \sqcup S_k = \mathcal{A}} \left[\prod_{i=1}^{k} N_i[S_i](P(M[A_{i-1}, A_i])) \right] \tag{3}$$

$$= \sum_{(A_i) \in \mathcal{C}^k_{\mathcal{A}}} \prod_{i=1}^{k} u_i^{|A_i - A_{i-1}|} v_i^{\mathrm{rk}(A_i) - \mathrm{rk}(A_{i-1})} \tag{4}$$

where $A_0 = \emptyset$ and $A_k = \mathcal{A}$ in the summation (4).

Proof of Theorem 4. First we make a change of coordinates by setting $u_{k+1} = 1$, $v_1 = b_1^{-1} \cdots b_k^{-1}$, $v_{k+1} = a_1 \cdots a_k$, for $i \in [k]$

$$u_i = \prod_{j=i}^{k} b_j,$$

and for $1 < i < k+1$

$$v_i = \prod_{j=i}^{k} b_j^{-1} \prod_{j=1}^{i-1} a_j.$$

Putting this change of coordinates into the formulation of $\mathcal{T}^{k+1}_{P(M)}$ in (4) we get

$$\mathcal{T}^{k+1}_{P(M)}((u_i);(v_i)) = \sum_{A_1, \ldots, A_{k+1} \in \mathcal{C}^{k+1}_{\mathcal{A}}} \prod_{i=1}^{k+1} u_i^{|A_i - A_{i-1}|} v_i^{\mathrm{rk}(A_i) - \mathrm{rk}(A_{i-1})}$$

$$= \sum_{A_1, \ldots, A_{k+1} \in \mathcal{C}^{k+1}_{\mathcal{A}}} \prod_{i=1}^{k} \left(\prod_{j=i}^{k} b_j \right)^{|A_i - A_{i-1}|} \left(\prod_{j=i}^{k} b_j^{-1} \prod_{j=1}^{i-1} a_j \right)^{\mathrm{rk}(A_i) - \mathrm{rk}(A_{i-1})} (a_1 \cdots a_k)^{\mathrm{rk}(A_{k+1}) - \mathrm{rk}(A_k)}.$$

In each of these terms we see that the exponent of a_i is

$$\sum_{j=i}^{k} \mathrm{rk}(A_{j+1}) - \mathrm{rk}(A_j) = \mathrm{rk}(M) - \mathrm{rk}(A_i)$$

since $A_{k+1} = \mathcal{A}$ and the exponent of b_i is

$$\sum_{j=1}^{i} \big(|A_j - A_{j-1}| - (\mathrm{rk}(A_j) - \mathrm{rk}(A_{j-1})) \big) = |A_i| - \mathrm{rk}(A_i).$$

Hence with this change of coordinates $\mathcal{T}^{k+1}_{P(M)}((u_i)_1^{k+1};(v_i)_1^{k+1})$ is the k^{th} chain Whitney polynomial $W^k_M((a_i)_1^k;(b_i)_1^k)$. Then restricting to the submonoid of matroids of **GP** using Proposition 25 we can conclude W^k_M and also T^k_M are matroid valuations. □

5 Chain Tutte evaluations and specializations

In this section we study various evaluations and specializations of chain Tutte polynomials. One idea we visit repeatedly is to find an evaluation that gives some combinatorial information and then apply the generalized deletion-contraction from Theorem 3. We start with some basic constant evaluations and then look at some polynomials. For most of this section we will just focus on T^2.

5.1 Constant evaluations

We begin by discussing some constant evaluations of T^k_M which are straightforward to describe combinatorially and are essentially obtained from the classical Tutte polynomial. The following can be concluded directly from the definition.

Proposition 27. *For any matroid M we have*

$$T^k_M(1,\dots,1;1,\dots,1)=|\mathcal{B}|$$

where $\mathcal{B}$ is the set of bases of M.

Next we look at an example of a different constant evaluation for a specific matroid.

Example 28. Let $\mathcal{B}_n$ be the Boolean matroid of rank n (a.k.a. the uniform matroid of rank n on n elements). Then $T^k_{\mathcal{B}_1}(2,\dots,2;1,\dots,1)=k+1$ since a flag of subsets of a set with one element is determined by how many empty sets are in the sequence (also just evaluate the expression given in Example 16). Using this and Theorem 15 we get that

$$T^k_{\mathcal{B}_n}(2,\dots,2;1,\dots,1)=(k+1)^n.$$

The Boolean matroids are fairly simple to understand since there is only one basis. However, in the $k=2$ case we can reformulate the result of Example 28 in general.

Proposition 29. *Let M be a matroid and I_m be the number of independent sets of size m in M. Then*

$$T^2_M(2,2;1,1)=\sum_{m=0}^{\mathrm{rk}(M)}2^m I_m.$$

If we swap the 1s and 2s in the above we get the 'dual' result. Recall that $T^1_M(1;2)$ is the number of spanning sets of the matroid. Now we examine $T^2_M(1,1;2,2)$ and again we need more notation.

Proposition 30. *Let SP be the set of spanning sets of $M = (\mathcal{A}, \mathrm{rk})$ and for $X \in SP$ let $NSP(X)$ be the number of subsets of X that are also spanning. Then*

$$\begin{aligned} T^2_M(1,1;2,2) &= \sum_{X \in SP} NSP(X) \\ &= \sum_{X \in SP} 2^{|\mathcal{A}|-|X|}. \end{aligned}$$

The evaluation $T^2_M(1,2;2,1)$ is the number of bases because in the summation the only terms that contribute are those which S_1 must be spanning and S_2 must be independent. If $S_1 \subseteq S_2$ with S_1 spanning and S_2 independent then $S_1 = S_2$ and they are a basis. Now we examine a similar evaluation of T^2_M.

Proposition 31. *For $X \in SP$ a spanning set let $NI(X)$ be the number of independent sets inside X. Then*

$$T^2_M(2,1;1,2) = \sum_{X \in SP} NI(X).$$

We can phrase Proposition 31 in terms of graphs.

Example 32. Let G be an undirected graph with M_G the associated matroid. Then for any spanning subgraph G' the number of independent sets inside this subgraph is the number of subforests. Hence

$$T^2_{M_G}(2,1;1,2) = \sum_{G' \text{ spanning}} NF(G')$$

where $NF(G')$ is the number of subforests in G'. Using the computer algebra system Sage [15] we compute these evaluations for complete graphs K_n and cycle graphs C_n in Tables 1 and 2.

Now we consider a few evaluations with $k > 0$ instead of just $k = 2$.

n	1	2	3	4	5
$T^2_{M_{K_n}}(2,1;1,2)$	1	2	19	523	36478

Table 1: The (2,1;1,2) evaluations for complete graphs

n	3	4	5	6	7
$T^2_{M_{C_n}}(2,1;1,2)$	19	47	111	255	575

Table 2: The (2,1;1,2) evaluations for cycle graphs

Proposition 33. *If M is a matroid with minimal flat $\hat{0}$ and maximal flat $\hat{1}$ then*

$$T^k_M(1,1,\dots,1;0,0,\dots,0) = \begin{cases} 1 & \textit{if } k \textit{ is even} \\ (-1)^{\mathrm{rk}(M)}\mu(\hat{0},\hat{1}) & \textit{if } k \textit{ is odd} \end{cases}.$$

Proof. We compute

$$T^k_M(1,1,\dots,1;0,0,\dots,0) = (-1)^{k\cdot\mathrm{rk}(M)} \sum_{\substack{S_1\subseteq\cdots\subseteq S_k\subseteq\mathcal{A} \\ \bigvee S_1=\cdots=\bigvee S_k=\hat{1}}} (-1)^{\sum|S_i|}.$$

Next we induct on k. The classic $k=1$ case can be found for example in [8, (6.21)] or can be deduced from Lemma 7. The $k=2$ base case is just Lemma 8 for $\mu(\hat{1},\hat{1})$. When k is even we have a telescoping decomposition

$$\sum_{\substack{S_1\subseteq\cdots\subseteq S_k\subseteq\mathcal{A} \\ \bigvee S_1=\cdots=\bigvee S_k=\hat{1}}} (-1)^{\sum|S_i|} = \sum_{\substack{S_{k-1}\subseteq S_k\subseteq\mathcal{A} \\ \bigvee S_{k-1}=\bigvee S_k=\hat{1}}} (-1)^{|S_{k-1}|+|S_k|} \left[\sum_{\substack{S_1\subseteq\cdots\subseteq S_{k-1} \\ \bigvee S_1=\cdots=\bigvee S_{k-1}}} (-1)^{\sum_{i=1}^{k-2}|S_i|} \right].$$

Then by induction the summation in the above bracket is 1. So, using Lemma 8 again we get the total is 1. Now suppose that $k>2$ is odd. Similarly we separate the sum

$$\sum_{\substack{S_1\subseteq\cdots\subseteq S_k\subseteq\mathcal{A} \\ \bigvee S_1=\cdots=\bigvee S_k=\hat{1}}} (-1)^{\sum|S_i|} = \sum_{\substack{S_k\subseteq\mathcal{A} \\ \bigvee S_k=\hat{1}}} (-1)^{|S_k|} \left[\sum_{\substack{S_1\subseteq\cdots\subseteq S_{k-1}\subseteq S_k \\ \bigvee S_1=\cdots=\bigvee S_k}} (-1)^{\sum_{i=1}^{k-1}|S_i|} \right].$$

Since $k-1$ is even the summation in the bracket above is 1 by induction and then the result follows from Lemma 7. □

Now switching the zeros and ones we examine the 'dual' result whose proof is very similar. Recall that $T^1_M(0;1)$ is, up to a sign, the reduced Euler characteristic of the independence complex of M (see [2, Section 7.7.1]).

Proposition 34. *If M is a matroid with χ the reduced Euler characteristic of the independence complex of M and $k \geqslant 1$ then*

$$T^k_M(0,0,\dots,0;1,1,\dots,1) = \begin{cases} 1 & \text{if } k \text{ is even} \\ (-1)^{\mathrm{rk}(M)}\chi & \text{if } k \text{ is odd} \end{cases}.$$

Proof. Assume that $k > 1$ is even. In this case there is again a telescoping of the sum but this time the sets are all independent:

$$T^k_M(0,0,\dots,0;1,1,\dots,1) = (-1)^{k\cdot\mathrm{rk}(M)} \sum_{\substack{S_1\subseteq\cdots\subseteq S_k\subseteq\mathcal{A} \\ \text{independent}}} (-1)^{\sum|S_i|}$$

$$= (-1)^{k\cdot\mathrm{rk}(M)} \sum_{\substack{S_k\subseteq\mathcal{A} \\ \text{independent}}} (-1)^{|S_k|} \sum_{\substack{S_{k-1}\subseteq S_k \\ \text{independent}}} (-1)^{|S_{k-1}|} \cdots \sum_{\substack{S_1\subseteq S_2 \\ \text{independent}}} (-1)^{|S_1|}.$$

The individual sum

$$\sum_{\substack{S_1\subseteq S_2 \\ \text{independent}}} (-1)^{|S_1|}$$

is 0 unless $S_2 = \emptyset$. If $S_2 = \emptyset$ then

$$\sum_{\substack{S_2\subseteq S_3 \\ \text{independent}}} (-1)^{|S_2|} \left[\sum_{\substack{S_1\subseteq S_2 \\ \text{independent}}} (-1)^{|S_1|} \right] = (-1)^{|\emptyset|} = 1.$$

Since k is even each pair reduces to 1 and we have the desired result. Finally if k is odd then the same decomposition has all the pairs inside the sum reduce to 1 and the last summation term

$$\sum_{\substack{S_k\subseteq\mathcal{A} \\ \text{independent}}} (-1)^{|S_k|}$$

remains and is exactly the reduced Euler characteristic (again see [2, Section 7.7.1]). This finishes the proof. □

5.2 The Möbius polynomial

The Möbius polynomial was used by Jurrius in [29] to study weight enumerators of error-correcting linear codes. Then recently Johnsen and Verdure studied the Möbius polynomial on error-correcting codes from a commutative algebra viewpoint in [27]. In this subsection we study the Möbius polynomial exclusively.

Proof of Theorem 6. First we compute the evaluation of the second chain Tutte polynomial:

$$\begin{aligned} T^2_{L(M)}(1-s,1-t;0,0) &= \sum_{A\subseteq B\subseteq E} (-s)^{\mathrm{crk}(A)}(-t)^{\mathrm{crk}(B)}(-1)^{|A|+|B|-\mathrm{rk}(A)-\mathrm{rk}(B)} \\ &= \sum_{A\subseteq B\subseteq E} (-1)^{|A|+|B|} s^{\mathrm{crk}(A)} t^{\mathrm{crk}(B)} \\ &= \sum_{X\leqslant Y\in L(M)} \left[\sum_{\substack{A\subseteq B\subseteq E\\ \bigvee A=X\\ \bigvee B=Y}} (-1)^{|A|+|B|} \right] s^{\mathrm{crk}(X)} t^{\mathrm{crk}(Y)}. \end{aligned}$$

Then by Lemma 8 we are done since that is exactly the definition of the Möbius polynomial (see (1)). We also get to conclude that the Möbius polynomial is a valuation from combining Theorem 5 and Proposition 6. □

We see that the Möbius polynomial is not a Tutte-Grothendieck invariant by using the evaluation in Proposition 6 applied to the sequence of matroids in Example 19. However, applying the generalized deletion-contraction from Theorem 3 and the evaluation for T^2 from Theorem 6 we get a new recursion for the Möbius polynomial.

Corollary 35. *If $M = (\mathcal{A}, \mathcal{I})$ is a matroid and $a \in \mathcal{A}$ is not a loop or coloop then*

$$\begin{aligned} \bar{\chi}_M(s,t) =& \bar{\chi}_{M\backslash a}(s,t) + \bar{\chi}_{M/a}(s,t) \\ &+ (-1)^{\mathrm{rk}(M/a)} \sum_{S\subseteq \mathcal{A}-a} (-1)^{\mathrm{rk}(M/S)+|S|} s^{\mathrm{rk}(M/S)} t^{\mathrm{rk}(M/a)-\mathrm{rk}(S)} \chi_{M\backslash[\mathcal{A}-S]}(s). \end{aligned}$$

Putting Proposition 6 and Proposition 33 together gives a quick proof that $\bar{\chi}_M(0,0) = 1$ (which is an easy consequence of $\mu(\hat{1},\hat{1}) = 1$).

5.3 Opposite characteristic polynomial

Let M be a matroid and $L(M)$ its lattice of flats. The order on $L(M)$ is given by inclusion. If we just reverse this inclusion order we get the lattice $L(M)^{\mathrm{op}}$ which is the same set of flats but with reverse inclusion order (in the case of a hyperplane arrangement L^{op} would be the regular inclusion order on intersections). This opposite order on flats is not necessarily atomic nor semimodular. However $L(M)^{\mathrm{op}}$ is still a ranked lattice and so we can consider its characteristic polynomial.

Definition 36. The *opposite characteristic polynomial* of a matroid M with top flat $\hat{1}$ is

$$\chi_M^{\mathrm{op}}(t) = \chi_{L(M)^{\mathrm{op}}}(t) = \sum_{X \in L(M)^{\mathrm{op}}} \mu(\hat{1}, X) t^{\mathrm{crk}^{\mathrm{op}}(X)}$$

where $\mathrm{crk}^{\mathrm{op}}(X) = \mathrm{rk}(X)$ (because of the opposite order).

Even this polynomial has an evaluation formula using the second chain Tutte polynomial.

Proposition 37. *If M is a matroid then*

$$T_M^2(1-t, 1; 0, 0) = t^{\mathrm{rk}(M)} \chi_{L(M)}^{\mathrm{op}}(t^{-1}).$$

Proof. We compute the evaluation

$$T_M^2(1-t, 1; 0, 0) = \sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ B \text{ spanning}}} (-1)^{\mathrm{rk}(M) - \mathrm{rk}(A) + |A| + |B| - \mathrm{rk}(A) - \mathrm{rk}(B)} t^{\mathrm{rk}(M) - \mathrm{rk}(A)}. \tag{5}$$

Since B is spanning we know that $\mathrm{rk}(B) = \mathrm{rk}(M)$ hence (5) becomes

$$\sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ B \text{ spanning}}} (-1)^{|A| + |B|} t^{\mathrm{rk}(M) - \mathrm{rk}(A)}. \tag{6}$$

Then we split the sum of (6) on A by flats as

$$\sum_{X \in L(M)} \left(\sum_{\substack{A \subseteq B \subseteq \mathcal{A} \\ \bigvee A = X \\ \bigvee B = \hat{1}}} (-1)^{|A| + |B|} \right) t^{\mathrm{rk}(M) - \mathrm{rk}(X)}. \tag{7}$$

Applying Lemma 8 to the inner sum of (7) we get

$$\sum_{X\in L(M)} \mu(X,\hat{1})t^{\mathrm{rk}(M)-\mathrm{rk}(X)}. \tag{8}$$

Since the Möbius function can be defined recursively from the top element $\hat{1}$ we have that (8) becomes

$$t^{\mathrm{rk}(M)} \sum_{X\in L(M)^{\mathrm{op}}} \mu(X,\hat{1})(t^{-1})^{\mathrm{rk}(X)}. \tag{9}$$

Since the corank in $L(M)^{\mathrm{op}}$ is actually the rank in $L(M)$ the expression in (9) is actually $t^{\mathrm{rk}(M)}\chi^{\mathrm{op}}_{L(M)}(t^{-1})$ and we have proved the result. □

Again we get to combine the evaluation result, in this case Proposition 37, with Theorem 5 to conclude the opposite characteristic polynomial is a valuation.

Corollary 38. *The opposite characteristic polynomial is a valuation on matroids.*

We also restrict the generalized deletion-contraction from the chain Tutte polynomial in Theorem 3 to the opposite characteristic polynomial. Again note that the opposite characteristic polynomial is not a Tutte-Grothendieck invariant (use the evaluation in Proposition 37 with Example 19).

Corollary 39. *If* $M = (\mathcal{A}, \mathrm{rk})$ *is a matroid,* $a \in \mathcal{A}$ *is not a loop or coloop, and* $\mathrm{SPD} = \{S \subseteq \mathcal{A} - a \mid \ S \text{ spans } M/a\}$ *then*

$$\chi^{\mathrm{op}}_M(t) = \chi^{\mathrm{op}}_{M\setminus a}(t) + \chi^{\mathrm{op}}_{M/a}(t) + (-1)^{\mathrm{rk}(M)-1}\sum_{S\in\mathrm{SPD}}(-1)^{|S|}\chi_{M|S}(t).$$

Proof. Because of the formula in Proposition 37 we plug that evaluation into Theorem 3. We simplify the split Tutte polynomial

$$\begin{aligned} sT^{2,1}_{M,a}(1-t,1;0,0) &= \sum_{S\subseteq\mathcal{A}-a} T^1_{M|S}(1-t;0)(0)^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(S)}(-1)^{|S|-\mathrm{rk}_{M/a}(S)}(-t)^{\mathrm{rk}(M/S)} \\ &= \sum_{S\in\mathrm{SPD}} \chi_{M|S}(t)(-1)^{|S|-\mathrm{rk}(M/a)}(-t)^0 \end{aligned}$$

which finishes the proof. □

From this recursive result we can deduce the next corollary (even though it can be derived from the definition and properties of the Möbius function).

Corollary 40. *If M is a simple matroid with* $\mathrm{rk}(M) > 0$ *then* $\chi_M^{\mathrm{op}}(1) = 0$.

Proof. We proceed by induction on the number of ground set elements of M. If there is one element and the rank is one then $\chi_M^{\mathrm{op}}(t) = t - 1$. Then for larger ground sets the formula in Corollary 39 finishes the proof. □

5.4 Generalized Möbius polynomial

In [28] a function J was studied that generalizes the Möbius function and it was used to define and study a generalized Möbius polynomial. Also in [28] it was conjectured that this generalized Möbius polynomial was a matroid valuation. We prove that conjecture in this subsection.

Definition 41. The J-function on a matroid M with ground set $\mathcal{A}$ is $J : \mathcal{F}l^3_{\mathcal{A}} \to R$ where R is a commutative ring and $\mathcal{F}l^3_{\mathcal{A}} = \{(X, Y, Z) \in L(M)^3 \mid X \leqslant Y \leqslant Z\}$ is 3-flags of flats in $L(M)$ defined recursively by $J(X, X, X) = 1$ for all $X \in L(M)$ and

$$\sum_{X \leqslant A \leqslant Y \leqslant B \leqslant Z} J(A, Y, B) = \delta_3(X, Y, Z)$$

for all $(X, Y, Z) \in \mathcal{F}l^3_{\mathcal{A}}$ where δ_3 is the 3-variable Kronecker delta function.

In [28] it was shown that this J-function satisfies various generalizations of the Möbius function. This led the authors there to define analogous generalized characteristic and Möbius polynomials.

Definition 42. The generalized J-Möbius polynomial of M is

$$\mathcal{M}_M(t) = \sum_{(X,Y,Z) \in \mathcal{F}l^3(L(M))} J(X, Y, Z) t^{\mathrm{crk}(X) + \mathrm{crk}(Y) + \mathrm{crk}(Z)}.$$

Also in [28, Proposition 6.10] it was shown that this generalized J-Möbius polynomial satisfies a decomposition with the classic characteristic polynomials.

Proposition 43 ([28])**.** *If M is a matroid then*

$$\mathcal{M}_M(t) = t^{\mathrm{rk}(M)} \sum_{y \in L(M)} t^{\mathrm{crk}(y)} \chi_{L(M)^y}(t) \chi_{(L(M)^{\mathrm{op}})^y}(t^{-1}).$$

Now putting together Proposition 43 and Corollary 38 we can answer [28, Question 6.26].

Corollary 44. *The generalized J-Möbius polynomial is a valuation on matroids.*

5.5 Expected codimension and Ford's S-polynomial

Understanding properties in general of the realization space of a matroid (a.k.a. matroid variety) is an elusive endeavor (see [42] and [37]). In [25] Ford studied the dimension of matroid varieties and when the dimensions could be computed combinatorially. To do this Ford defined the following.

Definition 45 ([25])**.** Let $M = (\mathcal{A}, \mathrm{rk})$ be a matroid and $a : 2^{\mathcal{A}} \to \mathbb{Z}$ be defined recursively by $a(\emptyset) = 0$ and

$$a(A) = \mathrm{crk}(A) - \sum_{B \subset A} a(B).$$

The *expected codimension of M* is

$$ec(M) = \sum_{A \subseteq \mathcal{A}} (\mathrm{rk}(M) - \mathrm{rk}(A))a(A).$$

The idea behind this definition is that counting Plücker coordinates gives a way to guess the codimension of the matroid variety. Then Ford in [25] proves that for positroids this expected codimension is actually the codimension of the positroid's realization space. The final result in Ford's paper on this expected dimension is that it is a valuative property. Hence a natural question is how we can produce this expected dimension from these chain Tutte polynomials. It turns out we can do this via another polynomial which Ford defines in [25, Lemma 5.4].

Definition 46 ([25])**.** Let $M = (\mathcal{A}, \mathrm{rk})$ be a matroid. The *Ford S-polynomial of M* is

$$S_M(x, y, z) = \sum_{A \subseteq B \subseteq \mathcal{A}} x^{|A| - \mathrm{rk}(A)} y^{\mathrm{rk}(M) - \mathrm{rk}(B)} z^{|B| - |A|}.$$

Next in [25] Ford proves that this polynomial $S_M(x, y, z)$ is a matroid valuation using a similar technique to Speyer's proof in [39] that the Tutte polynomial is a matroid valuation. It turns out that we can find this S_M polynomial in T^2 which gives another proof that it is a matroid valuation.

Proposition 47 ([25, Lemma 5.4])**.** *The Ford S-polynomial is a matroid valuation.*

Proof. Since T^2 is a matroid valuation and

$$T^2_M(z + 1, yz^{-1} + 1; xz^{-1} + 1, z + 1) = S_M(x, y, z)$$

we have concluded the result. □

Using the proof of [25, Theorem 5.3] we get the formula for expected codimension in terms of T^2.

Corollary 48. *For any matroid M*

$$ec(M) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[T^2_M(z+1, yz^{-1}+1; xz^{-1}+1, z+1)\right](1,1,-1).$$

Now examine the generalized deletion-contraction result on T^2, Theorem 3, applied to Ford's S-polynomial.

Proposition 49. *If $M = (\mathcal{A}, \mathrm{rk})$ is a matroid and $a \in \mathcal{A}$ is not a loop or coloop then*

$$\begin{aligned} S_M(x,y,z) = & S_{M\backslash a}(x,y,z) + S_{M/a}(x,y,z) \\ & + z \sum_{A\subseteq B\subseteq \mathcal{A}-a} x^{|A|-\mathrm{rk}(A)} y^{\mathrm{rk}(M/a)-\mathrm{rk}_{M/a}(B)} z^{|B|-|A|}\ . \end{aligned}$$

Note that the formula of Proposition 49 appears very close to $S_{M\backslash a}(x,y,z) + (1+z)S_{M/a}(x,y,z)$. However this is not correct and if it were correct then it would imply that $ec(M) = ec(M\backslash a)$ which is also not true in general.

Now putting together Proposition 49 and Corollary 48 we have a different presentation for the expected codimension of a matroid.

Corollary 50. *If $M = (\mathcal{A}, \mathrm{rk})$ is a matroid with $a \in \mathcal{A}$ not a loop or coloop then*

$$ec(M) = ec(M\backslash a) + ec(M/a) - \sum_{A\subseteq B\subseteq \mathcal{A}} (|A| - \mathrm{rk}(A))(\mathrm{rk}(M/a) - \mathrm{rk}_{M/a}(B))(-1)^{|B|-|A|}.$$

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