Tesler matrices and Lusztig data

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Abstract

The type A Kostant partition function (KPF) enumerates several families of objects that arise in representation theory and combinatorics, including Tesler matrices, Kostant pictures, Lusztig data, and integral flows. In this paper, we establish logarithmic asymptotics for several classes of KPF by linking them to integer partitions. To this end we introduce height diagrams, integral and row Tesler matrices. As an application, we obtain the logarithmic asymptotics of regular Tesler matrices. We also compare the known poset structures on the aforementioned KPF objects. We prove that the Lusztig data partial order induced on Kostant pictures refines the natural partial order on those, which we also show to be equivalent to the partial order on Tesler matrices.

Mathematics Subject Classifications: 05A16, 06A07, 60J10, 17B22, 17B37

1 Introduction

This paper is broadly concerned with Kostant's partition function (or KPF for short), which counts the number of ways an element of a semigroup known as the (type A) positive root cone can be decomposed (or partitioned) as a sum of distinguished generators known as the positive roots. We derive new insight from representation theory which we apply to solve a longstanding open problem in enumerative combinatorics.

In *part 1* (Section 2) we introduce the representation theory actors, Lusztig data and Kostant pictures (Section 2.1), followed by the enumerative combinatorics actors, Tesler matrices and integer flows on complete graphs (Section 2.2). Each of these sets is counted by the KPF. Using Kostant pictures we are able to draw a connection between regular (hook sum $(1, 1, \ldots, 1)$) and 'flat' Tesler matrices (hook sum $(n, 0, \ldots, 0)$), allowing us to find asymptotics for the number of regular Tesler matrices, improving the previous benchmark [9]. For this, we also develop a framework for counting 'flat' Tesler matrices, which connects KPF with integer partitions.

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Figure 1: Left-to-right: regular and flat weights

Theorem A (Theorem 27, Theorem 28, Theorem 29)*.* The asymptotics of 'flat' Tesler matrices is given by

$$\ln \mathrm{KPF}\left(H,\, \mathbf{0}^{W-1}, -H\right) \sim \begin{cases} W^2 \ln\left(\frac{\sqrt{H}}{W}\right) & \text{if } W \prec \sqrt{H} \\ \Theta\left(H\right) & \text{if } W \asymp \sqrt{H} \\ \sqrt{\frac{2}{3}}\pi W\sqrt{H} & \text{if } W \succ \sqrt{H} \end{cases}$$

In particular, this produces the formula for the regular Tesler matrices

$$\ln T\left(\mathbf{1}^n\right) \sim \pi\sqrt{\frac{8}{27}}\, n\sqrt{n}. \text{ (Theorem 37)}$$

In *part 2* (Section 3), we compare Tesler matrices and Lusztig data as posets, again relying on integral flows and Kostant pictures. The partial order on Tesler matrices is due to Armstrong, [5], while the partial order on Lusztig data is due to Lusztig, [6]. We find that:

Theorem B (Theorem 41)*.* The Tesler poset $\mathcal{T}(\mathbf{h})$ (or $\mathcal{I}(\mathbf{h})$) is isomorphic to a natural partial order on Kostant pictures $\mathcal{K}\left(\mathbf{h}, -\sum h_k\right)$, which we call the merge order.

In addition;

Theorem C (Theorem 43)*.* The poset coming from Lusztig data is a ***refinement*** of the poset from $\mathcal{K}(v)$. It can be concluded that the partial order on $A\left(\mathbf{h}, -\sum h_k\right)$ refines the partial order on $\mathcal{T}\left(\mathbf{h}\right)$ (Theorem 44).

Lusztig data arises in the representation theory of quantum groups and the theory of crystal bases. In particular, "multisegments", which have been shown to model the $B(\infty)$ crystal [8], are essentially Kostant pictures rotated 90 degrees and spread out. An important question, which we do not explore here, is whether this inherited crystal structure can be leveraged to study Tesler matrices in *their* original, representation theoretic context—the diagonal harmonics module.

One of our goals for the future is to better understand KPF posets. With this in mind, in *part 3* (Section 4), we study Markov chains in the Tesler poset. The statistics that arise can give us characteristic values for each poset structure. In addition, we consider ideas to partially generate the poset in order to reduce computing time, while still estimating certain values, for example the Möbius function, which can be used to study asymptotics [1].

2 Tesler matrices, Lusztig data, and their asymptotics

2.1 Lusztig data and Kostant's partition function

First of all, adopting the notation of [6] (Lusztig data), [5] (Tesler matrices), [9] (Integral flows and the Konstant partition function), we define the objects of our study.

Definition 1. Let $(e_1, e_2, \dots, e_{n+1})$ be the standard orthonormal vector basis of $\mathbb{R}^{n+1}$. A sum $\sum a_{ij}\alpha_{ij} = v$, where $\alpha_{ij} = e_i - e_{j+1}$, $1 \leqslant i \leqslant j \leqslant n$ and $a_{ij} \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$, is called a ***weight***. A ***Lustzig datum*** for v is any tuple

$$\mathbf{a} = (a_{11}, a_{12}, \dots, a_{1n}, a_{22}, a_{23}, \dots, a_{2n}, \dots, a_{nn}) \in (\mathbb{Z}_{\geqslant 0})^{\binom{n+1}{2}}$$

such that $\sum a_{ij}\alpha_{ij} = v$. This order on coefficients is the standard order; it corresponds to the standard reduced word for the longest element of S_{n+1}, the symmetric group on $n+1$ elements. The set of all possible Lusztig data for a given v is denoted as $A(v)$.

A more intuitive representation of a decomposition of a weight v due to [4] is as follows.

Definition 2. A ***Kostant picture*** for $v = \sum a_{ij}\alpha_{ij}$ is a diagram on n ordered vertices. In it, each term α_{ij} is represented by a loop around (or a bar above) vertices i through j. The set of all possible Kostant pictures for a given v is denoted by $\mathcal{K}(v)$, while their number $\mathrm{KPF}(v)$ defines the ***Kostant partition function***.

Example 3. For clarity, we shall tabulate every Lusztig data and its corresponding Kostant picture for $v = (1, 1, -1, -1)$ in Table 1.

Lusztig data	Kostant picture v	Decomposition
(1, 0, 0, 2, 0, 1)	$\overline{*}\,\overline{\overline{*}}\,\overline{*}$	$(\alpha_{11}) + 2(\alpha_{22}) + (\alpha_{33})$
(0, 1, 0, 1, 0, 1)	$\overline{*\,\overline{*}}\,\overline{*}$	$(\alpha_{12}) + (\alpha_{22}) + (\alpha_{33})$
(1, 0, 0, 1, 1, 0)	$\overline{*}\,\overline{\overline{*}\,*}$	$(\alpha_{11}) + (\alpha_{22}) + (\alpha_{23})$
(0, 1, 0, 0, 1, 0)	$\overline{*\,\overline{*}\,*}$	$(\alpha_{12}) + (\alpha_{23})$

Table 1: $A(v)$ and $\mathcal{K}(v)$ for $v = (1, 1, -1, -1)$

In this paper we introduce a 'combinatorially-friendly' alternative to weight called ***height***. To a weight $v = (v_1, v_2, \dots, v_{n+1})$ we associate the height $\psi = (\psi_1, \dots, \psi_n)$ such that $v = \sum \psi_i \alpha_{ii}$ (we can do such a decomposition since every $\alpha_{ij} = \alpha_{ii} + \alpha_{(i+1)(i+1)} + \dots + \alpha_{jj}$). We denote the set of all Kostant pictures with height ψ by $\mathcal{K}[\psi]$ putting height in square brackets. For example, we have $\mathcal{K}(1, 1, -1, -1) = \mathcal{K}[1, 2, 1]$.

Note 4. $\mathcal{K}[\psi_1, \dots, \psi_n]$ is the set of all Kostant pictures with n vertices, where each vertix k is contained in exactly ψ_k loops. Height can be easily obtained from weight: if $v = (v_1, \dots, v_{n+1})$, then $\psi = (v_1, v_1 + v_2, \dots, \sum_{k=1}^{n} v_k)$.

Height ψ can be visually represented by its ***height diagram***, which is simply the graph of $y = \psi(x)$ in the xy-plane. It is a discrete function that is zero on its entire domain except for a finite number of arguments (a subset of $\{1, 2, \dots, n\}$). That being said, we will draw it continuously in the next sections for the sake of clarity. The utility of height diagrams lies in the following.

Proposition 5. *Let ψ_1, ψ_2 be two heights such that $\psi_1(x) \geqslant \psi_2(x+a)$, $x \in \mathbb{Z}$ holds for some integer a. Then* $\mathrm{KPF}[\psi_1] \geqslant \mathrm{KPF}[\psi_2]$.

Proof. Consider the subset of $\mathcal{K}[\psi_1]$, which contains all the Kostant pictures with the number of loops α_{ii} at least $\psi_1(i) - \psi_2(i+a)$ for all i. It is easy to see that there is a bijection between this set and $\mathcal{K}[\psi_2]$. □

2.2 Tesler matrices and integral flows

Now, we shall define Tesler matrices. Let U_n be the set of $n \times n$ upper-triangular matrices with non-negative integer entries.

Definition 6. For a matrix $A = (a_{ij}) \in U_n$ we define its k^{th} ***hook sum*** h_k (Figure 2) as

$$h_k = \sum_{i=k}^{n} a_{ki} - \sum_{i=1}^{k-1} a_{ik},\ 1 \leqslant k \leqslant n,$$

and its ***hook sum vector*** as $\mathbf{h} = (h_1, h_2, \dots, h_n)$.

$$\begin{bmatrix} {\color{red}0} & {\color{red}0} & {\color{red}1} & {\color{red}0} & {\color{red}0} \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & {\color{blue}0} & 1 & 0 & 0 \\ & {\color{red}1} & {\color{red}0} & {\color{red}0} & {\color{red}0} \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & {\color{blue}1} & 0 & 0 \\ & 1 & {\color{blue}0} & 0 & 0 \\ & & {\color{red}0} & {\color{red}1} & {\color{red}1} \\ & & & 0 & 2 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & {\color{blue}0} & 0 \\ & 1 & 0 & {\color{blue}0} & 0 \\ & & 0 & {\color{blue}1} & 1 \\ & & & {\color{red}0} & {\color{red}2} \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & {\color{blue}0} \\ & 1 & 0 & 0 & {\color{blue}0} \\ & & 0 & 1 & {\color{blue}1} \\ & & & 0 & {\color{blue}2} \\ & & & & {\color{red}4} \end{bmatrix}$$

Figure 2: We add the red entries and subtract the blue ones to get $\mathbf{h} = (1, 1, 1, 1, 1)$

Definition 7. The set of all U_n matrices with a given hook sum vector $\mathbf{h}$ is called the set of ***(generalized) Tesler matrices with hook sum*** $\mathbf{h}$ and is denoted as $\mathcal{T}(\mathbf{h})$, while $T(\mathbf{h}) = |\mathcal{T}(\mathbf{h})|$.

Example 8. When the hook sum is $(1, 1, \dots, 1) = \mathbf{1}^n$, then $\mathcal{T}(\mathbf{1}^n)$ is called the set of ***regular Tesler matrices***. For $n = 2$ we have only two regular Tesler matrices:

$$\begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \text{ AND } \begin{bmatrix} 0 & 1 \\ & 2 \end{bmatrix}.$$

At first glance, Tesler matrices have little to do with Lusztig data or Kostant pictures. There is a representation that links both, though.

Definition 9. For a vector $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ with integer entries, we let $\mathcal{I}(\mathbf{h})$ denote the set of the ***integral flow graphs*** on $n+1$ vertices with the net flow $(\mathbf{h}, -\sum h_k)$ such that every flow on the edges is non-negative (Figure 3).

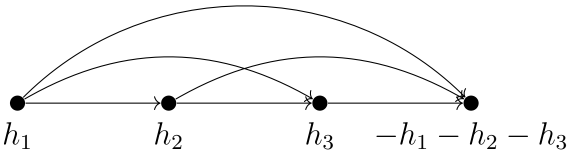


Figure 3: Integral flow graph for $n = 3$

It has been shown [12] that $T(\mathbf{h}) = |\mathcal{I}(\mathbf{h})| = \mathrm{KPF}(\mathbf{h}, -\sum h_i)$. This can be illustrated with bijections between the three structures (Figure 4).

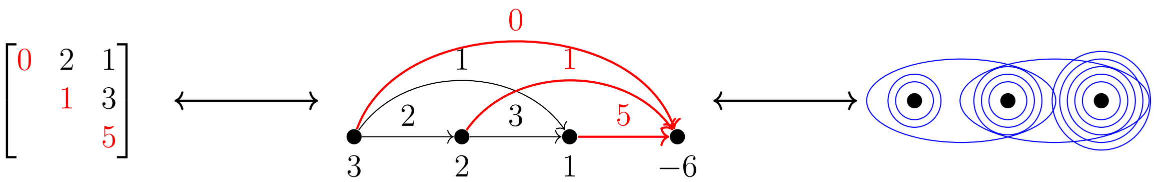


Figure 4: A Tesler matrix and an integral flow correspondence, $\mathbf{h} = (3, 2, 1)$

More precisely, given a Tesler matrix $A = (a_{ij})$, we construct an integral flow with an edge from the i^{th} vertex to the j^{th} vertex assigned the value a_{ij} for each $1 \leqslant i < j \leqslant n$, while an edge from the i^{th} vertex to the last vertex is assigned the value a_{ii}. Then, we draw a Kostant picture with the number of loops around vertices i through j equal to the flow value from the i^{th} vertex to the $(j+1)^{\text{th}}$ vertex.

Having done so, we can actually create a more 'combinatorially-friendly' Tesler matrices by getting rid of the subtraction in their definition. This is the first new definition of our paper.

Definition 10. Let ψ be some height. For each Lustzig datum $\mathbf{a} \in A[\psi]$, we rearrange its components into an upper-triangular matrix $[a_{ij}]$. The set of all such matrices we shall call the set of ***(generalized) integral Tesler matrices*** and denote as $\mathcal{T}[\psi]$.

Note 11. The difference in brackets is crucial in the paper, as we will often use different types of brackets to describe different objects, as in this instance, where square brackets are reserved for integral Tesler matrices, while parentheses are used for the ordinary ones.

Definition 12. Analogously, $T[\psi] = |\mathcal{T}[\psi]|$, and the same notation will be used for every other introduced bracket type.

Example 13. The hook sum vector $\mathbf{h} = \mathbf{1}^n$ corresponds to the height $\psi = (1, 2, \ldots, n)$ via the bijection between $\mathcal{T}(\mathbf{h})$ and $A[\psi]$ and so the regular Tesler matrices from Theorem 8 correspond to the integral Tesler matrices $\mathcal{T}[1, 2]$:

$$\begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \text{ AND } \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix}.$$

Note 14. The resulting bijection takes $[a_{ij}] \in \mathcal{T}(h_1, h_2, \ldots, h_k)$ and maps it into $[b_{ij}] \in \mathcal{T}[h_1, h_1 + h_2, \ldots, \sum h_k]$, where $b_{ij} = a_{i(j+1)}$, $j \neq n$ and $b_{in} = a_{ii}$.

Defining integral Tesler matrices this way, we can easily formulate an alternative definition.

Definition 15. Define the k^{th} ***integral hook sum*** s_k (Figure 5) of a matrix $A = (a_{ij}) \in U_n$ as

$$s_k = \sum_{i \leqslant k \leqslant j} a_{ij},\ 1 \leqslant k \leqslant n,$$

and its ***integral hook sum vector*** as $\mathbf{s} = (s_1, s_2, \ldots, s_n)$.

$$\begin{bmatrix} \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}0 \\ & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}0 \\ & \color{red}0 & \color{red}0 & \color{red}0 & \color{red}1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & \color{red}0 & \color{red}0 & \color{red}0 \\ & 0 & \color{red}0 & \color{red}0 & \color{red}1 \\ & & \color{red}1 & \color{red}1 & \color{red}0 \\ & & & 2 & 0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \color{red}0 & \color{red}0 \\ & 0 & 0 & \color{red}0 & \color{red}1 \\ & & 1 & \color{red}1 & \color{red}0 \\ & & & \color{red}2 & \color{red}0 \\ & & & & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \color{red}0 \\ & 0 & 0 & 0 & \color{red}1 \\ & & 1 & 1 & \color{red}0 \\ & & & 2 & \color{red}0 \\ & & & & \color{red}4 \end{bmatrix}$$

Figure 5: We add the red entries to get $\mathbf{s} = (1, 2, 3, 4, 5)$

Definition 16 (*alternative*)**.** The set of all U_n matrices with a given integral hook sum vector $\mathbf{s}$ is called the set of integral Tesler matrices and denoted as $\mathcal{T}[\mathbf{s}]$.

2.3 Asymptotics analysis via Tesler matrices and integer partitions

Definition 17. Let $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$. As $x \to \infty$, we write $f \sim g$ iff $f/g \to 1$; $f \prec g$ iff $f/g \to 0$; $f \asymp g$ iff $ag \leqslant f \leqslant bg$ for some $a, b \in \mathbb{R}^+$; $f \precsim g$ iff for each $a > 1$ as $x \to \infty$ eventually $f \leqslant ag$; $f \preccurlyeq g$ iff $f \leqslant ag$ for some $a \in \mathbb{R}^+$. Analogously are defined $f \succsim g$ and $f \succcurlyeq g$.

Note 18. $f \precsim g$ AND $f \succsim g \Longrightarrow f \sim g$.

In the following two sections, we will analyze the asymptotic behavior of the function defined as $T[\mathbf{s}(n)] : \mathbb{Z}^+ \to \mathbb{Z}^+$, where $\mathbf{s}(n)$ is a vector-valued function. For convenience, we define a binary operation $\times$ that maps integer pairs from $\mathbb{Z}^+ \times \mathbb{Z}^+$ into a vector

$$a \times b = \underbrace{(a, a, \ldots, a)}_{b \text{ times}} \in \left(\mathbb{Z}^+\right)^b.$$

For this section we shall study $\mathbf{s}(n) = H(n) \times W(n)$, where $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$. For the best previous asymptotical analysis of this type see [9]. Since such sequences tend to grow exponentially, from now on we will use the notation defined below.

Definition 19. $L[\mathbf{s}] = \ln T[\mathbf{s}]$ regardless of the bracket type, so $L(\mathbf{h}) = \ln T(\mathbf{h})$.

For example, it is easy to verify that $L[\mathbf{1}^n] = (n-1)\ln 2$. More importantly for us, it was shown (by a clever appeal to the Morris identity) in [2] that

$$L(1, 2, \ldots, n) = \sum_{k=1}^{n} \ln C_k \sim n^2 \ln 2,$$

where C_k denotes k^{th} Catalan number $\frac{1}{k+1}\binom{2k}{k}$. Regarding $T(\mathbf{1}^n)$ [OEIS], it was asked by Igor Pak whether $L(\mathbf{1}^n) \asymp n^2$ as with $L(1, 2, \ldots, n)$. This question led to the introduction of the Tesler poset [1, Proposition 6.6].

Now, before we can actually study integral Tesler matrices asymptotics for $\mathbf{s} = H \times W$ (hereinafter each sequence has index n implicitly), we shall first prove an important technical lemma.

Lemma 20. *Let $p(n;\ l,\ r)$ denote the number of partitions of n in which the size of each part s satisfies $l \leqslant s \leqslant r$, then for $l(n) \prec \sqrt{n} \prec r(n)$ holds*

$$\ln p(n;\ l,\ r) \sim \ln p(n;\ 1,\ n) \sim \sqrt{\frac{2}{3}}\pi\sqrt{n}.$$

Proof. The second equivalence is the famous Hardy-Ramanujan formula, so we shall derive the first one. Note that any partition of n could be sub-partitioned into 3 categories: with part sizes from 1 to $l-1$, from l to r, and from $r+1$ to n. If we denote the sum of all parts of the first type by n_1, the second as n_2, and the third as n_3, we shall obtain

$$p(n;\ 1,\ n) = \sum_{n_1+n_2+n_3=n} p(n_1;\ 1,\ l-1) \cdot p(n_2;\ l,\ r) \cdot p(n_3;\ r+1,\ n),$$

leading to the upper-bound

$$p(n;\ 1,\ n) \leqslant p(n;\ 1,\ l-1) \cdot p(n;\ l,\ r) \cdot p(n;\ r+1,\ n) \cdot \binom{n+2}{2}.$$

This gives us

$$\ln p(n;\ 1,\ n) \leqslant \ln p(n;\ 1,\ l-1) + \ln p(n;\ l,\ r) + \ln p(n;\ r+1,\ n) + \ln \binom{n+2}{2},$$

where the first summand is asymptotically negligible compared to $\sqrt{\frac{2}{3}}\pi\sqrt{n}$ on the left side since the formula for it from [10]. The same is with the last since $\ln \binom{n+2}{2} = 2\ln n + O(1) = o(\sqrt{n})$. It is now only left to prove the same for the third one. Note that every partition it counts has at most $n/(r+1) \leqslant n/r$ parts. Now, use the fact that the number of partitions of n into at most n/r parts is equal to that of partitions of n into parts not larger than n/r (see [11, Theorem 5.17, p. 108]). As in $n/r \prec \sqrt{n}$, the conclusion is the same as for the first summand. As $p(n;\ l,\ r) \leqslant p(n;\ 1,\ n)$, we can now derive

$$\ln p(n;\ l,\ r) \sim \ln p(n;\ 1,\ n)\,. \qquad \square$$

Now, we can start building a framework for studying the asymptotics of integral Tesler matrices. For one, condition on each integral hook sum is too complicated to account for in counting. To bypass this, we shall consider a universal set of integral Tesler matrices, which is far less restrictive.

Definition 21. The union of all $\mathcal{T}[s_1,\ s_2,\ \ldots,\ s_n]$ such that $\sum s_k = m$ is called ***integral Tesler matrices universe***, and denoted as $\mathcal{T}\{m,\ n\}$. Note $L\{m,\ n\} = \ln T\{m,\ n\}$

Answering the question of $L[H \times W]$ asymptotics requires finding the asymptotics of the corresponding universe $L\{HW,\ W\}$. This can be done by breaking the universe not into integral Tesler matrices, but into a much less complex structure when it comes to cardinality counting. That is where integer partitions emerge.

Definition 22. Define the k^{th} ***(weighted) row sum*** r_k of a matrix $A = (a_{ij}) \in U_n$ as

$$r_k = a_{kk} + 2a_{k(k+1)} + \cdots + (n-k+1)a_{kn} = \sum_{j=k}^{n} (j-k+1)\, a_{kj}$$

and its ***(weighted) row sum vector*** as $\mathbf{r} = (r_1,\ r_2,\ \ldots,\ r_n)$.

Definition 23. The set of all matrices with fixed row sum vector $\mathbf{r}$ shall be called the set of ***row Tesler matrices*** and denoted as $\mathcal{T}_r(\mathbf{r})$. As before, $T_r(\mathbf{r}) = |\mathcal{T}_r(\mathbf{r})|$, and $L_r(\mathbf{r}) = \ln T_r(\mathbf{r})\,.$

The motivation is as follows. Count the coefficients before each a_{ij} in the sum of all integral hook sums $\sum s_k$ (see Figure 6) to see that the coefficients in each row perfectly match those from the definition of r_i. Therefore, $\sum r_k = \sum s_k$, and so the proposition below holds.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 3 & 3 \\ & 1 & 2 & 2 & 2 \\ & & 1 & 1 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ & 1 & 2 & 3 & 3 \\ & & 1 & 2 & 2 \\ & & & 1 & 1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{bmatrix}$$

Figure 6: Counting coefficients before every a_{ij} in partial sums $s_1 + s_2 + \cdots + s_k$

Proposition 24. *Integral Tesler matrices universe is a universal set for $\mathcal{T}_r$:*

$$\mathcal{T}\{m,\, n\} = \bigcup_{r_1+r_2+\cdots+r_n=m} \mathcal{T}_r\left(r_1,\, r_2,\, \ldots,\, r_n\right).$$

As mentioned earlier, it is extremely easy to count row Tesler matrices.

Proposition 25. *The next formula is valid to count the number of $\mathcal{T}_r(\mathbf{r})$:*

$$T_r\left(r_1,\, r_2,\, \ldots,\, r_n\right) = \prod_{k=1}^{n} p\left(r_k;\; 1;\; n-k+1\right).$$

Proof. The k^{th} row of a matrix from $\mathcal{T}_r(\mathbf{r})$ can be interpreted as a partition of r_k with size of each part between 1 and $n-k+1$. Indeed, by definition we have $r_k = a_{kk} + 2a_{k(k+1)} + \cdots + (n-k+1)a_{kn}$. Therefore, $T_r(\mathbf{r})$ can be exrpessed as the number of ways to partition each $r_1,\, r_2,\, \ldots,\, r_n$. ☐

This can be used to actually get $L\{HW,\, W\}$ asymptotics. We shall first do so for $W^2 \succ H$.

Theorem 26. *Let $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $W^2 \succ H \succ 1$. Then*

$$L\{HW,\, W\} \sim L_r(H \times W) = L_r\underbrace{\left(H(n),\, H(n),\, \ldots,\, H(n)\right)}_{W(n)\ \textit{times}} \sim \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

Proof. First, let us show the second asymptotic equality. We shall do it using the following trick: fix some real number $q < 1$ and take the logarithm of Theorem 25 to get

$$L_r(H \times W) = \sum_{k=1}^{W} \ln p\left(H;\; 1,\, W-k+1\right) \geqslant \sum_{k=1}^{qW} \ln p\left(H;\; 1,\, W-k+1\right) \sim qW \cdot \sqrt{\frac{2}{3}}\pi\sqrt{H}.$$

We can do so since in the second sum $W-k+1 \geqslant (1-q)\,W \succ \sqrt{H}$ (as $W \succ \sqrt{H}$), and so Theorem 20 is applicable. As the statement holds for any $q < 1$, we can take $q \to 1$ to derive

$$L_r(H \times W) \gtrsim \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

Here $\geqslant$ naturally transitions into $\gtrsim$, which is the main advantage of using this notation.

To get the upper-bound we should note that $p\left(H;\; 1,\, r\right) \leqslant p\left(H;\; 1,\, H\right), r \in \mathbb{R}$ as we cannot have parts larger than H in a partition of H. This allows as to get the following:

$$L_r(H \times W) = \sum_{k=1}^{W} \ln p\left(H;\; 1,\, W-k+1\right) \leqslant \sum_{k=1}^{W} \ln p\left(H;\; 1,\, H\right) \sim W \cdot \sqrt{\frac{2}{3}}\pi\sqrt{H}.$$

Combining those two bounds, we have the desired second asymptotic equality.

As for the first one, we already have $L\{HW, W\} \gtrsim L_r(H \times W)$ as $\mathcal{T}\{HW, W\} \supseteq \mathcal{T}_r(H \times W)$ from Theorem 24. Now, we shall derive $L\{HW, W\} \lesssim \sqrt{\frac{2}{3}}\pi W\sqrt{H}$ using the same Proposition:

$$T\{HW, W\} = \sum_{r_1+\dots+r_W=HW} T_r(r_1, \dots, r_W) \leqslant \binom{HW+W-1}{W-1} \cdot \max T_r(r_1, \dots, r_W),$$

where the maximum is taken under $\sum r_k = HW$ and the inequality is valid since there are exactly $\binom{HW+W-1}{W-1}$ summands in our sum. Applying the formula for T_r and taking the logarithm we have:

$$L\{HW, W\} \leqslant \ln\binom{HW+W-1}{W-1} + \max\sum_{k=1}^{W} \ln p(r_k;\ 1,\ W-k+1).$$

Here, after expanding the first summand with Stirling's formula, we derive that it is insignificant:

$$\ln\binom{HW+W-1}{W-1} \sim W\ln(1+H) + HW\ln\left(1+\frac{1}{H}\right) \sim W\ln H \prec \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

What is left to show:

$$\begin{aligned}\max\sum_{k=1}^{W} \ln p(r_k;\ 1,\ W-k+1) &\leqslant \max\sum_{k=1}^{W} \ln p(r_k;\ 1,\ r_k)\\ &\sim \sqrt{\frac{2}{3}}\pi \cdot \max\sum_{k=1}^{W}\sqrt{r_k} = \sqrt{\frac{2}{3}}\pi W\sqrt{H},\end{aligned}$$

due to the fact that for fixed $\sum r_k$, the sum $\sum\sqrt{r_k}$ is maximized under $r_1 = r_2 = \dots = r_W$. □

As promised earlier, once we have $L\{HW, W\}$ asymptotics, we can derive it for $L[H \times W]$.

Theorem 27. *Let* $H, W : \mathbb{Z}^+ \to \mathbb{Z}^+$ *such that* $W^2 \succ H \succ 1$. *Then*

$$L[H \times W] \sim \sqrt{\frac{2}{3}}\pi W\sqrt{H}.$$

Proof. Before making everything formal, let us first lay down the motivation behind our steps. In the previous theorem we have shown that the universe is asymptotically equal to RHS, so we already have an upper-bound. Now we shall impose several restrictions on elements of $\mathcal{T}\{HW, W\}$ to get (almost) a subset of $\mathcal{T}[H \times W]$, thereby acquiring a lower-bound. What we want is for those restrictions to be loose enough to preserve the asymptotics.

Let us put our goal as follows: we want restrictions that make all the integral hook sums equal. This is difficult, in part, due to the corresponding regions of summation being different in shape. We, on the contrary, would like our construction to be as simple and symmetric as possible, and so have all the summation regions equal in shape.

The above dictates the first crucial restriction. In each matrix, non-zero entries shall be put only in a small portion of consecutive entries in each row. Those portions we shall call ***segments***. Each row will have one segment from $j = i + l - 1$ to $j = i + r - 1$ (see Figure 7). Of course, there will be a few last rows, which could not contain such segments. The idea is that there will be so few of them, they could be ignored.

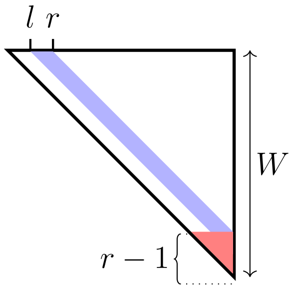


Figure 7: Segments are colored blue, the last segment-less rows red, zero entries white

This makes every summation region for s_k, $r \leqslant k \leqslant W - r + 1$ equal. Note that if we additionally made every segment the same, we would actually get the desired equality for integral hook sums. However, this is obviously an extremely harsh restriction that would drastically reduce our asymptotics. To bypass this, we shall divide every segment into some fixed number of ***blocks*** of the same length. On each such block we define weighed sum as $\sum (j - i + 1) a_{ij}$, and for each segment we compute the sequence of weighted sums over its blocks. If we now set these sequences equal across all segments, we would get all the integral hook sums almost equal to one another—in fact asymptotically equal (note that it can be said that all our matrices are now from $\mathcal{T}_r (H \times W)$). At the same time, at "microscopic level", every block could have its entries arranged independently of other blocks in all the ways which preserve the weighted sum, thereby keeping the asymptotics by allowing "enough chaos".

Let $l \sim \alpha^{-1}\sqrt{H}$, $r \sim \alpha\sqrt{H}$ be two functions for some $1 \prec \alpha \prec \min\left(W/\sqrt{H},\, \ln H\right)$. For now, it is significant that $l \prec \sqrt{H} \prec r \prec W$. Consider the subset $\mathcal{A} \subseteq \mathcal{T}_r (H \times W)$, in which every matrix $[a_{ij}]$ has $a_{ij} = 0$ for $1 \leqslant i \leqslant w - r + 1$ and $j - i + 1 \in [1,\, l) \cup (r,\, n]$. In other words, every its row (except the last irrelevant $r - 1$ ones) represents a partitions from $p\,(H;\, l,\, r)$. Note that $\ln |\mathcal{A}|$ is still asymptotically $\sqrt{\frac{2}{3}}\pi W\sqrt{H}$ since Theorem 25 and Theorem 20.

Now, we shall divide every segment $\left[a_{i(i+l-1)},\, a_{i(i+l)},\, \ldots,\, a_{i(i+r-1)}\right]$ into consecutive blocks of length $\gamma \sim \alpha^2 \ln H$. There are $\beta = (r - l + 1)/\gamma \sim \alpha^{-1}\sqrt{H}/\ln H$ of them in each segment, and the k^{th} block of the i^{th} segment can be written as

$$\left[a_{i(i+l+(k-1)\gamma-1)},\, a_{i(i+l+(k-1)\gamma)},\, \ldots,\, a_{i(i+l+k\gamma-2)}\right].$$

The weighted sum $\sum (j-i+1)\, a_{ij}$ over it we shall denote as m_{ik}. Note that $\sum_k m_{ik} = r_i = H$.

If by $\mathcal{A}_M \subseteq \mathcal{A}$ we now denote the subset with fixed block sum matrix $M = [m_{ik}]$, we get

$$\max |\mathcal{A}_M| \leqslant |\mathcal{A}| = \sum_M |\mathcal{A}_M| \leqslant \max |\mathcal{A}_M| \cdot \binom{H+\beta-1}{\beta-1}^{W-r+1}.$$

Note that the logarithm of the second multiplier is

$$(W-r+1)\ln \binom{H+\beta-1}{\beta-1} \sim \beta W \ln\left(1+\frac{H}{\beta}\right) + HW \ln\left(1+\frac{\beta}{H}\right)$$
$$\sim W\sqrt{H} \cdot \frac{\ln\left(\alpha\sqrt{H}\right)}{\alpha \ln H},$$

which is negligible compared to $\sqrt{\frac{2}{3}}\pi W\sqrt{H}$ since $\alpha \prec \ln H$.

Consider one segment and choose for it some weighted sums over blocks $m'_1, \ldots, m'_\beta$, which maximize the number of partitions $p\,(H;\ l,\ r)$. If we now let $M' = [m_{ik}]$ be a matrix such that $m_{ik} = m'_k$, we can guarantee $|\mathcal{A}_{M'}| = \max |\mathcal{A}_M|$, and so $\ln |\mathcal{A}_{M'}| \sim \sqrt{\frac{2}{3}}\pi W\sqrt{H}$.

Finally, we can bound integral hook sums s_k, $r \leqslant k \leqslant W-r+1$ for the matrices of $\mathcal{A}_{M'}$. To acquire the upper-bound for s_k, we shall count over all blocks, entries of which are at least partially counted in s_k. For lower-bound—which have all of their entries counted in s_k. For each such block, we can upper/lower-bound the sum over its entries as

$$s_k = \sum_{i\leqslant k\leqslant j} a_{ij} \leqslant \sum_{i\leqslant k\leqslant i+l+\gamma t-2} \frac{m_{it}}{l+\gamma\,(t-1)} = \sum_{t=1}^{\beta} \frac{l+\gamma t-1}{l+\gamma\,(t-1)} \cdot m'_t \sim \sum_{t=1}^{\beta} m'_t = H,$$
$$\sum_{i\leqslant k\leqslant j} a_{ij} \geqslant \sum_{i\leqslant k\leqslant i+l+\gamma(t-1)-1} \frac{m_{it}}{l+\gamma t-1} = \sum_{t=1}^{\beta} \frac{l+\gamma\,(t-1)}{l+\gamma t-1} \cdot m'_t \sim \sum_{t=1}^{\beta} m'_t = H.$$

The last asymptotical transition in each line is due to $l/\gamma \sim \alpha^{-3}\sqrt{H}/\ln H \succ 1$ as $\alpha \prec \ln H$.

So we have gotten $s_k \sim H$, $r \leqslant k \leqslant W-r+1$, thereby acquiring the almost-subset $\mathcal{A}_{M'}$ we wrote about at the beginning of the theorem. The last steps of this proof is to actually link $\mathcal{A}_{M'}$ to $\mathcal{T}\,[H \times W]$. The strategy is to use Theorem 5 as described below.

First, we shall restrict $\mathcal{A}_{M'}$ to just one integral hook sum vector (as for now it contains different matrices with different vectors). Generally, there are $\binom{HW+W-1}{W-1}$ different integral hook sum vectors in the universe, which is asymptotically negligible compared to $\sqrt{\frac{2}{3}}\pi W\sqrt{H}$ (it was shown in Theorem 26). Therefore we can choose $\mathcal{B} \subseteq \mathcal{A}_{M'}$, which itself is a subset of some $\mathcal{T}\,[\mathbf{s}]$ such that $\ln |\mathcal{B}| = \ln |\mathcal{A}_{M'}|$.

At last, the above holds for every H as long as $W^2 \succ H \succ 1$, so we can choose qH instead of H to get $\mathcal{B}$ with $s_k \sim qH$, $r \leqslant k \leqslant W-r+1$. Delete the first $r-1 \prec W$ rows and the last $r-1$ columns of each matrix of $\mathcal{B}$ so that $s_k \precsim qH$ would hold over all k. Recall that we are working with subset of $\mathcal{T}[H \times W]$, and so this operation would not affect asymptotics as we are deleting a negligibly small number of partitions. After deleting, however, $s_k(n) \leqslant H(n)$ for sufficiently large n, and so we can apply Theorem 5 to finish the proof, as $T[H \times W] \geqslant |\mathcal{B}|$ and so

$$L[H \times W] \gtrsim \sqrt{\frac{2}{3}}\pi W \sqrt{qH} \xrightarrow{q \to 1} \sqrt{\frac{2}{3}}\pi W \sqrt{H}. \qquad \square$$

Theorem 28. *Let H, $W : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $H \succ W^2 \succ 1$. Then*

$$L[H \times W] \sim L\{HW, W\} \sim L_r(H \times W) \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

Proof. Derivation of $L_r[H \times W]$ and $L\{HW, W\}$ is totally analogous to Theorem 26.

$$L_r[H \times W] = \sum_{k=1}^{w} \ln p(H;\ 1,\ k) \sim \sum_{k=1}^{w} k \ln\left(\frac{H}{k^2}\right)$$

since the $p(H;\ 1,\ k)$ asymptotics from [10]. This sum can be elementary approximated with

$$\sum_{k=1}^{w} k \ln\left(\frac{H}{k^2}\right) \sim \int_1^w k \ln\left(\frac{H}{k^2}\right) dk \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

To now get $L\{HW, W\}$, we shall only provide a sharp upper-bound via

$$\begin{aligned} T\{HW, W\} &= \sum_{r_1 + r_2 + \dots + r_W = HW} T_r(r_1,\ r_2,\ \dots,\ r_W) \\ &\leqslant \binom{HW + W - 1}{W - 1} \cdot \max T_r(r_1,\ r_2,\ \dots,\ r_W), \end{aligned}$$

where the maximum is taken under $\sum r_k = HW$. As for the first multiplier,

$$\ln\binom{HW + W - 1}{W - 1} \sim W \ln(1 + H) + HW \ln\left(1 + \frac{1}{H}\right) \sim W \ln H \prec W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

As for the second,

$$\max L_r(r_1,\ r_2,\ \dots,\ r_W) = \max \sum_{k=1}^{w} \ln p(H;\ 1,\ k) \sim \max \sum_{k=1}^{w} k \ln\left(\frac{r_k}{k^2}\right),$$

which is maximized under $r_k = 2kH/W$. Therefore

$$L\{HW, W\} \precsim \sum_{k=1}^{w} k \ln\left(\frac{2H}{kW}\right) \sim \int_1^W k \ln\left(\frac{2H}{kW}\right) dk \sim W^2 \ln\left(\frac{\sqrt{H}}{W}\right).$$

Finally, we shall provide a sharp lower-bound for $L\left[H\times W\right]$. To do this, consider all the Kostant pictures on W vertices with exactly H loops to derive

$$L\left[H\times W\right]\geqslant\ln\binom{H+W\left(W-1\right)/2-1}{H}\sim W^{2}\ln\left(\frac{\sqrt{H}}{W}\right).\qquad\square$$

While the asymptotics of $T\left[H\times W\right]$ has been researched, with an improved lower bound presented in [9, Theorem 1.1], it was done so for a constant $H\left(n\right)=H_{0}$ and $W\left(n\right)\to\infty$. Here, in contrast, both $H,\,W\to\infty$, allowing us to use asymptotics for the number of partitions to produce such sharp estimates.

Proposition 29. *Let $H,\,W:\mathbb{Z}^{+}\to\mathbb{Z}^{+}$ such that $H\asymp W^{2}\succ1$. Then*

$$L\left[H\times W\right]\asymp L\left\{HW,\,W\right\}\asymp L_{r}\left(H\times W\right)\asymp H.$$

Proof. Let $W_{1}\prec\sqrt{H}\prec W_{2}$, then we can bound all these functions by the previous theorems as

$$W_{1}^{2}\ln\left(\frac{\sqrt{H}}{W_{1}}\right)\precsim L\left[H\times W\right],\,L\left\{HW,\,W\right\},\,L_{r}\left(H\times W\right)\precsim\sqrt{\frac{2}{3}}\pi W_{2}\sqrt{H}.$$

If we now let $W_{1},\,W_{2}\to\sqrt{H}$, we would get the desired result. $\square$

2.4 Asymptotics analysis via Kostant pictures and height diagrams

In this section we shall use Kostant pictures to analyze asymptotics, and so we shall write $\psi\left(n\right)$ instead of $\mathbf{s}\left(n\right)$, and $L\left[\psi\right]$ instead of $L\left[\mathbf{s}\right]$.

In the previous section, asymptotics for the type $\psi=H\times W$ was derived. Now, we shall analyze the type $\psi\left(n\right)=\left(a_{1},\,a_{2},\,\ldots,\,a_{n}\right)$ for some fixed integer sequence $\left(a_{n}\right)_{n\geqslant1}$. In other words, one obtains $\psi\left(n+1\right)$ by appending a_{n+1} to $\psi\left(n\right)$. This will give us the asymptotic formula for the number of regular Tesler matrices on substituting $a_{n}=n$ (researched in [1, 9]).

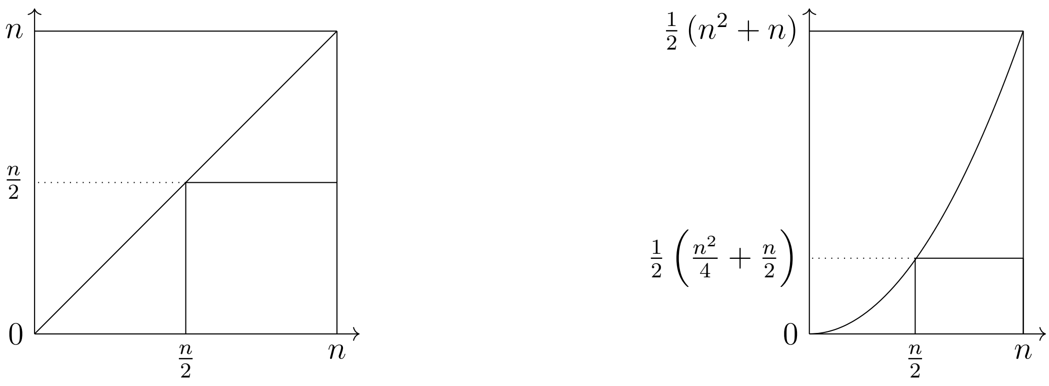
To do this, we shall utilize the Kostant pictures representation and height diagrams. The idea is to approximate complex height, which $\psi=\left(a_{1},\,a_{2},\,\ldots,\,a_{n}\right)$ is, with simpler type $\psi=H\times W$.

Proposition 30. $L\left(\mathbf{1}^{n}\right)\asymp L\left[n\times n\right]$ *AND* $L\left(1,\,2,\,\ldots,\,n\right)\asymp L\left[n^{2}\times n\right]$.

Proof. Note that weight $\mathbf{1}^{n}$ corresponds to the height $\psi\left(x\right)=x$, and so by looking at Figure 8a we can deduce (recall Theorem 5)

$$L\left[\frac{n}{2}\times\frac{n}{2}\right]\leqslant L\left(\mathbf{1}^{n}\right)\leqslant L\left[n\times n\right].$$

As $L\left[\frac{n}{2}\times\frac{n}{2}\right]\asymp L\left[n\times n\right]$, we get the desired result. The same can be stated for the second proposition (see Figure 8b). $\square$



(a) $L\left(\mathbf{1}^n\right) \asymp L\left[n \times n\right]$ (b) $L\left(1,\, 2,\, \ldots,\, n\right) \asymp L\left[n^2 \times n\right]$

Figure 8: Basic height diagrams usage

Before more precise analysis, as was in the previous section, we shall develop a helpful concept.

Definition 31. $\mathcal{K}^k\left[\psi\right]$ denotes the set of Kostant pictures with height ψ, in which every loop is colored with one of $\lfloor k \rfloor$ distinct colors. Also, $T^k\left[\psi\right] = \left|\mathcal{K}^k\left[\psi\right]\right|$ and $L^k\left[\psi\right] = \ln T^k\left[\psi\right]$.

Definition 32. $C_k\left[\psi\right]$ denotes the maximal number of colorings a Kostant picture from $\mathcal{K}\left[\psi\right]$ can have into $\lfloor k \rfloor$ distinct colors.

Proposition 33. $\max\left(T\left[\psi\right],\, C_k\left[\psi\right]\right) \leqslant T^k\left[\psi\right] \leqslant T\left[\psi\right] \cdot C_k\left[\psi\right]$.

Theorem 34 (Decomposition Theorem)**.** *Let $\psi = \left(\psi_1,\, \psi_2,\, \ldots,\, \psi_{kW}\right)$ be some height. Define sequences $\phi_t = \left(\psi_{(t-1)W+i}\right)_{1 \leqslant i \leqslant W}$, $1 \leqslant t \leqslant k$ and*

$$S = \sum_{i=1}^{k} T\left[\psi_{(i-1)W+1},\, \psi_{(i-1)W+2},\, \ldots,\, \psi_{iW}\right].$$

Then we can lower-bound $T\left[\psi\right]$ by

$$\max\left(S,\, T^{\frac{W^2}{4}}\left[\min\left(\min\phi_1,\, \min\phi_2\right),\, \ldots,\, \min\left(\min\phi_{k-1},\, \min\phi_k\right)\right]\right),$$

and upper-bound it by

$$S \cdot T^{W^2}\left[\min\left(\max\phi_1,\, \max\phi_2\right),\, \min\left(\max\phi_2,\, \max\phi_3\right),\, \ldots,\, \min\left(\max\phi_{k-1},\, \max\phi_k\right)\right].$$

Proof. Decompose ψ into k divisions of length W each (Figure 9). If we ignore all possible loops α_{ij} that are contained within several such divisions, we have $T\left[\psi\right] \geqslant S$.

Now, consider all such missing α_{ij} as determined by two parameters: the set of divisions the loop covers and the relative positions of its ends in the first and the last divisions. Note that this situation can be modeled by

$$\mathcal{K}\left[\min\left(\min\phi_1,\, \min\phi_2\right),\, \min\left(\min\phi_2,\, \min\phi_3\right),\, \ldots,\, \min\left(\min\phi_{k-1},\, \min\phi_k\right)\right]$$

with each loop having one of $\left(\frac{W}{2}\right)^2$ possible colors ($\frac{W}{2}$ possible choices per loop end so that the loops with disjoint division sets do not cross) in order to produce a lower bound.

If, instead, we allow all W possible choices per end, we achieve the upper-bound. $\square$

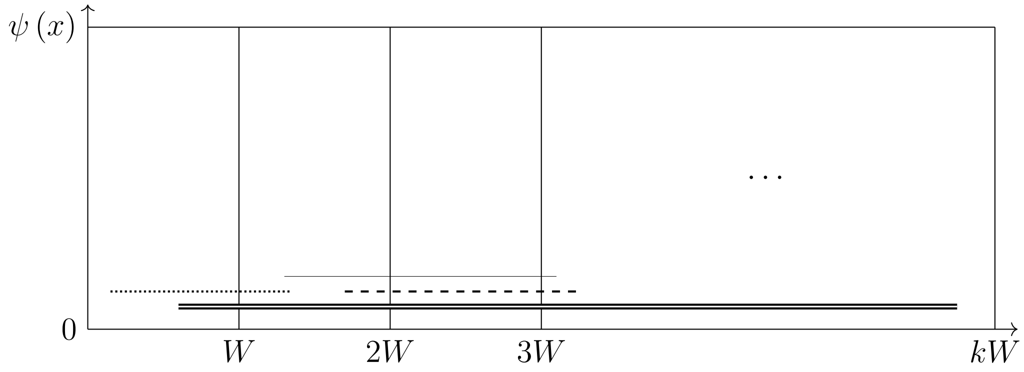


Figure 9: Decomposition of $T[\psi]$

Lemma 35. *Let ψ, $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ be two function such that $k \succ \psi^2 \succ 1$, then*

$$\ln C_k[\psi(1), \psi(2), \ldots, \psi(n)] \sim \sum_{i=1}^{n} \psi(i) \ln\left(\frac{k}{\psi(i)}\right).$$

Proof. Let $A = [a_{ij}] \in \mathcal{T}[\psi(1), \psi(2), \ldots, \psi(n)]$ correspond to the Kostant picture that maximizes the number of colorings into k distinct colors, in other words,

$$C_k[\psi(1), \psi(2), \ldots, \psi(n)] = \prod_{v \in [a_{ij}]} \binom{v+k-1}{k-1}.$$

Note that in every row of A we have entry values in descending order since otherwise we could permute values to get the same value of the product above, while lowering some integral hook sums. Now, let $a = a_{ii}$, $b = a_{i(i+1)}$, $c = a_{(i+1)(i+1)}$. In the sum of integral hook sums $\sum s_i$ those entries are counted as $a + 2b + c$ (see Figure 6). So, if we denote $\text{sum} = a + 2b + c$, then we shall demand maximization of

$$\binom{a+k-1}{k-1} \cdot \binom{c+k-1}{k-1} \cdot \binom{\frac{\text{sum}-a-c}{2}+k-1}{k-1}.$$

The expression above is maximized under $b \sim a^2/k \sim c^2/k \prec 1$, and so we can assume values only on the main diagonal, which leads to

$$\ln C_k[\psi(1), \psi(2), \ldots, \psi(n)] \sim \sum_{i=1}^{n} \ln\binom{\psi(i)+k-1}{k-1} \sim \sum_{i=1}^{n} \psi(i) \ln\left(\frac{k}{\psi(i)}\right). \quad \square$$

Theorem 36. *Let* $\psi : \mathbb{Z}^+ \to \mathbb{Z}^+$ *be a function, satisfying both* $n^2 \succ \psi(n) \succ 1$ *and* $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ \psi(n)$*. Then*

$$L[\psi(1), \psi(2), \ldots, \psi(n)] \sim \sqrt{\frac{2}{3}}\pi \sum_{i=1}^{n} \sqrt{\psi(i)}.$$

Proof. Let W be a function such that $n \succ W(n) \succ \sqrt{\psi(n)}$. We shall now use Theorem 34 with $n = kW$. In it, we can estimate the coloring term by Theorem 33 and the previous Lemma as

$$L^{W^2}[\psi(2W), \psi(3W), \ldots, \psi(n)] \precsim L[\psi(2W), \psi(3W), \ldots, \psi(n)] + \sum_{i=1}^{k-1} \psi(iW) \ln\left(\frac{W^2}{\psi(iW)}\right).$$

The first term can be upper-bounded by $L[\psi(n) \times k]$, for which we already derived asymptotics in the previous section. Now,

$$\sum_{i=1}^{n} \sqrt{\psi(i)} \succ n \succ n \cdot \frac{\sqrt{\psi(n)}}{W} = k\sqrt{\psi(n)},$$

$$\sum_{i=1}^{n} \sqrt{\psi(i)} \succ \psi(n) = k^2 \cdot \frac{\psi(n)}{k^2} \geqslant k^2 \ln\left(\frac{\psi(n)}{k^2}\right).$$

The second term is asymptotically negligible compared to

$$\sum_{i=1}^{k-1} W\sqrt{\psi(iW)} \preccurlyeq \sqrt{\frac{2}{3}}\pi \sum_{i=1}^{n} \sqrt{\psi(i)}.$$

Therefore, only S is left, for which we can write

$$\sqrt{\frac{2}{3}}\pi W \sum_{i=1}^{k-1} \sqrt{\psi(iW)} \precsim S \precsim \sqrt{\frac{2}{3}}\pi W \sum_{i=1}^{k} \sqrt{\psi(iW)}.$$

Since $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ \psi(n)$, we can choose W to satisfy $\sum_{i=1}^{n} \sqrt{\psi(i)} \succ W\sqrt{\psi(n)}$, and so get the proposed asymptotics. □

Corollary 37. $L(\mathbf{1}^n) \sim \sqrt{\frac{8}{27}}\pi n\sqrt{n}$.

Note that the asymptotics presented in Theorem 37 drastically improves the previous one in [9, Theorem 1.2]. We also shall present here some analysis of $C_k[H \times W]$.

Proposition 38. *Let* H*,* W*,* $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ *such that* $k \succ 1$*. Then*

$$\ln C_k[H \times W] \asymp \begin{cases} HW \ln\left(\frac{k}{H}\right) & \text{if } H \prec k \\ W\sqrt{kH} & \text{if } k \preccurlyeq H \preccurlyeq kW^2 \\ kW^2 \ln\left(\frac{H}{kW^2}\right) & \text{if } H \succ kW^2 \end{cases}$$

Proof. Let $U = [u_{ij}] \in \mathcal{T}[H \times W]$, then

$$\ln C_k [H \times W] = \max \sum_{c \in [u_{ij}]} \ln \binom{c+k-1}{k-1} \sim \max \sum_{0 \neq c \in [u_{ij}]} \left[c \ln \left(1 + \frac{k}{c}\right) + k \ln \left(1 + \frac{c}{k}\right) \right].$$

This last sum that we want to maximize we shall denote as $S(U)$.

Take an integral Tesler matrix $A \in \mathcal{T}[H \times W]$ such that $S(A)$ is maximized. Now, recall how we counted the coefficients in Figure 6. Define the m^{th} diagonal as the set of entries with coefficients m. Bring every non-zero entry of A to a vacant place on a lower diagonal if there are any left. After this, replace every entry with the arithmetic average on its diagonal. As the result, we have some other matrix $A' = [a'_{ij}]$, for which $S(A') \geqslant S(A)$. We will later show that in fact $S(A') \asymp S(A)$, but for now we shall find $S(A')$.

If we want $S(A')$ to be maximized, we shall demand that in every row of A' we have entry values in descending order. This means that we can in fact delete the last non-zero diagonal, preserving the asymptotics of $S(A')$ in terms of $\asymp$. This implies $a'_{ij} \geqslant 1$ for non-zero entries, which we shall call the ***realisticity*** condition. In the same way we are asymptotically allowed to delete the entries a'_{ij} with either $i > W - \Delta W$, $j \leqslant \Delta W$, or $j - i \geqslant \Delta W = \left\lfloor \frac{1}{3} \cdot W \right\rfloor$. This leaves us with a trapezoid matrix (Figure 10).

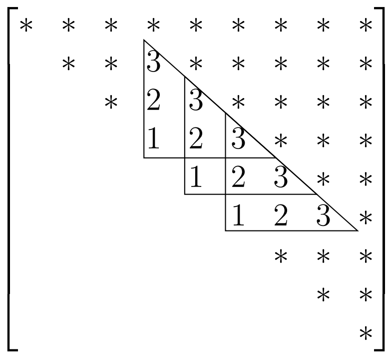


Figure 10: Diagonals and integral hook sums $s_i \preccurlyeq H$ of a trapezoid matrix

Now we shall construct a matrix $B = [b_{ij}]$ which maximizes $S(B)$ and is precisely of the form described above. That is, B is a trapezoid matrix with shared entry value on each diagonal, with the only difference being that we do not require B to be realistic. Since it should upper-bound A', for it we let $s_i \asymp H$, $\Delta W < i \leqslant W - \Delta W$. Let c_m denote the value entries share at the m^{th} diagonal. It is only left to say that B has at most $D \asymp \min\left(W, \sqrt{H}\right)$ non-zero diagonals to produce

$$S(B) \asymp W \sum_{m=1}^{D} c_m \ln \left(1 + \frac{k}{c_m}\right) + kW \sum_{m=1}^{D} \ln \left(1 + \frac{c_m}{k}\right).$$

The right-hand side is maximized iff $\ln\left(1+\frac{k}{c_m}\right) = m\ln\left(1+\frac{k}{c_1}\right)$. Let us denote $a = 1 + \frac{k}{c_1}$, then $c_m = \frac{k}{a^m-1}$, allowing us to rewrite everything as

$$S(B) \asymp HW\ln a + kW\sum_{m=1}^{\infty}\frac{1}{m}\cdot\frac{1-a^{-mD}}{a^m-1}, \sum_{m=1}^{D}\frac{m}{a^m-1} = \frac{H}{k},$$

which is solved by cases.

(i) If $H \prec k$, then

$$\frac{H}{k} = \sum_{m=1}^{D}\frac{m}{a^m-1} \sim \frac{1}{a}.$$

Substituting this, we get $S(B) \asymp HW\ln\left(\frac{k}{H}\right)$. Here the realisticity condition can be easily achieved if we let $c_1 = H$ and $c_m = 0$, $m \geqslant 2$.

(ii) If $H \asymp k$, then by similar analysis $a - 1 \asymp 1$, leading to $S(B) \asymp HW$. The same claim about realisticity holds.

(iii) If $k \prec H \preccurlyeq kW^2$, then we shall utilize the following bounds:

$$\int_1^{D+1}\frac{x\,dx}{a^x-1} \leqslant \sum_{m=1}^{D}\frac{m}{a^m-1} \leqslant \int_0^{D}\frac{x\,dx}{a^x-1}.$$

This leads to $\ln a \asymp \sqrt{\frac{k}{H}}$ and $S(B) \asymp W\sqrt{kH}$. At the same time, those integrals could be taken up to $D' = \sqrt{\frac{H}{k}}$ instead of D without changing the resulting $\ln a$, and so the realisticity condition can be achieved since $c_{D'} \asymp k$.

(iv) If $H \succ kW^2$, then $D \asymp W$ and

$$\frac{H}{k} = \sum_{m=1}^{D}\frac{m}{a^m-1} \sim \frac{D}{\ln a}.$$

This leads to $S(B) \asymp kW^2\ln\left(\frac{H}{kW^2}\right)$. The realisticity holds automatically.

To end the theorem, consider $B' = [b'_{ij}]$ such that $b'_{ij} = \lfloor b_{ij}\rfloor$. Clearly, $S(B') \preccurlyeq S(A) \preccurlyeq S(B)$. □

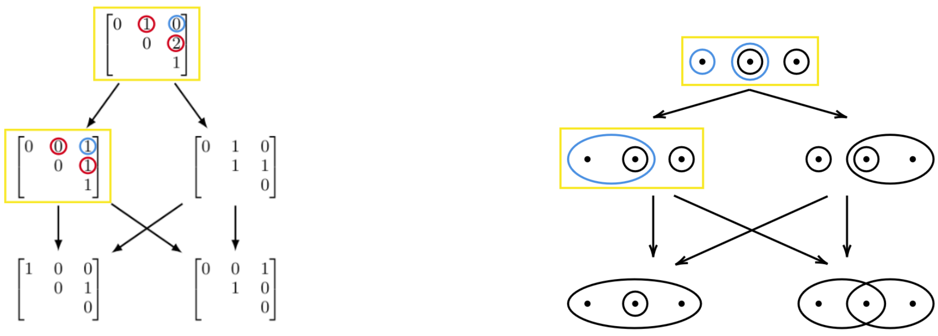
3 Partial order on KPF-sets

3.1 Merge order

Definition 39 ([1])**.** To define a ***poset on Tesler matrices***, first fix a hook sum vector $\mathbf{h}$ and define a covering relation $(b_{ij}) \lessdot (a_{ij})$ in $\mathcal{T}(\mathbf{h})$ iff they have the same entries except

$$a_{ij} = b_{ij}+1,\ a_{jk} = b_{jk}+1,\ a_{ik} = b_{ik}-1 \text{ for a unique triple } i<j<k$$
$$\text{or } a_{ij} = b_{ij}+1,\ a_{jj} = b_{jj}+1,\ a_{ii} = b_{ii}-1 \text{ for a unique pair } i<j.$$

An example can be seen in Figure 11a. Since $\mathcal{I}(\mathbf{h})$ is in bijection with $\mathcal{T}(\mathbf{h})$, we can transport this partial order to the integer flow setting where it becomes the order defined by the covering relation seen in Figure 12. We refer to this partial order (in either setting) as the ***merge*** order.



(a) The $\mathcal{T}(1, 1, -1)$ poset

(b) The $\mathcal{K}(1, 1, -1, -1)$ poset

Figure 11: Merge order on Tesler matrices and Kostant pictures

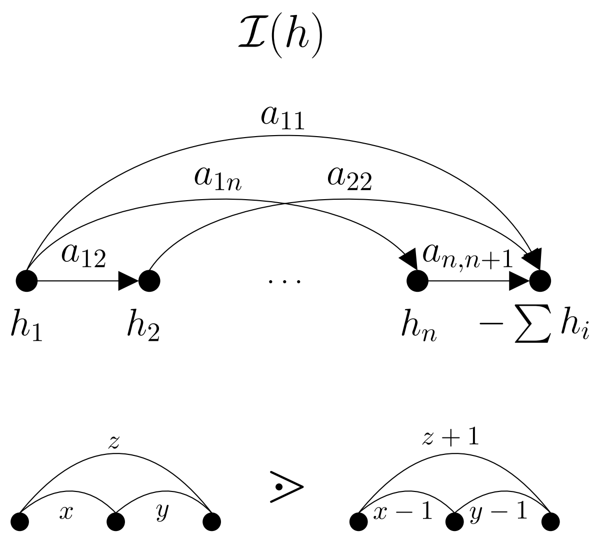


Figure 12: The corresponding integral flow poset

Now we shall define the merge order on Kostant pictures.

Definition 40. We define the ***poset on Kostant pictures*** by letting one picture be covered by another iff it can be acquire from the second one by merging two of its loops into one (Figure 11b).

It is worth noting that these partial orders are ranked. As such, the rank functions are comfortable to work with. For example, the rank functions, Möbius functions, and etc. can all be calculated.

Theorem 41. *The merge order on $\mathcal{I}(\mathbf{h})$ and $\mathcal{K}\left(\mathbf{h}, -\sum h_k\right)$ are isomorphic.*

Proof. By construction there is already a bijection. However, it is not obvious whether the partial order will hold.

Let i, k, and j be any natural numbers such that $1 \leqslant i < k < j \leqslant n$. Then consider the diagram for the integral flow with its partial order on $\mathcal{I}(\mathbf{h})$. Now we construct the corresponding diagrams as Kostant pictures. As we can see, the two extra loops disappear and create a larger one. This is equivalent to the partial order on $\mathcal{K}\left(\mathbf{h}, -\sum h_k\right)$ (Figure 13).

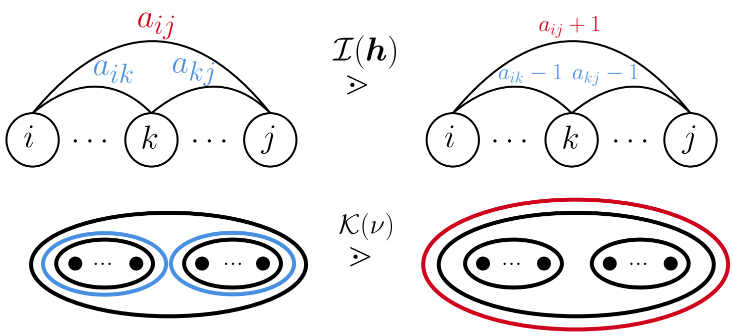


Figure 13: The merge flow covering relation correspondance

Note that it is important to check the other way around (some other posets might not work), however, in this case it's analogous. Also, consider the case $j = n+1$ as the same except in the diagrams a_{ij} is a_{ii} and a_{kj} is a_{kk}. $\square$

3.2 Two-sided dictionary order, how is it different?

In this section we aim to compare the Tesler matrices and the Lusztig data through their posets. We begin by defining the poset on $A(\nu)$.

Definition 42 ([3])**.** The ***two-sided dictionary*** partial order on $A(\nu)$ is defined by $\mathbf{a} \leqslant \mathbf{a}'$ if there can be found two integers $l \leqslant r$ such that $a'_l > a_l$, $a'_r > a_r$, $a'_i = a_i$ for all $i < l$ and $i > r$.

It was conjectured in [7] that Theorem 42 and the partial order from Theorem 40 are equivalent.

Theorem 43. *The poset coming from Lusztig data is a* ***refinement*** *(the cardinalities are equal, however, the edges are preserved only in one direction) of the poset from $\mathcal{K}(\nu)$.*

Proof. Let's consider the merge order on Kostant pictures and read off its Lusztig data. We get that $(\dots, \alpha_{ij}, \dots, \alpha_{ik}, \dots, \alpha_{j+1k}, \dots) \succeq (\dots, \alpha_{ij}-1, \dots, \alpha_{ik}+1, \dots, \alpha_{j+1k}-1, \dots)$ under $\mathcal{K}(\nu)$. Additionally, the partial order holds under $A(\nu)$ (Figure 15).

That means every covering in the Kostant pictures is a covering in the Lusztig data. However, the reverse isn't true. An example can be shown for $\nu = (1, 1, -1, -1)$ (Figure 16). $\square$

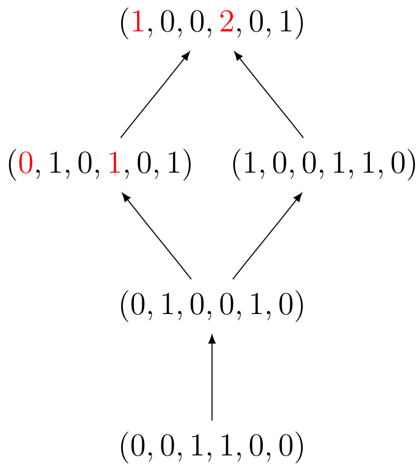


Figure 14: Poset on $A\,(1,\ 1,\ -1,\ -1)$

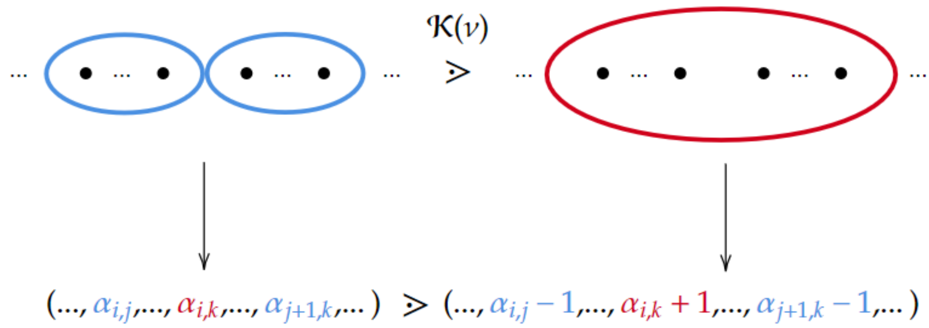


Figure 15: Merge order induced on Lusztig data

Corollary 44. *The partial order on* $A\,(\mathbf{h}, -\sum h_k)$ *refines the partial order on* $\mathcal{T}\,(\mathbf{h})$*.*

Naturally we would like to upgrade the merging order so that it will be equivalent to the two-sided dictionary order. Although an intuitive new order hasn't been made, there were some ideas below.

- The ***excess-merge*** order works on cases similar to Figure 16 by allowing any two intersecting loops cover two replaced loops—one is the intersection and the other is the union. However, there exists a counterexample, for example $\mathbf{h} = (1, 1, -1, -1)$ (Figure 17).

- Taking into account Figure 17, the ***excess-merge-shift*** order also allows three (or more) adjacent loops shifting to form two (or more) adjacent loops. In this case, there also exists a counterexample, $\mathbf{h} = (2, 1, 2, 1)$.

Additionally, it should be noted that, unlike the merge orders, the two-sided dictionary order is global. In other words, each covering relation depends on other values (in this case—the outermost elements), which are not changed during the covering.

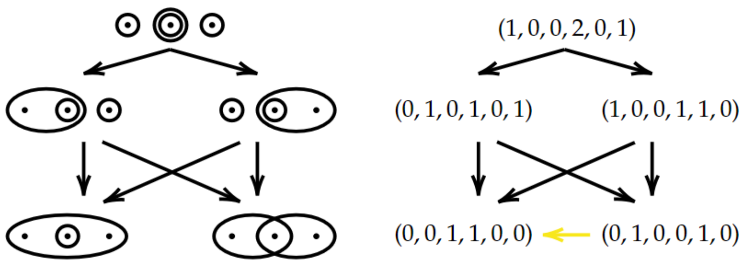


Figure 16: Hasse diagrams for $v = (1, 1, -1, -1)$

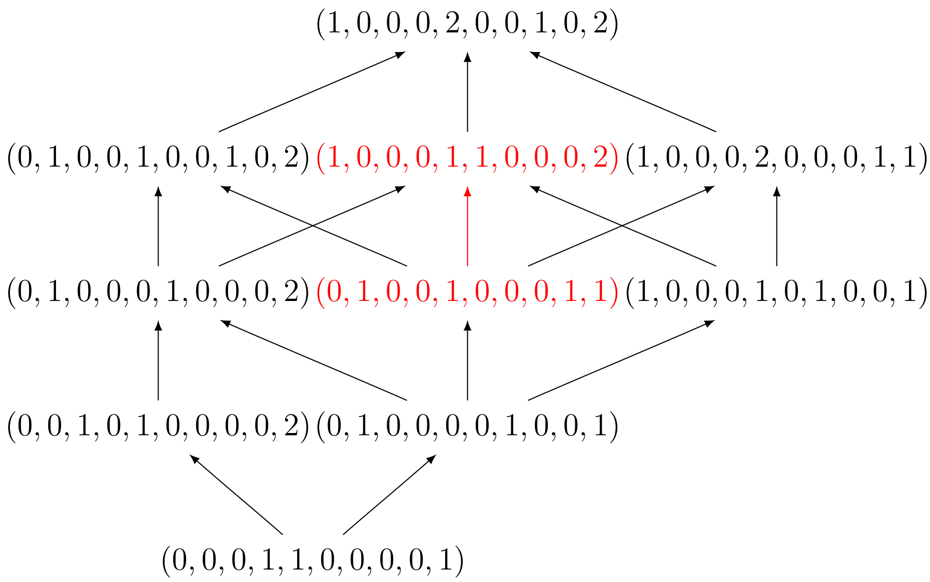


Figure 17: Excess-merge order counterexample with covering in red

In the representation theory of canonical bases (which are in particular crystals and therefore indexed by Lusztig data) the two-sided dictionary order is used to show that Lusztig's canonical basis is in fact a basis. This is done by showing that the change of basis from a standard (PBW) basis $\{F^a\}$ to Lusztig's canonical basis $\{\bar{F}^a\}$ is unitriangular with respect to the two-sided dictionary order (Theorem 45).

Theorem 45 ([3])**.** *For every Lusztig data a,*

$$\bar{F}^a = F^a + \sum_{a' \prec a} p^a_{a'}(q) F^{a'} \tag{1}$$

where the $p^a_{a'}(q)$ are Laurent polynomials in q.

The orders we consider lead us to ask if the change of basis holds true for any coarsening of the two-sided dictionary order.

4 Probabilistic Tesler Matrices

Consider a system of n states and probabilities t_{ij} (going from state i to j) connecting them. The sum of all probabilities going from a state is equal to 1. A step consists of going from one state to another randomly. Such a system is known as a ***Markov chain***.

Definition 46. The ***Markov property*** states that the probability of the next state after step $s+1$ depends only on its state after step s.

We recall that a Markov chain can be represented by a labeled graph as well as by a matrix (Figure 18) (the adjacency matrix of the graph).

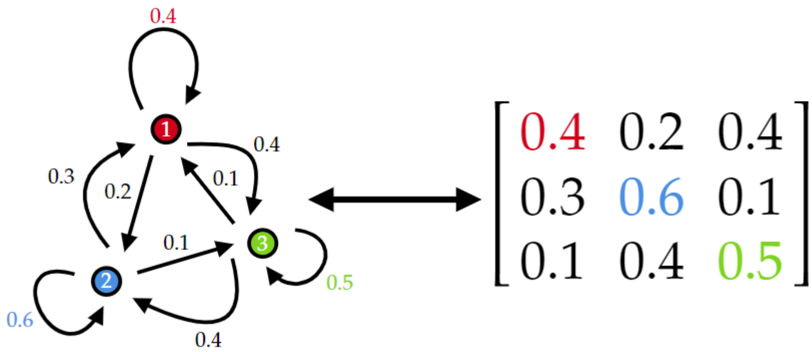


Figure 18: Graph and matrix representations of a Markov chain

Definition 47. The matrix T representing a Markov chain is known as the ***transition matrix***. Its entries t_{ij} are the probabilities connecting the states.

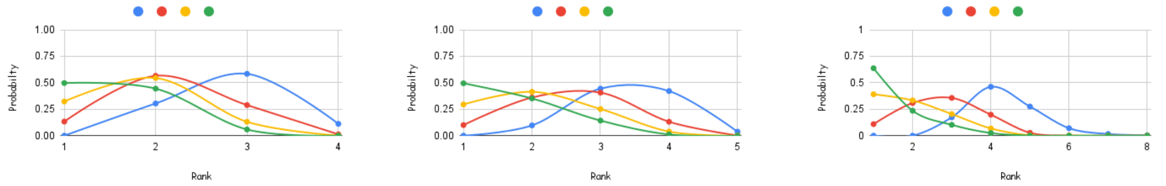
Definition 48. Our ***state vector***, V_s, is the distribution between the states after s steps. When $s = 0$, the state vector is ***initial***. Each state vector follows the condition

$$V_s = V_0 \cdot T^s.$$

4.1 Algorithm

To talk about the distribution of the elements in a poset, we must establish the probability of reaching a given element using our algorithm for traversal. We initiate our traverse at the maximal element, because it is unique, and we proceed to move down the poset (along covering relations) if possible.

The algorithm (see Appendix) takes as **inputs** the size of the matrix s, the number of random steps r, the number of overall trials t, and the hook sum $\mathbf{h}$. Each step replaces a matrix by one of the matrices it covers or itself (uniformly at random). The **output** consists of the ***probability distribution over all ranks*** (where a rank is essentially an indicator of how high an element is in the poset), the ***average ranks***, the ***rank ratios***



2, 4, 6, 8 random steps   3, 6, 9, 12 random steps   5, 10, 15, 20 random steps

Figure 19: Rank distributions of posets $\mathcal{T}(1,\,1,\,1)$, $\mathcal{T}(1,\,2,\,1)$, $\mathcal{T}(3,\,1,\,2)$, left to right

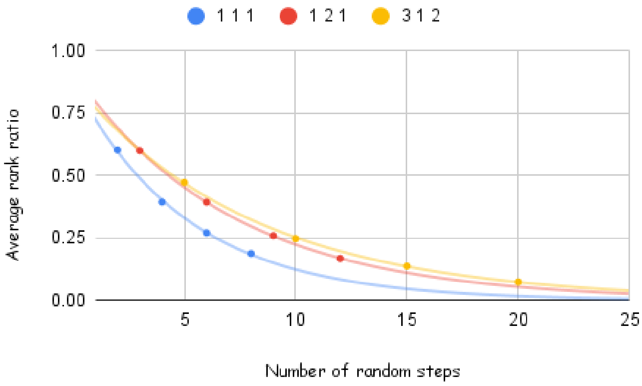


Figure 20: Rank ratio graphs with their logarithmic trends

$\left(\frac{\text{rank}-1}{\text{range}}\right)$, and the ***average step efficiency*** (a step that goes down is efficient and we count until a minimum element is reached) after s steps.

Each step proceeds as follows. Choose a random element in the upper-right part of the current matrix. If it is on the main diagonal, then another element is chosen on the same diagonal and a downward covering is checked. Otherwise, we record the coordinates of the initial element $(a_x,\,a_y)$ and then choose a number from the range $1 \leqslant \cdots < a_y < \cdots < a_x < \ldots \leqslant s$. Those 3 numbers are the same as i, j, k in definition Theorem 39. From here we can decide whether a downward covering relation is available, and change or stay.

As an example, consider the Tesler posets $\mathcal{T}(1,\,1,\,1)$, $\mathcal{T}(1,\,2,\,1)$, $\mathcal{T}(3,\,1,\,2)$. Overall, the rank distribution should go to 1 when r, the number of random steps, goes to infinity. Figure 19 shows us how this would look like when r is increased linearly. Since the distribution gets shifted to 1, when $r \to \infty$, the average step efficiency reaches a certain limit. For the earlier used posets, we get 0.404788, 0.359631, and 0.428762, respectively. Lastly, different posets can be compared by looking at their rank ratio graphs. Each of these graphs start at 1, when $r = 0$, and a larger poset corresponds to a "higher" graph. This can all be seen on Figure 20. We note that each of these graphs grow demonstrate logarithmic growth.

5 Future questions

Finally, we shall formulate some questions that we believe deserve further exploration.

Clearly present one is the problem of asymptotical evaluation of $L[H \times W]$ (also $L\{HW, W\}$ and $L_r(H \times W)$) for $W \asymp \sqrt{H}$. Since there is no simple formula for the number of partitions in this case, however, it is unlikely that there is a simple one for $L[H \times W]$. The other question is whether we can somehow generalize Theorem 36, for example, if $\psi(n) \succ n^2$?

Another natural question pertains to the two-sided dictionary order. Does it refine the merge order on Kostant pictures in a conceptually intuitive way? Dually, we can ask if the two-sided dictionary order can be coarsened (to an order which naturally refines the merge order on Kostant pictures) without affecting the change of basis formula in Equation (1); that is, *is the unitriangular change of basis formula of* Theorem 45 redundant in the sense that $p^a_{a'}(q) = 0$ for some $a' \lessdot a$?

Looking at Markov chains in the Tesler poset, an important question is how can we easily predetermine data such as the average step efficiency limit and the rank ratio graph? In other words, given the Tesler poset for a fixed $\mathbf{h}$, is there a simple way to figure out these values?

In particular, can we easily compute a good approximation of the Möbius function (without generating the entire poset). Experimentally, we observe that some edges, when removed, barely influence the value of μ (these would be more "isolated", Figure 21). How do we generate only the Möbius-important part of a poset systematically, leaving the "isolated" edges out of sight?

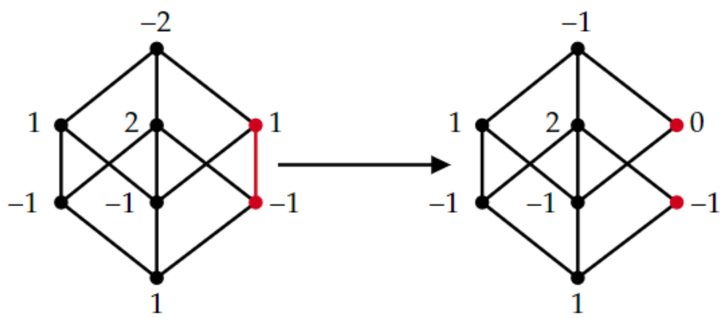


Figure 21: $\max_1 \left|\mu\left(\hat{0},\, A\right)\right| = \max_2 \left|\mu\left(\hat{0},\, A\right)\right| = 2$

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Appendix: Pseudocode

```
s ← size of the matrix;
r ← number of random steps;
t ← number of trials;            /* t is large in order to have accurate data */
h_i ← corresponding hook value input;
Calculate hook_sum and max_sum (sum of the elements in the maximal matrix);
Initialize rank_array[i] that holds the number of times the final element was at each
  rank;
for t trials do
    for r random steps do
        current_sum ← the sum of all the elements in the matrix;
        if current_sum = hook_sum then
            The current trial reached the bottom of the poset;
        end
        a_x ← x coordinate of randomly chosen element;
        a_y ← y coordinate of randomly chosen element;                 /* a_x ⩾ a_y */
        if a_x = a_y then
            b_x ← x coordinate of a different randomly chosen element on the diagonal;
            b_y ← y coordinate of a different randomly chosen element on the diagonal;
            if covering does not work then
                if we have not reached the end then
                    unsuccessful_attempts++;
                end
                Go back and do the next random step;
            end
            Do the covering and do the next random step;
        end
        Randomly choose a number from 1 ⩽ ··· < a_y < ··· < a_x < ... ⩽ s;
        Rename coordinates a_x, a_y and the number as i, j, k;
        if covering does not work then
            if we have not reached the end then
                unsuccessful_attempts++;
            end
            Go back and do the next random step;
        end
        Do the covering and do the next random step;
    end
    final_sum ← the sum of all the elements in the matrix after r randoms steps;
    useful_steps = max_sum − final_sum;
    step_efficiency = useful_steps / (useful_steps+unsuccessful_attempts);
    current_rank = final_sum − hook_sum + 1;
    rank_array[current_rank]++;
end
Calculate the average step efficiency over t trials and average_rank_value;
```