On the nonexistence of almost Moore digraphs with self-repeats

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Abstract

An almost Moore digraph is a diregular digraph of degree $d > 1$, diameter $k > 1$ and order $d + d^2 + \cdots + d^k$. Their existence has only been shown for $k = 2$. It has also been conjectured that there are no more almost Moore digraphs, but so far their nonexistence has only been proven for $k = 3, 4$ and for $d = 2, 3$ when $k \geqslant 3$.

In this paper we study the structure of the subdigraphs of an almost Moore digraph induced by the vertices fixed by an automorphism determined by a power of the permutation r of repeats of the digraph. We deduce that each almost Moore digraph of degree d and diameter k with self-repeats has such a subdigraph whose vertices have order $\leqslant d - 1$ under r. From this, we extend the results about the nonexistence of almost Moore digraphs with self-repeats of degrees 4 and 5 to those whose diameter is large enough with respect to the degree. More precisely, we prove their nonexistence when $k \geqslant 2(d-1)$ if k is odd and when $k \geqslant 2(d-1)^2$ if k is even. We also show that these findings jointly with other results imply that there are no almost Moore digraphs with self-repeats for degrees d, $6 \leqslant d \leqslant 12$, and $k > 2$.

Mathematics Subject Classifications: 05C35, 05C20, 05C50

1 Introduction

Any digraph with maximum out-degree d and diameter k has at most $1 + d + \cdots + d^k$ vertices. This is the well-known Moore bound. However, Plesník and Znám [24] in 1974 and later Bridges and Toueg [7] in 1980 proved the nonexistence of digraphs which reach this bound unless $d = 1$ (directed cycles of length $k + 1$) or $k = 1$ (complete symmetric digraphs). Then, the existence of almost Moore digraphs, also called (d, k)-digraphs, that is, digraphs of maximum out-degree $d > 1$, diameter $k > 1$ and order one less than the Moore bound is an interesting and hard question, which appears in the context of the degree/diameter problem. For a detailed overview of this problem, see the survey given by Miller and Širáň [22] in 2005.

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An almost Moore digraph G of degree d and diameter k has the property that each vertex $v \in V(G)$ has a unique vertex $u \in V(G)$, called the *repeat* of v and denoted by $r(v)$, such that there are exactly two walks from v to $r(v)$ of length $\leqslant k$, at least one of them being of length k. If $r(v) = v$, the vertex v is called a *self-repeat* of G. Moreover, almost Moore digraphs are known to be diregular (both in- and out-regular), a result proved by Miller et al. [21] in 2000.

Concerning the existence of almost Moore digraphs it seems they do only exist for diameter $k = 2$. In 1983 Fiol et al. [14] proved that the line digraph LK_{d+1} of the complete digraph K_{d+1}, that is, the Kautz digraph $K(d, 2)$, is a $(d, 2)$-digraph. In 2001, Gimbert [16] finalised the complete classification showing there is only one $(d, 2)$-digraph, unless the case $d = 2$ for which there are two more digraphs.

Nevertheless, their nonexistence for $k > 2$ has only been proven in a few cases. Miller and Friš [20] proved in 1992 that there are no $(2, k)$-digraphs for $k \geqslant 3$ and Baskoro et al. [4, 5] proved in 2005 the nonexistence of $(3, k)$-digraphs for $k \geqslant 3$. Regarding the diameter, Conde et al. [11, 12] showed the nonexistence of $(d, 3)$ and $(d, 4)$-digraphs for $d > 2$ in 2008 and 2013, respectively. A decade later, in 2023 López et al. [19] proved that $(4, k)$ and $(5, k)$-digraphs with self-repeats do not exist for any $k \geqslant 5$. In the case without self-repeats, Miret et al. [23] have reduced the number of possible permutation cycle structures such digraphs can have.

There exist two relevant conjectures related to the nonexistence of (d, k)-digraphs for $k > 2$. If either of them is true, then the nonexistence of (d, k)-digraphs for any $d > 1$ and $k > 2$ is proven. The first one was given by Cholily [8] in 2011 and it is based on the structure of the out-neighbours of a distinguished type of vertices, the so called *k-type* vertices [9]. The second one, the so called *cyclotomic conjecture*, was given by Gimbert [15] in 1999 and is related to the factorization of certain polynomials appearing in the characteristic polynomial of the adjacency matrix of any (d, k)-digraph. In [10] the nonexistence of almost Moore digraphs is proven assuming that the cyclotomic conjecture is true.

In this paper, we prove that (d, k)-digraphs with self-repeats do not exist when the diameter k is large with respect to the degree d. More precisely, for the pairs (d, k) such that either k is odd with $k \geqslant 2(d - 1)$ or k is even with $k \geqslant 2(d - 1)^2$ there are no (d, k)-digraphs with self-repeats. To do this, we study first the structure of certain subdigraphs of almost Moore digraphs, which are induced by the vertices fixed by an automorphism. Then, we prove that any (d, k)-digraph with self-repeats has such a (d', k)-subdigraph, $d' \leqslant d$, with permutation cycle structure of the form $(k, \dots, m_{d'-1}, 0 \dots, 0)$. Studying these digraphs in conjunction with well-known results from the literature further properties are deduced. Using these properties together with those obtained when studying the spectra of their adjacency matrix we can extend the results given in [19] for degrees 4 and 5 to show there are no almost Moore digraphs with self-repeats when the diameter is large enough with respect to the degree. Finally, combining these results with the validation (verified computationally) that the cyclotomic conjecture holds for certain small values, we conclude that for degree d, $6 \leqslant d \leqslant 12$, there are no almost Moore digraphs with self-repeats.

2 Almost Moore digraphs

A digraph G is a finite nonempty set $V(G)$ of objects called *vertices* together with a set $E(G)$ of ordered pairs of distinct vertices called *arcs*. The *order* of G is the cardinality of $V(G)$ denoted by $|V(G)|$. The set of vertices which are adjacent from, respectively to, a given vertex v is denoted by $N^+(v)$, respectively $N^-(v)$, and its cardinality is the *out-degree* of v, $d^+(v) = |N^+(v)|$, respectively *in-degree* of v, $d^-(v) = |N^-(v)|$. A digraph is *diregular* of degree d if for any vertex v, $d^+(v) = d^-(v) = d$. The length of a shortest walk $u \to v$ is the (walk) *distance* from u to v. Its maximum over all pairs of vertices is called the *diameter* k of the digraph. Besides, we denote by $N_i^+(v)$, $i \geqslant 1$, the set of vertices at distance i (in G) from v. Note that $N_1^+(v) = N^+(v)$. If we need to consider the vertices with these properties but belonging to a given subdigraph H of G, it will be denoted by $N_{i,H}^+(v)$, that is, the set of vertices at distance i (in H) from v.

This paper focuses on almost Moore digraphs, that is, diregular digraphs of degree $d > 1$, diameter $k > 1$ and whose order is one less than the Moore bound, that is,

$$N = d + d^2 + \cdots + d^k.$$

These digraphs are said to be of *defect* 1 since its order is one less than the Moore bound, that is $N - 1$.

The permutation

$$r : V(G) \longrightarrow V(G),$$

which assigns to each $v \in V(G)$ its repeat $r(v)$, is an automorphism of G (see [2]). The smallest integer $t \geqslant 1$ such that $r^t(v) = v$ is called the order of v and it is denoted by $ord_r(v)$.

Given a (d, k)-digraph G, its adjacency matrix $\boldsymbol{A}$ satisfies the equation

$$\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^k = \boldsymbol{J} + \boldsymbol{P} \tag{1}$$

where $\boldsymbol{J}$ denotes the all-one matrix and $\boldsymbol{P} = (p_{ij})$ is the $(0, 1)$-matrix associated with the permutation r of the set of vertices $V(G) = \{v_1, \ldots, v_N\}$, or, equivalently, $p_{ij} = 1$ if $r(v_i) = v_j$ and $p_{ij} = 0$ otherwise. Since G has no cycles of length less than k (see [7, 24]), its adjacency matrix $\boldsymbol{A}$ satisfies

$$\mathrm{Tr}(\boldsymbol{A}^\ell) = 0,\ \ell = 1, 2, \ldots, k-1, \text{ and } \mathrm{Tr}(\boldsymbol{A}^k) = \mathrm{Tr}(\boldsymbol{P}). \tag{2}$$

The case $\boldsymbol{P} = \boldsymbol{I}$ was studied by Bosák [6] who proved that when $k \geqslant 3$ there are no such (d, k)-digraphs. Later, Filipovski and Jajcay studied in [13] the digraphs of outdegree at most d and diameter k having order $N - \delta$ containing only self-repeat vertices, namely, those whose adjacency matrix satisfies the equation $\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^k = \boldsymbol{J} + \delta\boldsymbol{I}$, showing their nonexistence for $d \geqslant \delta \geqslant k + 1 \geqslant 4$.

In general, for $k \geqslant 3$, a (d, k)-digraph G contains either no cycles of length k or exactly one cycle of length k consisting of self-repeat vertices (see [2]). Therefore, when G has self-repeats, it has exactly k self-repeats and

$$\mathrm{Tr}(\boldsymbol{A}^k) = k. \tag{3}$$

The permutation r as a digraph automorphism decomposes into disjoint cycles. The number of such permutation cycles of each length $i \leqslant N$, will be denoted by m_i and the vector

$$(m_1, m_2, \ldots, m_N), \quad \sum_{i=1}^{N} i m_i = N,$$

will be referred to as the *permutation cycle structure* of G associated to r. Note that in the case G is a (d,k)-digraph with self-repeats it turns out $m_1 = k$.

2.1 The cyclotomic conjecture

Let G be a (d,k)-digraph with permutation matrix P. The factorization of the characteristic polynomial of $\boldsymbol{J}+\boldsymbol{P}$ in $\mathbb{Q}[x]$ can be given in terms of the cyclotomic polynomials

$$\Phi_i(x) = \prod_{1 \leqslant h \leqslant \varphi(i)} (x - \zeta_h^i),$$

where ζ_h^i ranges over the primitive roots of unity of order i and where $\varphi(i)$ stands for Euler's function.

From (1), the problem of the factorization of the characteristic polynomial of $\boldsymbol{A}$ in $\mathbb{Q}[x]$ is connected to the study of the irreducibility in $\mathbb{Q}[x]$ of the polynomials

$$F_{i,k}(x) = \Phi_i(1 + x + \cdots + x^k),$$

k being the diameter of G.

In [15], Gimbert proved that $F_{2,k}(x)$ is irreducible and gave for $i > 2$ the following conjecture on the irreducibility of these polynomials:

Conjecture 1 (Cyclotomic conjecture)**.** Let $i > 2$ and $k > 1$ be integers. Then

i) If k is even, then $F_{i,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $i \mid (k+2)$, in which case $F_{i,k}(x)$ has just two factors.

ii) If k is odd, then $F_{i,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $i \mid 2(k+2)$, in which case $F_{i,k}(x)$ has just two factors.

This conjecture is known as cyclotomic conjecture [10]. The case $k = 2$ was proved by Lenstra Jr. and Poonen [18]. The cases $k = 3$ and $k = 4$ were proved by Conde et al. in [11] and [12], respectively.

Remark 2. We have computationally checked the factorization of $F_{i,k}(x)$ in $\mathbb{Q}[x]$ for $2 < i \leqslant 12$ and $4 < k \leqslant 200$, verifying Conjecture 1 for all these values.

3 Subdigraphs of a (d,k)-digraph induced by the fixed vertices of an automorphism

The main aim of this section is to show that any almost Moore digraph contains a subdigraph which is also an almost Moore digraph having a specific permutation cycle structure. To do so we analyse the subdigraph of fixed vertices of some automorphism following a similar line of reasoning as in [25]. The next lemma will be useful to determine the permutation cycle structure of such a subdigraph.

Lemma 3. *Let G be a (d,k)-digraph and let φ be an automorphism of G. If u, v are two distinct fixed vertices by φ, then any walk of length at most k connecting u with v is fixed by φ^2.*

Proof. Since G is connected by hypothesis and its diameter is k, there exists at least one shortest walk connecting u with v of length at most k. Let W be such a walk. We distinguish two cases: either $v \neq r(u)$ and then such a walk is unique, or $v = r(u)$ and then there exists exactly one other walk W' of length at most k connecting u with v.

In the first case, when $v \neq r(u)$, $\varphi(W)$ is a path of length equal to the length of W and it connects $\varphi(u) = u$ with $\varphi(v) = v$. Therefore, by the uniqueness of W, we must have $\varphi(W) = W$. Hence, $W = \varphi^2(W)$.

In the second case, when $v = r(u)$, $\varphi(W)$ is a walk of length at most k connecting $\varphi(u) = u$ and $\varphi(v) = v$, so either $\varphi(W) = W$, and then $\varphi^2(W) = W$ holds as we wanted to show, or, otherwise, $\varphi(W) = W'$. When $\varphi(W) = W'$, W and W' must both be of length k, and either $\varphi(W') = W'$ or $\varphi(W') = W$. However, it cannot be $\varphi(W') = W'$ because $W' = \varphi(W)$, so φ would not be a digraph automorphism. Therefore we must have $\varphi(W') = W$ which implies $W = \varphi(\varphi(W)) = \varphi^2(W)$. $\square$

3.1 The subdigraphs H_α

We will study certain subdigraphs of a (d,k)-digraph G denoted by H_α. Indeed, for each positive integer $\alpha > 1$ we consider the subdigraph H_α of G induced by the following set of vertices:

$$\{v \in G : \text{there exists an integer } t \geqslant 0 \text{ such that } ord_r(v) \mid 2^t\alpha\}, \tag{4}$$

where r is the permutation of repeats of G.

Notice two important facts about the subdigraphs H_α. The first one is that if G contains self-repeats, the subdigraph of self-repeats is a directed cycle C_k and it is always contained in H_α for any α. That is,

$$C_k \subseteq H_\alpha, \quad \forall \alpha > 1.$$

The second important observation is, since $ord_r(v) = ord_r(r(v))$, that

$$v \in H_\alpha \Longleftrightarrow r(v) \in H_\alpha. \tag{5}$$

Now we introduce the concept of an (r,k)-*closed subdigraph* which is a compact notation that is useful later for the following results.

Definition 4. A subdigraph H of a (d,k)-digraph G is (r,k)-closed when for every two distinct vertices u,v from H any path of length at most k connecting u with v is also contained in H and, furthermore, $r(H) \subseteq H$.

Let $j,h \geqslant 1$ be positive integers. Since r is an automorphism, Lemma 3 implies that any walk connecting two fixed points u, v of the automorphism $r^{2^{j-1}h}$ of length at most k is fixed by $r^{2^j h}$. This property has a key implication: all fixed points by $r^{2^j h}$ with $j \geqslant 0$ is a (r,k)-closed subdigraph of G. The next proposition states and proves this result.

Proposition 5. *Any subdigraph H_α of a (d,k)-digraph G is (r,k)-closed.*

Proof. Let $u,v \in H_\alpha$ and let W a walk of length at most k connecting u with v. By the assumption $u,v \in H_\alpha$, we know that there exist non-negative integer values e_u, e_v and strictly positive integer values α_u, α_v with $\alpha_u, \alpha_v \mid \alpha$ such that:

$$ord_r(u) = 2^{e_u}\alpha_u, \qquad ord_r(v) = 2^{e_v}\alpha_v.$$

Then consider the automorphism $\varphi = r^{2^{\max(e_u,e_v)}\alpha}$. Of course, $\varphi(u) = u$ and $\varphi(v) = v$ because $ord_r(u), ord_r(v) \mid 2^{\max(e_u,e_v)\alpha}$. By Lemma 3, for any $w \in W$, $w = \varphi^2(w)$. Therefore

$$ord_r(w) \mid 2 \cdot 2^{\max(e_u,e_v)}\alpha = 2^{1+\max(e_u,e_v)}\alpha.$$

This clearly implies $w \in H_\alpha$, too. Finally, if $z \in H_\alpha$ then $ord_r(z) \mid 2^t\alpha$ for some integer $t \geqslant 0$. Then $ord_r(r(z)) \mid 2^t\alpha$ because $ord_r(r(z)) = ord_r(z)$ by (5). From here we get $r(H_\alpha) \subseteq H_\alpha$ and this completes the proof. □

3.2 Regularity of H_α

Let $v \in V(G)$ be a vertex of G and label the set of the neighbours from v, $N^+(v)$, in the following way:

$$N^+(v) = \{v_1, v_2, \ldots, v_d\}.$$

We denote by $T(v_j)$ the subset of vertices w from G such that there exists a shortest walk from v to w that goes through v_j, with $1 \leqslant j \leqslant d$. By definition of an almost Moore digraph,

$$T(v_1), T(v_2), \ldots, T(v_d)$$

are pairwise disjoint except for at most two indices i,j such that $T(v_i) \cap T(v_j) = \{r(v)\}$.

Now, define for each j with $1 \leqslant j \leqslant d$ the following values:

$$n(j) = \#\{(z,v) \in E(G) \mid z \in T(v_j)\}.$$

The distance from v_j to v is either $k-1$ if $v = r(v)$ and $v_j = r(v_j)$ or exactly k. Then, for each j with $1 \leqslant j \leqslant d$, it holds that $n(j) \geqslant 1$. Furthermore, $n(j) \leqslant 2$. Otherwise the order of G would be less than or equal to $d + d^2 + \cdots + d^k - 1$. Furthermore,

$$n(j) = 2 \Longleftrightarrow v = r(v_j),$$

so there is at most one subindex j with $1 \leqslant j \leqslant d$ such that $n(j) = 2$. Then write $v_{-j} \in V(G)$ to be the vertex from $T(v_j)$ such that

$$(v_{-j}, v) \in E(G) \text{ when } n(j) = 1.$$

And let $v_{-j,1}, v_{-j,2}$ be the two distinct vertices from $T(v_j)$ such that

$$(v_{-j,1}, v), (v_{-j,2}, v) \in E(G) \text{ when } n(j) = 2.$$

We distinguish the following cases:

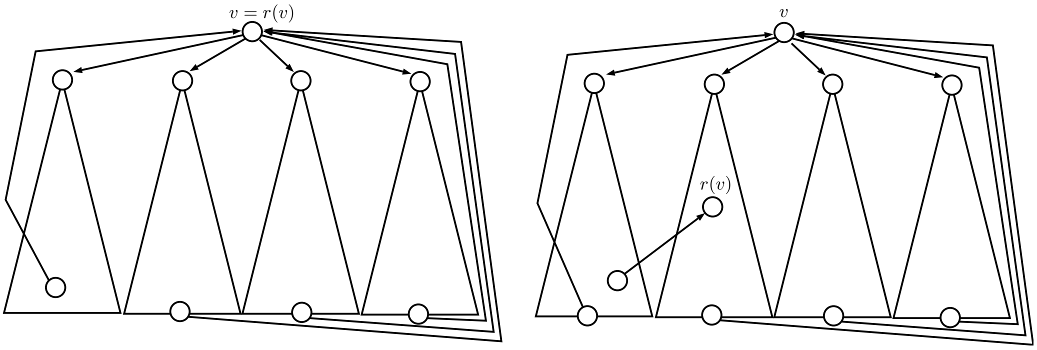

Figure 1: The case $n_j(1)$ for all j.

I) $n(j) = 1$ for all j, with $1 \leqslant j \leqslant d$. In this case it must happen that $v_{-a} \neq v_{-b}$ for any distinct values a, b with $1 \leqslant a, b \leqslant d$, because G is diregular of degree d (see [21]). We further distinguish two subcases:

   i) $n(j) = 1$ for all j, with $1 \leqslant j \leqslant d$, and v is a self-repeat (left picture from Figure 1).

   ii) $n(j) = 1$ for all j, with $1 \leqslant j \leqslant d$, and v is not a self-repeat (right picture from Figure 1).

II) $n(j) = 1$ for all j with $1 \leqslant j \leqslant d$ except for one subindex h for which $n(h) = 2$. Since G is d-regular, in this scenario there are only two possibilities:

   i) $v_{-a} = v_{-b}$ for exactly one pair of distinct values a, b with $1 \leqslant a, b \leqslant d$ and $a, b \neq h$ (left picture from Figure 2).

   ii) $v_{-a} = v_{-h,1}$ or $v_{-a} = v_{-h,2}$ for exactly one value a with $1 \leqslant a \leqslant d$ and $a \neq h$ (right picture from Figure 2).

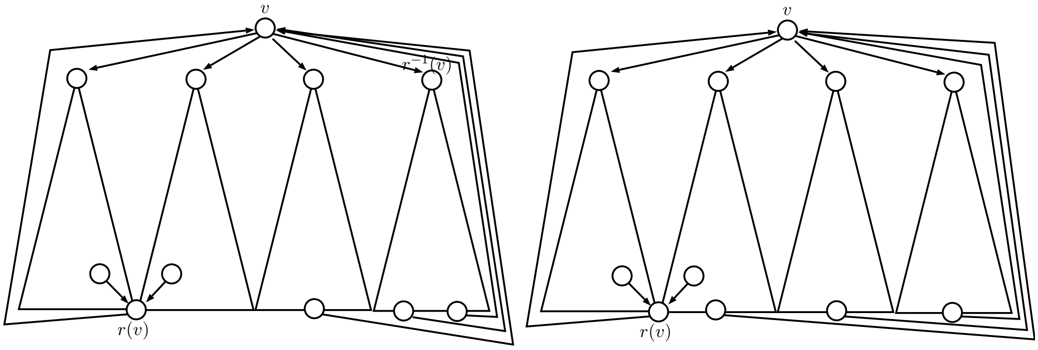

Figure 2: $n(j) = 1$ for all j except for on subindex.

As a consequence of this we have the following result.

Corollary 6. *Let H_α be a subdigraph of a (d,k)-digraph G and assume $v \in H_\alpha$. If $(z,v) \in E(G)$ is such that $z \in T(v_j)$ with $1 \leqslant j \leqslant d$, then*

$$v_j \in H_\alpha \Longleftrightarrow z \in H_\alpha .$$

Proof. By definition, both v_j and z belong to $T(v_j)$. Therefore, if $r(v) = v$ and $z \in N^+_{k-1}(v)$ the walk W_j connecting v_j with z is of length $k-2$, otherwise it has length $k-1$. Taking this into account, if $v_j \in H_\alpha$, then the walk that starts at v_j follows W_j and then uses the arc (z,v) has length at most $(k-1)+1 = k$ and goes through z. Since H_α is (r,k)-closed and both v, v_j belong to H_α then $z \in H_\alpha$, too. The converse follows similarly. □

As a direct consequence of this result we obtain:

Corollary 7. *If v is a vertex of H_α of a (d,k)-digraph G, then*

$$d^+_{H_\alpha}(v) = d^-_{H_\alpha}(v).$$

Proof. Let J be the collection of subindices j with $1 \leqslant j \leqslant d$ such that $v_j \in H_\alpha \cap N^+(v)$. Similarly, let J' be the collection of subindices j such that $v_{-j} \in H_\alpha \cap N^-(v)$. We will distinguish the following cases:

I) Case $n(j) = 1$ for all j: By Corollary 6 we get $J = J'$ implying our claim.

II Case $n(j) = 1$ for all j except for h:

  i) Since H_α is (r,k)-closed and $v \in H_\alpha$, then $v_{-a} = v_{-b} = r(v) \in H_\alpha$ and so $v_a, v_b \in H_\alpha$ as well using Corollary 6. Similarly, $v_h = r^{-1}(v) \in H_\alpha$ because, again, H_α is (r,k)-closed and $v \in H_\alpha$, so $v_{-h,1}, v_{-h,2} \in H_\alpha$ as well using Corollary 6. Therefore:

$$\begin{aligned} N^+_{H_\alpha}(v) &= \{v_j \mid j \in J\} = \{v_j \mid j \in J \setminus \{a,b,h\}\} \cup \{v_a, v_b, v_h\}, \\ N^-_{H_\alpha}(v) &= \{v_{-j} \mid j \in J'\} \cup \{v_{-h,1}, v_{-h,2}\} \\ &= \{v_{-j} \mid j \in J' \setminus \{a,b\}\} \cup \{v_{-a}, v_{-h,1}, v_{-h,2}\}. \end{aligned}$$

By Corollary 6 we have $J \setminus \{a, b, h\} = J' \setminus \{a, b\}$.

ii) Here $v_h = r^{-1}(v) \in H_\alpha$ because H_α is (r, k)-closed and $v \in H_\alpha$, so $v_{-h,1}, v_{-h,2} \in H_\alpha$ as well using Corollary 6. Since $v_{-a} = v_{-h,1}$ or $v_{-a} = v_{-h,2}$ in this case then $v_a \in H_\alpha$ as well. Therefore:

$$\begin{aligned} N^+_{H_\alpha}(v) &= \{v_j \mid j \in J\} = \{v_j \mid j \in J \setminus \{a, h\}\} \cup \{v_a, v_h\}\,, \\ N^-_{H_\alpha}(v) &= \{v_{-j} \mid j \in J' \setminus \{a\}\} \cup \{v_{-h,1}, v_{-h,2}\}\,. \end{aligned}$$

By Corollary 6, we have $J \setminus \{a, h\} = J' \setminus \{a\}$.

In both cases the conclusion can be derived easily.

Therefore, the out-degree of v is equal to its in-degree and the claim follows. □

From this, we can prove the subdigraphs H_α are diregular.

Proposition 8. *Any subdigraph H_α of a (d, k)-digraph G is diregular.*

Proof. Let $v \in H_\alpha$ and $N^+(v) = \{v_1, \ldots, v_d\}$. Without loss of generality, let $v_1, \ldots, v_s \in H_\alpha$ and $v_{s+1}, \ldots, v_d \notin H_\alpha$. By Corollary 7 it is enough to show that $d^-_{H_\alpha}(v_1) = s$ which in turn is the same as showing

$$|N^-(v_1) \cap \overline{H}_\alpha| = d - s,$$

where $\overline{H}_\alpha = G \setminus H_\alpha$. Now let us introduce the following sets (see Figure 3):

$$W_1(j) = \{(w, v_1) \in E(G) \mid w \in T(v_j)\}, \quad 1 \leqslant j \leqslant d.$$

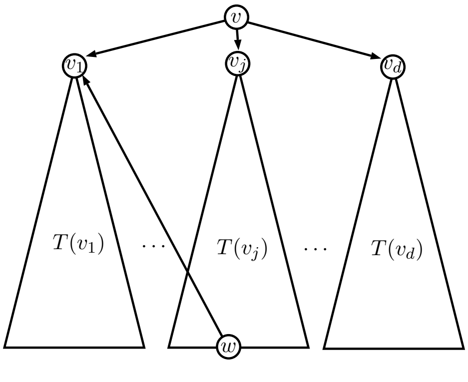

Figure 3: The connection between w and v_1.

Some key observations:

i) $|W_1(j)| \leqslant 2$ for every $j \in \{2, \ldots, d\}$ because otherwise there would be at most $N-1$ vertices in G, again, contradicting the definition of an almost Moore digraph.

ii) Moreover, if $|W_1(j)| = 2$ with $2 \leqslant j \leqslant d$ then $v_1 = r(v_j)$ implying $j \leqslant s$ because $r(H_\alpha) \subseteq H_\alpha$ since H_α is (r, k)-closed.

iii) If $|W_1(j)| = 0$ then $v = r(v)$ and $v_j = r(v_j)$ implying $j \leqslant s$ because $v \in H_\alpha$ together with the definition of H_α.

iv) If $W_1(j_1) \cap W_1(j_2) \neq \emptyset$ for some $j_1 \neq j_2$ then $T(v_{j_1}) \cap T(v_{j_2}) \neq \emptyset$. Thus we must have $T(v_{j_1}) \cap T(v_{j_2}) = \{r(v)\}$. Hence $j_1, j_2 \leqslant s$, because $v \in H_\alpha$. Moreover $r(H_\alpha) \subseteq H_\alpha$ by definition of H_α.

v) For any $w_j \in W_1(j)$, with j such that $1 \leqslant j \leqslant d$, the shortest walks given by

$$v_j \to \cdots \to w_j \to v_1 \quad \text{and} \quad v \to v_j \to \cdots \to w_j$$

have length at most k. Therefore, if $w_j \in H_\alpha$ then $v_j \in H_\alpha$ because both v, w_j belong to H_α and H_α is (r,k)-closed. Similarly, if $v_j \in H_\alpha$ then $w_j \in H_\alpha$ because both $v_1, v_j \in H_\alpha$ and H_α is (r,k)-closed.

Therefore, by (i), (ii) and (iii) every j with $s < j \leqslant d$ must satisfy $|W_1(j)| = 1$. Then, consider

$$W_1(j) = \{w_j\}, \quad s < j \leqslant d.$$

Thus, by (iv), $w_{j_1} \neq w_{j_2}$ for any two distinct values j_1, j_2 such that $s < j_1, j_2 \leqslant d$. By (v), $w_j \in \overline{H}_\alpha$. Therefore, we have that $|N^-(v_1) \cap \overline{H}_\alpha| \geqslant d - s$. On the other hand, by (v) if $w_j \in N^-(v_1) \cap \overline{H}_\alpha$ then $v_j \in \overline{H}_\alpha$, which is the same as saying $s < j \leqslant d$. From here $d - s \geqslant |N^-(v_1) \cap \overline{H}_\alpha|$. Combining these results we reach to the conclusion that $|N^-(v_1) \cap \overline{H}_\alpha| = d - s$ as claimed. □

3.3 The subdigraphs H_α different from C_k are almost Moore digraphs

Now we show that any subdigraph H_α of a (d,k)-digraph, $H_\alpha \neq C_k$, is also a (d',k)-digraph, with $d' \leqslant d$.

Lemma 9. *Suppose that H_α is d'-regular. Then for any vertex $u \in H_\alpha$ there exist d'^j vertices of H_α at distance j from u in G if $j < k$ and $d'^k - 1$ vertices of H_α at distance k from u in G.*

Proof. This claim follows easily by applying induction. Indeed, if there are d'^j vertices at distance $1 \leqslant j < k-1$ from u, by Proposition 8, H_α is d'-regular, so there are $d' \cdot d'^j = d'^{j+1}$ vertices at distance $j+1$ from u. Therefore, in particular, there are d'^{k-1} vertices at distance $k-1$ from u, implying that there are $d' \cdot d'^{k-1} - 1 = d'^k - 1$ vertices at distance k from u because $r(u) \in H_\alpha$ since $ord_r(u) = ord_r(r(u))$. □

Proposition 10. *Any subdigraph H_α of a (d,k)-digraph G has diameter at most k.*

Proof. Let u, v be vertices of H_α. Since $H_\alpha \subseteq G$ the distance from u to v is at most k in G, implying that there exists a walk W connecting u with v of length at most k. Since H_α is (r,k)-closed by Proposition 5 then W is contained inside H_α, too. Therefore, the distance from u to v inside H_α is at most k as well. □

Corollary 11. *If $H_\alpha \neq C_k$ then it has diameter k.*

Proof. If $H_\alpha \neq C_k$ then there exists $d' \geqslant 2$ such that H_α is diregular of degree d' by Proposition 8. Now, let u be any vertex from H_α. From Lemma 9 we know that there exist $d'^k - 1 \geqslant d' - 1$ vertices of H_α inside $N_k^+(u)$. Therefore, if $d' \geqslant 2$ there exists at least one vertex v of H_α at distance exactly k from u. Then $d_{H_\alpha}(u,v) \geqslant d_G(u,v) = k$ implying the conclusion. ☐

Now, from previous results we end up deducing the main result of this section.

Theorem 12. *Any subdigraph H_α of a (d,k)-digraph G different from C_k is a (d',k)-digraph, with $d' \leqslant d$.*

Proof. We know, by Proposition 8, that H_α is diregular with degree $d' \leqslant d$ and its diameter, from Corollary 11, is k. Then it is straightforward by Lemma 9 to derive that H_α has order $d' + d'^2 + \cdots + d'^k$. Hence, it is a (d',k)-digraph. ☐

4 Permutation cycle structure

In this section we will study certain permutation cycle structures that some digraphs H_α can have.

Definition 13. A permutation cycle structure $(m_1, \ldots, m_j, \ldots, m_N)$ is said to be 2-critical if there exists a unique subindex $\alpha > 1$ for which $m_\alpha \neq 0$ and the subindices $j > 1$ such that $m_j \neq 0$ are those of the form $j = 2^t\alpha$ for some $t \geqslant 0$.

This definition is crucial because as we shall see that at least one subdigraph H_α of a given almost Moore digraph has this type of permutation cycle. Note that if α is a prime number then H_α is either the subdigraph C_k or it has a permutation cycle structure which is 2-critical.

Proposition 14. *Any (d,k)-digraph G with self-repeats contains a subdigraph H_α which has a 2-critical permutation cycle structure.*

Proof. For any positive integer x, let $\hat{x}$ be the greatest odd divisor of x. Among the values $\alpha > 1$ with $m_\alpha > 0$ choose the one minimising $\hat{\alpha}$. Then the subdigraph H_α is different from C_k and according to the definition (13) its permutation cycle structure is 2-critical. Indeed, if $v \in H_\alpha$ with $\beta = ord_r(v)$, then $\beta \mid 2^t\alpha$ and $\hat{\beta} \mid \hat{\alpha}$. Hence $\hat{\beta} \leqslant \hat{\alpha}$. But obviously $m_\beta > 0$ by definition and then by construction of α we have $\hat{\beta} = \hat{\alpha}$, which is equivalent to our claim. ☐

Now we take a look at a reasoning used by Baskoro, Cholily and Miller in [1] which is extremely useful to reach the final conclusion of this section.

Proposition 15. *([1, Thm. 1]) Let v_0 be a self-repeat of a (d,k)-digraph G. Let $S_1 = \{s_0, \ldots, s_t\}$ be the set containing all the orders of vertices in $N_1^+(v_0)$ and let S_m, $m \geqslant 2$, be the subset of values* $\operatorname{lcm}(s_i, s_j)$*, with $s_i, s_j \in S_{m-1}$. If $v \in N_j^+(v_0)$, $2 \leqslant j \leqslant k$, then $ord_r(v) \in S_j$.*

Then, considering Proposition 14 together with the Proposition 15 we get the following result.

Theorem 16. *Any (d,k)-digraph G with self-repeats and 2-critical structure satisfies*

$$ord_r(v) \leqslant d-1, \quad \forall v \in G.$$

Proof. Let v_0 be a self-repeat of G. Since $r(N_1^+(v_0)) = N_1^+(r(v_0))$ (see [4]), we can consider the permutation r on $N_1^+(v_0)$. As in Proposition 15, let $S_1 = \{s_0, s_1, \ldots, s_h\}$ be the set of orders of vertices in $N_1^+(v_0)$. We assume that the permutation cycle structure of G is 2-critical, that is, there exists a value α such that $m_\alpha \neq 0$ and all the values j with $m_j \neq 0$ with $j > 1$ are of the form $j = 2^t\alpha$, with $t \geqslant 0$. Then, S_1 contains the values $2^{t_1}\alpha, 2^{t_2}\alpha, \ldots, 2^{t_h}\alpha$ with $t_1, \ldots, t_h \geqslant 0$ apart from the value 1. But in this case, $\operatorname{lcm}(2^{t_a}\alpha, 2^{t_b}\alpha) = 2^{\max(t_a,t_b)}\alpha$ and $\operatorname{lcm}(2^{t_c}\alpha, 1) = 2^{t_c}\alpha$. Therefore, $S_m \subseteq S_1$ for all $m \geqslant 1$. Hence, it is clear that

$$d = 1 + \sum_{i=1}^{h} n_i s_i, \tag{6}$$

where n_i is the multiplicity of s_i, and the claim follows. □

Corollary 17. *Any (d,k)-digraph G with self-repeats contains a (d',k)-subdigraph H_α, $d' \leqslant d$, whose permutation cycle structure is 2-critical with $m_j = 0$ for $j > d'-1$. Moreover, $\alpha \mid (d'-1)$.*

Proof. From Proposition 14 the digraph G contains a (d',k)-subdigraph H_α, $d' \leqslant d$, whose permutation cycle structure $(k, \ldots, m_j, \ldots)$ is 2-critical. Then, by Theorem 16, $ord_r(v) \leqslant d'-1$, $\forall v \in H_\alpha$, implying that $m_j = 0$, $\forall j \geqslant d'$. In this case, equality (6) can be rewritten as $d' = 1 + \sum_{i=1}^{h} n_i' 2^{t_1}\alpha$. Then, clearly $\alpha \mid (d'-1)$. □

5 Matrix approach

Now we study the spectrum of almost Moore digraphs with self-repeats and exploit the properties from the previous sections to dig deeper and obtain the main results of this paper.

Let G be a (d,k)-digraph with permutation cycle structure $(k, \ldots, m_j, \ldots)$. The characteristic polynomial of the matrix $\boldsymbol{P}$ associated to the permutation r of repeats is given by

$$c(\boldsymbol{P}, x) = (x-1)^k \prod_{m_j \neq 0} (x^j - 1)^{m_j}.$$

Its factorization in $\mathbb{Q}[x]$ in terms of cyclotomic polynomials $\Phi_i(x)$, since $x^j - 1 = \prod_{i|j} \Phi_i(x)$, is:

$$c(\boldsymbol{P}, x) = (x-1)^{m(1)} \prod_{i|j,\ m_j \neq 0} \Phi_i(x)^{m(i)}, \tag{7}$$

where $m(i) = \sum_{i|j} m_j$ represents the total number of permutation cycles of order multiple of i. Thus the matrix $\boldsymbol{P}$ is diagonalizable with eigenvalues:

- 1 with multiplicity $m(1)$;
- ζ_h^i, $1 \leqslant h \leqslant \varphi(i)$, primitive roots of unity of order i with multiplicity $m(i)$.

Therefore, the characteristic polynomial of the matrix $\boldsymbol{J}+\boldsymbol{P}$ (see [3]) is

$$c(\boldsymbol{J}+\boldsymbol{P},x) = (x-(N+1))(x-1)^{m(1)-1} \prod_{i|j,\; m_j\neq 0} \Phi_i(x)^{m(i)}. \tag{8}$$

Its eigenvalues are the same eigenvalues as the ζ_h^i of $\boldsymbol{P}$, except 1 which has multiplicity $m(1)-1$, and the eigenvalue $N+1$ with multiplicity 1.

From Lemma 1 in [19], the adjacency matrix $\boldsymbol{A}$ of G, which satisfies $\boldsymbol{I}+\boldsymbol{A}+\cdots+\boldsymbol{A}^k = \boldsymbol{J}+\boldsymbol{P}$, is also diagonalizable with eigenvalues:

- d with multiplicity 1;
- λ_h, $1 \leqslant h \leqslant m(1)-1$, roots of the factor $x^k+\cdots+x^2+x$;
- $\alpha_{h,n}^i$, $1 \leqslant n \leqslant m(i)$, roots of the factor $x^k+\cdots+x^2+x+1-\zeta_h^i$.

Note we can take a basis of eigenvectors of $\boldsymbol{A}$ which is also a basis of eigenvectors of $\boldsymbol{P}$. More precisely, the eigenvectors of $\boldsymbol{A}$ corresponding to eigenvalues $\alpha_{h,n}^i$, $1 \leqslant n \leqslant m(i)$, are eigenvectors of $\boldsymbol{P}$ corresponding to eigenvalue ζ_h^i. And the eigenvectors of $\boldsymbol{A}$ corresponding to eigenvalues d and λ_h, $1 \leqslant h \leqslant m(1)-1$, are eigenvectors of $\boldsymbol{P}$ corresponding to eigenvalue 1.

5.1 Computing some traces of $\boldsymbol{P}^j\boldsymbol{A}^\ell$

The traces of $\boldsymbol{A}^\ell$, $1 \leqslant \ell \leqslant k$, are well-known and given in (2) and (3). In order to study the traces of $\boldsymbol{P}^j\boldsymbol{A}^\ell$, for $j \geqslant 1$, we consider the sets

$$R_{\ell,j}(G) = \left\{ v \in V(G) \mid \text{there is a } r^j(v) \to v \text{ walk of length } \ell \right\},$$

which are a generalisation of $R_\ell(G)$ introduced in [19]. Note that $R_\ell(G) = R_{\ell,1}(G)$.

Proposition 18. *Let* $(k,\ldots,m_{d-1},0,\ldots,0)$ *be the permutation cycle structure of a* (d,k)*-digraph with self-repeats. If* $\ell(d-1) < k+1$ *then* $R_{\ell,j}(G) = \emptyset$.

Proof. The case $j=1$ follows from Proposition 3 in [19]. Since r is a graph automorphism, given W a path of length ℓ connecting $r^j(v)$ with v, $r^{tj}(W)$ with t such that $1 \leqslant t \leqslant ord_r(v)$ are also paths of length ℓ connecting $r^{tj}(v)$ with $r^{(t-1)j}(v)$. More precisely, given such a path W, as long as $\ell t < k$, since G does not contain cycles of length $< k$, concatenating the following directed paths of length ℓ gives another path of length $\ell t < k$ connecting $r^{tj}(v)$ with v:

$$r^{tj}(W) \to r^{(t-1)j}(W) \to \cdots \to r^j(W) \to W.$$

Therefore, since $ord_r(v) \leqslant d-1$, then when $\ell(d-1) < k+1$ we have $|R_{\ell,j}(G)| = 0$. □

Corollary 19. *Let $\boldsymbol{A}$ be the adjacency matrix of a (d,k)-digraph with permutation matrix $\boldsymbol{P}$ and permutation cycle structure $(k,\dots,m_{d-1},0,\dots,0)$. Then, for $j>0$,*

$$\mathrm{Tr}(\boldsymbol{P}^j\boldsymbol{A}^\ell)=0,\quad 1\leqslant \ell<\frac{k+1}{d-1}.$$

Proof. The case $j=1$ follows from Corollary 1 in [19]. For $j>1$, since the cardinality of the sets $R_{\ell,j}(G)$ have the following interpretation:

$$|R_{\ell,j}(G)|=\mathrm{Tr}(\boldsymbol{P}^j\boldsymbol{A}^\ell),$$

the claim follows directly by Proposition 18. □

Using this result we deduce the next proposition:

Proposition 20. *Let G be a (d,k)-digraph with self-repeats and permutation cycle structure $(k,\dots,m_{d-1},0,\dots,0)$. Then, the eigenvalues λ_h, $1\leqslant h\leqslant m(1)-1$, of its adjacency matrix $\boldsymbol{A}$ which are roots of $x^k+\cdots+x^2+x$ satisfy:*

$$0=d^\ell+\sum_{h=1}^{m(1)-1}\lambda_h^\ell,\quad 1\leqslant \ell<\frac{k+1}{d-1}. \tag{9}$$

Proof. According to the eigenvalues of $\boldsymbol{A}$ given at the beginning of this section, for each ℓ satisfying $1\leqslant\ell<\dfrac{k+1}{d-1}$ it turns out that

$$0=\mathrm{Tr}(\boldsymbol{A}^\ell)=d^\ell+\sum_{h=1}^{m(1)-1}\lambda_h^\ell+\sum_{m(i)\neq 0}\Big(\sum_{1\leqslant h\leqslant\varphi(i)}\Big(\sum_{1\leqslant n\leqslant m(i)}\alpha^i_{h,n}\Big)\Big).$$

Taking into account the eigenvalues of $\boldsymbol{P}$, together with Corollary 19, we have

$$0=\mathrm{Tr}(\boldsymbol{P}\boldsymbol{A}^\ell)=d^\ell+\sum_{h=1}^{m(1)-1}\lambda_h^\ell+\sum_{m(i)\neq 0}\Big(\sum_{1\leqslant h\leqslant\varphi(i)}\zeta_h^i\Big(\sum_{1\leqslant n\leqslant m(i)}\alpha^i_{h,n}\Big)\Big).$$

Consider now the greatest integer s such that $m_s\neq 0$. Again from Corollary 19

$$0=\mathrm{Tr}(\boldsymbol{A}^\ell)+\mathrm{Tr}(\boldsymbol{P}\boldsymbol{A}^\ell)+\cdots+\mathrm{Tr}(\boldsymbol{P}^{s-1}\boldsymbol{A}^\ell)=\mathrm{Tr}((\boldsymbol{I}+\boldsymbol{P}+\cdots+\boldsymbol{P}^{s-1})\boldsymbol{A}^\ell).$$

Therefore

$$0=s\Big(d^\ell+\sum_{h=1}^{m(1)-1}\lambda_h^\ell\Big)+\sum_{m(i)\neq 0}\Big(\sum_{1\leqslant h\leqslant\varphi(i)}(1+\zeta_h^i+\cdots+(\zeta_h^i)^{s-1})\Big(\sum_{1\leqslant n\leqslant m(i)}\alpha^i_{h,n}\Big)\Big).$$

Since $1+\zeta_h^i+\cdots+(\zeta_h^i)^{s-1}=0$ for each ζ_h^i root of unity of order a divisor of s, the claim follows. □

6 Nonexistence of (d,k)-digraphs with self-repeats having large diameter k with respect to the degree d

Now we will see, taking into account previous results, that (d,k)-digraphs do not exist when the diameter k is large enough with respect to the degree d.

In order to prove this, note that as in [19], the eigenvalues λ_h^ℓ, $1 \leqslant h \leqslant m(1)-1$, of $\boldsymbol{A}$ satisfying (9) are roots of the factor $x^k + \cdots + x^2 + x$, which can be expressed as

$$x \prod_{n>1,\, n|k} \Phi_n(x).$$

Then, it turns out that x and $\Phi_n(x)$, with $n>1$ and $n \mid k$, are factors of the characteristic polynomial of $\boldsymbol{A}$ with multiplicities a_0 and a_n, respectively, satisfying

$$a_0 + \sum_{n>1,\, n|k} \varphi(n) a_n = m(1) - 1.$$

Therefore

$$\sum_{h=1}^{m(1)-1} \lambda_h^\ell = \sum_{n>1,\, n|k} a_n S_\ell(\Phi_n(x)), \tag{10}$$

where $S_\ell(a(x))$ denotes the sum of the ℓth powers of all roots of $a(x)$.

The sums $S_\ell(\Phi_n(x))$, called *Ramanujan sums*, can be computed as (see [17])

$$S_\ell(\Phi_n(x)) = \sum_{j|\gcd(n,\ell)} \mu\left(\frac{n}{j}\right) j, \tag{11}$$

where $\mu(n)$ is the Möbius function.

Proposition 21. *Given a pair of integers (d,k), $d>1$, $k>1$, such that there exists a prime ℓ with $\ell(d-1) < k+1$ and $\gcd(\ell,k)=1$, then there cannot exist any (d,k)-digraph with self-repeats and permutation cycle structure $(k,\ldots,m_{d-1},0,\ldots,0)$.*

Proof. From Proposition 20, together with expression (10), if such a digraph exists then its adjacency matrix $\boldsymbol{A}$ would satisfy:

$$0 = d^\ell + \sum_{n>1,\, n|k} a_n S_\ell(\Phi_n(x)), \quad 1 \leqslant \ell < \frac{k+1}{d-1}. \tag{12}$$

If ℓ and k are relatively prime, then for every $n \mid k$ we have by (11):

$$S_\ell(\Phi_n(x)) = \mu(n).$$

Thus, if there exists a prime ℓ such that $\gcd(\ell,k)=1$ and $1 < \ell < \frac{k+1}{d-1}$, then $S_\ell(\Phi_n(x)) = S_1(\Phi_n(x))$ for all n with $n \mid k$, which would imply that

$$d^\ell - d = 0,$$

but this is not possible except $d=1$ or $\ell=1$. □

The existence of a prime ℓ satisfying Proposition 21 allows us to prove the nonexistence of (d,k)-digraphs with self-repeats when (d,k) fulfills the following conditions.

Theorem 22. *For $d \geqslant 6$, there do not exist (d,k)-digraphs with self-repeats when either*

- *$k \geqslant 2(d-1)$ and k odd, or*
- *$k \geqslant 2(d-1)^2$ and k even.*

Proof. We first prove that in these conditions there are no (d,k)-digraphs with self-repeats and permutation cycle structure $(k,\ldots,m_{d-1},0,\ldots,0)$. Indeed, according to Proposition 21, it is enough to find a prime value ℓ such that

$$\gcd(\ell,k)=1 \text{ and } 1<\ell<\frac{k+1}{d-1}. \tag{13}$$

Consider the distinct consecutive prime numbers up to $(k+1)/(d-1)$:

$$2=p_1<p_2<\cdots<p_s<\frac{k+1}{d-1}\leqslant p_{s+1}. \tag{14}$$

If there exists a j with $1\leqslant j\leqslant s$ such that $p_j\nmid k$ from Proposition 21 it turns out there are no (d,k)-digraphs for such d and k. In particular, if k is odd, taking $\ell=2$, we deduce that for $k\geqslant 2(d-1)$ there are no (d,k)-digraphs with structure $(k,\ldots,m_{d-1},0,\ldots,0)$. Otherwise, k is even and

$$p_1p_2\ldots p_s \mid k.$$

Now, assume $k\geqslant 2(d-1)^2$. Since there exists at least a prime number between p_s and $2p_s$, it turns out that

$$2(d-1)<\frac{k+1}{d-1}\leqslant p_{s+1}<2p_s. \tag{15}$$

Then $d-1<p_s$ and

$$k<2p_s(d-1)<2p_s^2.$$

If $s\geqslant 4$, since $p_{s-1}<p_s<2p_{s-1}$, we have

$$2p_s^2\leqslant p_{s-3}(2p_{s-1})p_s\leqslant p_{s-3}p_{s-2}p_{s-1}p_s\leqslant k<2p_s^2.$$

Thus, we must have $s\leqslant 3$ and, taking into account (14) and (15), we get

$$2(d-1)^2\leqslant k<p_4(d-1)=7(d-1).$$

This implies $2(d-1)<7$, which is a contradiction since $d\geqslant 6$. Therefore, if k is even it turns out that for $k\geqslant 2(d-1)^2$ there are no (d,k)-digraphs with self-repeats and $m_j=0$, $\forall j\geqslant d$.

Assume now there exists a (d,k)-digraph with self-repeats G such that the pair (d,k) satisfies the conditions of the claim. According to Corollary (17), G contains a (d',k)-subdigraph H_α, $d'\leqslant d$, whose permutation cycle structure is 2-critical with $m_j=0$ for all $j>d'-1$. Since either $k\geqslant 2(d-1)\geqslant 2(d'-1)$ if k is odd or $k\geqslant 2(d-1)^2\geqslant 2(d'-1)^2$ if k is even, H_α can not exist. Therefore, neither does G. □

Combining Theorem 22 together with Remark 2, we can prove the nonexistence of (d, k)-digraphs with self-repeats for degrees d, $6 \leqslant d \leqslant 12$.

Lemma 23. *Let $d > 1$ and $k > 2$ be two integers, and let $(m_1, \ldots, m_i, \ldots, m_N)$ be a N-tuple of nonnegative integers with*

$$\sum_{i=1}^{N} i m_i = N = d + d^2 + \cdots + d^k.$$

If Conjecture 1 is true for the polynomials $F_{i,k}(x)$, with $2 < i \leqslant N$ and $m(i) = \sum_{j|i} m_j \neq 0$, then (d, k)-digraphs with permutation cycle structure $(m_1, \ldots, m_i, \ldots, m_N)$ do not exist.

Proof. It follows straightfordwarly from Theorem 2 in [10]. ☐

Corollary 24. *There are no (d, k)-digraphs, $k > 2$, with self-repeats for*

$$d \in \{6, 7, 8, 9, 10, 11, 12\}.$$

Proof. By Theorem 22, (d, k)-digraphs with self-repeats do not exist when $k \geqslant 2(d-1)^2$. Since (d, k)-digraphs with self-repeats contain a subdigraph H_α whose permutation cycle structure satisfies $m_i = 0$, for $i > d - 1$ (see Corollary 17), it is straightforward to show that they do not exist for degrees d, $6 \leqslant d \leqslant 11$, and $2 < k < 2(d-1)^2$, either. Indeed, the polynomials $F_{i,k}(x)$, $2 < i \leqslant 10$ and $2 < k \leqslant 200$, factorize in $\mathbb{Q}[x]$ according to Conjecture 1. As we stated in Remark 2, this property has already been checked computationally. Therefore, by Lemma 23 (d, k)-digraphs do not exist for degrees d such that $5 \leqslant d - 1 \leqslant 10$ and $k > 2$.

For degree $d = 12$, again by Remark 2 and Lemma 23 such digraphs do not exist when $2 < k \leqslant 200$ and by Theorem 22 when $k \geqslant 2(d-1)^2 = 242$. Besides, if k is odd they do exist neither when $k \geqslant 2(d-1) = 22$. Then, it is enough to see that the $(12, k)$-digraphs do not exist for k even in the interval $[202, 240]$. But for each k in this range one can find a prime ℓ satisfying conditions (13). ☐

Finally, as a summary, Figure 4 shows the values of the degree d and the diameter k for which the existence or nonexistence of (d, k)-digraphs with self-repeats has been proven.

We have drawn in blue the line $k = 2$ for which we know the existence of $(d, 2)$-digraphs. The area in red corresponds to the values $d \in [2, 5]$ and $k \in [3, 4]$, where we already knew the nonexistence of (d, k)-digraphs with self-repeats. And the area determined by $6 \leqslant d \leqslant 12$, as well as the area $k \geqslant 2(d-1)^2$ together with lines k odd between $[2(d-1), 2(d-1)^2]$, all of them drawn in grey, are the new values for which the nonexistence has also been proven.

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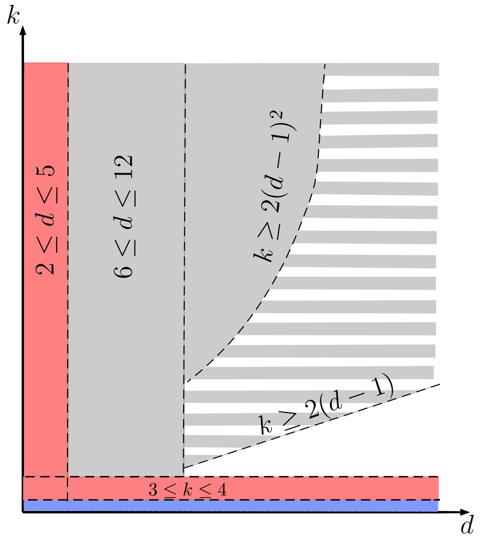

Figure 4: Values of d and k for the nonexistence of (d, k)-digraphs with self-repeats.

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