A lower bound theorem for d-polytopes with $2d + 2$ vertices

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Abstract

We establish a lower bound theorem for the number of k-faces ($1 \leqslant k \leqslant d-2$) in a d-dimensional polytope P (abbreviated as a d-polytope) with $2d+2$ vertices, extending the previously known case for $k=1$. We identify all minimisers for $d \leqslant 5$. Two distinct lower bounds emerge, depending on the number of facets of P. When P has precisely $d+2$ facets, the lower bound is tight when d is odd. If P has at least $d+3$ facets, the lower bound is always tight, and equality holds for some $1 \leqslant k \leqslant d-2$ only when P has precisely $d+3$ facets.

Moreover, for $1 \leqslant k \leqslant \lceil d/3 \rceil - 2$, the minimisers among d-polytopes with $2d+2$ vertices have precisely $d+3$ facets, while for $\lfloor 0.4d \rfloor \leqslant k \leqslant d-1$, the lower bound arises from d-polytopes with $d+2$ facets.

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1 Introduction

Two (convex) polytopes are combinatorially equivalent if their face lattices are isomorphic. We do not distinguish between combinatorially equivalent polytopes.

Grünbaum [5, Sec. 10.2] conjectured that the function

$$\theta_k(d+s,d) := \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}, \text{ for } 1 \leqslant s \leqslant d \tag{1}$$

gives the minimum number of k-faces of a d-polytope with $d+s$ vertices and proved the conjecture for $s = 2, 3, 4$. Xue [13] then completed the proof, including a characterisation of the unique minimisers for $k \in [1 \ldots d-2]$. Earlier, Pineda-Villavicencio et. al [9] had settled the conjecture for some values of k.

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When $k \in [1 \dots d-2]$ (the set of integers between 1 and $d-2$), each minimiser of (1) is combinatorially equivalent to a $(d-s)$-fold pyramid over a simplicial s-prism for $s \in [1 \dots d]$; we call such a polytope an *(s,d − s)-triplex* and denote it by $M(s, d-s)$.

Let P be a d-polytope in $\mathbb{R}^d$, let F be a face of P, and let K be a closed halfspace in $\mathbb{R}^d$ such that the vertices of P not in K are the vertices of F. A polytope P' is obtained by *truncating the face* F of P if $P' = P \cap K$.

For the parameters $d \geqslant 2$ and $s \in [2 \dots d]$, if we truncate a *simple* vertex (a vertex contained in exactly d facets) from the $(s, d-s)$-triplex, we obtain a d-polytope $J(s,d)$ with $2d+s-1$ vertices, $d+3$ facets, and whose number of k-faces with $k \in [1 \dots d-1]$ is given by the function

$$\eta_k(2d+s-1, d) := \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d+1-s}{k+1}. \tag{2}$$

The paper [12] refers to $J(2,d)$ as a *d-pentasm* and the polytope $J(3,d)$ as $B(d)$.

The *Cartesian product* of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ is the Cartesian product of the sets P and P': $P \times P' = \{(p,p')^t \in \mathbb{R}^{d+d'} \mid p \in P, p' \in P'\}$. We denote by $T(d)$ the d-simplex and by f_k the number of k-faces in a polytope.

Lemma 1 (McMullen 1971, [6, Sec. 3])**.** Let P be a d-dimensional polytope with $d+2$ facets, where $d \geqslant 2$. Then, there exist integers $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$ such that P is combinatorially equivalent to a $(d-a)$-fold pyramid over $T(m) \times T(a-m)$. The number of k-faces of P is

$$\rho_k(a,m,d) := \binom{d+2}{k+2} - \binom{d-a+m+1}{k+2} - \binom{d-m+1}{k+2} + \binom{d-a+1}{k+2}. \tag{3}$$

In particular, $f_0(P) = d+1+m(a-m)$.

Pineda-Villavicencio [7, Prob. 8.7.11] conjectured $J(s,d)$ and some d-polytopes with $d+2$ facets minimise the number of faces in polytopes with up to $3d-1$ vertices. Our results provide further evidence in support of this conjecture.

Conjecture 2 ([7, Prob. 8.7.11])**.** Let $d \geqslant 3$ and $s \in [2 \dots d]$. Let P be a d-polytope with $2d+s-1$ vertices.

(i) If P has at least $d+3$ facets, then $f_k(P) \geqslant f_k(J(s,d))$ for each $k \in [1 \dots d-2]$.

(ii) If P has $d+2$ facets, then it is combinatorially equivalent to a $(d-a)$-fold pyramid over $T(m) \times T(a-m)$ for some $a \in [2 \dots d]$ and $2 \leqslant m \leqslant \lfloor a/2 \rfloor$.

The case $s = 2$ in Conjecture 2 was settled independently by Pineda-Villavicencio and Yost [12], and by Xue [14]. Here we verify the case of $s = 3$. Let

$$\tau_k(2d+2, d) := \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{\lceil (d+1)/2 \rceil - 1}{k+1} - \binom{\lceil (d+1)/2 \rceil - 2}{k+1}.$$

See also Lemma 7. Additionally, denote by $A(d)$ a polytope obtained by truncating a nonsimple vertex of a $(2, d-2)$-triplex. Our main theorem reads as follows.

Theorem 3. *Let $d \geqslant 3$, let P be a d-polytope with $2d+2$ vertices, and let $k \in [1 \ldots d-2]$.*

(i) *If P has at least $d+3$ facets, then $f_k(P) \geqslant f_k(J(3,d)) = f_k(A(d)) = \eta_k(2d+2, d)$. Moreover, equality holds for some k only if P has exactly $d+3$ facets.*

(ii) *If P has $d+2$ facets, then $f_k(P) \geqslant \tau_k(2d+2, d)$. Furthermore, the equality is tight for every $k \in [1 \ldots d-2]$ when d is odd, $a = \lfloor (d+3)/2 \rfloor + 1$, and P is combinatorially equivalent to a $(d-a)$-fold pyramid over $T(2) \times T(a-2)$.*

We remark that, when $d \geqslant 9$, another dichotomy manifests:

(i) If $1 \leqslant k \leqslant \lceil d/3 \rceil - 2$, then $\eta_k(2d+2, d) < \tau_k(2d+2, d)$ (Lemma 23(i)), so $\eta_k(2d+2, d)$ is the lower bound for the number of k-faces in d-polytopes with $2d+2$ vertices.

(ii) If $\lfloor 0.4d \rfloor \leqslant k \leqslant d-1$, then $\eta_k(2d+2, d) > \tau_k(2d+2, d)$ (Lemma 23(ii)), making $\tau_k(2d+2, d)$ the lower bound for k-faces in d-polytopes with $2d+2$ vertices.

It is worth noting that a d-polytope with $2d+2$ vertices and $d+2$ facets exists (if and) only if $d+1$ is composite and $d \neq 3$. This is an easy consequence of the last sentence of Lemma 1.

The proof of Part (i) of Theorem 3 (restated as Theorem 18) is lengthy, involving case analysis, induction, properties of *super-Kirkman* polytopes—where every pair of facets intersects in a ridge—and a key counting result of Xue [13, Prop. 3.1], restated in Corollary 12, which estimates the number of faces containing at least one vertex from a set of at most d vertices.

2 Polytopes with $d+2$ vertices or facets

By duality, the following lemma for d-polytopes with $d+2$ vertices is equivalent to Lemma 1. Historically, Lemma 4 was proved first, and Lemma 1 deduced from it later. The *direct sum* $P \oplus P'$ of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ with the origin in their relative interiors is the $(d+d')$-polytope:

$$P \oplus P' = \operatorname{conv}\left(\left\{ \begin{pmatrix} p \\ 0_{d'} \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, p \in P \right\} \bigcup \left\{ \begin{pmatrix} 0_d \\ p' \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, p' \in P' \right\} \right).$$

Here, 0_d denotes the zero vector in $\mathbb{R}^d$.

Lemma 4. [5, Sec. 6.1] Let P be a d-dimensional polytope with $d+2$ vertices, where $d \geqslant 2$. Then, there exist integers $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$ such that P is combinatorially equivalent to a $(d-a)$-fold pyramid over $T(m) \oplus T(a-m)$. The number of k-dimensional faces of P is

$$\binom{d+2}{d-k+1} - \binom{d-a+m+1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.$$

In particular, $f_{d-1}(P) = d+1+m(a-m)$.

As in Grünbaum [5, Thm. 6.1.4], we denote the $(d-a)$-fold pyramid over $T(m) \oplus T(a-m)$ by $T_m^{d,d-a}$. Grünbaum [5, p. 101] established inequalities for the number of k-faces in d-polytopes with $d+2$ vertices.

Lemma 5. For $k \in [0 \ldots d-1]$, the following statements hold:

(i) If $2 \leqslant a \leqslant d$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor - 1$, then $f_k(T_m^{d,d-a}) \leqslant f_k(T_{m+1}^{d,d-a})$, with strict inequality if and only if $m \leqslant k$.

(ii) If $2 \leqslant a \leqslant d-1$ and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$, then $f_k(T_m^{d,d-a}) \leqslant f_k(T_m^{d,d-(a+1)})$, with strict inequality if and only if $a-m \leqslant k$.

The dual polytope $(T_m^{d,d-a})^*$ of $T_m^{d,d-a}$ is combinatorially equivalent to a $(d-a)$-fold pyramid over $T(m) \times T(a-m)$. We list the $d+2$ facets of $(T_m^{d,d-a})^*$ below.

Remark 6. For $d \geqslant 2$, $2 \leqslant a \leqslant d$, and $1 \leqslant m \leqslant \lfloor a/2 \rfloor$, the $d+2$ facets of the $(d-a)$-fold pyramid P over $T(m) \times T(a-m)$ are as follows:

(i) $m+1$ facets that are $(d-a)$-fold pyramids over $T(m-1) \times T(a-m)$, whose number of vertices is $f_0(P) - (a-m+1)$;

(ii) $a-m+1$ facets that are $(d-a)$-fold pyramids over $T(m) \times T(a-m-1)$, whose number of vertices is $f_0(P) - (m+1)$;

(iii) $d-a$ facets that are $(d-a-1)$-fold pyramids over $T(m) \times T(a-m)$, whose number of vertices is $f_0(P) - 1$.

The next lemma, whose proof is embedded in the proof of [14, Thm. 4.1], settles Theorem 3(ii). We write a proof for the sake of completeness.

Lemma 7. For $d \geqslant 4$, $s \in [2 \ldots d-2]$, $a = \lfloor (d+s)/2 \rfloor + 1$, and $1 \leqslant k \leqslant d-2$, the number

$$\begin{aligned}\tau_k(2d+s-1,d) &:= \binom{d+2}{k+2} - \binom{d-a+3}{k+2} - \binom{d-1}{k+2} + \binom{d-a+1}{k+2} \\ &= \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-a+2}{k+1} - \binom{d-a+1}{k+1}\end{aligned}$$

of k-faces of the d-polytope $(T_2^{d,d-a})^*$ is a lower bound on the number of k-faces of d-polytopes with exactly $d+2$ facets and **at least** $2d+s-1$ vertices.

Proof. We reason as in [14, Sec. 4]. Let $P := (T_2^{d,d-a})^*$. Then $f_0(P) = d+2\lfloor (d+s)/2 \rfloor -1$. Now consider a d-polytope Q, distinct from P, with $d+2$ facets and at least $2d+s-1$ vertices. By Lemma 1, Q must have the form $Q = (T_m^{d,d-b})^*$, where $2 \leqslant b \leqslant d$ and $1 \leqslant m \leqslant \lfloor b/2 \rfloor$. Since $f_0(Q) = d+1+m(b-m) \geqslant 2d+s-1$ and $b \leqslant d$, we must have $m \geqslant 2$.

Assume that $f_0(Q) \geqslant f_0(P)$. We want to prove that $f_k(P) \leqslant f_k(Q)$ for $1 \leqslant k \leqslant d-2$. We work in the dual setting, proving instead that $f_k(P^*) \leqslant f_k(Q^*)$ for $1 \leqslant k \leqslant d-2$. From $f_{d-1}(Q^*) \geqslant f_{d-1}(P^*)$, we get

$$m(b-m) \geqslant 2(a-2). \tag{4}$$

Suppose that $m = 2$. Then $b \geqslant a$ and Lemma 5(ii) yields that $f_k(P^*) \leqslant f_k(Q^*)$ for $1 \leqslant k \leqslant d-2$, as desired. Consequently, we assume that $m \geqslant 3$. Hence $b \geqslant 6$.

If $b \geqslant a$ then Lemma 5(ii) again yields that $f_k(T_m^{d,d-a}) \leqslant f_k(Q^*)$ for $1 \leqslant k \leqslant d-2$. Additionally, $f_k(T_m^{d,d-a}) \geqslant f_k(P^*)$ for $1 \leqslant k \leqslant d-2$ by Lemma 5(i). Hence, we are home in this case. Consequently, we may assume that $a > b$.

$$\begin{aligned} f_k(Q^*) - f_k(P^*) = & \underbrace{\left[-\binom{d-b+m+1}{d-k+1} + \binom{d-a+3}{d-k+1} \right]}_{=A} \\ & + \underbrace{\left[-\binom{d-m+1}{d-k+1} + \binom{d-1}{d-k+1} \right]}_{=B} \\ & + \underbrace{\left[\binom{d-b+1}{d-k+1} - \binom{d-a+1}{d-k+1} \right]}_{=C}. \end{aligned} \tag{5}$$

Using Lemma 22(iii), we write A, B, C as follows:

$$A = -\sum_{\ell=1}^{(a-b)+m-2} \binom{d-b+m+1-\ell}{d-k}, \; B = \sum_{\ell=1}^{m-2} \binom{d-1-\ell}{d-k},$$
$$C = \sum_{\ell=1}^{a-b} \binom{d-b+1-\ell}{d-k}.$$

From this we get expressions for $A+C$ and $A+B+C$.

$$\begin{aligned}
A+C &= -\sum_{\ell=1}^{(a-b)+m-2} \binom{d-b+m+1-\ell}{d-k} + \sum_{\ell=1}^{a-b} \binom{d-b+1-\ell}{d-k} \\
&= -\sum_{\ell=1}^{m} \binom{d-b+m+1-\ell}{d-k} + \sum_{\ell=1}^{2} \binom{d-a+\ell}{d-k} \\
&\quad + \underbrace{\left[-\sum_{\ell=m+1}^{a-b+m-2} \binom{d-b+m+1-\ell}{d-k} + \sum_{\ell=1}^{a-b-2} \binom{d-b+1-\ell}{d-k} \right]}_{=0} \\
&= -\sum_{\ell=1}^{m} \binom{d-b+\ell}{d-k} + \sum_{\ell=1}^{2} \binom{d-a+\ell}{d-k} \\
&= -\sum_{\ell=1}^{m-2} \binom{d-b+\ell+2}{d-k} - \sum_{\ell=1}^{2} \left[\binom{d-b+\ell}{d-k} - \binom{d-a+\ell}{d-k} \right].
\end{aligned}$$

$$\begin{aligned}
A+C+B &= -\sum_{\ell=1}^{m-2} \binom{d-b+\ell+2}{d-k} - \sum_{\ell=1}^{2} \left[\binom{d-b+\ell}{d-k} - \binom{d-a+\ell}{d-k} \right] \\
&\quad + \sum_{\ell=1}^{m-2} \binom{d-1-\ell}{d-k} \\
&= \sum_{\ell=1}^{m-2} \left[\binom{d-1-\ell}{d-k} - \binom{d-b+\ell+2}{d-k} \right] - \sum_{\ell=1}^{2} \left[\binom{d-b+\ell}{d-k} - \binom{d-a+\ell}{d-k} \right].
\end{aligned}$$

Using Lemma 22(iii), we further get

$$A+C+B = \underbrace{\sum_{\ell=1}^{m-2} \left[\sum_{h=1}^{b-3-2\ell} \binom{d-1-\ell-h}{d-k-1} \right]}_{=D} - \underbrace{\sum_{\ell=1}^{2} \left[\sum_{h=1}^{a-b} \binom{d-b+\ell-h}{d-k-1} \right]}_{=E}.$$

The coefficient $\binom{d-b+3}{d-k-1}$ is the smallest among those in D. Thus,

$$D \geqslant \sum_{\ell=1}^{m-2} (b-3-2\ell) \binom{d-b+3}{d-k-1}. \tag{6}$$

The coefficient $\binom{d-b+1}{d-k-1}$ is the largest among those in E. Thus,

$$E \leqslant 2(a-b) \binom{d-b+1}{d-k-1}. \tag{7}$$

Combining (6) and (7), we get that

$$\begin{aligned} A+B+C &\geqslant \sum_{\ell=1}^{m-2}(b-3-2\ell)\binom{d-b+3}{d-k-1}-2(a-b)\binom{d-b+1}{d-k-1} \\ &= (m-2)(b-2-m)\binom{d-b+3}{d-k-1}-2(a-b)\binom{d-b+1}{d-k-1}. \end{aligned}$$

Furthermore, (4) is equivalent to $(m-2)(b-2-m)\geqslant 2(a-b)$, which implies that

$$A+B+C \geqslant 2(a-b)\left[\binom{d-b+3}{d-k-1}-\binom{d-b+1}{d-k-1}\right]\geqslant 0, \text{ (since } a>b). \tag{8}$$

The lemma follows. □

The function $\tau_k(2d+s-1,d)$ in Lemma 7 is obtained by setting $a=\lfloor (d+s)/2\rfloor+1$ and $m=2$ in the function $\rho_k(a,m,d)$ from Lemma 1.

3 Known lower bounds for polytopes with few vertices and auxiliary results

We define a vertex figure of a polytope P at a vertex v as the polytope $P/v := H\cap P$ where H is a hyperplane separating v from the other vertices of P. There is a bijection between the k-faces of P that contain v and the $(k-1)$-faces of P/v. We require a result by Xue [13, Prop. 3.1]; see also [12, Prop. 12].

Proposition 8 (Xue 2021)**.** Let $d\geqslant 2$ and let P be a d-polytope. In addition, suppose that $r\leqslant d+1$ is given and that $S:=(v_1,v_2,\ldots,v_r)$ is a sequence of distinct vertices in P. Then the following statements hold:

(i) There is a sequence $F_1,F_2,\ldots,F_r$ of faces of P such that each F_i has dimension $d-i+1$ and contains v_i, but does not contain any v_j with $j<i$.

(ii) For each $k\geqslant 1$, the number of k-faces of P that contain at least one of the vertices in S is bounded from below by

$$\sum_{i=1}^{r} f_{k-1}(F_i/v_i).$$

Xue's lower bound theorem for d-polytopes with at most $2d$ vertices (see [13]) reads as follows:

Theorem 9 (***d***-polytopes with at most **2*d*** vertices, [13])**.** *Let $d\geqslant 2$ and $1\leqslant s\leqslant d$. If P is a d-polytope with $d+s$ vertices, then*

$$f_k(P)\geqslant \theta_k(d+s,d), \text{ for all } k\in[1\ldots d-1].$$

Also, if $f_k(P)=\theta_k(d+s,d)$ for some $k\in[1\ldots d-2]$, then P is combinatorially equivalent to the $(s,d-s)$-triplex.

We proceed with a lower bound theorem for d-polytopes with $2d+1$ vertices, independently proved by Pineda-Villavicencio and Yost [12], and by Xue [14]. Denote by $f(P)$ the f-vector of a polytope P, and denote by $\Sigma(d)$ a d-polytope combinatorially equivalent to the convex hull of

$$\{0, e_1, e_1+e_k, e_2, e_2+e_k, e_1+e_2, e_1+e_2+2e_k : 3 \leqslant k \leqslant d\},$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^d$.

Theorem 10 (Pineda-Villavicencio and Yost 2022; Xue 2023)**.** *Let $d \geqslant 3$, P a d-polytope with* ***at least*** *$2d+1$ vertices, and $k \in [1 \dots d-2]$.*

(i) *Let $d=3$. If P is $\Sigma(3)$ or a 3-pentasm, then $f(P)=(7,11,6)$. Otherwise, $f_1(P) > 11$ and $f_2(P) > 6$.*

(ii) *If $d \geqslant 4$ and P has at least $d+3$ facets, then $f_k(P) \geqslant f_k(J(2,d))$, with equality for some $k \in [1 \dots d-2]$ only if $P = J(2,d)$.*

(iii) *If $d \geqslant 4$ and P has $d+2$ facets, then $f_k(P) \geqslant f_k((T_2^{d,d-(\lfloor d/2 \rfloor+2)})^*)$. If d is even then $(T_2^{d,d-(\lfloor d/2 \rfloor+2)})^*$ has $2d+1$ vertices.*

A corollary of Theorems 9 and 10 extends to d-polytopes with at least $d+s$ vertices.

Corollary 11. For each $d \geqslant 2$, each $k \in [1 \dots d-1]$, and each $s \in [1 \dots d]$, if P is a d-polytope with **at least** $d+s$ vertices, then $f_k(P) \geqslant \theta_k(d+s,d)$.

Moreover, if $f_k(P) = \theta_k(d+s,d)$ for some $k \in [1 \dots d-2]$, then P has $d+2$ facets.

Proof. The case $d=2$ is trivial. Let $d \geqslant 3$. If P has at most $2d$ vertices, the claim follows from Theorem 9. Hence assume $f_0(P) \geqslant 2d+1$. Since $\theta_k(d+s,d)$ is an increasing function in s, it suffices to show that $f_k(P) \geqslant \theta_k(2d,d) = \binom{d+1}{k+1} + \binom{d}{k+1}$ for each $k \in [1 \dots d-2]$. If P has at least $d+3$ facets, then by Theorem 10(ii) we have

$$f_k(P) \geqslant \eta_k(2d+1,d) = \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k} > \theta_k(2d,d).$$

If P has exactly $d+2$ facets, then, by Theorem 10(iii) (see also Lemma 7),

$$\begin{aligned} f_k(P) \geqslant f_k((T_2^{d,d-(\lfloor d/2 \rfloor+2)})^*) &= \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{\lceil d/2 \rceil}{k+1} \\ &\quad - \binom{\lceil d/2 \rceil - 1}{k+1} \\ &> \binom{d+1}{k+1} + \binom{d}{k+1} = \theta_k(2d,d). \end{aligned}$$

If d is even, note that $d-1 = \lceil d/2 \rceil + \lceil d/2 \rceil - 1$, so the strict inequality is rather straightforward. If d is odd, the polytope $(T_2^{d,d-(\lfloor d/2 \rfloor+2)})^*$ has exactly $2d$ vertices, and the strict inequality follows from Theorem 9.

Additionally, if $f_k(P) = \theta_k(d+s,d)$, then by Theorem 9, P must be combinatorially equivalent to a triplex with at least $d+s$ vertices, which has $d+2$ facets. This completes the proof. □

It is convenient to write a corollary of Proposition 8 and Corollary 11.

Corollary 12. Let $d \geqslant 2$ and let P be a d-polytope. In addition, suppose that $r \leqslant d+1$ is given and that $S := (v_1, v_2, \ldots, v_r)$ is a sequence of distinct vertices in P. Then the following hold:

(i) There is a sequence $F_1, F_2, \ldots, F_r$ of faces of P such that each F_i has dimension $d-i+1$ and contains v_i, but does not contain any v_j with $j < i$.

(ii) For each $1 \leqslant i \leqslant r$, let s_i satisfy $1 \leqslant s_i \leqslant d-i+1$, and suppose that $\deg_{F_i}(v_i) \geqslant d-i+1+s_i$. Then, for each $k \geqslant 1$, the number of k-faces of P that contain at least one of the vertices in S is bounded below by

$$\sum_{i=1}^{r} f_{k-1}(F_i/v_i) \geqslant \sum_{i=1}^{r} \theta_{k-1}(d-i+1+s_i, d-i).$$

4 Polytopes with $2d+2$ vertices: Small cases and pyramids over simple polytopes

Denote by $C(d)$ a polytope obtained by truncating a *simple edge*, an edge with two simple vertices, of a $(2, d-2)$-triplex. For $d \geqslant 3$, it has $3d-2$ vertices and $d+3$ facets. Obviously $C(2)$ is just another quadrilateral.

Theorem 13 (Small cases)**.** *For $d = 3, 4, 5$, the d-polytopes with exactly $2d+2$ vertices that minimise the number of k-faces for each $k \in [1 \ldots d-1]$ are as follows.*

(i) *For $d = 3$, the minimisers are combinatorially equivalent to the polytopes $A(3)$ (the cube) and $J(3,3)$, each with precisely $\eta_k(8,3)$ k-faces for $k = 1, 2$.*

(ii) *For $d = 4$, the minimisers are combinatorially equivalent to the four polytopes $A(4)$, $J(3,4)$, $C(4)$, and $\Sigma(4)$, all with f-vector $(10, 21, 18, 7)$; equivalently, each has precisely $\eta_k(10,4)$ k-faces for $k = 1, 2, 3$.*

(iii) *For $d = 5$, the minimiser with seven facets is combinatorially equivalent to $T(2) \times T(3)$, with f-vector $(12, 30, 34, 21, 7)$.*

Among the 5-polytopes with at least eight facets, the minimisers include polytopes combinatorially equivalent to $A(5)$ and $J(3,5)$, both with f-vector $(12, 32, 39, 25, 8)$. Equivalently, each has precisely $\eta_k(12,5)$ k-faces for $k = 1, 2, 3, 4$. These two are the only minimisers for $k = 1, 2$, but not for $k = 3, 4$.

Moreover, in each case, for every $k \in [1 \ldots d-1]$, the minimisers have either $d+2$ or $d+3$ facets.

Proof. (i)–(ii) The case $d = 3$ of Theorem 13 is easy to check; one may also consult the catalogue [3]. The 4-polytopes with 10 vertices and the minimum number of edges, namely 21, are the polytopes $A(4)$, $J(3,4)$, $C(4)$, and $\Sigma(4)$ [8, Thm. 6.1]. The Euler–Poincaré-Schläfli equation for $d = 4$ [7, Thm. 2.12.17] yields that

$$f_0 - f_1 + f_2 - f_3 = 0.$$

If $f_0 = 10$ and $f_1 \geqslant 22$, this equation implies that $f_2 - f_3 \geqslant 12$, or equivalently that $f_2 \geqslant 12 + f_3$. Since no 4-polytope with 10 vertices has 6 facets (Lemma 1), the inequality $f_3 \geqslant 7$ gives that $f_2 \geqslant 19$. This settles the case $d = 4$.

(iii) Suppose that $d = 5$. The unique 5-polytope with 7 facets and 12 vertices is $T(2) \times T(3)$ (Lemma 1). Consider a 5-polytope P with 12 vertices and at least 8 facets. We show the following.

(a) $f_1(P) \geqslant 32$, with equality only for $A(5)$ and $J(3,5)$;

(b) $f_2(P) \geqslant 39$, with equality only for $A(5)$ and $J(3,5)$;

(c) $f_3(P) \geqslant 25$ but there are other minimisers.

(a) According to [10, Thm. 13], the 5-polytopes with 12 vertices, at least 8 facets, and the minimum number of edges, namely 32, are the polytopes $A(5)$ and $J(3,5)$.

(c) We consider the dual setting: if P has 12 vertices and at least 8 facets, then the dual polytope P' has 12 facets and at least 8 vertices. Suppose first that $f_0(P') = 8$. According to Theorem 9, the unique 5-polytope with 8 vertices and at most $\theta_1(8,5) = 22$ edges is the $(3,2)$-triplex, which has 7 facets. Furthermore, by reasoning with Gale transforms [7, Sec. 2.14] or consulting the catalogues in [4], we can deduce that the only 5-polytopes with 8 vertices and exactly $\theta_1(8,5) + 1 = 23$ edges are the 3-fold pyramid over the pentagon and the 2-fold pyramid over the tetragonal antiwedge, each with 8 facets. The *tetragonal antiwedge* is the unique nonpyramidal 3-polytope with six vertices and six 2-faces. The d-polytopes with $d + 3$ vertices and $\theta_1(d + 3, d) + 2$ edges are characterised in [12, Prop. 16 (i)–(vi)]; they all have $2d + 1$ or fewer facets; for the particular case of $d = 5$, we can also consult the catalogues in [4]. Dually, we see that a 5-polytope with 8 facets and no more than $\theta_1(8,5) + 2 = 24$ ridges has at most 11 vertices. Thus, $f_3(P) \geqslant 25 = f_3(A(5)) = f_3(J(3,5))$, and also $f_3(P) - f_4(P) \geqslant 25 - 8 = 17$.

Suppose now that P has 9 facets. Then, combining Theorem 9 and [9, Thm. 20] we get that P' has at least $\theta_1(9,5)+2 = 26$ edges. If instead P has 10 facets, then, combining Theorem 9 and [9, Prop. 17] we get that the dual P' has at least $\theta_1(10,5)+2 = 27$ edges. Hence, $f_3(P) \geqslant 26$ and $f_3(P) - f_4(P) \geqslant 17$ in both cases. Finally, suppose that P has at least 11 facets. In this case, the dual graph of P informs us that $2f_3(P) \geqslant 5f_4(P) \geqslant 55$; that is, $f_3 \geqslant 28$ and $f_3 - f_4 \geqslant 1.5f_4 \geqslant 16.5$. From this analysis, regardless of the number of facets in P, we get that $f_3(P) \geqslant 25$ again. Note for subsequent use that

$$f_3(P) - f_4(P) \geqslant 17 \tag{9}$$

in each case.

Examination of the catalogues in [4] reveals that there are exactly three other minimisers of f_3, all with $f = (12, 33, 40, 25, 8)$. The easiest one to describe is the pyramid over $J(4,4)$. The catalogue also reveals 12 other 5-polytopes with $f_0 = 12$, $f_4 = 8$ but $f_3 \geqslant 26$.

(b) If P is not combinatorially equivalent to $A(5)$ or $J(3,5)$, then $f_1(P) \geqslant 33$ from (a), in which case (9) and the Euler–Poincaré-Schläfli equation yield that

$$f_2 = f_1 - f_0 + f_3 - f_4 + 2 \geqslant 33 - 12 + 17 + 2 = 40.$$

This concludes the proof of the theorem. □

4.1 Pyramids with at least $d + 3$ facets and simple base polytope

We require the lower bound theorem for simple polytopes [1, 2] and the notion of *truncation polytopes*, polytopes obtained from simplices by successive truncation of vertices.

Remark 14. The smallest vertex counts of truncation d-polytopes are $d + 1$, $2d$, $3d - 1$, and $4d - 2$.

Theorem 15 (Simple polytopes, Barnette (1971–73)). *Let $d \geqslant 2$ and let P be a simple d-polytope with f_{d-1} facets. Then*

$$f_k(P) \geqslant \begin{cases} (d-1)f_{d-1} - (d+1)(d-2), & \text{if } k = 0; \\ \binom{d}{k+1} f_{d-1} - \binom{d+1}{k+1}(d-1-k), & \text{if } k \in [1 \ldots d-2]. \end{cases} \tag{10}$$

If, for $d \geqslant 4$, $f_k(P)$ achieves equality for some $k \in [0 \ldots d-2]$, then P must be combinatorially equivalent to a truncation polytope. For $d = 2, 3$, equality holds for every simple d-polytope.

Lemma 16. Let $d \geqslant 3$ and $0 \leqslant t \leqslant d - 2$. Additionally, let P be a t-fold pyramid over a simple $(d - t)$-polytope and suppose P has **at least** $2d + 2$ vertices and **at least** $d + 3$ facets. Then, for each $k \in [1 \ldots d - 2]$, the following statements hold:

(i) If $t = 0$, then $f_0(P) \geqslant 3d - 1$ and $f_k(P) \geqslant \eta_k(3d - 1, d)$. Equality holds for some $k \in [1 \ldots d - 2]$ if and only if P is either a cube or the simple d-polytope $J(d, d)$ ($d \geqslant 3$).

(ii) If $t \geqslant 1$, then $f_k(P) \geqslant \eta_k(2d+2, d)$. Moreover equality holds if and only if $k = d-2$, and P is combinatorially equivalent to a t-fold pyramid over the simple $(d - t)$-polytope $J(d - t, d - t)$ with t satisfying $f_0(P) = 3d - 2t - 1 \geqslant 2d + 2$.

Proof. (i) First suppose that $t = 0$. Since $f_{d-1}(P) \geqslant d + 3$, the lower bound theorem for simple polytopes yields that $f_0(P) \geqslant 3d - 1$ and, for $k \geqslant 1$, that

$$\begin{aligned} f_k(P) &\geqslant \binom{d}{k+1}(d+3) - \binom{d+1}{k+1}(d-1-k) \\ &= 2\binom{d}{k+1} + \binom{d+1}{k+1} + \underbrace{\left[(d+1)\binom{d}{k+1} - \binom{d+1}{k+1}(d-k)\right]}_{=0} \\ &= 2\binom{d}{k+1} + \binom{d+1}{k+1} = \eta_k(3d-1, d) + \binom{1}{k+1} = \eta_k(3d-1, d). \end{aligned} \tag{11}$$

If $d=3$, $f_1(P)=\eta_1(3d-1,d)$ for all simple 3-polytopes with $3d-1$ vertices and $d+3$ facets; the cube and $J(3,3)$ are the only such polytopes [8, Lem. 2.19(v)]. Suppose $d\geqslant 4$. By [8, Lem. 2.19(v)], $J(d,d)$ is the only simple d-polytope with $3d-1$ vertices and $d+3$ facets. If $f_0(P)>3d-1$, then, by the lower bound theorem for simple polytopes, (11) is a strict inequality, since a truncation d-polytope with $d+3$ facets has $3d-1$ vertices; see also Remark 14.

(ii) We first deal with $d=3$. If $t=2,3$, then P is a 3-simplex, which has fewer than 8 vertices. If $t=1$, then P is a pyramid over an n-gon with $n\geqslant 7$, in which case we have that $f_1(P)=2n>12=\eta_1(8,3)$ and $f_2(P)=n+1>d+3$. Hence the case $d=3$ is settled.

Suppose $d=4$. We reason as in the case $d=3$. If $t=3,4$, then P is a 4-simplex, which has fewer than 10 vertices. If instead $t=2$, then P is a two-fold pyramid over an n-gon with $n\geqslant 8$, in which case we have that $f_1(P)=3n+1>21=\eta_1(10,4)$ and $f_2(P)=3n+1>18=\eta_2(10,4)$. If $t=1$, then P is a pyramid over a simple 3-polytope F with at least ten vertices and at least six faces (there are no simple 3-polytopes with nine vertices). By Euler's polyhedral formula, $f_1(F)\geqslant 14$. This implies that $f_k(P)>\eta_k(10,4)$ for $k=1,2$. We remark that if P were a pyramid over a cube or $J(3,3)$, it would have nine vertices, contrary to our assumption on the number of vertices. Thus we can start our induction on d for all $t\geqslant 0$. Assume $d\geqslant 5$.

Claim 16. If Q is combinatorially equivalent to a t-fold pyramid over $J(d-t,d-t)$, then

$$f_k(Q)=\binom{d+1}{k+1}+2\binom{d}{k+1}-2\binom{t+1}{k+1}.$$

(Of course the latter term is 0 when $k>t$.) Furthermore, if $f_0(Q)=3d-2t-1\geqslant 2d+2$, then $f_{d-2}(Q)=\eta_{d-2}(2d+2,d)$, while $f_k(Q)>\eta_k(2d+2,d)$ for all $1\leqslant k\leqslant d-3$.

Proof of claim. The face numbers of Q follow directly from the multifold-pyramid formula; see, for instance, [5, Thm. 4.2.2]. If $k>t$, $f_{d-2}(Q)=\eta_{d-2}(2d+2,d)$; moreover, $f_k(Q)>\eta_k(2d+2,d)$ for each $1\leqslant k\leqslant d-3$. Suppose $1\leqslant k\leqslant t$. Since $2t+1\leqslant d-2$, we must have $1\leqslant k<d-3$, in which case

$$f_k(Q)-\eta_k(2d+2,d)=\binom{d-2}{k+1}-2\binom{t+1}{k+1}\geqslant\binom{2t+1}{k+1}-2\binom{t+1}{k+1}>0.$$

Applying Vandermonde's identity to $\binom{2t+1}{k+1}$ (Lemma 22(v)) gives the last strict inequality. □

Since $t\geqslant 1$, the polytope P is a pyramid over a $(d-1)$-polytope F, itself a $(t-1)$-fold pyramid with at least $2(d-1)+2$ vertices and at least $(d-1)+3$ facets. Thus our induction hypothesis applies to F. It also follows that

$$f_k(P)=f_k(F)+f_{k-1}(F).$$

Suppose that $t\geqslant 2$. By the induction hypothesis, we get that

$$f_k(F)\geqslant\eta_k(2(d-1)+2,d-1)=\binom{d}{k+1}+2\binom{d-1}{k+1}-\binom{d-3}{k+1},\tag{12}$$

with equality if and only if $k = d-3$, F is combinatorially equivalent to a $(t-1)$-fold pyramid over $J(d-t, d-t)$, and $f_0(F) \geqslant 2(d-1)+3 > 2(d-1)+2$.

Since $f_0(F) = f_0(P) - 1$, $f_0(P) \geqslant 2d+2$, and $d \geqslant 5$, the induction hypothesis gives

$$\begin{aligned} f_1(P) &> \eta_1(2(d-1)+2, d-1) + 2d + 1 \\ &= \left[\binom{d}{2} + 2\binom{d-1}{2} - \binom{d-3}{2}\right] + \left[\binom{d}{1} + 2\binom{d-1}{1} - \binom{d-3}{1}\right] \\ &= \eta_1(2d+2, d). \end{aligned}$$

Furthermore, for $k \in [2 \dots d-2]$, we have

$$f_k(P) \geqslant \eta_k(2(d-1)+2, d-1) + \eta_{k-1}(2(d-1)+2, d-1) = \eta_k(2d+2, d).$$

For $2 \leqslant k \leqslant d-3$, $f_{k-1}(F) > \eta_{k-1}(2(d-1)+2, d-1)$ by the induction hypothesis, whence $f_k(P) > \eta_k(2d+2, d)$.

If $k = d-2$, the equality $f_{d-2}(P) = \eta_{d-2}(2d+2, d)$ would then imply that $f_{d-3}(F) = \eta_{d-3}(2(d-1)+2, d-1)$, which, by the induction hypothesis, holds if and only if F is a $(t-1)$-fold pyramid over $J(d-t, d-t)$ and $f_0(F) \geqslant 2d$; observe that $f_{d-2}(F) = d+2 = \eta_{d-2}(2(d-1)+2, d-1)$. Therefore, $f_{d-2}(P) = \eta_{d-2}(2d+2, d)$ only when P is a t-fold pyramid over $J(d-t, d-t)$.

Assume that $t = 1$. Then F is a simple $(d-1)$-polytope, in which case (i) yields that $f_0(F) \geqslant 3(d-1)-1$ and $f_0(P) \geqslant 3d-3$. Moreover, as $f_0(F) \geqslant 3(d-1)-1$, Part (i) yields that $f_k(F) \geqslant \eta_k(3d-4, d-1)$ for each $k \in [1 \dots d-2]$. Therefore, for $2 \leqslant k \leqslant d-2$, we find that

$$\begin{aligned} f_k(P) &= f_k(F) + f_{k-1}(F) \\ &\geqslant \eta_k(3(d-1)-1, d-1) + \eta_{k-1}(3(d-1)-1, d-1) \\ &= \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{2}{k+1} \geqslant \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1} \\ &= \eta_k(2d+2, d). \end{aligned}$$

The last inequality is tight only for $k = d-2$. By Part (i), if $d \geqslant 5$, then the first inequality is tight only when $F = J(d-1, d-1)$. Hence in the equality case, P is combinatorially equivalent to a pyramid over $J(d-1, d-1)$. Additionally,

$$\begin{aligned} f_1(P) \geqslant f_1(F) + 3d - 4 &\geqslant \binom{d}{2} + 2\binom{d-1}{2} + \binom{d}{1} + 2\binom{d-1}{1} - 2 \\ &= \eta_1(3d-3, d) + 1 > \eta_1(2d+2, d). \end{aligned}$$

This concludes the proof of the lemma. ☐

There is a basic bound for the number of k-faces in a polytope.

Lemma 17. If F is a facet of a d-polytope P, then, for each $k \in [0 \dots d-1]$, $f_k(P) \geqslant f_k(F) + f_{k-1}(F)$,

Proof. For $k \geqslant 0$, each $(k-1)$-face F_0 in F is contained in a k-face F_1 not in F such that $F_0 = F_1 \cap F$. The result now follows. ☐

5 Proof of the main theorem

We restate Part (i) of Theorem 3 for convenience.

Theorem 18 (Restatement of Theorem 3(i))**.** *Let $d \geqslant 3$, and let P be a d-polytope with $2d+2$ vertices. If P has at least $d+3$ facets, then $f_k(P) \geqslant f_k(J(3,d)) = f_k(A(d)) = \eta_k(2d+2,d)$, for each $k \in [1 \dots d-2]$. Moreover, equality holds for some k only if P has exactly $d+3$ facets.*

We prove the inequality first and address the equality afterwards. The cases $d = 3,4,5$ are settled in Theorem 13. Assume that $d \geqslant 6$. The paper [10] settled the case $k=1$ for all d. Our proof of the inequality part proceeds by induction on d for all $k \in [2 \dots d-2]$.

5.1 Claim 18: A facet with $f_0(P) - 2$ vertices

We begin with an overview of the proof. Let F be a facet with $f_0(P)-2$ vertices, and let $X := \{v_1, v_2\}$ be the vertices outside F. We distinguish two cases according to the number of facets of F.

1. ***F* has at least *$d+2$* facets**. We apply the induction hypothesis together with Lemma 17 to get the result.

2. ***F* has exactly *$d+1$* facets**. Here $f_k(F) \geqslant \tau_k(2(d-1)+2, d-1)$ by Lemma 7. Each vertex in X is the apex of a pyramid over a $(d-2)$-face of F. By analysing the $(d-2)$-faces of F (see Remark 6), we conclude that each vertex in X has degree at least $d+2$ in some facet of P and at least $d+3$ in P.

   We use Corollary 12 to find faces F_1, F_2 in P such that each F_i has dimension $d-i+1$, contains $v_i \in X$, and excludes any $v_j \in X$ with $j < i$. Note that $F_1 = P$. Hence the number of k-faces of P that contain at least one vertex in X is bounded below by

   $$\sum_{i=1}^{2} f_{k-1}(F_i/v_i) \geqslant \theta_{k-1}(d+3, d-1) + \theta_{k-1}(d+2, d-2).$$

   We use Corollary 11 to bound $f_{k-1}(F_i/v_i)$. These elements settle the subcase.

We now proceed with the details.

Claim 18. If P has a facet F with $2(d-1)+2 = 2d$ vertices, then $f_k(P) \geqslant \eta_k(2d+2,d)$.

Proof of claim. We consider two cases according to the number of $(d-2)$-faces in F.

Suppose that $f_{d-2}(F) \geqslant d-1+3$. For $k \in [2 \dots d-2]$, combining the induction hypothesis on $d-1$ and Lemma 17 gives that

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + f_{k-1}(F) \\ &\geqslant \left[\binom{d}{k+1} + 2\binom{d-1}{k+1} - \binom{d-3}{k+1}\right] + \left[\binom{d}{k} + 2\binom{d-1}{k} - \binom{d-3}{k}\right] \qquad (13) \\ &= \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1} = \eta_k(2d+2,d). \end{aligned}$$

Assume that $f_{d-2}(F) = d-1+2$. It follows that F is a $(d-1-a)$-fold pyramid over $T(m)\times T(a-m)$ for some $2 \leqslant a \leqslant d-1$ and $2 \leqslant m \leqslant \lfloor a/2 \rfloor$ (Lemma 1). There are two vertices outside F, say v_1 and v_2. A facet F_2 containing v_2 but not v_1 must be a pyramid over a $(d-2)$-face R of F; the same applies to a facet containing v_1 but not v_2. According to Remark 6, for any $(d-2)$-face R of F, there are just three possible values for $f_0(R)$, namely $2d-(a-m+1)$ (which is $\geqslant d+2$), $2d-(m+1)$ (which is $\geqslant d+3$), and $2d-1$. This implies that $\deg_{F_2}(v_2) \geqslant d+2$ and $\deg_P(v_1), \deg_P(v_2) \geqslant d+3$. Consequently, thanks to Proposition 8, the k-faces of P include the k-faces in F, at least $\tau_k(2(d-1)+2, d-1)$ by Lemma 7, the k-faces in P containing v_1, at least $\theta_{k-1}(d-1+4, d-1)$ by Corollary 11, plus the k-faces containing v_2 inside F_2, at least $\theta_{k-1}(d-2+4, d-2)$ by Corollary 11. When $k \geqslant 1$, this analysis yields the following:

$$\begin{aligned}
f_k(P) &\geqslant f_k(F) + \sum_{i=1}^{2} \theta_{k-1}(d-i+4, d-i)\\
&\geqslant \tau_k(2(d-1)+2, d-1) + \left[\binom{d}{k} + \binom{d-1}{k} - \binom{d-4}{k}\right]\\
&\quad + \left[\binom{d-1}{k} + \binom{d-2}{k} - \binom{d-5}{k}\right]\\
&= \eta_k(2d+2, d) + \binom{d-2}{k+1} - \binom{\lceil d/2\rceil - 1}{k+1} - \binom{\lceil d/2\rceil - 2}{k+1} - \binom{d-4}{k}\\
&\quad - \binom{d-5}{k}\\
&\geqslant \eta_k(2d+2, d).
\end{aligned} \tag{14}$$

From Lemma 23(iii), we have that $\binom{d-2}{k+1} - \binom{\lceil d/2\rceil-1}{k+1} - \binom{\lceil d/2\rceil-2}{k+1} - \binom{d-4}{k} - \binom{d-5}{k} > 0$ when $k \leqslant d-3$ and that $\binom{d-2}{k+1} - \binom{\lceil d/2\rceil-1}{k+1} - \binom{\lceil d/2\rceil-2}{k+1} - \binom{d-4}{k} - \binom{d-5}{k} = 0$ for $k = d-2$. This settles this case. □

5.2 Claim 18: A facet with between d and $2d-1$ vertices

We begin with an overview of the proof. Let F be facet of P with $f_0(F) = (d-1)+r$ for some $r \in [1 \ldots d]$. If $1 \leqslant r \leqslant d-1$, then we apply Theorem 9 to obtain $f_k(F) \geqslant \theta_k(d-1+r, d-1)$. If $r = d$ and $f_{d-2}(F) \geqslant d-1+3$, then we use $f_k(F) \geqslant \eta_k(2(d-1)+1, d-1)$ by Theorem 10. If instead $r = d$ and $f_{d-2}(F) = d+1$, we invoke Lemma 7 to get $f_k(F) \geqslant \tau_k(2(d-1)+1, d-1)$.

Let $X := \{v_1, \ldots, v_{d-r+3}\}$ be the set of vertices outside F. We apply Corollary 12 to find faces $F_1, \ldots, F_{d-r+3}$ in P such that each F_i has dimension $d-i+1$, contains $v_i \in X$, and excludes any $v_j \in X$ with $j < i$. Thus, the number of k-faces of P that contain at least one vertex in X is bounded below by

$$\sum_{i=1}^{d-r+3} f_{k-1}(F_i/v_i). \tag{15}$$

By Lemma 16, we could assume that P contains a vertex that is both nonpyramidal and nonsimple. Let v_1 be such a vertex of maximum degree, and let F be a facet not containing v_1. The main difficulty with this approach lies in bounding $f_{k-1}(P/v_1)$, for which we only have the lower bound $\theta_{k-1}(d+1, d-1)$. The lower bounds for $f_k(F)$ and $f_{k-1}(P/v_1)$, combined with (15), are insufficient to get the desired inequality. Therefore, we refine the argument by applying Corollary 12 twice: first to vertices inside a second facet $J_1 \neq F$, and then to vertices outside $F \cup J_1$. For this approach to yield a larger lower bound than (15), we need at least two vertices outside $F \cup J_1$. We now proceed with the details.

Claim 18. If P has facets F and J_1 such that $f_0(F) \leqslant 2d-1$ and there are at least two vertices outside $F \cup J_1$, then $f_k(P) \geqslant \eta_k(2d+2, d)$.

Proof of claim. Assume that $f_0(F) = (d-1) + r$ for $r \in [1 \dots d]$ and let $X := \{v_1, \dots, v_{d-r+3}\}$ be the set of vertices outside F. Additionally, let $J_1 \cap X = \{v'_1, \dots, v'_t\}$. Then $1 \leqslant t \leqslant \#X - 2 = d - r + 1$. By Corollary 12, there is a sequence $F'_1, \dots, F'_t$ of faces within J_1 such that each F'_i has dimension $d-1-i+1$ and contains v'_i but does not contain any v'_j with $j < i$. The number of k-faces of J_1 that contain at least one of the vertices in $\{v'_1, \dots, v'_t\}$ is bounded from below by

$$\sum_{i=1}^{t} f_{k-1}(F'_i/v'_i) \geqslant \sum_{i=1}^{t} \binom{d-i}{k}.$$

There are still $\#X - t = d - r + 3 - t \geqslant 2$ vertices outside $F \cup J_1$. Thus, by Corollary 12, we have the following:

$$f_k(P) \geqslant f_k(F) + \sum_{i=1}^{t} \binom{d-i}{k} + \sum_{j=1}^{d-r+3-t} \binom{d+1-j}{k}. \tag{16}$$

In the case of $1 \leqslant r \leqslant d-1$, from Theorem 9, we get that

$$f_k(F) \geqslant \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-r}{k+1}. \tag{17}$$

Combining (16) and (17), we get

$$f_k(P) \geqslant \left[\binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-r}{k+1}\right] + \sum_{i=1}^{t} \binom{d-i}{k} + \sum_{j=1}^{d-r+3-t} \binom{d+1-j}{k}.$$

Furthermore, for $k \geqslant 2$, we can use Lemma 22 to write this expression as

$$\begin{aligned} &= \eta_k(2d+2, d) - \sum_{\ell=1}^{d-r-1} \binom{d-r-\ell}{k} - \binom{d-2}{k} + \sum_{i=1}^{t-1} \binom{d-1-i}{k} \\ &\quad + \sum_{j=1}^{d-r+1-t} \binom{d-1-j}{k}. \end{aligned} \tag{18}$$

In the case of $r = d$, if $f_{d-2}(F) \geqslant d - 1 + 3$, Theorem 10 gives

$$f_k(F) \geqslant \eta_k(2(d-1)+1, d-1) = \binom{d}{k+1} + 2\binom{d-1}{k+1} - \binom{d-2}{k+1}. \tag{19}$$

If instead $f_{d-2}(F) = d - 1 + 2$, then Lemma 7 give that

$$\begin{aligned} f_k(F) &\geqslant \tau_k(2(d-1)+1, d-1) \\ &= \binom{d}{k+1} + \binom{d-1}{k+1} + \binom{d-2}{k+1} - \binom{\lceil (d-1)/2 \rceil}{k+1} - \binom{\lceil (d-1)/2 \rceil - 1}{k+1}. \end{aligned} \tag{20}$$

We analyse (18) for all $1 \leqslant r \leqslant d-1$, and treat the case $r = d$ by combining (16) with either (19) or (20).

Case 1. $1 \leqslant r \leqslant d-1$ and $2 \leqslant t \leqslant d-1$ (recall that $t \leqslant d - r + 1$).

Here (18) becomes

$$\begin{aligned} f_k(P) \geqslant \eta_k(2d+2, d) &+ \left[\sum_{j=1}^{d-r+1-t} \underbrace{\binom{d-1-j}{k} - \binom{d-r-j}{k}}_{\geqslant 0 \text{ (for } r \geqslant 1)} \right] \\ &+ \left[\sum_{i=1}^{t-2} \underbrace{\binom{d-2-i}{k} - \binom{t-1-i}{k}}_{\geqslant 0 \text{ (as } t \leqslant d-1)} \right] \\ &\geqslant \eta_k(2d+2, d). \end{aligned} \tag{21}$$

Equality holds precisely in the following scenarios: when $t = d-1$ and $r \in \{1, 2\}$ (for all k); when $t = 2$ and $r = 1$ (for all k); and when $r = d + 1 - t$, $t \in [2 \dots d-1]$, and $k = d - 2$.

Case 2. $2 \leqslant r \leqslant d-1$ and $t = 1$.

Here (18) becomes

$$\begin{aligned} f_k(P) \geqslant \eta_k(2d+2, d) &+ \left[\sum_{j=1}^{d-r-1} \underbrace{\binom{d-2-j}{k} - \binom{d-r-j}{k}}_{\geqslant 0 \text{ (for } r \geqslant 2)} \right] \\ &\geqslant \eta_k(2d+2, d). \end{aligned} \tag{22}$$

Equality occurs only when $r = 2$ (for all k), $r = d-1$ (for all k), or $k = d-2$ (for all r).

Case 3. $r = d$.

We must have that $t = 1$. We consider two scenarios according to the number of $(d-2)$-faces in F.

First suppose that $\boldsymbol{f_{d-2}(F) \geqslant d-1+3}$. With the help of Theorem 10, (16) becomes

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + \sum_{i=1}^{3} f_{k-1}(F_i/v_i) \\ &\geqslant \left[\binom{d}{k+1} + 2\binom{d-1}{k+1} - \binom{d-2}{k+1}\right] + \left[\binom{d}{k} + 2\binom{d-1}{k}\right] \\ &= \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1} = \eta_k(2d+2, d). \end{aligned} \tag{23}$$

Assume now that $\boldsymbol{f_{d-2}(F) = d-1+2}$. It follows that F is a $(d-1-a)$-fold pyramid over $T(m) \times T(a-m)$ for some $2 \leqslant a \leqslant d-1$ and $2 \leqslant m \leqslant \lfloor a/2 \rfloor$ (Lemma 1). According to Remark 6, for any $(d-2)$-face R contained in F, there are only three possible values for $f_0(R)$: $2d-1-(a-m+1)$ (which is $\geqslant d+1$), $2d-1-(m+1)$ (which is $\geqslant d+2$), or $2d-2$. Since $t = 1$, J_1 must be a pyramid intersecting F in a $(d-2)$-face, which has at least $d+1$ vertices. Thus J_1 has at least $d+2$ vertices; that is, $f_0(J_1) = d-1+r'$ where $3 \leqslant r' \leqslant d$.

For $3 \leqslant r' \leqslant d-1$, if we swap the roles of J_1 and F and assume that F contains t' vertices from the vertices outside J_1, then $2 \leqslant t' \leqslant d-2$. Consequently, the relevant pairs (r', t') are covered in case 1.

It remains to consider $r' = d$. In this situation, both F and J_1 are pyramids. The vertex in $J_1 \setminus F$, say v_1, has degree $2d-2$ in J_1, and thus, degree at least $2d-1$ in P. Let v_2 and v_3 be the other two vertices outside F. By Corollary 12, there is a sequence F_1, F_2, F_3 of faces of P such that each F_i has dimension $d-i+1$ and contains v_i, but does not contain any v_j with $j < i$.

If the number of $(d-2)$-faces in F_1/v_1 is at least $(d-1)+3$, then Theorem 10 yields, for $k \geqslant 2$, that

$$\sum_{i=1}^{3} f_{k-1}(F_i/v_i) \geqslant \eta_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k}.$$

In this case, Lemma 7 gives that

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + \sum_{i=1}^{3} f_{k-1}(F_i/v_i) \\ &\geqslant \tau_k(2(d-1)+1, d-1) + \eta_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k}, \end{aligned} \tag{24}$$

and Lemma 23(iv) gives that

$$f_k(P) > \eta_k(2d+2, d).$$

Suppose the number of $(d-2)$-faces in F_1/v_1 is $(d-1)+2$. Then Lemma 7 yields, for $k \geqslant 2$, that

$$\sum_{i=1}^{3} f_{k-1}(F_i/v_i) \geqslant \tau_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k}.$$

Thus

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + \sum_{i=1}^{3} f_{k-1}(F_i/v_i) \\ &\geqslant \tau_k(2(d-1)+1, d-1) + \tau_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k}. \end{aligned} \tag{25}$$

Lemma 23(v) now gives that

$$f_k(P) > \eta_k(2d+2, d).$$

Case 4. $r = 1$ and $t = d$.

In this case, F is a simplex and $\#X = d+2$. If the facet J_1 is not a simplex, then $\dim(J_1 \cap F) \geqslant 0$ and $d+1 \leqslant f_0(J_1) \leqslant 2d-1$. Thus, by swapping the roles of J_1 and F, we have between 1 and $d-1$ vertices in $F \setminus J_1$. As a result, the computation for (r,t) where $2 \leqslant r \leqslant d$ and $1 \leqslant t \leqslant d-1$ would apply, namely Cases 1–3. Hence we may assume that J_1 is a simplex. It follows that J_1 is disjoint from F.

Let w_1 and w_2 be the two vertices outside $F \cup J_1$. Let F_1 be a facet containing w_1 but not w_2. Then $f_0(F_1) \leqslant 2d-1$. If F_1 is not a simplex, then J_1 must share vertices with F_1, and thus $J_1 \setminus F_1$ contains between 1 and $d-1$ vertices. Additionally, there are (at least) two vertices outside $F_1 \cup J_1$. Thus, if F_1 played the role of F, then the computation for (r,t) where $2 \leqslant r \leqslant d$ and $1 \leqslant t \leqslant d-1$ would apply. Hence we may assume that F_1 is a simplex.

If $0 \leqslant \dim(F_1 \cap F) < d-2$, then J_1 would share vertices with F_1, and $J_1 \setminus F_1$ would contain between 2 and $d-1$ vertices. Again, if F_1 played the role of F, then the computation for (r,t) where $r=1$ and $2 \leqslant t \leqslant d-1$ would apply. As a consequence, we may assume that F_1 is a simplex intersecting F in either a ridge of P or the empty set. By symmetry, we may also assume F_1 intersects J_1 in either a ridge of P or the empty set. The same reasoning applies to the facets in P containing w_2 but not w_1. Let F_2 be a facet containing w_2 but not w_1.

Suppose that both F_1 and F_2 intersect F in a ridge of P. Then the number of k-faces in P is at least the sum of the following contributions: the number of k-faces of F_i containing w_i, namely $\binom{d-1}{k}$; the number of k-faces of F, namely $\binom{d}{k+1}$; and the number of k-faces in P containing at least one vertex in J_1, which is at least $\sum_{i=1}^{d}\binom{d+1-i}{k} = \binom{d+1}{k+1}$ by Proposition 8. Thus

$$\begin{aligned} f_k(P) &\geqslant 2\binom{d-1}{k} + \binom{d}{k+1} + \binom{d+1}{k+1} = \eta_k(2d+2,d) + \binom{d-1}{k} - \binom{d-2}{k} \\ &> \eta_k(2d+2,d). \end{aligned} \tag{26}$$

An analogous argument applies if both F_1 and F_2 intersect J_1 in a ridge. We may therefore assume that F_1 intersects F in a ridge and F_2 intersects J_1 in a ridge. If there were another facet F_1' containing w_1 but not w_2 that intersects J_1 in a ridge, then both F_2 and F_1' would intersect J_1 in a ridge. Hence, without loss of generality, we may now assume that every facet containing w_1 but not w_2 intersects F in a ridge, and every facet containing w_2 but not w_1 intersects J_1 in a ridge.

If there were no facet containing both w_1 and w_2, then there would be precisely d facets containing w_1 and the collection of these facets and F would form the boundary complex of a d-simplex, a contradiction.

Let F_{12} be a facet containing w_1 and w_2. If $f_0(F_{12}) = 2d$, then we find a scenario settled by Claim 18. Thus, we may assume that $f_0(F_{12}) \leqslant 2d - 1$. Without loss of generality, we may further assume that $\dim(F_{12} \cap J_1) \geqslant \dim(F_{12} \cap F) \geqslant -1$, which implies that $\dim(F_{12} \cap J_1) \geqslant 1$ for $d \geqslant 6$. When $\dim(F_{12} \cap F) = d - 2$, we find that $f_0(F_{12}) = 2d$. Thus, we may further assume that $\dim(F_{12} \cap F) \leqslant d - 3$. Hence there are at least two vertices outside $F_{12} \cup J_1$, vertices in $F \setminus F_{12}$.

Suppose first that F_{12} is not a simplex. If F_{12} played the role of F, then $J_1 \setminus F_{12}$ would contain between 1 and $d-2$ vertices, and the computation for (r, t) where $2 \leqslant r \leqslant d$ and $1 \leqslant t \leqslant d - 2$ would apply. Now suppose that F_{12} is a simplex. If F_{12} played the role of F, then $J_1 \setminus F_{12}$ would contain between 2 and $d - 2$ vertices, and the computation for (r, t) where $r = 1$ and $2 \leqslant t \leqslant d - 2$ would apply. In either situation we are reduced to previously treated cases, which completes the proof of this case.

Case 5. $r = 1$ and $t = 1$.

In this case, F is a simplex and $\#X = d + 2$. Consider a facet N_1 with $d - 1 + r'$ vertices that does not contain the vertex $v_1 \in X$ but does contain some vertex in $X \setminus \{v_1\}$, say v_2. If N_1 contained t' vertices from $X \setminus \{v_1\}$ with $2 \leqslant t' \leqslant d$, then there would be at least two vertices outside $N_1 \cup F$. If N_1 played the role of J_1, we would then be in one of the previously treated cases with $(r, t) = (1, t')$. Therefore we may assume that N_1 contains either exactly one vertex from X (namely v_2) or every vertex in $X \setminus \{v_1\}$.

Suppose that N_1 contains only v_2 from X. In this case both N_1 and F are simplices, and $N_1 \cap F$ is a $(d - 2)$-face. Thus we may take N_1 as the facet J_1. Now consider a facet N_2 different from F that does not contain $v_2 \in X$. Then N_2 must contain some vertex in $X \setminus \{v_2\}$, say v_p. By reasoning as before, we may assume that N_2 contains either exactly one vertex from X (namely v_p) or every vertex in $X \setminus \{v_2\}$. First suppose that $\dim(N_2 \cap F) = d-2$. If N_2 contains only v_p from X, then, by Proposition 8, the number of k-faces of P containing at least one vertex in $X \setminus \{v_p, v_2\}$ is at least $\sum_{i=1}^{d} \binom{d+1-i}{k} = \binom{d+1}{k+1}$ (since $k \geqslant 1$). The number of k-faces of N_1 containing v_2 is $\binom{d-1}{k}$ and the number of k-faces of N_2 containing v_p is $\binom{d-1}{k}$. Thus

$$\begin{aligned} f_k(P) &\geqslant f_k(F) + 2\binom{d-1}{k} + \binom{d+1}{k+1} = \eta_k(2d+2, d) + \binom{d-1}{k} - \binom{d-2}{k} \\ &> \eta_k(2d+2, d). \end{aligned} \tag{27}$$

Moreover, if N_2 contains every vertex in $X \setminus \{v_2\}$, then $f_0(N_2) = d+1+d-1 = 2d$, which reduces to Claim 18. Hence we may assume that such a facet N_2 does not intersect F in a ridge of P, which implies that every facet intersecting F in a ridge of P contains v_2. However, this is impossible: dually, this would mean that every neighbour of the vertex v_F conjugate to F lies in the facet K_{v_2} conjugate to v_2, but the facet K_{v_2} is not the base of a pyramid in the dual polytope of P. This settles this scenario.

We may now assume that N_1 contains every vertex in $X \setminus \{v_1\}$. If $N_1 \cap F$ is a $(d-2)$-face, then $f_0(N_1) = d+1+d-1 = 2d$, which reduces to Claim 18. Thus we may assume that $N_1 \cap F$ is not a $(d-2)$-face. Consequently, we may finally assume that every facet intersecting F in a ridge of P contains v_1; however, this is impossible by the preceding argument. This contradiction settles Claim 18. □

5.3 Final part for the inequality statement of the theorem.

By virtue of Claim 18, we may assume that P has no facet with $2d$ vertices. We claim that the intersection of any two facets of P must be a ridge. This is easy to prove if one of them is the base of a pyramid, so let F and F_1 be any two facets of P such that $f_0(F) \leqslant 2d-1$ and $f_0(F_1) \leqslant 2d-1$. By virtue of Claim 18, we may also assume that $F \cup F_1$ misses at most one vertex of P. Suppose that $\dim(F_1 \cap F) \in [-1 \ldots d-3]$. Then there is a facet K other than F_1 and F that contains $F_1 \cap F$. Furthermore, we see that K cannot miss two vertices in $F \setminus F_1$ and similarly cannot miss two vertices in $F_1 \setminus F$. This implies that there is a unique vertex in $F \setminus K$ and a unique vertex in $F_1 \setminus K$, and no vertices at all outside $F \cup F_1 \cup K$. Thus K has $2d$ vertices in total, contrary to our assumption. Hence $\dim(F_1 \cap F) = d-2$.

Thus, P is a super-Kirkman polytope, meaning that every pair of facets intersects in a ridge of the polytope [11]. Since P is super-Kirkman and has at least $d+3$ facets, every facet of P must contain at least $d+2$ ridges of P. The smallest simple $(d-1)$-polytope with at least $d+2$ $(d-2)$-faces contains at least $3d-4$ vertices (Theorem 15), which is strictly larger than $2d+1$ for $d \geqslant 6$. The following is now plain.

Remark 19. Every facet of P is a nonsimple polytope: it has either between $d+1$ and $2d-1$ vertices or $2d+1$ vertices.

If every vertex in P is either pyramidal or simple, then P is a multifold pyramid over a simple polytope. In this instance, the result follows from Lemma 16. Thus we may assume that P have a nonpyramidal, nonsimple vertex v_1. The vertex v_1 has degree at least $d+4$ in P, as every nonsimple vertex in a super-Kirkman d-polytope has degree at least $d+4$ [11, Thm. 3.9]. Let F be a facet in P that does not contain v_1, and let X be the set of vertices outside F. From Remark 19, we get that $f_0(F) = d-1+r$ with $r \in [2 \ldots d]$.

Suppose that $r = d$. There are three vertices outside F, say v_1, v_2, and v_3. By Corollary 12, there is a sequence F_1, F_2, F_3 of faces of P such that each F_i has dimension $d-i+1$ and contains v_i, but does not contain any v_j with $j < i$. The number of k-faces of P that contain v_1 is at least $\theta_{k-1}(d+4, d-1)$ (by Corollary 11), so the number of k-faces that contain at least one vertex in $\{v_1, v_2, v_3\}$ is bounded below by

$$\begin{aligned}\sum_{i=1}^{3} f_{k-1}(F_i/v_i) &\geqslant \theta_{k-1}(d+4, d-1) + \theta_{k-1}(d-1, d-2) + \theta_{k-1}(d-2, d-3)\\ &= \binom{d}{k} + \binom{d-1}{k} - \binom{d-5}{k} + \binom{d-1}{k} + \binom{d-2}{k}\\ &= \binom{d}{k} + 2\binom{d-1}{k} + \sum_{i=1}^{3}\binom{d-2-i}{k-1}.\end{aligned}$$

In this case, Theorem 10 yields that

$$\begin{aligned}f_k(P) &\geqslant f_k(F) + \sum_{i=1}^{3} f_{k-1}(F_i/v_i)\\ &\geqslant \left[\binom{d}{k+1} + 2\binom{d-1}{k+1} - \binom{d-2}{k+1}\right] + \binom{d}{k} + 2\binom{d-1}{k}\\ &\quad + \sum_{i=1}^{3}\binom{d-2-i}{k-1}\\ &= \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1} + \sum_{i=1}^{3}\binom{d-2-i}{k-1}\\ &= \eta_k(2d+2, d) + \sum_{i=1}^{3}\binom{d-2-i}{k-1}\\ &> \eta_k(2d+2, d).\end{aligned} \tag{28}$$

Suppose that $3 \leqslant r \leqslant d-1$. There are $d-r+3$ vertices outside F, including v_1. By Corollary 12, the number of k-faces of P that contain at least one of the vertices outside F is bounded from below by

$$\binom{d}{k} + \binom{d-1}{k} - \binom{d-5}{k} + \sum_{i=1}^{d-r+2}\binom{d-i}{k}.$$

Additionally, the facet F is not a triplex, as it contains $d+2$ ridges of P (Theorem 9).

Thus $f_k(F) > \theta_k(d-1+r, d-1)$. We now have that

$$\begin{aligned}
f_k(P) &\geqslant f_k(F) + \binom{d}{k} + \binom{d-1}{k} - \binom{d-5}{k} + \sum_{i=1}^{d-r+2} \binom{d-i}{k} \\
&> \left[\binom{d}{k+1} + \binom{d-1}{k+1} - \binom{d-r}{k+1}\right] + \binom{d}{k} + \binom{d-1}{k} - \binom{d-5}{k} \\
&\quad + \sum_{i=1}^{d-r+2} \binom{d-i}{k} \\
&= \eta_k(2d+2, d) - \binom{d-r}{k+1} - \binom{d-5}{k} + \sum_{i=3}^{d-r+2} \binom{d-i}{k} \\
&= \eta_k(2d+2, d) - \sum_{\ell=1}^{d-r-1} \binom{d-r-\ell}{k} - \binom{d-5}{k} + \sum_{i=3}^{d-r+2} \binom{d-i}{k} \\
&= \eta_k(2d+2, d) + \underbrace{\left[\sum_{\ell=1}^{d-r-1} \binom{d-3-\ell}{k} - \binom{d-r-\ell}{k}\right]}_{\geqslant 0 \text{ (for } r \geqslant 3)} \\
&\quad + \underbrace{\left[\binom{d-3}{k} - \binom{d-5}{k}\right]}_{\geqslant 0} \\
&\geqslant \eta_k(2d+2, d).
\end{aligned} \tag{29}$$

Hence $f_k(P) > \eta_k(2d+2, d)$ for all $k \in [2 \dots d-2]$.

Finally, **assume that $r = 2$**. A facet F_1 other than F not containing v_1 must contain $\#X - 1$ vertices from X by Claim 18. Furthermore, $\dim(F_1 \cap F) = d-2$, implying that $f_0(F_1 \cap F) = d-1$ or d. In the former case, $f_0(F_1) = 2d-1$, while in the latter case $f_0(F_1) = 2d$. Both scenarios have already been covered. This completes the proof of the theorem.

5.4 The equality statement of the theorem

The equality part is addressed in Claim 19, where we examine all instances in Claim 18, Claim 18, and section 5.3 in which the inequalities can hold with equality, relying on the analysis developed in the inequality part.

Claim 19. Let P be a d-polytope with $2d+2$ vertices, at least $d+3$ facets, and $f_k(P) = \eta_k(2d+2, d)$ for some $k \in [1 \dots d-2]$. Then P has exactly $d+3$ facets.

Proof of claim. We proceed by induction on $d \geqslant 3$ for all $k \in [1 \dots d-2]$. The case $k = 1$ was settled in [10, Thm. 22] for all d. For $d = 3, 4, 5$, Theorem 13 ensures that equality holds for some $k \in [2 \dots d-2]$ if and only if P has exactly $d+3$ facets.

Let $d \geqslant 6$, and let P be a d-polytope with $2d+2$ vertices and at least $d+3$ facets such that $f_k(P) = \eta_k(2d+2, d)$ for some $k \in [2 \dots d-2]$. We examine the possible equality cases from the proof of Theorem 18, beginning with those in Claim 18.

(i) Claim 18: P has a facet with $f_0(P) - 2$ vertices.

Let v_1, v_2 be the vertices outside F, and consider two cases based on the number of $(d-2)$-faces in F.

Suppose F has at least $d+2$ $(d-2)$-faces. If $f_k(P) = \eta_k(2d+2, d)$ for some $k \in [2 \dots d-2]$ in (13), then $f_k(F) = \eta_k(2d, d-1)$, $f_{k-1}(F) = \eta_{k-1}(2d, d-1)$, and each k-face of P is either a k-face of F or a k-face intersecting F in a $(k-1)$-face. By induction, F has exactly $d+2$ $(d-2)$-faces. If there were a facet F' not intersecting F in a ridge of P, then F' would be a two-fold pyramid over a $(d-3)$-face of F. However, this would imply the existence of a k-face in P that is a two-fold pyramid over a $(k-2)$-face of F, contradicting our equality condition. Thus every facet of P intersects F in a $(d-2)$-face, which implies that P has exactly $d+3$ facets.

Suppose F has exactly $d+1$ $(d-2)$-faces. If $f_k(P) = \eta_k(2d+2, d)$ for some $k \in [2 \dots d-2]$ in (14), then $f_k(F) = \tau_k(2d, d-1)$; the number of k-faces containing v_1 is exactly $\theta_{k-1}(d+3, d-1)$; and there is a facet F_2 containing v_2 but not v_1, where v_2 lies in exactly $\theta_{k-1}(d+2, d-2)$ k-faces. We need the following basic fact, inspired by [13, Sec. 4]; we omit its proof.

Fact 20. Let F' and F'' be two distinct facets of a d-polytope that both contain a vertex v, and let $1 \leqslant k \leqslant d-1$. Then there exists a k-face of F' containing v that is not a k-face of F''.

If there were another facet $F_2' \neq F_2$ containing v_2 but not v_1, then, by Fact 20, the number of k-faces containing v_2 but not v_1 would be strictly larger than $\theta_{k-1}(d+2, d-2)$, contradicting our equality condition. Thus, apart from the facet F, P has exactly $d+1$ facets containing v_1 (by Corollary 11) and a unique facet containing v_2 but not v_1. Hence P has exactly $d+3$ facets.

(ii) Claim 18: P has facets F and J_1 such that $f_0(F) \leqslant 2d-1$ and there are at least two vertices outside $F \cup J_1$.

As in the proof of the claim, let $1 \leqslant r \leqslant d$ and $1 \leqslant t \leqslant d-r+1$, and assume that $f_0(F) = (d-1)+r$. Additionally, let $X = \{v_1, v_2, \dots, v_{d-r+3}\}$ be the set of vertices outside F, let $J_1 \cap X = \{v_1', \dots, v_t'\}$, and let $X \setminus J_1 = \{v_1'', \dots, v_{d-r+3-t}''\}$. We begin with a simple fact.

Fact 21. If the $d-r+3-t$ vertices in $X \setminus J_1$ are contained in precisely $\sum_{i=1}^{d-r+3-t} \binom{d+1-i}{k}$ k-faces of P, then the number of k-faces in P that contain the vertex v_i'' and no vertex v_j'' with $j < i$ is precisely $\binom{d+1-i}{k}$.

Proof of fact. By Corollary 12, there is a sequence $Y_1, \dots, Y_{d-r+3-t}$ of faces in P such that each Y_i has dimension $d+1-i$ and contains v_i'' but does not contain any v_j'' with $j < i$.

The minimum number of k-faces within the $(d+1-i)$-face Y_i that contain the vertex v_i'' is attained when the vertex figure Y_i/v_i'' is a $(d-i)$-simplex. Since Y_i excludes each v_j'' with $j < i$ and the number of k-faces of P containing some vertex in $X \setminus J_1$ is precisely $\sum_{i=1}^{d-r+3-t} \binom{d+1-i}{k}$, each vertex figure must be a minimiser. See also Fact 20. □

Following the proof of Claim 18, there are five cases to consider based on the pair (r, t).

Case 1. $1 \leqslant r \leqslant d-1$ and $2 \leqslant t \leqslant d-1$.

If $f_k(P) = \eta_k(2d+2, d)$ in (16) (see also (21)), then (a) the number of k-faces of P containing some vertices in $J_1 \cap X$ but no vertex in $X \setminus J_1$ is exactly $\sum_{i=1}^{t} \binom{d-i}{k}$; (b) every vertex in $J_1 \cap X$ is simple in J_1; (c) the number of k-faces of P containing some vertex in $X \setminus J_1$ is exactly $\sum_{j=1}^{d-r+3-t} \binom{d+1-j}{k}$; and (d) every vertex in $X \setminus J_1$ is simple in P.

Since v_1'' is simple in P, it is contained in precisely d facets of P. There is a unique facet I_2 of P containing v_2'' but not v_1''. Indeed, suppose that there are two facets I_2 and I_2' of P containing v_2'' but not v_1''. Then, by Fact 20, v_2'' would contribute more than $\binom{d-1}{k}$ to $\sum_{j=1}^{d-r+3-t} \binom{d+1-j}{k}$, contradicting Fact 21.

If there were a facet of P different from J_1 that contains a vertex in $J_1 \cap X$ but no vertex from $X \setminus J_1$, say J_1', then we could select such a vertex as v_1'. The contribution of the k-faces containing v_1' in $J_1 \cup J_1'$ would then exceed $\binom{d-1}{k}$ (see Fact 20), yielding more than $\sum_{i=1}^{t} \binom{d-i}{k}$ k-faces of P containing a vertex in $J_1 \cap X$ but no vertex in $X \setminus J_1$. Thus P has exactly $d+3$ facets: the d facets containing v_1'', together with I_2, J_1, and F.

Case 2. $2 \leqslant r \leqslant d-1$ and $t = 1$.

If $f_k(P) = \eta_k(2d+2, d)$ in (16) (see also (22)), then the same conditions as in case 1 apply, and therefore so does the same argument. Thus P has exactly $d+3$ facets.

Case 3. $r = d$ (and $1 \leqslant t \leqslant d-r+1$).

We consider two subcases according to the number of $(d-2)$-faces of F. If $f_{d-2}(F) \geqslant d-1+3$ and $f_k(P) = \eta_k(2d+2, d)$ in (16) (see also (22)), then the same conditions and argument in case 1 apply, and therefore P has exactly $d+3$ facets.

Suppose $f_{d-2}(F) = d-1+2$. Then F is a $(d-1-a)$-fold pyramid over $T(m) \times T(a-m)$ for some $2 \leqslant a \leqslant d-1$ and $2 \leqslant m \leqslant \lfloor a/2 \rfloor$ (Lemma 1). Since $t = 1$, J_1 must be a pyramid intersecting F in a $(d-2)$-face, which has at least $d+1$ vertices because $m \geqslant 2$ (Remark 6). Thus $f_0(J_1) = d-1+r'$ where $3 \leqslant r' \leqslant d$. For $3 \leqslant r' \leqslant d-1$, if we swap the roles of J_1 and F and assume that F contains t' vertices from the vertices outside J_1, then $2 \leqslant t' \leqslant d-2$. Consequently, the pairs (r', t') are covered in case 1. If $r' = d$, then we have strict inequalities in all the scenarios; see (24) and (25).

Case 4. $r = 1$ and $t = d$.

Here F is a simplex. If the facet J_1 were not a simplex, then $\dim(J_1 \cap F) \geqslant 0$ and $d+1 \leqslant f_0(J_1) \leqslant 2d-1$. Thus, by swapping the roles of J_1 and F, there would have between 1 and $d-1$ vertices in $F \setminus J_1$, reducing this scenario to Cases 1–3. If J_1 is a simplex, then we are reduced to Claim 18, to a previously treated case in Claim 18, or we obtain strict inequality in (26).

Case 5. $r = 1$ and $t = 1$.

Here F is also a simplex and J_1 is a pyramid. In this case, the analysis yields strict inequality (27), a reduction to Claim 18, or a reduction to a previously treated case in Claim 18.

This concludes the analysis of the equality cases arising from Claim 18. We now deal with cases considered in section 5.3.

By Claim 18, we may assume that P has no facet with $2d$ vertices. We also know that P is a super-Kirkman polytope. Additionally, every facet of P is nonsimple, and its number of vertices is either $2d+1$ or in the range $[d+1 \dots 2d-1]$ (see Remark 19).

If every vertex in P is either pyramidal or simple, then P is a multifold pyramid over a simple polytope. By Lemma 16, P has exactly $d+3$ facets. Suppose now that there is a vertex v_1 that is nonpyramidal and nonsimple, and let F be a facet that does not contain v_1, where $f_0(F) = d-1+r$ with $r \in [2 \dots d]$. If $r = d$, we have strict inequality in (28), while if $3 \leqslant r \leqslant d-1$, we have strict inequality in (29). If $r = 2$, then there is a facet $F_1 \neq F$ not containing v_1 with $f_0(F_1) \in \{2d-1, 2d\}$. Both scenarios have already been covered. This completes the proof of the claim. □

The proof of Theorem 18 is now complete.

6 Combinatorial identities and proofs

We start with a combinatorial lemma whose three first expressions are from [13].

Lemma 22 (Combinatorial equalities and inequalities)**.** For all integers $d \geqslant 2$, $k \in [1 \dots d-1]$, the following statements hold:

(i) If $2 \leqslant r \leqslant s \leqslant d$ are integers, then $\theta_k(d+s-r, d-1) + \sum_{i=1}^{r} \binom{d+1-i}{k} \geqslant \theta_k(d+s, d)$, with equality only if $r = 2$ or $r = s$.

(ii) If $n \geqslant c$ are positive integers, then $\binom{n}{c} = \binom{n-1}{c-1} + \binom{n-1}{c}$.

(iii) If $n \geqslant c$ and $n \geqslant a$ are positive integers, then $\binom{n}{c} - \binom{n-a}{c} = \sum_{i=1}^{a} \binom{n-i}{c-1}$.

(iv) If n, c are positive integers, then $\binom{n}{c} = \sum_{i=1}^{n} \binom{n-i}{c-1}$.

(v) (Vandermonde's identity) If n, a, c are nonnegative integers, then

$$\binom{n+a}{c} = \sum_{i=0}^{c} \binom{n}{i}\binom{a}{c-i}.$$

Proof. The proof of (i) is embedded in the proof of [13, Thm. 3.2]; it is also spelled out in [7, Claim 1]. Repeated applications of (ii) yield (iii) and (iv), while (v) is well known. □

Lemma 23. The following inequalities hold.

(i) If $d \geqslant 9$ and $1 \leqslant k \leqslant \lceil d/3 \rceil - 2$, then $\eta_k(2d+2, d) < \tau_k(2d+2, d)$.

(ii) If $d \geqslant 9$ and $\lfloor 0.4d \rfloor \leqslant k \leqslant d-1$, then $\eta_k(2d+2,d) > \tau_k(2d+2,d)$.

(iii) If $d \geqslant 6$ and $1 \leqslant k \leqslant d-3$, then

$$\binom{d-2}{k+1} - \binom{\lceil d/2 \rceil - 1}{k+1} - \binom{\lceil d/2 \rceil - 2}{k+1} - \binom{d-4}{k} - \binom{d-5}{k} > 0.$$

If $k = d-2$ or $d-1$, then the expression is zero.

(iv) If $d \geqslant 4$ and $2 \leqslant k \leqslant d-2$, then

$$\eta_k(2d+2,d) < \tau_k(2(d-1)+1,d-1) + \eta_{k-1}(2(d-1)+1,d-1) + \binom{d-1}{k} + \binom{d-2}{k}.$$

(v) If $d \geqslant 5$ and $1 \leqslant k \leqslant d-2$, then

$$\eta_k(2d+2,d) < \tau_k(2(d-1)+1,d-1) + \tau_{k-1}(2(d-1)+1,d-1) + \binom{d-1}{k} + \binom{d-2}{k}.$$

Proof. We reason as in [14, Prop. 6.1].

(i) Write $\varphi_k(d) := \tau_k(2d+2,d) - \eta_k(2d+2,d)$. We can verify that $\varphi_1(9) = 4$. We prove the statement by induction on $d \geqslant 9$ for all $1 \leqslant k \leqslant \lceil d/3 \rceil - 2$. Assume that $d \geqslant 10$. We have that

$$\begin{aligned}\tau_k(2d+2,d) &= \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{\lceil (d+1)/2 \rceil - 1}{k+1} \\ &\quad - \binom{\lceil (d+1)/2 \rceil - 2}{k+1} \\ \eta_k(2d+2,d) &= \binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1},\end{aligned}$$

which, after some simplifications, gives that

$$\varphi_k(d) = \binom{d-2}{k+1} - \binom{d-1}{k} - \binom{\lceil (d+1)/2 \rceil - 1}{k+1} - \binom{\lceil (d+1)/2 \rceil - 2}{k+1}.$$

Since

$$\begin{aligned}\varphi_k(d-1) &= \binom{d-3}{k+1} - \binom{d-2}{k} - \binom{\lceil d/2 \rceil - 1}{k+1} - \binom{\lceil d/2 \rceil - 2}{k+1} \\ \varphi_{k-1}(d-1) &= \binom{d-3}{k} - \binom{d-2}{k-1} - \binom{\lceil d/2 \rceil - 1}{k} - \binom{\lceil d/2 \rceil - 2}{k},\end{aligned}$$

from Lemma 22(ii) it follows that

$$\varphi_k(d-1) + \varphi_{k-1}(d-1) = \binom{d-2}{k+1} - \binom{d-1}{k} - \binom{\lceil d/2 \rceil}{k+1} - \binom{\lceil d/2 \rceil - 1}{k+1}. \tag{30}$$

If d is even, then $\varphi_k(d) = \varphi_k(d-1) + \varphi_{k-1}(d-1)$. If $d = 2p+1$ for some integer p, then

$$\varphi_k(d) - (\varphi_k(d-1) + \varphi_{k-1}(d-1)) = \binom{p-1}{k} + \binom{p}{k} > 0. \tag{31}$$

If both $k-1, k \leqslant \lceil (d-1)/3 \rceil - 2$, combining (30) and (31) and the induction hypothesis on $d-1$ give that $\varphi_k(d) > 0$. Thus, it remains to consider the case

$$\lceil (d-1)/3 \rceil - 1 \leqslant k \leqslant \lceil d/3 \rceil - 2,$$

which reduces to the pairs $(d,k) = (6p+1, 2p-1), (6p+4, 2p)$ for some integer $p \geqslant 1$. Consider the pair $(6p+1, 2p-1)$.

$$\begin{aligned}
\varphi_{2p-1}(6p+1) &= \binom{6p-1}{2p} - \binom{6p}{2p-1} - \binom{3p}{2p} - \binom{3p-1}{2p} \\
&= \frac{p+1}{3p}\binom{6p}{2p-1} - 4\binom{3p-1}{2p} \\
&= \frac{(3p-1)!}{(2p)!(p-1)!}\left[2 \cdot \frac{3p+1}{p+2} \cdots \frac{6p}{4p+1} - 4\right] \\
&= \binom{3p-1}{2p}\left[2 \cdot \frac{3p+1}{p+2} \cdots \frac{6p}{4p+1} - 4\right].
\end{aligned}$$

From the last expression, we can gather that $\varphi_{2p-1}(6p+1) > 0$.

Now consider the pair $(6p+4, 2p)$.

$$\begin{aligned}
\varphi_{2p}(6p+4) &= \binom{6p+2}{2p+1} - \binom{6p+3}{2p} - \binom{3p+2}{2p+1} - \binom{3p+1}{2p+1} \\
&= \frac{2p+3}{2(4p+3)}\binom{6p+2}{2p+1} - \frac{4p+3}{p+1}\binom{3p+1}{2p+1} \\
&= \binom{3p+1}{2p+1}\frac{4p+3}{p+1}\left[\frac{2p+3}{2(4p+3)} \cdot \frac{3p+2}{p+2} \cdots \frac{4p+2}{2p+2} \cdot \frac{4p+4}{2p+3} \cdots \frac{6p+2}{4p+1} - 1\right] \\
&= \binom{3p+1}{2p+1}\frac{4p+3}{p+1}\left[\frac{2p+3}{2p+3} \cdot \frac{1}{2} \cdot \frac{3p+2}{p+2} \cdots \frac{4p+2}{2p+2} \cdot \frac{4p+4}{4p+3} \cdots \frac{6p+2}{4p+1} - 1\right].
\end{aligned}$$

From the last expression, we can gather that $\varphi_{2p}(6p+4) > 0$.

(ii) Write $\varphi_k(d) := \eta_k(2d+2, d) - \tau_k(2d+2, d)$. After some simplification, we have

$$\varphi_k(d) = \binom{d-1}{k} - \binom{d-2}{k+1} + \binom{\lceil (d+1)/2 \rceil - 1}{k+1} + \binom{\lceil (d+1)/2 \rceil - 2}{k+1}.$$

To show that $\varphi_k(d) > 0$ for $d \geqslant 9$, it suffices to prove that $\varphi'_k(d) := \binom{d-1}{k} - \binom{d-2}{k+1} > 0$.

For $k = d-2, d-1$, it is straightforward that $\varphi'_k(d) > 0$. Thus, assume that $k \leqslant d-3$.

$$\begin{aligned}\varphi'_k(d) =& \binom{d-1}{k} - \binom{d-2}{k+1} \\ =& \frac{(d-1)!}{(d-k-1)!k!} - \frac{(d-2)!}{(d-k-3)!(k+1)!} \\ =& \frac{(d-2)!}{(d-k-3)!k!}\left(\frac{d-1}{(d-k-1)(d-k-2)} - \frac{1}{k+1}\right) \\ =& \frac{(d-2)!}{(d-k-3)!k!}\left(\frac{(d-1)(k+1)-(d-k-1)(d-k-2)}{(d-k-1)(d-k-2)(k+1)}\right) \\ =& \frac{(d-2)!}{(d-k-3)!k!}\left(\frac{-k^2+(3d-4)k+(-d^2+4d-3)}{(d-k-1)(d-k-2)(k+1)}\right).\end{aligned}$$

The expression $-k^2 + (3d-4)k + (-d^2+4d-3)$ is positive when k is between $k_1 := (3d-4-\sqrt{5d^2-8d+4})/2$ and $k_2 := (3d-4+\sqrt{5d^2-8d+4})/2$. Since $k_1 < \lfloor 0.4d \rfloor$ and $k_2 > d-1$ for all $d \geqslant 9$, it follows that $\varphi'_k(d) > 0$ for $\lfloor 0.4d \rfloor \leqslant k \leqslant d-3$.

(iii) Write $\varphi_k(d) := \binom{d-2}{k+1} - \binom{\lceil d/2 \rceil - 1}{k+1} - \binom{\lceil d/2 \rceil - 2}{k+1} - \binom{d-4}{k} - \binom{d-5}{k}$. From Lemma 22 we get that

$$\varphi_k(d) = \left[\binom{d-5}{k+1} - \binom{\lceil d/2 \rceil - 2}{k+1}\right] + \left[\binom{d-3}{k} - \binom{\lceil d/2 \rceil - 1}{k+1}\right]. \tag{32}$$

Suppose that $d = 2p+1$ for some $p \geqslant 3$. Then

$$\varphi_k(d) = \left[\binom{2p-4}{k+1} - \binom{p-1}{k+1}\right] + \left[\binom{2p-2}{k} - \binom{p}{k+1}\right].$$

For $k+1 \geqslant p$, we have $\varphi_k(d) > 0$. Hence assume that $k+1 < p$. If $p \leqslant 4$, then we can easily verify that $\varphi_k(d) > 0$ for all k. So assume $p \geqslant 5$. On one hand,

$$\binom{2p-4}{k+1} - \binom{p-1}{k+1} = \sum_{\ell=1}^{p-3}\binom{2p-4-\ell}{k} > (p-3)\binom{p-1}{k},$$

and on the other hand,

$$\left[\binom{2p-2}{k} - \binom{p}{k+1}\right] > \binom{p-1}{k} - \binom{p}{k+1} = \binom{p-1}{k} - \frac{p}{k+1}\binom{p-1}{k}.$$

Thus, we have

$$\begin{aligned}\varphi_k(d) &> (p-2)\binom{p-1}{k} - \frac{p}{k+1}\binom{p-1}{k} \geqslant (p-2)\binom{p-1}{k} - \frac{p}{2}\binom{p-1}{k} \\ &\geqslant \frac{p-4}{2}\binom{p-1}{k} > 0.\end{aligned}$$

Suppose that $d = 2p$ for some $p \geqslant 3$. Then

$$\varphi_k(d) = \left[\binom{2p-5}{k+1} - \binom{p-2}{k+1}\right] + \left[\binom{2p-3}{k} - \binom{p-1}{k+1}\right].$$

For $k+2 \geqslant p$, we have $\varphi_k(d) > 0$. Hence assume that $k+2 < p$. If $p \leqslant 4$, then we can easily verify that $\varphi_k(d) > 0$ for all k. So assume $p \geqslant 5$. On one hand,

$$\binom{2p-5}{k+1} - \binom{p-2}{k+1} = \sum_{\ell=1}^{p-3} \binom{2p-5-\ell}{k} > (p-3)\binom{p-2}{k},$$

and on the other hand,

$$\binom{2p-3}{k} - \binom{p-1}{k+1} > \binom{p-2}{k} - \binom{p-1}{k+1} = -\binom{p-2}{k+1} = -\frac{p-2-k}{k+1}\binom{p-2}{k}.$$

Thus, we have

$$\varphi_k(d) > (p-3)\binom{p-2}{k} - \frac{p-2-k}{k+1}\binom{p-2}{k} > 0.$$

(iv) We reason as in (i), and write

$$\varphi_k(d) := \tau_k(2(d-1)+1, d-1) + \eta_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k} - \eta_k(2d+2, d).$$

We verify that $\varphi_1(4) = 1, \varphi_2(4) = 2$. We prove the statement by induction on $d \geqslant 4$ for all $1 \leqslant k \leqslant d-2$. Assume that $d \geqslant 5$. Applications of Lemma 9(iv) give that

$$\begin{aligned}\varphi_k(d) = &\left[\binom{d}{k+1} + \binom{d-1}{k+1} + \binom{d-2}{k+1} - \binom{d - \lfloor (d+1)/2 \rfloor}{k+1}\right.\\ &\left. - \binom{d - \lfloor (d+1)/2 \rfloor - 1}{k+1}\right] + \left[\binom{d}{k} + 2\binom{d-1}{k} - \binom{d-2}{k}\right] + \binom{d-1}{k}\\ &+ \binom{d-2}{k} - \left[\binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1}\right]\\ =&2\binom{d-2}{k+1} + 2\binom{d-1}{k} - \binom{d}{k+1} - \binom{\lceil (d-1)/2 \rceil}{k+1} - \binom{\lceil (d-1)/2 \rceil - 1}{k+1}.\end{aligned}$$

The induction hypothesis on $d-1$ yields that $\varphi_k(d-1) > 0$ and $\varphi_{k-1}(d-1) > 0$ for all $1 \leqslant k \leqslant d-3$.

$$\begin{aligned}\varphi_k(d-1) &= 2\binom{d-3}{k+1} + 2\binom{d-2}{k} - \binom{d-1}{k+1} - \binom{\lceil d/2 \rceil - 1}{k+1} - \binom{\lceil d/2 \rceil - 2}{k+1}\\ \varphi_{k-1}(d-1) &= 2\binom{d-3}{k} + 2\binom{d-2}{k-1} - \binom{d-1}{k} - \binom{\lceil d/2 \rceil - 1}{k} - \binom{\lceil d/2 \rceil - 2}{k}.\end{aligned}$$

If d is even, then $\varphi_k(d) = \varphi_{k-1}(d-1) + \varphi_k(d-1)$, which implies that $\varphi_k(d) > 0$ for $1 \leqslant k \leqslant d-3$. In the case of odd d, $\varphi_k(d) \geqslant \varphi_{k-1}(d-1) + \varphi_k(d-1)$, which implies that $\varphi_k(d) > 0$ for $1 \leqslant k \leqslant d-3$. For $k = d-2$,

$$\begin{aligned}\varphi_{d-2}(d) =& 2\binom{d-2}{d-1} + 2\binom{d-1}{d-2} - \binom{d}{d-1} - \binom{\lceil (d-1)/2 \rceil}{d-1} - \binom{\lceil (d-1)/2 \rceil - 1}{d-1} \\ =& 0 + 2(d-1) - d - 0 - 0 \\ =& d - 2 > 0.\end{aligned}$$

This part is now proved.

(v) We reason as in (i), and write

$$\varphi_k(d) := \tau_k(2(d-1)+1, d-1) + \tau_{k-1}(2(d-1)+1, d-1) + \binom{d-1}{k} + \binom{d-2}{k} - \eta_k(2d+2, d).$$

Applications of Lemma 9(iv) give that

$$\begin{aligned}\varphi_k(d) =& \left[\binom{d}{k+1} + \binom{d-1}{k+1} + \binom{d-2}{k+1} - \binom{\lceil (d-1)/2 \rceil}{k+1} - \binom{\lceil (d-1)/2 \rceil - 1}{k+1}\right] \\ &+ \left[\binom{d}{k} + \binom{d-1}{k} + \binom{d-2}{k} - \binom{\lceil (d-1)/2 \rceil}{k} - \binom{\lceil (d-1)/2 \rceil - 1}{k}\right] \\ &+ \binom{d-1}{k} + \binom{d-2}{k} - \left[\binom{d+1}{k+1} + 2\binom{d}{k+1} - \binom{d-2}{k+1}\right] \\ =& \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{\lceil (d-1)/2 \rceil + 1}{k+1} - \binom{\lceil (d-1)/2 \rceil}{k+1} \\ &+ \binom{d-1}{k} + \binom{d-2}{k} - \binom{d+1}{k+1} - 2\binom{d}{k+1} + \binom{d-2}{k+1} \\ =& \binom{d-1}{k+1} - \binom{\lceil (d-1)/2 \rceil + 1}{k+1} - \binom{\lceil (d-1)/2 \rceil}{k+1}.\end{aligned}$$

We show that $\varphi_k(d)$ is strictly increasing in d for all k: For $p \geqslant 2$,

$$\begin{aligned}\varphi_k(2p) &= \binom{2p-1}{k+1} - \binom{p+1}{k+1} - \binom{p}{k+1} \\ \varphi_k(2p+1) &= \binom{2p}{k+1} - \binom{p+1}{k+1} - \binom{p}{k+1},\end{aligned}$$

which implies

$$\begin{aligned}\varphi_k(2p+1) - \varphi_k(2p) &= \binom{2p}{k+1} - \binom{2p-1}{k+1} = \binom{2p}{k} > 0 \\ \varphi_k(2p+2) - \varphi_k(2p+1) &= \binom{2p+1}{k+1} - \binom{2p}{k+1} - \binom{p+2}{k+1} + \binom{p}{k+1} \\ &= \binom{2p}{k} - \binom{p+1}{k} - \binom{p}{k} > 0.\end{aligned}$$

Since $\varphi_k(d)$ is increasing in d for all k, and $\varphi_1(5) = 2, \varphi_2(5) = 3, \varphi_3(5) = 1$, it follows that $\varphi_k(d) > 0$ for all $k \in [1 \ldots d-2]$. This part is now proved, which completes the proof of the lemma. □

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References

[1] D. W. Barnette, *The minimum number of vertices of a simple polytope*, Isr. J. Math. **10**, pages 121–125, 1971.

[2] D. W. Barnette, *A proof of the lower bound conjecture for convex polytopes*, Pac. J. Math. **46**, pages 349–354, 1973.

[3] D. Britton and J. D. Dunitz, *A complete catalogue of polyhedra with eight or fewer vertices.* Acta Cryst. Ser. A **29**, pages 362–371, 1973.

[4] K. Fukuda, H. Miyata, and S. Moriyama, *Classification of oriented matroids*, https://polydb.org/#collection=Polytopes.Combinatorial.CombinatorialTypes, 2013.

[5] B. Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. **221**, Springer-Verlag, New York, 2003.

[6] P. McMullen, *The minimum number of facets of a convex polytope*, J. London Math. Soc. (2) **3**, pages 350–354, 1971.

[7] G. Pineda-Villavicencio, *Polytopes and graphs*, Cambridge Studies in Advanced Mathematics, vol. **211**, Cambridge Univ. Press, Cambridge, 2024.

[8] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *The excess degree of a polytope*, SIAM J. Discrete Math. **32**, no. 3, pages 2011–2046, 2018.

[9] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Lower bound theorems for general polytopes*, Eur. J. Comb. **79**, pages 27–45, 2019.

[10] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Minimum number of edges of d-polytopes with $2d + 2$ vertices*, Electron. J. Combin., #P3.18, 2022.

[11] G. Pineda-Villavicencio, J. Wang, and D. Yost, *Polytopes with low excess degree*, Discrete Math., to appear, arXiv:2405.16838, 2024.

[12] G. Pineda-Villavicencio and D. Yost, *A lower bound theorem for d-polytopes with $2d + 1$ vertices*, SIAM J. Discrete Math. **36**, pages 2920–2941, 2022.

[13] L. Xue, *A proof of Grünbaum's lower bound conjecture for general polytopes*, Isr. J. Math. **245**, pages 991–1000, 2021.

[14] L. Xue, *A lower bound theorem for strongly regular CW spheres with up to $2d + 1$ vertices*, Discrete Comput. Geom. **72**, pages 1348–1368, 2024.