A bijection between non-separable planar maps and fighting fish

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Abstract

The class of fighting fish is a recently introduced model of branching surfaces constructed by gluing square cells in a directed way, that generalizes the standard combinatorial class of parallelogram polyominoes. Fighting fish are enumerated with respect to their half-perimeter by the sequence $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$, also known for counting non-separable rooted planar maps w.r.t the number of edges. Another common feature of these two combinatorial classes is that fighting fish can be seen as particular excursions on the square lattice confined to the quarter plane, while excursions on the quarter plane encode tree-rooted planar maps via Mullin's encoding.

We build on this point of view and on recursive decompositions of fish and maps to show that fighting fish can be identified with the Lehman-Lenormand codes of non-separable rooted planar maps, which is obtained by endowing maps with their rightmost depth first search tree before applying Mullin's encoding. The resulting direct bijection yields a simple characterization of fighting fish as minimal non-separable excursions in the quarter plane.

We then also introduce a natural extension of fighting fish that we call *generalized fighting fish* and we follow again the same approach to show that they correspond to Lehman-Lenormand codes of (general) rooted planar maps.

Mathematics Subject Classifications: 05A19, 05C10, 05C30

1 Introduction

Fighting fish were first defined in 2016 by Duchi et al. in [7] as a model of branching surfaces that is a generalization of parallelogram polyominoes. While parallelogram polyominoes are enumerated according to their half-perimeter by the Catalan numbers (see [6]), fighting fish are counted by the sequence $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}, n \geqslant 1$. It is the sequence $A000139$ of the OEIS [21] and it counts some other combinatorial objects: non-separable

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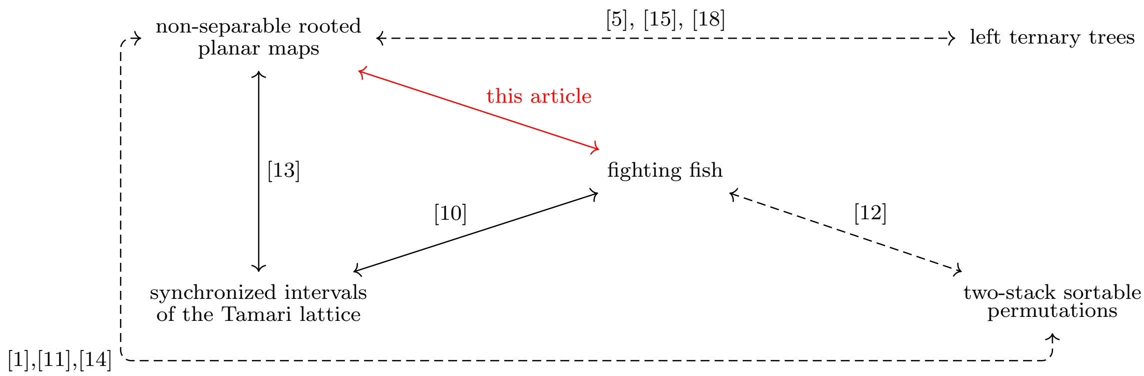

Figure 1: Some bijections between non-separable rooted maps and related combinatorial structures

planar maps, synchronized intervals of the Tamari lattice, left ternary trees, two-stack sortable permutations. In the last few years, a growing number of articles has been dealing with the problem of connecting bijectively these different classes of objects in order to have a better combinatorial understanding of this family. The following diagram summarizes the currently known bijections, dashed (resp. plain) arrows indicating recursive (resp. direct) bijections.

The present article is the second chapter of an extended version of a 12 page article presented at the FPSAC 2022 conference [9]: the first chapter [10] deals with the bijective link between fighting fish and intervals of the Tamari lattice, and introduces a notion of *extended fighting fish*. The objective of the current writing is twofold. The first purpose is to obtain a direct bijection between fighting fish and non-separable planar maps. This bijection relies on the restriction of a construction proposed by Mullin 50 years ago, that encodes any planar map endowed with a distinguished spanning tree by the shuffle of the contour word of the spanning tree with the parenthesis encoding of the edges not in the spanning tree. Our main result is indeed to show that Mullin's bijection is mapping non-separable planar maps endowed with their rightmost depth-first search spanning tree to fighting fish. The second purpose is to introduce another natural extension of fighting fish that we call *generalized fighting fish*, different from the extended fighting fish of [10]: we prove that these generalized fighting fish are exactly the Mullin encodings of (general) rooted planar maps endowed with their rightmost depth-first search spanning tree. In this case Mullin's code is also known as Lehman-Lenormand code, see [4, Chapter 2, II.3] and the discussion below.

The article is organized as follows. In Section 2 we first introduce fighting fish as surfaces obtained by directed gluings of unitary 45 degree tilted squares, and we show that, via the counterclockwise tour of their boundary, they are identified with certain excursions with steps $\{E, N, W, S\}$ on the quarter plane. We then introduce planar maps, and we state Mullin's theorem that tree-rooted planar maps with n edges are in bijection with excursions of length $2n$ on the square lattice. We show that in the case of non-separable planar maps canonically endowed with their rightmost DFS spanning tree the associated class of excursions can be characterized as minimal and non-separable. In Section 3 we first give a recursive decomposition for fighting fish, which we show to be isomorphic to Tutte's recursive decomposition of non-separable planar maps. A careful analysis of the resulting recursive bijection allows us to prove our main result, Theorem 9, stating that it can in fact be described as a direct (non-recursive) bijection using Mullin's encoding of non-separable planar map canonically endowed with their rightmost DFS spanning tree. This direct bijection, together with the previously mentioned characterization of excursions associated with non-separable maps, yields the corollary that fighting fish correspond to minimal and non-separable excursions. In Section 4 we introduce the class of generalized fighting fish: these objects are counted by the same numbers as (general) rooted planar maps, and we give them a recursive decomposition isomorphic to Tutte's decomposition of rooted planar maps. We then show, in Theorem 21 that this recursive bijection can again be described in a direct way by Mullin's encoding of planar maps canonically endowed with their rightmost DFS spanning tree. In the conclusion we discuss applications of our result to random generation.

2 Preliminaries

2.1 Fighting fish

A complete presentation of fighting fish has been made in the introductory articles [7] and [8], and we restate here their definitions in the language of cell gluings used in these papers.

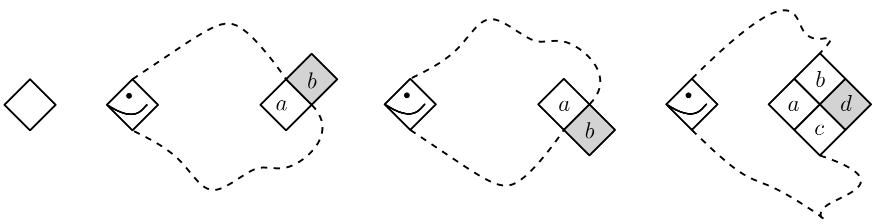

Figure 2: A cell and operations of upper, lower and double gluing.

A *cell* is a unit square on a 45 degree tilted lattice whose boundary is made out of four edges called *left lower edge*, *right lower edge*, *right upper edge* and *left upper edge*. We intend to build fighting fish as sets of cells glued together along some of their edges, so we define an edge of a cell to be *free* if it is not glued to the edge of another cell. A *fighting fish* is a finite set of cells constructed starting with an initial cell (the *head*), then attaching one by one new cells using one of the three following operations (represented in Figure 2):

- *Upper gluing*: Let a be a cell in a fish whose right upper edge is free; we glue the left lower edge of a new cell b to the right upper edge of a.
- *Lower gluing*: Let a be a cell in a fish whose right lower edge is free; we glue the left upper edge of a new cell b to the right lower edge of a.
- *Double gluing*: Let a, b and c be three cells in a fish such that b (resp. c) has its left lower (resp. upper) edge glued to the right upper (resp. lower) edge of a, and the right lower (resp. upper) edge of b (resp. c) is free; we glue the left upper and left lower edges of a new cell d respectively to the right lower edge of b and to the right upper edge of c.

While the description of fighting fish is iterative, we are interested in these objects independently of the order in which they are constructed. There can then be multiple ways to grow a given fighting fish with these operations. We also want to emphasize that fighting fish are not planar objects in the sense that we cannot always fit them in the plane because some unit squares would represent two or more different cells (see Figure 3(b)). Still, we will present them in our two-dimensional pictures by taking care of showing which cells are glued together (Figure 3(a) provides an example of two possible representations of the same fighting fish).

The *size* of a fighting fish F, denoted size(F), is the number of its free lower edges (or equivalently the number of its free upper edges), that is half of the length of its boundary.

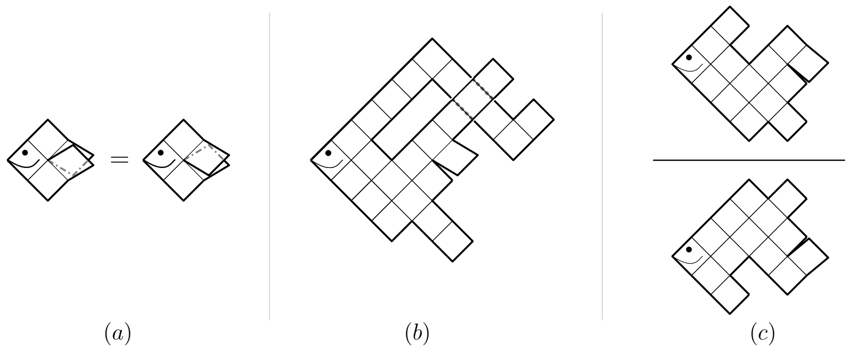

Figure 3: (a) Two representations of the same fighting fish of size 6; (b) a fighting fish of size 23; (c) duality on fighting fish.

Let us call *nose* the leftmost point of the head of a fighting fish. Another way to think of a fighting fish is to perform its counterclockwise tour: if we follow the boundary of a fighting fish counterclockwise, starting from the nose, we encounter all its free edges once until getting back to the nose. We now encode each type of free edge by a step in $\{E, N, W, S\}$: $E = (1, 0)$ (resp. $S = (0, -1)$) for a left lower (resp. upper) one and $N = (0, 1)$ (resp. $W = (-1, 0)$) for a right lower (resp. upper) one. Any word on the alphabet $\{E, N, W, S\}$ corresponds uniquely to a walk starting at the origin $(0, 0)$

on the (45-degree-tilted) square lattice $\mathbb{Z}^2$, hence we will make no difference between such words and walks. An *excursion* is a square lattice walk confined to the positive quadrant $\{x, y \geqslant 0\}$ and ending at the origin. Hence, performing their counterclockwise tour, fighting fish are exactly excursions obtained from the word $ENWS$ using the three operations (see Figure 4 for an example):

- Upper gluing: replace a factor W by NWS.
- Lower gluing: replace a factor N by ENW.
- Double gluing: replace a factor WN by NW.

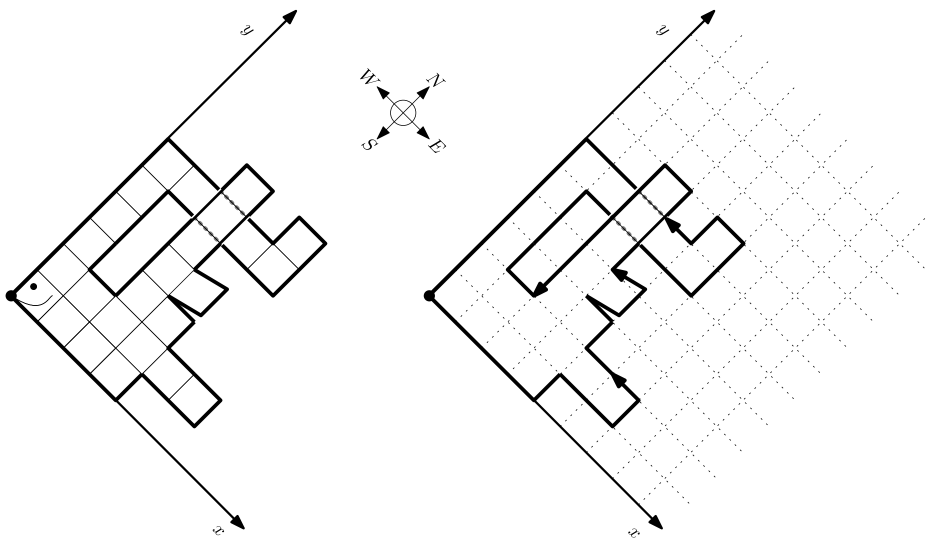

Figure 4: A fighting fish and its associated excursion on the 45 degree tilted lattice.

For a walk $w \in \{E, N, W, S\}^*$, we define its *longitude* and *latitude* to be the x and y coordinates of the endpoint. We then have

$$\begin{aligned} \mathrm{long}(w) &= |w|_E - |w|_W \\ \mathrm{lat}(w) &= |w|_N - |w|_S \,, \end{aligned}$$

where $|w|_a$ is the number of occurrences of the letter a in w, for $a \in \{E, N, W, S\}$.

The *dual* of a word in $\{E, N, W, S\}^*$ is obtained by reversing the word and replacing every occurrence of E (resp. N, W, S) by an occurrence of S (resp. W, N, E): it corresponds to reversing the timeline of the corresponding walk on $\mathbb{Z}^2$ while symmetrizing it with respect to the line $y = x$. For excursions corresponding to fish, duality corresponds to duality on fish, that is mirror symmetry w.r.t the horizontal axis, as illustrated on fighting fish in Figure 3(c).

2.2 Planar maps

Planar maps are classical objects in combinatorics (see [20] for more details) and while many equivalent definitions are possible, we present them as proper connected multigraph embeddings on the plane in order to be as graphical as possible in our setting.

A *planar map* is a proper embedding of a connected multigraph on the plane, defined up to continuous deformations preserving the orientation. A planar map splits the plane into *edges*, *vertices* and *faces*, where faces are the connected components of the plane deprived of edges and vertices. The *size* of a planar map is the number of its edges. A *corner* is an incidence between a vertex and a face. A *bridge* is an edge whose deletion disconnects the map. A *loop* is an edge whose ends are incident to the same vertex. As usual we consider *rooted planar maps*, where a corner on the boundary of the infinite face is distinguished. This corner is called the *root*, while the infinite face is also called the *root face*, the edge preceding the root in counterclockwise order on the boundary of the infinite face is called the *root edge* and the vertex in the root corner is called the *root vertex*. The *reduced outer degree* out(M) of a rooted planar map M is the number of non-root corners incident to its root face. We refer to Figure 5(a) for an example of a rooted planar map. A rooted planar map is said to be *separable* if its set E of edges can be partitioned into

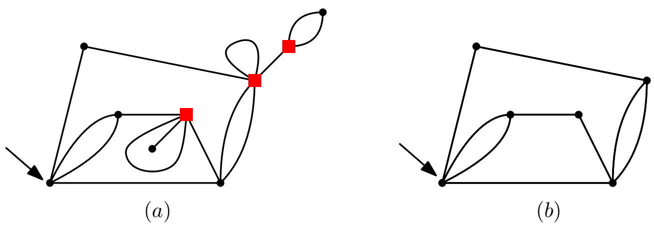

Figure 5: (a) A rooted planar map with its cut vertices shown as red squares; (b) a non-separable rooted planar map.

two disjoint non-empty subsets E_1 and E_2 so that exactly one vertex v is adjacent to edges of both sets (v is said to be a *cut-vertex*). A *non-separable planar map* is a rooted planar map with at least two edges that is not separable (see Figure 5(b) for an example).

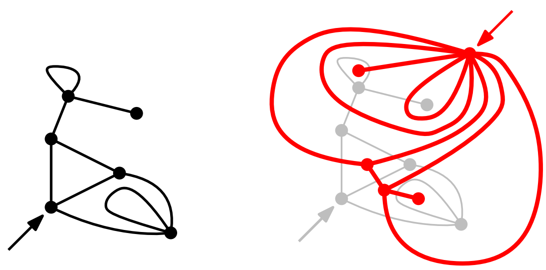

Figure 6: A rooted planar map in black then gray with its dual map drawn over it in thick red.

For a rooted planar map M, its *dual map* M^* (see Figure 6) is the planar map whose vertices correspond to the faces of M and whose set of edges is built in the following way: to each edge e of M, if we denote by f_1 and f_2 the two (not necessarily distinct) faces adjacent to e, there is an edge e^* in M^* linking the vertices v_{f_1} and v_{f_2} respectively corresponding to faces f_1 and f_2 of M, such that e^* crosses only e among edges in M and only once. Since a corner corresponds to an incidence between a vertex and a face, the root corner of M^* is chosen to be the same vertex-face incidence taken as root for M (interchanging the roles of face and vertex).

Let us remark that because it is embedded on the plane, a planar map defines a cyclic ordering of the edges and the corners around each vertex and the same is true around each face. Hence the edge e following a corner c in counterclockwise direction around a vertex v is well-defined as well as the corner c' on the other side of that edge e around the same vertex v. We will call e and c' respectively the *vertex-following edge* and the *vertex-following corner* of the corner c. We define similarly the *face-following edge* e' and the *face-following corner* c'' of the corner c, but this time in a clockwise direction around the face. Since vertex-following edge e and face-following edge e' of a corner coincide ($e = e'$), we will use the simpler terminology of *following edge*. We refer to Figure 7 for an illustration of these notions.

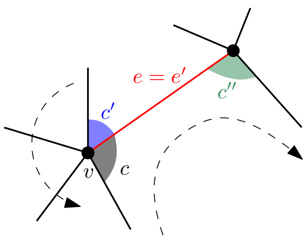

Figure 7: A corner c, its following edge e, its vertex-following corner c', its face-following edge $e' = e$, and its face-following corner c''.

In the following sections, we will endow rooted planar maps with a particular spanning tree called the *rightmost depth-first search spanning tree* and we will denote it for short as the rightmost DFS spanning tree. For a rooted planar map M, it is the spanning tree T obtained by an exploration of the corners of M using the following procedure (see Figure 8 for an example):

- Initialization: Set the tree T as the tree containing the root vertex of M and no edge, and set the active corner to be the root corner. Set also the set of visited edges to be empty.

- Core: Consider the active corner c: it is incident to a unique vertex v and to a unique face f, and we denote by e the following edge of c. There are 4 possible cases:

  – if e is not a visited edge and the face-following corner c' of c is incident to a vertex that does not belong to T, then add the edge e to T and to the set of visited edges and set the active corner to be c';

- if e is not a visited edge and the face-following corner of c is incident to a vertex that belongs to T, then add the edge e to the set of visited edges and set the active corner to be the vertex-following corner of c; *here the edge e is NOT inserted in T*;
- if e is a visited edge and e is an edge of T, then set the active corner to be the face-following corner of c;
- if e is a visited edge and e is not an edge of T, then set the active corner to be the vertex-following corner of c.

Repeat until the active corner is the root corner again.

- End: Return T.

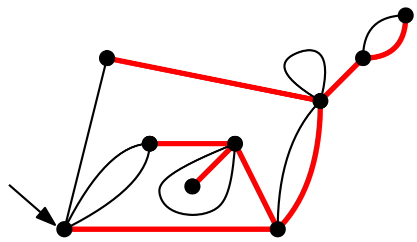

Figure 8: A rooted planar map and its rightmost DFS spanning tree highlighted in thick red.

Let us prove that T is indeed a spanning tree of the map M. First we observe that T remains connected during the construction since each time a vertex not yet in T is discovered, an edge connecting it to a vertex already in T is added to T. Moreover T is a tree, because every edge inserted in it during the procedure leads to a vertex that was not visited before, hence creates no cycle. Observe also that if an edge is visited during the procedure then both its endpoints have been visited. To prove that T is a spanning tree, let us color the edges of T and replay the exploration process: by construction the process visits all the corners of the map M that are incident to vertices of T, while turning in counterclockwise order around T; if T was not a spanning tree of M then by connectivity of M there would be an edge connecting a vertex v of T to a vertex not in T, hence an edge that was not visited. But this edge should have been discovered since all the corners around v have been visited.

2.3 The counterclockwise code of tree-rooted planar maps

We now recall an encoding of tree-rooted planar maps due to Lehman [16], see also [24], by shuffles of two Dyck words, or parenthesis-integer systems in their terminology. To be precise about the origin of the encoding, Mullin was the first to notice that tree-rooted planar maps could be encoded as intertwined Catalan structures in his article [17] about the enumeration of tree-rooted planar maps. Our presentation of the result follows closely the approach of Schaeffer in a book chapter about planar maps [20].

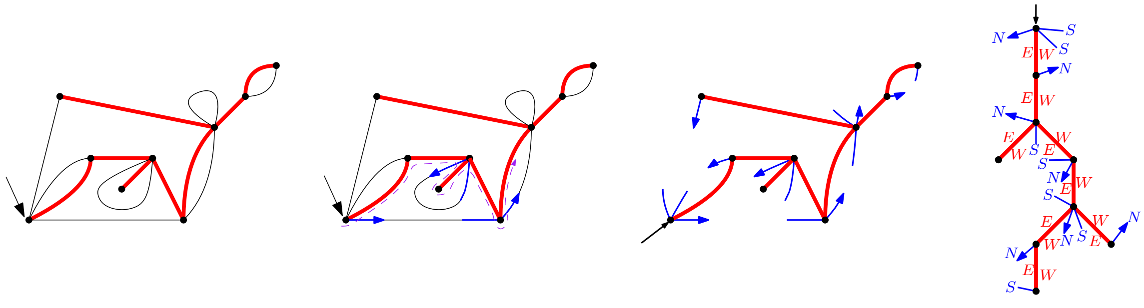

Figure 9: (a) A tree-rooted planar map; (b) the beginning of the counterclockwise tour; (c) the associated blossoming tree; (d) the blossoming tree with the letters of the counterclockwise code.

Let M be a rooted planar map, that we endow with a spanning tree T which we consider rooted at the same corner as M. The couple (M, T) is called a *tree-rooted planar map*. The *dual* of a tree-rooted planar map (M, T) is the tree-rooted planar map (M^*, T^*), with M^* being the dual map of M and T^* the spanning tree of M^* formed by the edges dual to the edges of $\mathrm{M} \setminus T$. We create the *blossoming tree* associated to the couple (M, T) by performing the *counterclockwise tour* of M around T (see the construction in Figure 9): we start from the root corner of M then visit all the corners of M by traveling on the border of the tree T in counterclockwise direction until we get back to the root corner. Each time we encounter an edge that does not belong to T we put an opening or closing stem and we continue traveling around the stem: more precisely when we encounter such an edge for the first time, we put an opening stem on the considered vertex along the considered edge, otherwise we put a closing stem in the same manner. We then forget the edges not belonging to T to get the blossoming tree associated to (M, T): it is the rooted plane tree T decorated on its vertices with opening and closing stems. We define then the *counterclockwise code* $\Gamma(\mathrm{M}, T)$ of the tree-rooted planar map (M, T) as the word in $\{E, N, W, S\}^*$ obtained via the following procedure: perform the counterclockwise tour of the associated blossoming tree starting from the root corner, then encode the first visit of an edge in T by E, the second visit of an edge in T by W, an opening stem by N and a closing stem by S, and stop when returning at the root corner. Such a word in $\{E, N, W, S\}^*$ can be seen as a walk on the square lattice $\mathbb{Z}^2$ by sending each letter to its corresponding unit cardinal step. Let us remark that it can also be seen as a shuffle of two Dyck words, one on the letters $\{E, W\}$, the other on the letters $\{N, S\}$, and so any N step pairs with its matching S step, and any E step pairs with its matching W step.

While the initial result was presented by Mullin in [17] in a more word theoretic fashion, we state an equivalent formulation of his result here, and we refer to the review of Schaeffer in [20] (subsection 1.2.1) for a more in-depth treatment of the subject.

Theorem 1. *(Mullin [17]) The counterclockwise code* Γ *is a bijection between tree-rooted planar maps with* n *edges and excursions of length* $2n$ *on the square lattice. This bijection also preserves duality, which means that the following diagram commutes:*

$$\begin{array}{ccc} (\mathrm{M}, T) & \xmapsto{\textit{map duality}} & (\mathrm{M}^*, T^*) \\ \Gamma \big\downarrow & & \big\downarrow \Gamma \\ \Gamma(\mathrm{M}, T) & \xmapsto{\textit{excursion duality}} & \Gamma(\mathrm{M}^*, T^*) \end{array}$$

where the excursion duality is defined in Section 2.1.

Concerning statistics, a tree-rooted planar map (M, T) *having* $p+1$ *vertices,* $q+1$ *faces has a counterclockwise code with* p E *steps and* q N *steps.*

2.4 Characterizations of some counterclockwise codes

We now prove some characterizations of counterclockwise codes of some subclasses of tree-rooted planar maps. They will enable us to give simple characterizations of (generalized) fighting fish.

Recall that an excursion on $\{E, S, W, N\}$ is a shuffle of a $\{E, W\}$-Dyck word with a $\{N, S\}$-Dyck word (*i.e.* any prefix of an excursion contains at least as many E as W and at least as many N as S). We say that an excursion w is *minimal* if there is no decomposition $w = w_1 N w_2 E w_3 S w_4 W w_5$, where the N matches with the S in the underlying $\{N, S\}$-Dyck word, and the E matches with the W in the underlying $\{E, W\}$-Dyck word, and with $w_1, w_2, w_3, w_4, w_5 \in \{E, N, W, S\}^*$.

Proposition 2. *An excursion* w *is minimal if it contains no factor of the form* $Ew'S$ *with* w' *a (possible empty) excursion.*

Proof. If w is not minimal, let $w = w_1 N w_2 E w_3 S w_4 W w_5$ be among the decompositions of the previous form, the one such that E is rightmost among those such that S is leftmost (in w). Then in the factor Ew_3S, w_3 is a (possibly empty) excursion, as it is a shuffle of a $\{E, W\}$-Dyck and a $\{N, S\}$-Dyck words. Indeed each S in w_3 has to be matched by a N on its left and in w_3: if the matching N was not in w_3 there would be another decomposition $w = w_1' N w_2' E w_3' S w_4' W w_5'$ with S further on the left, therefore we would have a contradiction. Moreover, each N has to be matched by a S in w_3 otherwise the S following w_3 would be matched by an N of w_3, in contradiction with the initial hypothesis. Symmetrically, we use the same arguments for the E and W to conclude that w_3 is an excursion. Conversely, if w can be decomposed as $w_i E w' S w_f$ with w' a possibly empty excursion, then the N matching the final S step of $Ew'S$ necessarily belongs to w_i since w' is an excursion. In a similar way, we have that the W matching the initial E step of $Ew'S$ belongs to w_f. Hence w writes $w = w_1 N w_2 E w' S w_3 W w_4$ with $w_1, w_2, w', w_3, w_4 \in \{E, N, W, S\}^*$, so w is not minimal. □

We first show that the counterclockwise codes of rooted planar maps endowed with their rightmost DFS spanning tree are exactly *minimal* excursions. This result has been known since the work of Walsh and Lehman ([24]). Apparently it was first described by Lehman in an unpublished work [16], and then was proved by his PhD student Walsh in his thesis ([23], pages 92-93). For completeness we provide a short proof here.

Proposition 3. *An excursion is minimal if and only if it is the counterclockwise code of a rooted planar map endowed with its rightmost DFS spanning tree.*

Proof. Let (M, T) be a rooted planar map endowed with a spanning tree which is not its rightmost DFS spanning tree T'. Consider the first edge e' encountered in the counterclockwise tour of (M, T) that belongs to T' but not to T. Note that every edge e encountered before e' in the counterclockwise tour of (M, T) is in T if and only if it is in T'. Indeed, the reverse implication (e in T' implies e in T) is true by definition of e'. Also, the direct implication (e in T implies e in T') follows directly from the fact that T' is the rightmost DFS tree of M. Let us denote by v and $v' \neq v$ the two vertices incident to e' such that v is the parent of v' in T'. Since T is a spanning tree of M, there exists a unique edge $e \neq e'$ in T incident to v' and $v'' \neq v'$ such that v'' is the parent of v' in T. Now we remark that e is encountered after e' in the counterclockwise tour of (M, T): if e had been encountered before e', then e would belong to T' which is impossible since T' contains only one edge having child v' and it is e'. It follows that in the counterclockwise code of (M, T), we encounter an N for the first visit of e', then an E for the first visit of e, then an S for the second visit of e' and then a W for the second visit of e. Hence the counterclockwise code of (M, T) is not minimal.

Conversely, let T be the rightmost DFS spanning tree of a rooted planar map M. Let e be an edge of M not in T. In the counterclockwise code w of (M, T), e gives rise to a matching pair N/S, and we let v (resp. v') be the vertex incident to e corresponding to N (resp. corresponding to S). Since T is the rightmost DFS spanning tree of M, v' is an ancestor of v in T. Then any edge e' in T which is visited for the first time between the first and second visit of e in the counterclockwise tour of (M, T) is also visited for the second time between the first and second visit of e, so that w cannot be decomposed in $w_1 N w_2 E w_3 S w_4 W w_5$, with $w_1, w_2, w_3, w_4, w_5 \in \{E, N, W, S\}^*$. Hence w is minimal. ☐

Let us define an excursion w to be *separable* if there exists a decomposition $w = w_1 w_2 w_3$ with w_2 and $w_2' = w_1 w_3$ being non-empty excursions, and *non-separable* when it is not separable and it has half-length at least 2.

Proposition 4. *An excursion is separable if and only if it is the counterclockwise code of a tree-rooted separable planar map.*

Proof. Let w_2 and w_2' be two non-empty excursions, and let $(\mathrm{M}_2, T_2) = \Gamma^{-1}(w_2)$ and $(\mathrm{M}_2', T_2') = \Gamma^{-1}(w_2')$ be the corresponding tree-rooted planar maps. Let $w_2' = w_1 w_3$ be a decomposition of w_2' and let c be the corresponding corner of M_2', that is the corner such that w_1 (resp. w_3) contains all the steps of w_2' registered before c is encountered (resp. after c is encountered) during the counterclockwise tour of (M_2', T_2'). Let us denote $(\mathrm{M}, T) = \Gamma^{-1}(w_1 w_2 w_3)$. Then the rooted planar map M is obtained by gluing the root corner of M_2 to the corner c of M_2', and taking the root corner of M_2' as root corner.

Hence, if E_2 (resp. E_2') is the set of edges of M_2 (resp. M_2'), then E_2 and E_2' partition the set E of edges of M, and the only vertex incident to edges of both E_2 and E_2' is the vertex where the gluing is done. Hence M is separable.

Conversely, any separable planar map can be built by gluing the root corner of a map to a corner of another planar map, hence giving rise to a separable excursion. □

Applying Proposition 3 and Proposition 4, we get:

Proposition 5. *An excursion is minimal and non-separable if and only if it is the counterclockwise code of a non-separable rooted planar map endowed with its rightmost DFS spanning tree.*

3 A bijection between fighting fish and non-separable planar maps

The purpose of this section is to present a bijection between fighting fish and non-separable planar maps, already announced in [9].

3.1 A decomposition of fighting fish

For a fighting fish F, its *jaw* jaw(F) is, starting from the nose in counterclockwise order, the length of the first sequence of left lower free edges. For example, in Figure 10, $\mathrm{jaw}(\mathrm{F}_1) = 2$ and $\mathrm{jaw}(\mathrm{F}_2) = 3$.

We now present a decomposition of fighting fish that has been introduced in a companion paper exploring the connection between fighting fish and synchronized Tamari intervals [10]. Let us present the two operations involved in this decomposition:

- the *concatenation* $\mathrm{F}_1 \odot \mathrm{F}_2$ (see Figure 10) of two fighting fish F_1 and F_2 is obtained by gluing the right lower free edge ending the jaw of F_1 to the leftmost upper free edge of F_2.

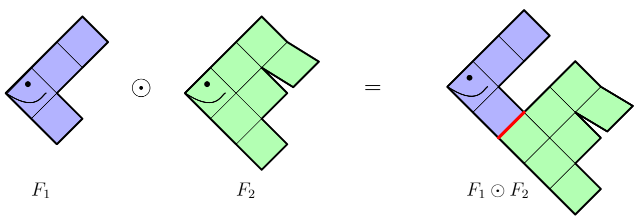

Figure 10: Concatenation of fighting fish.

- the *i-augmentation* $\maltese_i(\mathrm{F})$ of a fighting fish F (see Figure 11), for i an integer between 1 and jaw(F), is obtained in the following way: we glue the left lower free edge of each one of the first i cells of the jaw of F to a new cell and then glue the right lower edge and the left upper edge of all pairs of adjacent new cells.

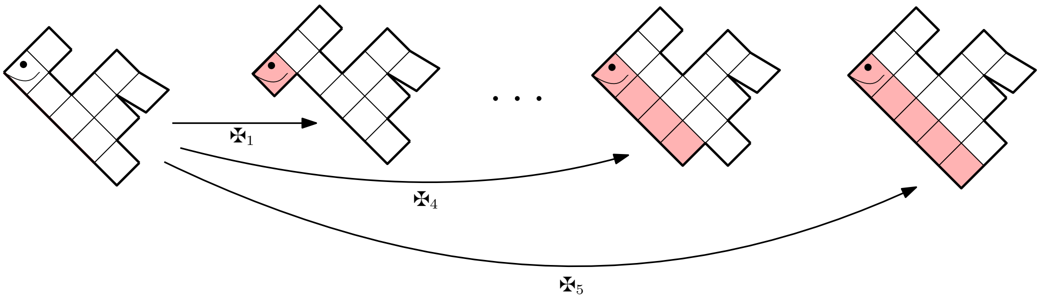

Figure 11: Some augmentations of a fighting fish.

Proposition 6. *For all fighting fish* F, F_1 *and* F_2*, and all* $1 \leqslant i \leqslant \mathrm{jaw(F)}$*, we have:*

$$\begin{aligned}
\mathrm{size}(\maltese_i(\mathrm{F})) &= \mathrm{size}(\mathrm{F}) + 1,\\
p(\maltese_i(\mathrm{F})) &= p(\mathrm{F}),\\
q(\maltese_i(\mathrm{F})) &= q(\mathrm{F}) + 1,\\
\mathrm{jaw}(\maltese_i(\mathrm{F})) &= i,\\
\mathrm{size}(\mathrm{F}_1 \odot \mathrm{F}_2) &= \mathrm{size}(\mathrm{F}_1) + \mathrm{size}(\mathrm{F}_2) - 1,\\
p(\mathrm{F}_1 \odot \mathrm{F}_2) &= p(\mathrm{F}_1) + p(\mathrm{F}_2),\\
q(\mathrm{F}_1 \odot \mathrm{F}_2) &= q(\mathrm{F}_1) + q(\mathrm{F}_2) - 1,\\
\mathrm{jaw}(\mathrm{F}_1 \odot \mathrm{F}_2) &= \mathrm{jaw}(\mathrm{F}_1) + \mathrm{jaw}(\mathrm{F}_2).
\end{aligned}$$

where $p(\mathrm{F}) = |\,\mathrm{F}\,|_E$ *and* $q(\mathrm{F}) = |\,\mathrm{F}\,|_N$*.*

Proof. The proof is straightforward and follows directly from the previous operations. □

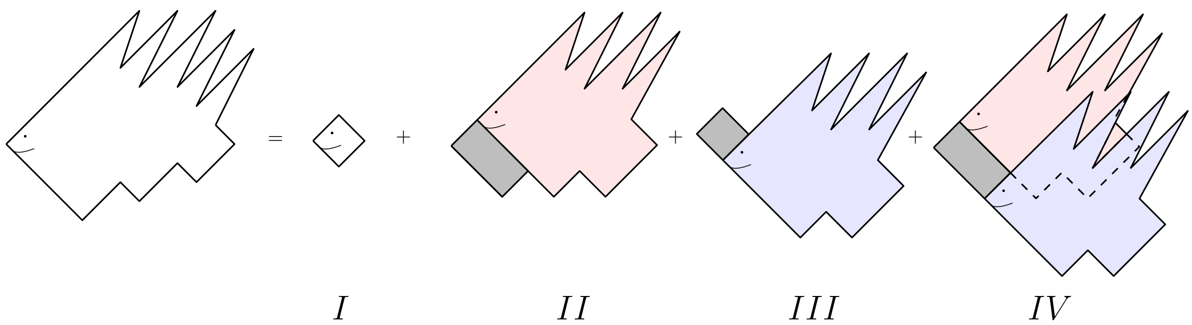

Figure 12: A decomposition of fighting fish.

Then the following proposition is derived from Theorem 1 in [10]:

Proposition 7. *Let* F *be a fighting fish. Then exactly one of the following cases occurs (see Figure 12):*

- *Case I:* $\mathrm{F} = ENWS$ *is the Head.*

- *Case II:* $\mathrm{F} = \maltese_i(\mathrm{F}_1)$ *for some fighting fish* F_1 *and some* $1 \leqslant i \leqslant \mathrm{jaw}(\mathrm{F}_1)$.
- *Case III:* $\mathrm{F} = ENWS \odot \mathrm{F}_2$ *for some fighting fish* F_2.
- *Case IV:* $\mathrm{F} = \maltese_i(\mathrm{F}_1) \odot \mathrm{F}_2$ *for some fighting fish* F_1 *and* F_2 *and some* $1 \leqslant i \leqslant \mathrm{jaw}(\mathrm{F}_1)$.

3.2 A decomposition of non-separable planar maps

Note that we can assign an integer from 1 to $\mathrm{out}(\mathrm{M})$ to each non-root corner in the root face of a map M according to its position in the counterclockwise tour of the root face (observe that for a non-separable planar map $\mathrm{out}(\mathrm{M})$ is also the number of non-root vertices in the outer face). We define now two operations on non-separable planar maps (note that the operations are valid for planar maps in general, but we choose to restrict them since this section is about non-separable planar maps):

- the *concatenation* $\mathrm{M}_1 \odot \mathrm{M}_2$ (see Figure 13) of two non-separable planar maps M_1 and M_2 having respective root edges $e_1 = u_1 \to v_1$ and $e_2 = u_2 \to v_2$ is the non-separable planar map obtained by identifying v_2 and u_1, then deleting the root edges of M_1 and M_2 and creating a new root edge from u_2 to v_1.

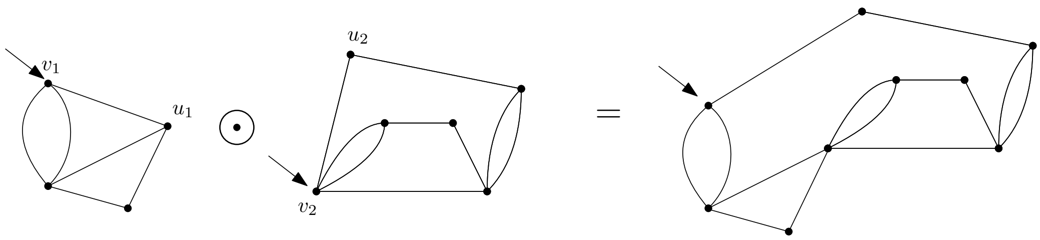

Figure 13: Concatenation of non-separable planar maps.

- the *i-augmentation* $\maltese_i(\mathrm{M})$ of a non-separable planar map M (see Figure 14), for i an integer between 1 and $\mathrm{out}(\mathrm{M})$ is the non-separable planar map obtained by adding an edge between the i^{th} non-root vertex in counterclockwise order after the root in the root face and the root vertex so that the root edge of M is not incident to the root face anymore, and the added edge is the new root edge.

Proposition 8. *For all non-separable planar maps* M, M_1 *and* M_2, *and all* $1 \leqslant i \leqslant$

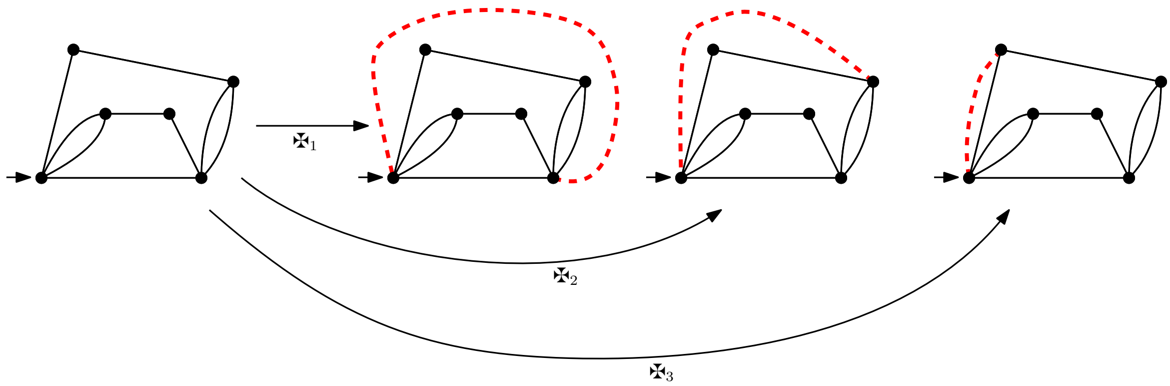

Figure 14: The three possible augmentations of a non-separable planar map.

out(M), *we have:*

$$\begin{aligned}
\mathrm{size}(✠_i(\mathrm{M})) &= \mathrm{size}(\mathrm{M}) + 1,\\
p(✠_i(\mathrm{M})) &= p(\mathrm{M}),\\
q(✠_i(\mathrm{M})) &= q(\mathrm{M}) + 1,\\
\mathrm{out}(✠_i(\mathrm{M})) &= i,\\
\mathrm{size}(\mathrm{M}_1 \odot \mathrm{M}_2) &= \mathrm{size}(\mathrm{M}_1) + \mathrm{size}(\mathrm{M}_2) - 1,\\
p(\mathrm{M}_1 \odot \mathrm{M}_2) &= p(\mathrm{M}_1) + p(\mathrm{M}_2),\\
q(\mathrm{M}_1 \odot \mathrm{M}_2) &= q(\mathrm{M}_1) + q(\mathrm{M}_2) - 1,\\
\mathrm{out}(\mathrm{M}_1 \odot \mathrm{M}_2) &= \mathrm{out}(\mathrm{M}_1) + \mathrm{out}(\mathrm{M}_2).
\end{aligned}$$

where $p(\mathrm{M})$ *is the number of non-root vertices and* $q(\mathrm{M})$ *the number of non-root faces of* M*.*

Proof. The proof is straightforward and follows directly from the previous operations. □

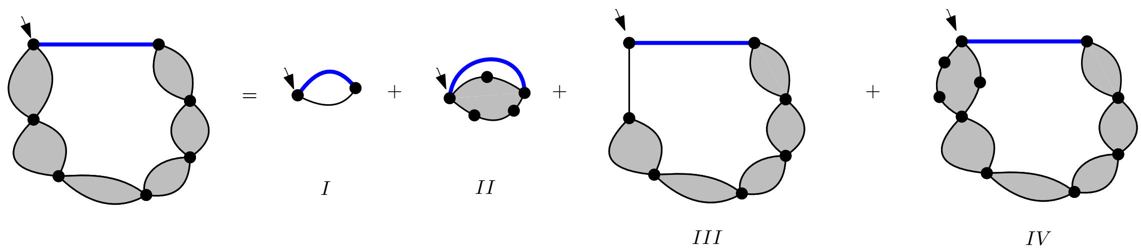

Figure 15: The series decomposition on non-separable planar maps.

We now recall a classical decomposition of non-separable planar maps called the *series decomposition* (see Figure 15). It was first described by Brown in [2]. Consider a non-separable planar map M and delete its root edge to obtain a planar map M′ with two marked corners c and c' where the two ends of the root edge were inserted. This map may have cut vertices, that are linearly ordered from c to c'. If there is at least one such cut point, by cutting M' at its first cut vertex, we obtain a pair of planar maps with two marked corners in the outer face each: the first map is either non-separable or reduced to the one edge planar map with two distinct vertices (that we call *bridge*), the second corresponds to the result of the root deletion on another non-separable map. Then exactly one of the following four cases occurs:

- Case I: M is the only non-separable map with two edges (that we will denote by D in the following) (*i.e.* M' has no cutpoint and it is reduced to a bridge).
- Case II: $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ for some non-separable planar map M_1 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$ (*i.e.* M' has no cutpoint but it is not a bridge).
- Case III: $\mathrm{M} = D \odot \mathrm{M}_2$ for some non-separable planar map M_2 (M' has at least one cutpoint and the first planar map of the pair is a bridge).
- Case IV: $\mathrm{M} = \maltese_i(\mathrm{M}_1) \odot \mathrm{M}_2$ for some non-separable planar maps M_1 and M_2 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$ (M' has more than two components and the first one is not a bridge).

3.3 The bijection

In the last two subsections, we presented two recursive decompositions for fighting fish and for non-separable planar maps. Since these two decompositions are isomorphic, we are now able to define a recursive bijection Φ from non-separable planar maps of size n and reduced outer degree k to fighting fish of size n and jaw length k by setting, for M a non-separable planar map:

- Case I: if $\mathrm{M} = D$, we set $\Phi(\mathrm{M}) = ENWS$.
- Case II: if $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ for some non-separable planar map M_1 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$, we set $\Phi(\mathrm{M}) = \maltese_i(\Phi(\mathrm{M}_1))$, which is well defined since $\mathrm{jaw}(\Phi(\mathrm{M}_1)) = \mathrm{out}(\mathrm{M}_1)$.
- Case III: if $\mathrm{M} = D \odot \mathrm{M}_2$ for some non-separable planar map M_2, we set $\Phi(\mathrm{M}) = ENWS \odot \Phi(\mathrm{M}_2)$.
- Case IV: if $\mathrm{M} = \maltese_i(\mathrm{M}_1) \odot \mathrm{M}_2$ for some non-separable planar maps M_1 and M_2 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$, we set $\Phi(\mathrm{M}) = \maltese_i(\Phi(\mathrm{M}_1)) \odot \Phi(\mathrm{M}_2)$, again using $\mathrm{jaw}(\Phi(\mathrm{M}_1)) = \mathrm{out}(\mathrm{M}_1)$.

Here comes our main contribution: the recursive bijection Φ obtained by identification of the two natural decompositions of our objects can in fact be described in a direct fashion using the counterclockwise code of a non-separable planar map endowed with its rightmost depth-first-search spanning tree.

Theorem 9. *The mapping Φ is a bijection between non-separable planar maps with n edges, $p+1$ vertices, $q+1$ faces and reduced outer degree k and fighting fish of size n with p left lower free edges, q right lower free edges and jaw length k. Moreover, the image $\Phi(\mathrm{M})$ of a non-separable planar map M by Φ is the counterclockwise code $\Gamma(\mathrm{M}, T)$ of the tree-rooted planar map (M, T) where T is the rightmost DFS spanning tree of M.*

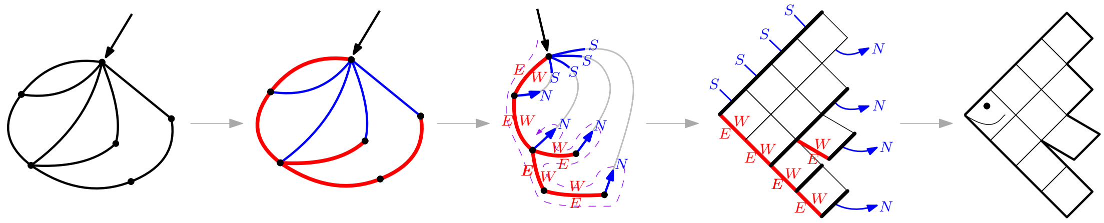

Figure 16: The bijection Φ applied to a non-separable planar map.

Proof. Since the two recursive decompositions are isomorphic, Φ is a well-defined bijection between non-separable planar maps with n edges and reduced outer degree k and fighting fish of size n and jaw length k. We proceed by induction on the size of the non-separable planar map M to prove that $\Phi(\mathrm{M}) = \Gamma(\mathrm{M}, T)$, where T is the rightmost DFS spanning tree of M. Let M be a non-separable planar map. We distinguish between the four possible cases:

- Case I: if $\mathrm{M} = D$, then $\Phi(\mathrm{M}) = ENWS$ is indeed the counterclockwise code of (M, T) where T is the rightmost DFS tree of M.

- Case II: if $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ for some non-separable planar map M_1 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$, then by the induction hypothesis, $\Phi(\mathrm{M}_1)$ is the counterclockwise code of M_1 endowed with its rightmost DFS spanning tree T_1. Then the rightmost DFS spanning tree of M is also T_1 since M is obtained by adding an edge towards the root, that is an already visited vertex. Hence, if we write $\Phi(\mathrm{M}_1) = E^{k_1}\,\mathrm{G}$ with $k_1 = \mathrm{out}(\mathrm{M}_1)$, the counterclockwise code of (M, T_1) is $E^i N E^{k_1 - i}\,\mathrm{G}\,S$, that is exactly $\maltese_i(\Phi(\mathrm{M}_1)) = \Phi(\mathrm{M})$.

- Case III: if $\mathrm{M} = D \odot \mathrm{M}_2$ for some non-separable planar map M_2, then by the induction hypothesis, $\Phi(\mathrm{M}_2)$ is the counterclockwise code of M_2 endowed with its rightmost DFS spanning tree T_2. The rightmost DFS spanning tree T of M consists of a root vertex with a pending edge on which T_2 is attached. Writing $\Phi(\mathrm{M}_2) = \mathrm{G}\,S$ with $\mathrm{G} \in \{E, N, W, S\}^*$, the counterclockwise code of (M, T) is then $E\,\mathrm{G}\,WS$, that is exactly $ENWS \odot \Phi(\mathrm{M}_2) = \Phi(\mathrm{M})$.

- Case IV: if $\mathrm{M} = \maltese_i(\mathrm{M}_1) \odot \mathrm{M}_2$ for some non-separable planar maps M_1 and M_2 and some $1 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1)$, then by the induction hypothesis, $\Phi(\mathrm{M}_1)$ and $\Phi(\mathrm{M}_2)$ are the respective counterclockwise codes of M_1 and M_2 endowed with their respective rightmost DFS spanning trees T_1 and T_2. Then the rightmost DFS spanning tree T

of M is obtained by gluing the root of T_2 onto the i-th corner of the rightmost branch of T_1, with $i + \mathrm{out}(M_2) = \mathrm{out}(M)$. Let us write $\Phi(\mathrm{M}_1) = E^i\,\mathrm{G}_1$ and $\Phi(\mathrm{M}_2) = \mathrm{G}_2\,S$, with $\mathrm{G}_1, \mathrm{G}_2 \in \{E, N, W, S\}^*$. Performing the counterclockwise tour of M around T, we get that the counterclockwise code of (M, T) is $E^i\,\mathrm{G}_2\,\mathrm{G}_1\,S$, which is exactly $\maltese_i(\Phi(\mathrm{M}_1)) \odot \Phi(\mathrm{M}_2) = \Phi(\mathrm{M})$.

In each case the parameters n, p, q, and k of the theorem are easily checked to behave as announced. □

Corollary 10. *The bijection Φ preserves duality, in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{M} & \xmapsto{\textit{map duality}} & \mathrm{M}^* \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ \Phi(\mathrm{M}) & \xmapsto{\textit{excursion duality}} & \Phi(\mathrm{M}^*) \end{array}$$

In particular if (M, T) is a non-separable map endowed with its rightmost depth first search spanning tree, then the distinguished spanning tree T^* in the dual tree-rooted map (M^*, T^*) is the rightmost first search spanning tree of the dual map M^*.

Proof. According to Theorem 1, Γ is bijective between tree-rooted maps and excursions, so in particular for each fighting fish F there is a unique tree-rooted map $(\mathrm{M}, T) = \Gamma^{-1}(\mathrm{F})$, and moreover, $(\mathrm{M}^*, T^*) = \Gamma^{-1}(\mathrm{F}^*)$, since Γ commutes with duality. According to Theorem 9, the maps M and M^* are then also the preimages of F and F^* by Φ, that is to say Φ^{-1} commutes with duality, and hence, so does Φ. Finally according to Theorem 9 the rightmost depth first search spanning tree T' of M^* satisfies $\Gamma(\mathrm{M}^*, T') = \Phi(\mathrm{M}^*)$, so $T' = T^*$ again because Γ is bijective between tree-rooted maps and excursions. □

Using Proposition 5, we get a characterization of fighting fish, which is new to the best of our knowledge:

Corollary 11. *An excursion $w \in \{E, N, W, S\}^*$ is a fighting fish if and only if it is non-separable and contains no factor ES.*

Proof. By Theorem 9 and Proposition 5, fighting fish are exactly minimal non-separable excursions. But for a non-separable excursion, an occurrence of a factor EwS, with w an excursion, can happen only with w being empty by non-separability. Hence a non-separable excursion is minimal if and only if it contains no factor ES. □

3.4 Enumeration results

For the sake of completeness, we recall now two enumeration results, which have been known since the work of Brown and Tutte for non-separable planar maps (see [3]), and which have been proven combinatorially by Schaeffer in his PhD ([18]). Concerning fighting fish, the two same counting formulas have been proven analytically in [8]. While those results are not ours, our bijection sheds some light on a combinatorial way to derive the formulas for fighting fish.

Theorem 12. *The number of non-separable rooted planar maps having $n+1$ edges, or equivalently the number of fighting fish of size $n+1$ is:*

$$\frac{2(3n)!}{(n+1)!(2n+1)!}.$$

The number of non-separable rooted planar maps having $p+1$ vertices and $q+1$ faces, or equivalently the number of fighting fish with p E steps and q N steps is:

$$\frac{(2p+q-2)!(2q+p-2)!}{p!q!(2p-1)!(2q-1)!}.$$

4 A bijection between generalized fighting fish and rooted planar maps

4.1 Generalized fighting fish

Let us now present an alternative definition of fighting fish in terms that will enable us to generalize it. The following proposition is illustrated by Figure 17.

Proposition 13. *Fighting fish are exactly finite sets of cells that can be built starting from the head, then attaching new cells using one of the following two operations:*

- Upper strip gluing*: Let $a_1, a_2, \ldots, a_k$, $k \geqslant 1$ be cells of a fish such that a_i has its right lower edge glued to the left upper edge of a_{i+1} for all $1 \leqslant i \leqslant k-1$, and such that all a_i have their right upper edge free; for every i, we glue the right upper edge of a_i to the left lower edge of a new cell b_i while gluing the right lower edge of b_i to the left upper edge of b_{i+1} for $1 \leqslant i \leqslant k-1$.*
- Lower strip gluing*: Let $a_1, a_2, \ldots, a_k$, $k \geqslant 1$ be cells of a fish such that a_i has its right upper edge glued to the left lower edge of a_{i+1} for all $1 \leqslant i \leqslant k-1$, and such that all a_i have their right lower edge free; for every i, we glue the right lower edge of a_i to the left upper edge of a new cell b_i while gluing the right upper edge of b_i to the left lower edge of b_{i+1} for $1 \leqslant i \leqslant k-1$.*

Proof. First, the upper strip gluing of k cells $b_1, \ldots, b_k$ to $a_1, \ldots, a_k$ can be obtained by performing an upper gluing of b_1 on a_1, then performing $k-1$ double gluings of b_{i+1} on a_{i+1} and b_i, for $1 \leqslant i \leqslant k-1$. Similarly, a lower strip gluing of k cells can be obtained with a lower gluing and $k-1$ double gluings. Hence every finite set of cells that can be obtained starting from the head and performing upper and lower strip gluings is a fighting fish.

We will now prove that every fighting fish can be obtained starting from the head and performing upper and lower strip gluings by induction on the number m of cells in the fighting fish. For $m=1$, the only fighting fish is the head, so the property is verified. Suppose that the property is true for fighting fish having strictly less than $m \geqslant 2$ cells and

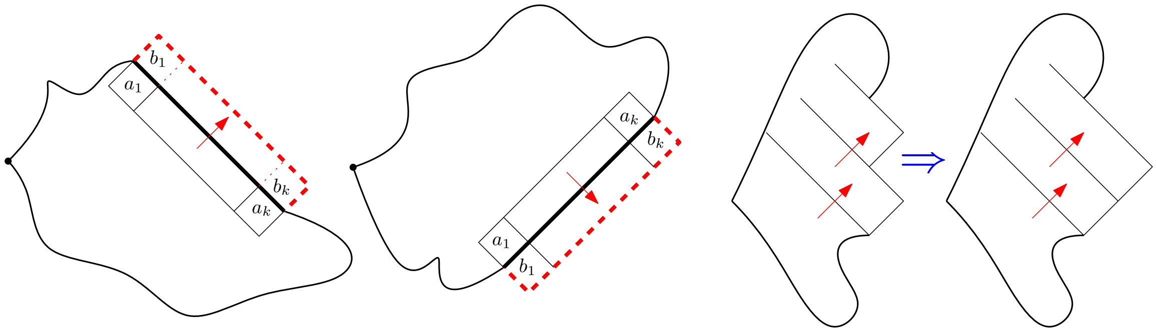

Figure 17: Upper and Lower strip gluing, and an example of double gluing realized by extending an anterior upper strip gluing.

let F be a fighting fish having m cells. If F is obtained by applying an upper gluing on a fighting fish F_1, then F_1 has $m-1$ cells, so it is constructible using upper and lower strip gluings, and so is F by performing one additional upper strip gluing of 1 cell. The case where F is obtained with a lower gluing on a fighting fish F_2 is similar. If F is obtained with a double gluing of a cell b on two cells a (on the left upper edge of b) and a' (on the left lower edge of b) of a fighting fish F_3, then F_3 has $m-1$ cells so it is constructible from the head using upper and lower strip gluings. In this construction, either a is added using an upper strip gluing or a' is added using a lower strip gluing, because in F_3 (and in F), the left lower edge of a and the left upper edge of a' are glued respectively to the right upper edge and to the right lower edge of the same cell c: indeed if a was attached by a lower strip gluing then c would have had to be attached at the same time, and similarly if a' was attached by an upper strip gluing then c would have had to be attached at the same time, but c is attached only once. So either a was attached by an upper gluing or a' by a lower gluing, or both. If only a is added using an upper strip gluing in F_3, then F can be obtained using the same strip construction as for F_3, except we perform an upper strip gluing with the extra cell b when we glue the upper strip containing a. The symmetrical statement is true if only a' is added using a lower strip gluing. If both a is added using upper strip gluing and a' using lower strip gluing then extend the gluing that was done last. In any case, F is constructible using upper and lower strip gluings and the proposition follows. □

If we state this proposition in the word setting, we obtain the following: fighting fish are exactly the finite words on the alphabet $\{E, N, W, S\}$ that can be obtained from the word $ENWS$ using a sequence of the two operations:

- Upper strip gluing: replace a factor W^k, $k \geqslant 1$, by NW^kS.
- Lower strip gluing: replace a factor N^k, $k \geqslant 1$, by EN^kW.

Building on this last formulation we can naturally define a larger class of excursions that includes fighting fish by allowing k to be 0 in the operations of strip gluing.

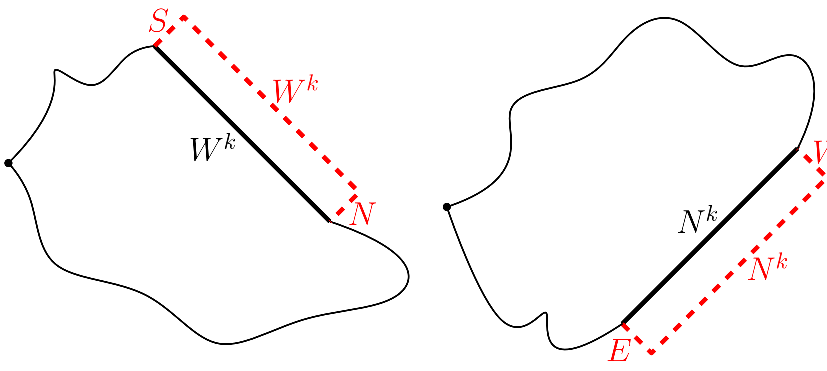

Figure 18: The operations $\triangle_k$ and $\bigtriangledown_k$ as rewritings.

Definition 14. The class of *generalized fighting fish*, denoted $\mathcal{G}$ is the set of words or walks inductively defined from the empty word ε with the help of the following two types of operations (see Figure 18 for an example):

- **Operation $\triangle_k$, $k \geqslant 0$**: replace a factor W^k by NW^kS. The operation $\triangle_k$ at position i, denoted $\triangle_{k,i}$ is *valid* on a generalized fish F if $\mathrm{F} = \mathrm{G}\,W^k\,\mathrm{H}$ with $|\,\mathrm{G}\,| = i$, and then $\triangle_{k,i}(\mathrm{F}) = \mathrm{G}\,NW^kS\,\mathrm{H}$,
- **Operation $\bigtriangledown_k$, $k \geqslant 0$**: replace a factor N^k by EN^kW. The operation $\bigtriangledown_k$ at position i, denoted $\bigtriangledown_{k,i}$ is *valid* on a generalized fish F if $\mathrm{F} = \mathrm{G}\,N^k\,\mathrm{H}$ with $|\,\mathrm{G}\,| = i$, and then $\bigtriangledown_{k,i}(\mathrm{F}) = \mathrm{G}\,EN^kW\,\mathrm{H}$.

In particular a sequence $\sigma_1, \ldots, \sigma_m$ of strip gluings is *valid* if for all ℓ, σ_ℓ is either a valid $\triangle_{k,i}$ operation or a valid $\bigtriangledown_{k,i}$ operation on $\sigma_{\ell-1} \circ \cdots \circ \sigma_1(\varepsilon)$. The set $\mathcal{G}$ is the set of images of ε through all finite valid strip gluing sequences.

Remark that the class of fighting fish is given by those generalized fighting fish that can be obtained from $ENWS$ using only operations $\bigtriangledown_k$ and $\triangle_k$, with $k \geqslant 1$.

Generalized fighting fish are defined above in terms of words/walks and we will mostly work in this setting in the rest of this article. Nonetheless the geometric interpretation of fighting fish in terms of branching surfaces made of cells can be extended to generalized fighting fish: while the operation $\triangle_k$ and $\bigtriangledown_k$ for $k \geqslant 1$ grow bands of width k, one should consider that $\triangle_0$ and $\bigtriangledown_0$ similarly grow bands of infinitesimal width similar to the rubber band edges of rubber band graphs. From this point of view a generalized fighting fish can still be considered as the boundary of a simply connected surface.

As for fighting fish, the *size* of a generalized fighting fish F is half of its length, and we have:

$$\mathrm{size}(\mathrm{F}) = \frac{1}{2}(|F|_E + |F|_N + |F|_W + |F|_S) = |F|_E + |F|_N = |F|_W + |F|_S$$

The *jaw* of a non-empty generalized fighting fish F, denoted by jaw(F) is the number of decompositions $\mathrm{F} = \mathrm{F}_1\,\mathrm{F}_2$ with F_1 and F_2 non-empty words having both latitude 0 (we recall that the latitude of a word $w \in \{E, N, W, S\}^*$ is the quantity $|w|_N - |w|_S$, while the longitude is $|w|_E - |w|_W$). By convention the jaw of the empty generalized fish is -1. For (non-generalized) fighting fish, this definition agrees with the one given in the preceding section because the excursion of a fighting fish can only return to latitude 0 at the last step after leaving it, so that only the initial sequence of E steps contributes to the jaw. As opposed to that in the case of generalized fighting fish, the excursion is allowed to return arbitrarily often to latitude 0 so the jaw of a generalized fighting fish can be larger than the length of the longest prefix E^k of the associated excursion (see for instance the rightmost fish in Figure 19). Remark that there are exactly $\mathrm{jaw}(\mathrm{F}) + 2$ ways to decompose $\mathrm{F} = \mathrm{F}_1\,\mathrm{F}_2$ with F_1 and F_2 of latitude 0 if we allow F_1 and F_2 to be empty. Let us denote by $\ell(\mathrm{F}) = \mathrm{jaw}(\mathrm{F}) + 1$, then we can order those decompositions according to the length of F_1 and we will number them from 0 (when F_1 is empty) to $\ell(\mathrm{F})$ (when F_2 is empty), and we will call them i^{th}*-return decomposition* for $i = 0, \ldots, \ell(\mathrm{F})$.

By definition generalized fighting fish are described by sequences of $\triangle_k$ and $\bigtriangledown_k$ operations: each operation is applied *to* a fish *at* a position i. While the construction of a fish using $\triangle_k$ and $\bigtriangledown_k$ operations is in general not unique, it will be useful to show that the sequence of operations to construct a given fish can be arranged to construct one after the other the successive factors between two returns at latitude 0.

Lemma 15. *Let* F *be a generalized fighting fish, and let* $\mathrm{F}_{(i)}\,\mathrm{F}^{(i)}$ *be its* i^{th}*-return decomposition for* $0 \leqslant i \leqslant \mathrm{jaw}(\mathrm{F}) + 1$*. Then* F *can be built from the empty fish* ε *using operations* $\triangle_k$ *and* $\bigtriangledown_k$*,* $k \geqslant 0$*, in such a way that the construction constructs the following* $\mathrm{jaw}(\mathrm{F}) + 2$ *generalized fighting fish:* $\mathrm{F}_{(0)}\,W^{\mathrm{long}(\mathrm{F}_{(0)})}$*, then later* $\mathrm{F}_{(1)}\,W^{\mathrm{long}(\mathrm{F}_{(1)})}$*, and so on until* $\mathrm{F}_{(\mathrm{jaw}(\mathrm{F})+1)}\,W^{\mathrm{long}(\mathrm{F}_{(\mathrm{jaw}(\mathrm{F})+1)})}$*, in this order. In other terms, there exists a valid sequence* $\sigma_1, \ldots, \sigma_m$ *that produces* F *and a weakly increasing sequence* $(m_j)_{j=0,\ldots,\mathrm{jaw}(\mathrm{F})+1}$ *of indices such that*

$$\sigma_{m_j} \circ \cdots \circ \sigma_1(\varepsilon) = \mathrm{F}_{(j)}\,W^{\mathrm{long}(\mathrm{F}_{(j)})}.$$

and for all $\ell > m_j$*,* $\sigma_\ell = \triangle_{k,i}$ *or* $\sigma_\ell = \bigtriangledown_{k,i}$ *with* $i \geqslant |\,\mathrm{F}_{(j)}\,|$*. We will call such a construction an* in-order construction *of* F*.*

In particular once an intermediary fish $\mathrm{F}_{(j)}\,W^{\mathrm{long}(\mathrm{F}_{(j)})}$ has been constructed in an in-order construction of a fish F, no later operations perform further modification of the prefix $\mathrm{F}_{(j)}$.

Proof of Lemma 15. We proceed by induction on the length of the generalized fighting fish. First, the empty word trivially admits an (empty) in-order construction. We just then need to prove that the property is inherited by the application of operations $\triangle_k$ and $\bigtriangledown_k$, $k \geqslant 0$. Let F be a generalized fighting fish admitting an in-order construction, and consider all the possible fighting fish F' that can be obtained from F:

- The first case is $\mathrm{F}' = \mathrm{G}\,NW^kS\,\mathrm{H}$, a generalized fighting fish obtained by applying a last operation $\triangle_{k,\ell}$ to $\mathrm{F} = \mathrm{G}\,W^k\,\mathrm{H}$ with $\ell = |\,\mathrm{G}\,|$. Let $0 \leqslant j \leqslant \mathrm{jaw}(\mathrm{F}) + 1$ be the smallest integer such that $\mathrm{G}\,W^k$ is a prefix of $\mathrm{F}_{(j)}$, *i.e.* $\mathrm{F}_{(j)} = \mathrm{G}\,W^k\,\mathrm{K}$. Then inserting the operation $\triangle_{k,\ell}$ into an in-order construction $\sigma_m, \cdots, \sigma_1$ of F just after getting

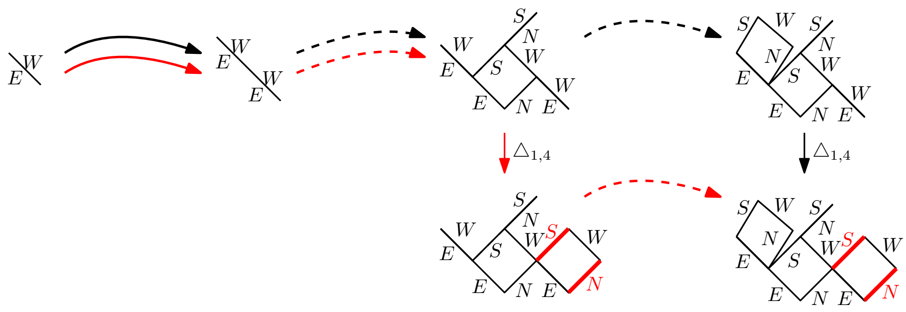

Figure 19: Illustration of the proof of Lemma 15 for the operation $\triangle_1$. Black arrows summarize the growth between the intermediary steps $\mathrm{F}_{(j)}\, W^{\mathrm{long}(\mathrm{F}_{(j)})}$ in the in-order construction of $\mathrm{F} = E^2NEWWNSSNWS$: $\mathrm{jaw}(\mathrm{F}) = 3$ and $\mathrm{F}_{(1)} = E$, $\mathrm{F}_{(2)} = EE$, $\mathrm{F}_{(3)} = E^2NEWWNSS$. Red arrows do the same for $\mathrm{F}' = \triangle_{1,4}(\mathrm{F}) = E^2NE\underline{N}W\underline{S}WNSSNWS$, with $\mathrm{jaw}(\mathrm{F}') = 3$ and $\mathrm{F}'_{(1)} = \mathrm{F}_{(1)}$, $\mathrm{F}'_{(2)} = \mathrm{F}_{(2)}$ as well, but $\mathrm{F}'_{(3)} = E^2NE\underline{N}W\underline{S}WNSS = \triangle_{1,4}(\mathrm{F}_{(3)})$: the in-order construction of F' is obtained from that of F by inserting a $\triangle_1$ operation at the 4th position of the prefix $\mathrm{F}_{(3)}$ to get $\mathrm{F}'_{(3)}$.

$\mathrm{F}_{(j)}\, W^{\mathrm{long}(\mathrm{F}_{(j)})} = \sigma_{m_j} \circ \ldots \circ \sigma_1(\varepsilon)$ produces an in-order construction $\triangle_{k,\ell}, \sigma_{m_j}, \ldots, \sigma_1$ of $\mathrm{G}\, NW^kS\, \mathrm{K}\, W^{\mathrm{long}(\mathrm{F}_{(j)})}$ (see Figure 19 for an example). This in-order construction can then be extended to a valid construction $\sigma_m^{+2}, \ldots, \sigma_{m_j+1}^{+2}, \triangle_{k,\ell}, \sigma_{m_j}, \ldots, \sigma_1$ of F', where $\sigma^{+2} = \triangle_{k',\ell'+2}$ for all $\sigma = \triangle_{k',\ell'}$ and similarly for $\sigma = \triangledown_{k',\ell'}$: indeed these steps σ_i with $i > m_j$ of the in-order construction of F do not touch $\mathrm{F}_{(j)}$, so that, upon taking into account the shift induced by the two inserted letters, they can be applied to $\mathrm{F}'_{(j)}\, W^{\mathrm{long}(\mathrm{F}'_{(j)})}$ to produce F'. Finally observe that the resulting construction is in-order since all the factorizations of F' are factorizations of F (the insertion and the +2-shift may only duplicates the jth factorization if $k = 0$ and $\mathrm{lat}(\mathrm{G}) = 0$, and do not create other new factorizations). Observe that in this case the jaw (or the number of factorization) of F' is thus smaller or equal to the jaw of of F plus one.

- The other case is $\mathrm{F}' = \mathrm{G}\, EN^kW\, \mathrm{H}$, a generalized fighting fish obtained by applying a last operation $\triangledown_{k,\ell}$ to $\mathrm{F} = \mathrm{G}\, N^k\, \mathrm{H}$ with $\ell = |\,\mathrm{G}\,|$. Let $0 \leqslant j \leqslant \mathrm{jaw}(\mathrm{F}) + 1$ be the smallest integer such that $\mathrm{G}\, N^k$ is a prefix of $\mathrm{F}_{(j)}$, *i.e.* $\mathrm{F}_{(j)} = \mathrm{G}\, N^k\, \mathrm{K}$. Then we insert an operation $\triangledown_{k,\ell}$ into an in-order construction $\sigma_m, \ldots, \sigma_1$ of F just after getting $\mathrm{F}_{(j)}\, W^{\mathrm{long}(\mathrm{F}_{(j)})} = \sigma_{m_j} \circ \ldots \circ \sigma_1(\varepsilon)$ to produce an in-order construction $\triangledown_{k,\ell}, \sigma_{m_j}, \ldots, \sigma_1$ of $\mathrm{G}\, EN^kW\, \mathrm{K}\, W^{\mathrm{long}(\mathrm{F}_{(j)})}$, and again this in-order construction can be extended to produce F' in order as above, with the only difference that if $k = 0$ and $\mathrm{lat}(\mathrm{G}) = 0$ then two new factorizations appear in F' that were not in F at position $|\,\mathrm{G}\,| + 1$ and $|\,\mathrm{G}\,| + 2$. In particular in this case the jaw of F' is the jaw of F plus 0, 1 or 2 depending whether how many of k and $\mathrm{lat}(\mathrm{G})$ are equal to 0.

Since all the generalized fighting fish that can be produced from F can be obtained through an in-order construction, the induction step is proven. □

4.2 A decomposition for generalized fighting fish

In order to present a decomposition for generalized fighting fish we need to introduce the following operations:

- The concatenation $\oplus$ of two generalized fighting fish F_1 and F_2, that is given by $\mathrm{F}_1 \oplus \mathrm{F}_2 = \mathrm{F}_1\, E\, \mathrm{F}_2\, W$ (see Figure 20).

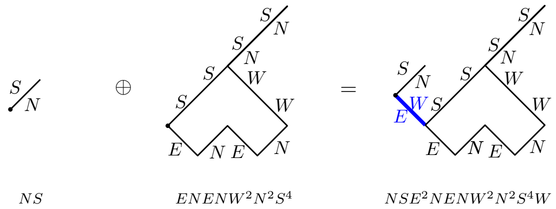

Figure 20: Concatenation of generalized fighting fish.

- Let us take a generalized fighting fish F and let i be an integer between 0 and $\mathrm{jaw}(\mathrm{F}) + 1$, and let $\mathrm{F} = \mathrm{F}_{(i)}\,\mathrm{F}^{(i)}$ be the i^{th}-return decomposition of F. Then the *i-augmentation* $\maltese_i(\mathrm{F})$ of F (see Figure 21), is $\maltese_i(\mathrm{F}) = \mathrm{F}_{(i)}\, N\, \mathrm{F}^{(i)}\, S$.

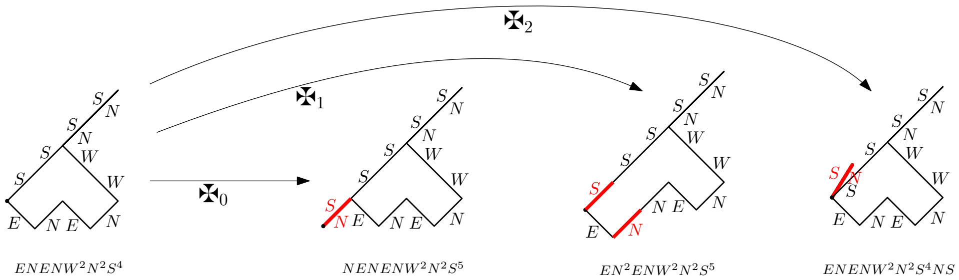

Figure 21: i-augmentation of a generalized fighting fish.

Proposition 16. *The concatenation and the augmentation of generalized fighting fish both produce valid generalized fighting fish.*

Proof. To generate the concatenation $\mathrm{F}_1 \oplus \mathrm{F}_2$ via Δ_k and ∇_k, it suffices to generate F_1, then apply ∇_0 at the end of the walk to get $\mathrm{F}_1\, EW$ and finally apply the operations used to build F_2, starting between the final E and final W of $\mathrm{F}_1\, EW$. This yields $\mathrm{F}_1\, E\, \mathrm{F}_2\, W = \mathrm{F}_1 \oplus \mathrm{F}_2$.

For the augmentation, we use Lemma 15. A construction of $\maltese_i(\mathrm{F})$ is obtained by inserting an operation $\triangle_0$ into an in-order construction of F just after getting $\mathrm{F}_{(i)}\, W^{\mathrm{long}(\mathrm{F}_{(i)})}$, hence producing $\mathrm{F}_{(i)}\, NW^{\mathrm{long}(\mathrm{F}_{(i)})}S$, and the rest of the in-order construction of F can be applied unchanged to this fish to produce $\maltese_i(\mathrm{F})$. □

Proposition 17. *For all fighting fish* F *and all* $0 \leqslant i \leqslant \mathrm{jaw}(\mathrm{F})+1$*, we have:*

$$\begin{aligned}
\mathrm{size}(\maltese_i(\mathrm{F})) &= \mathrm{size}(\mathrm{F})+1,\\
p(\maltese_i(\mathrm{F})) &= p(\mathrm{F}),\\
q(\maltese_i(\mathrm{F})) &= q(\mathrm{F})+1,\\
\mathrm{jaw}(\maltese_i(\mathrm{F})) &= i,\\
\mathrm{size}(\mathrm{F}_1 \oplus \mathrm{F}_2) &= \mathrm{size}(\mathrm{F}_1)+\mathrm{size}(\mathrm{F}_2)+1,\\
p(\mathrm{F}_1 \oplus \mathrm{F}_2) &= p(\mathrm{F}_1)+p(\mathrm{F}_2)+1,\\
q(\mathrm{F}_1 \oplus \mathrm{F}_2) &= q(\mathrm{F}_1)+q(\mathrm{F}_2),\\
\mathrm{jaw}(\mathrm{F}_1 \oplus \mathrm{F}_2) &= \mathrm{jaw}(\mathrm{F}_1)+\mathrm{jaw}(\mathrm{F}_2)+3.
\end{aligned}$$

where $p(\mathrm{F}) = |\,\mathrm{F}\,|_E$ *and* $q(\mathrm{F}) = |\,\mathrm{F}\,|_N$*.*

Proof. The proof is straightforward and follows directly from the previous operations. □

Remark that the i-augmentation of a generalized fighting fish is a generalization of $\maltese_i$ defined for fighting fish in Subsection 3.1.

Proposition 18. *Let* F *be a generalized fighting fish. Then exactly one of the following cases occurs (see Figure 22):*

- *Case I:* $\mathrm{F} = \varepsilon$ *is the empty word.*
- *Case II:* $\mathrm{F} = \mathrm{F}_1 \oplus \mathrm{F}_2$ *for some generalized fighting fish* F_1 *and* F_2*.*
- *Case III:* $\mathrm{F} = \maltese_i(\mathrm{F}_1)$ *for some generalized fighting fish* F_1 *and some* $0 \leqslant i \leqslant \mathrm{jaw}(\mathrm{F}_1)+1$*.*

Proof. Let F be a non-empty generalized fighting fish.

- If F ends with a W, then the final W and the E step matching this final W were created by a $\triangledown_k$ operation, and no later operation $\vartriangle_\ell$ operation was applied to this final W since it lies at latitude 0. For the same reason k is necessarily 0 and the matching E step also lies at latitude 0. Hence, if we write $\mathrm{F} = \mathrm{F}_1\,E\,\mathrm{F}_2\,W$ with the E matching the W, an in-order construction of F as in Lemma 15 goes through F_1, proving that F_1 is a valid generalized fighting fish. Pursuing this in-order construction, $\mathrm{F}_1\,EW$ is created, and from this one gets $\mathrm{F} = \mathrm{F}_1\,E\,\mathrm{F}_2\,W$ with no operation involving $\mathrm{F}_1\,E$ and the final W, hence we get an in-order construction for F_2. Finally F_1 and F_2 are valid generalized fighting fish and $\mathrm{F} = \mathrm{F}_1 \oplus \mathrm{F}_2$.
- If F ends with an S step, then we can write $\mathrm{F} = \mathrm{F}_1\,N\,\mathrm{F}_2\,S$, where the N matches the final S step. An in-order construction of F goes through $\mathrm{F}_1\,W^{\mathrm{long}(\mathrm{F}_1)}$, and the next operation in this in-order construction is necessarily $\vartriangle_{\mathrm{long}(\mathrm{F}_1)}$ giving rise to $\mathrm{F}_1\,NW^{\mathrm{long}(\mathrm{F}_1)}S$. The end of the in-order construction of F involves no operation acting on $\mathrm{F}_1\,N$ or on the final S, hence an in-order construction of $\mathrm{F}_1\,\mathrm{F}_2$ can be obtained from an in-order construction of F by forgetting the $\vartriangle_{\mathrm{long}(\mathrm{F}_1)}$ operation. Finally $\mathrm{F}_1\,\mathrm{F}_2$ is a generalized fighting fish and $\mathrm{F} = \maltese_{\mathrm{jaw}(\mathrm{F})}(\mathrm{F}_1\,\mathrm{F}_2)$.

□

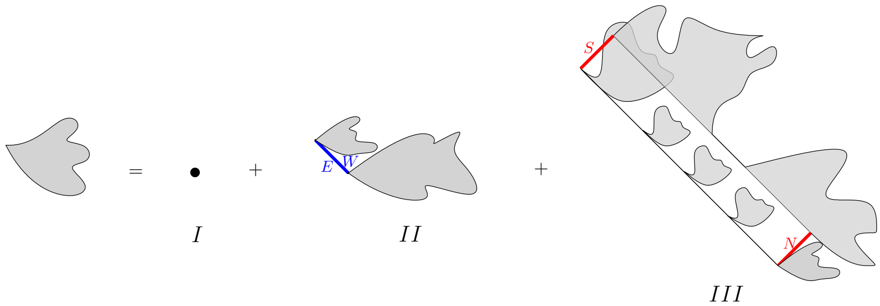

Figure 22: Decomposition of generalized fighting fish.

4.3 A classical decomposition of rooted planar maps

In order to present Tutte's classical *root edge deletion* decomposition of a rooted planar map M we need to introduce the following operations: (recall that out(M) denotes the reduced outer degree of M)

- the concatenation $\mathrm{M}_1 \oplus \mathrm{M}_2$ of two planar maps M_1 and M_2 is obtained by adding a edge from the root corner of M_1 to the root corner of M_2 and by keeping the root of M_1 (see Figure 23).

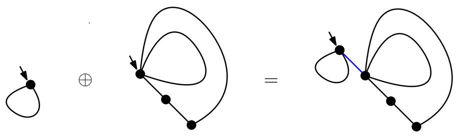

Figure 23: Concatenation $\oplus$ of rooted planar maps.

- the *i-augmentation* $\maltese_i(\mathrm{M})$ of M (see Figure 24), for i an integer between 0 and $\mathrm{out}(\mathrm{M}) + 1$, is obtained by adding a edge from the i^{th} corner of the root face of M (we order the corners in counterclockwise direction around the root face) to the right of the root of M (observe that we consider two corners around the root, the one on the left and the one on the right of the arrow indicating the root corner).

Proposition 19. *For all rooted planar maps* M, M_1 *and* M_2*, and all* $0 \leqslant i \leqslant \mathrm{out}(\mathrm{M}) + 1$,

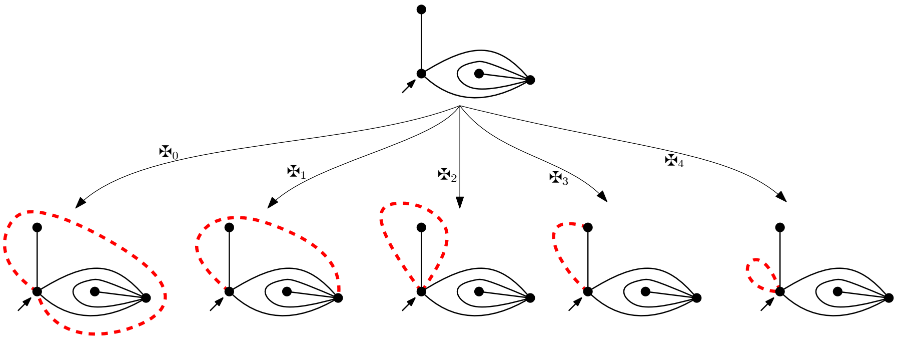

Figure 24: i-augmentations of a rooted planar map.

we have:

$$\begin{aligned}
\text{size}(\maltese_i(\mathrm{M})) &= \text{size}(\mathrm{M}) + 1, \\
p(\maltese_i(\mathrm{M})) &= p(\mathrm{M}), \\
q(\maltese_i(\mathrm{M})) &= q(\mathrm{M}) + 1, \\
\text{out}(\maltese_i(\mathrm{M})) &= i, \\
\text{size}(\mathrm{M}_1 \oplus \mathrm{M}_2) &= \text{size}(\mathrm{M}_1) + \text{size}(\mathrm{M}_2) + 1, \\
p(\mathrm{M}_1 \oplus \mathrm{M}_2) &= p(\mathrm{M}_1) + p(\mathrm{M}_2) + 1, \\
q(\mathrm{M}_1 \oplus \mathrm{M}_2) &= q(\mathrm{M}_1) + q(\mathrm{M}_2), \\
\text{out}(\mathrm{M}_1 \oplus \mathrm{M}_2) &= \text{out}(\mathrm{M}_1) + \text{out}(\mathrm{M}_2) + 3
\end{aligned}$$

where $p(\mathrm{M})$ *is the number of non-root vertices and* $q(\mathrm{M})$ *the number of non-root faces of* M*.*

Proof. The proof is straightforward and follows directly from the previous operations. □

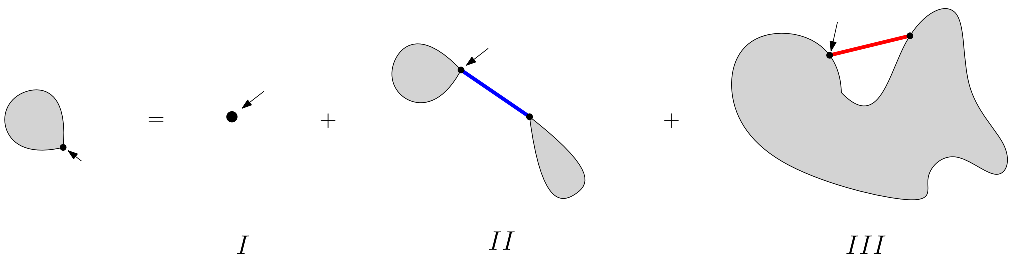

Figure 25: Tutte's decomposition of rooted planar maps.

Tutte's classical *root edge deletion* decomposition ([22]) of rooted planar maps consists in removing the root edge. Then we have the following proposition:

Proposition 20. *Let* M *be a rooted planar map. Then exactly one of the following cases occurs (see Figure 25):*

- *Case I:* M *is the rooted planar map with* 0 *edge.*
- *Case II:* $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_2$ *for some rooted planar maps* M_1 *and* M_2*.*
- *Case III:* $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ *for some rooted planar map* M_1 *and some* $0 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1) + 1$*.*

4.4 The bijection

In the last two subsections, we presented a recursive decomposition for generalized fighting fish and recalled the classical decomposition for rooted planar maps. Since these two decompositions are isomorphic, we are now able to define a bijection ξ from rooted planar maps of size n and reduced outer degree k to generalized fighting fish of size n and jaw k. Let M be a rooted planar map, we examine all possible cases:

- Case I: if M is the zero edge planar map, then $\xi(\mathrm{M})$ is the empty word.
- Case II: the root edge e of M is a disconnecting one, that is, if removed it disconnects M into two rooted planar maps M_1 and M_2, then $\xi(\mathrm{M}) = \xi(\mathrm{M}_1)E\xi(\mathrm{M}_2)W$.
- Case III: the root edge e of M is not a disconnecting one, that is $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ for some rooted planar map M_1 and some $0 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1) + 1$, then we set $\xi(\mathrm{M}) = \maltese_i(\xi(\mathrm{M}_1))$, which is well defined since $\mathrm{jaw}(\xi(\mathrm{M}_1)) = \mathrm{out}(\mathrm{M}_1)$.

We are now in position to state the counterpart of Theorem 9 for generalized fighting fish and general planar maps:

Theorem 21. *The mapping ξ is a bijection between rooted planar maps with n edges, $p+1$ vertices, $q+1$ faces and reduced outer degree k and generalized fighting fish of size n with p left lower free edges (E steps), q right lower free edges (N steps), and jaw k. Moreover, for a rooted planar map* M*, $\xi(\mathrm{M})$ is the counterclockwise code of the tree-rooted planar map (M, T) where T is the rightmost DFS spanning tree of* M*. As a consequence, ξ preserves duality (in the same sense as in Corollary 10) and its restriction to non-separable planar maps is Φ.*

Proof. Since the two recursive decompositions are isomorphic, ξ is a well-defined bijection between rooted planar maps with n edges and $k+1$ corners in the infinite face and generalized fighting fish of size n and $k+1$ points at latitude 0. In fact the more precise statement holds, that if the ith visited corner in the rightmost DFS traversal of M lies in the outerface then the factorization $\xi(\mathrm{M}) = \mathrm{F}' \cdot \mathrm{F}''$ with $|\,\mathrm{F}'\,| = i$ occurs at latitude 0 (that is $\mathrm{lat}(\mathrm{F}') = 0$). We proceed by induction on the size of the rooted planar map M to prove this together with the fact that $\Phi(\mathrm{M}) = \Gamma(\mathrm{M}, T)$, where T is the rightmost DFS spanning tree of M. Let M be a rooted planar map. We distinguish between the three possible cases:

- Case I: The counterclockwise code of the zero edge planar map endowed with its rightmost DFS spanning tree is indeed the empty word.

- Case II: The root edge e of M is a disconnecting one, so that $\xi(\mathrm{M}) = \xi(\mathrm{M}_1)E\xi(\mathrm{M}_2)W$. Then, by induction hypothesis, $\xi(\mathrm{M}_1)$ and $\xi(\mathrm{M}_2)$ are the respective counterclockwise codes of M_1 and M_2 endowed with their respective rightmost DFS spanning trees T_1 and T_2. Now recall that the root edge e is the edge following the root corner in clockwise direction so that the rightmost DFS spanning tree algorithm applied to M will by definition first perform a tour of M_1 which will extract T_1, then insert the edge e and then explore M_2 to get T_2: the counterclockwise code of (M, T) is exactly $\xi(\mathrm{M}_1)E\xi(\mathrm{M}_2)W$, that is $\xi(\mathrm{M})$.

- Case III: if $\mathrm{M} = \maltese_i(\mathrm{M}_1)$ for some rooted planar map M_1 and some $0 \leqslant i \leqslant \mathrm{out}(\mathrm{M}_1) + 1$, then by the induction hypothesis, $\xi(\mathrm{M}_1)$ is the counterclockwise code of M_1 endowed with its rightmost DFS spanning tree T_1. Then the rightmost DFS spanning tree of M is also T_1, because the added edge has one endpoint which is the last half edge of the root vertex and it is discovered from its other side in the DFS. Performing the counterclockwise tour of M around T, we get that the counterclockwise code of (M, T) is exactly $\maltese_i(\xi(\mathrm{M}_1))$, that is $\xi(\mathrm{M})$.

□

We also get a characterization of generalized fighting fish by using Proposition 3:

Corollary 22. *An excursion $w \in \{E, N, W, S\}$ is a generalized fighting fish if and only if it is minimal, that is it contains no factor of the form $Ew'S$ with w' an excursion.*

On the enumerative side, we then get a formula for the number of generalized fighting fish of size n, since the number of rooted planar maps with n edges is known (see [18] for a combinatorial proof). This number is equal to

$$\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}.$$

Let us also remark that the bijection ξ preserves the following statistics:

- The number of *down bridges* in the generalized fighting fish F, *i.e.* the number of ways to decompose F as $\mathrm{F} = \mathrm{F}_1\, E\, \mathrm{G}\, W\, \mathrm{F}_2$ where $\mathrm{F}_1\, \mathrm{F}_2$ and G are (possibly empty) generalized fighting fish, corresponds to the number of bridges in the planar map M.

- The number of *up bridges* in the generalized fighting fish F, *i.e.* the number of ways to decompose F as $\mathrm{F} = \mathrm{F}_1\, N\, \mathrm{G}\, S\, \mathrm{F}_2$ where $\mathrm{F}_1\, \mathrm{F}_2$ and G are (possibly empty) generalized fighting fish, corresponds to the number of loops in the planar map M.

We then get the following proposition:

Proposition 23. *The previous bijection ξ specializes into bijections between the set of loopless planar maps (resp. bridgeless planar maps) and the set of generalized fighting fish without up bridges (resp. generalized fighting fish without down bridges).*

In particular, following [25], these four families are counted by the numbers

$$\frac{1}{(n+1)(2n+1)}\binom{4n+2}{n},$$

where n is the number of edges for the maps, and the size for generalized fighting fish.

Let us remark that generalized fighting fish without up bridges (resp. without down bridges) are exactly the excursions obtained from the empty word using only operations $\triangledown_k$ with $k \geqslant 0$ and $\triangle_\ell$ with $\ell \geqslant 1$ (resp. using only operations $\triangledown_k$ with $k \geqslant 1$ and $\triangle_\ell$ with $\ell \geqslant 0$).

5 Conclusion and perspectives

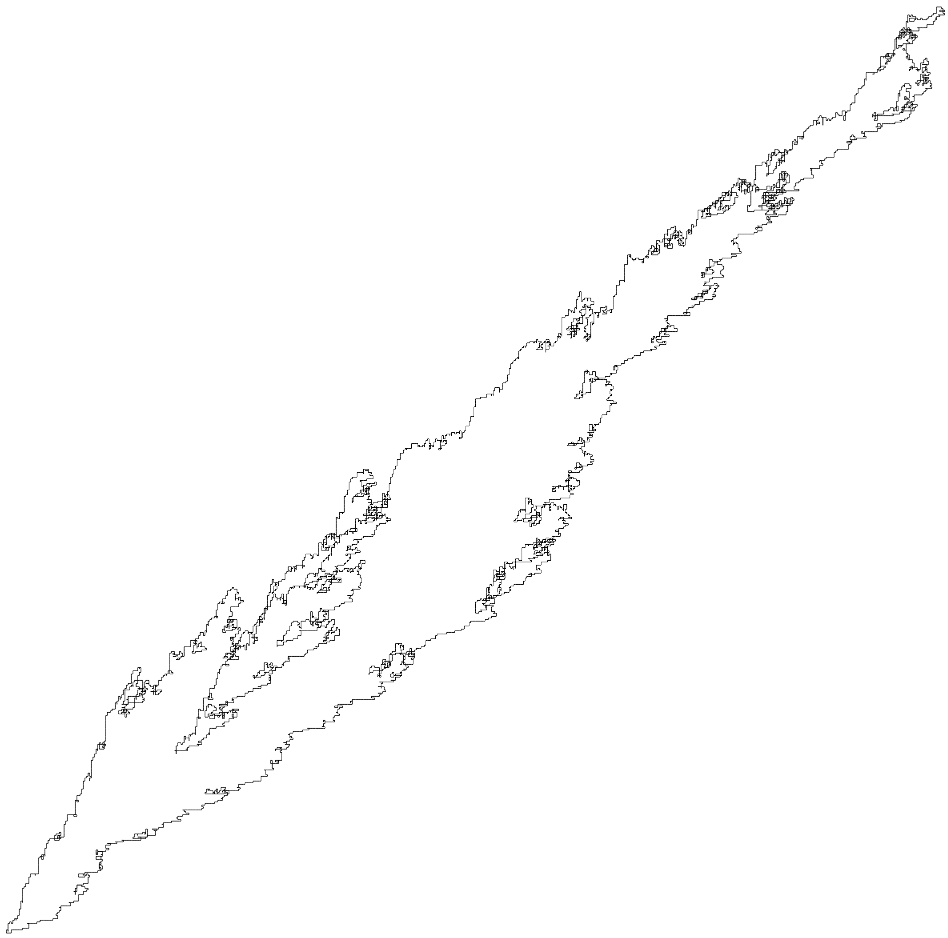

Figure 26: A uniform random fighting fish of size 10000.

We have shown that the counterclockwise code induces a direct bijection between rooted planar maps (endowed with their rightmost DFS spanning tree) and generalized fighting fish, that further specializes in a bijection between non-separable rooted planar maps and fighting fish. We observe that the counterclockwise code on the rightmost DFS spanning tree being based on a single DFS, it can be performed in linear time: therefore the random generators for non-separable planar maps and rooted planar maps due to Schaeffer [19] (available at https://www.lix.polytechnique.fr/Labo/Gilles.Schaeffer/PagesWeb/PlanarMap/) immediately give linear random generators for fighting fish and generalized fighting fish. In Figure 26 we show a random fighting fish of size 10000 obtained by adapting this generator.

While Mullin/Lehman-Lenormand codes date back to the early seventies, the main contribution of this article is to enlighten that the counterclockwise codes of (non-separable) rooted planar maps endowed with their rightmost DFS spanning trees can be given a very natural geometric interpretation as a branching surface obtained by gluing unit cells. This geometric point of view based on fighting fish already helped in [10] to prove a simple formula for the distance of intervals in the Tamari lattice, using the correspondence of the area statistic on fighting fish, where the area of a (generalized) fighting fish is the number of cells it contains. This raises the problem of finding a natural definition of the corresponding statistic on rooted (non-separable) planar maps.

Another perspective would be to unify the two extensions of fighting fish we presented in this paper and the preceding one. In [10], the class of extended fighting fish is defined by allowing the additional operation of replacing every occurrence of the substring WN by a vertical step V in the definition of fighting fish. It would be nice to have a general model of fish including the two classes of generalized and extended fighting fish, and that would fit nicely with the bijections we defined, either with Tamari intervals and rooted planar maps.

Finally it is worth observing that in the present article we use a similar recursive decomposition of fighting fish as in [10] and a similar recursive decomposition of non-separable planar maps as in [13]. However the way Tamari intervals are encoded in [10] and in [13] is not directly compatible (in particular in [10] the lower path of each interval is encoded by its return vector, while it is encoded by its ascending runs in [13], therefore the composition of these two bijections does not immediately yields a direct bijection between rooted non-separable maps and fighting fish as the one we describe here.

Acknowledgements

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