Prime factorization of meanders

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Abstract

This paper introduces a prime factorization of open meanders, articulated through the framework of 2-colored operads. We demonstrate that each open meander can be canonically constructed from building blocks of two types: iterated snakes and irreducible meanders. We find out that iterated snakes allow efficient enumeration, and thus the problem of enumerating meanders reduces to the problem of enumerating irreducible meanders. Additionally, we present some results concerning the asymptotics of meanders of both classes.

Mathematics Subject Classifications: 57K99, 05A99

1 Introduction

A meander is a configuration of a pair of simple curves in a disk (the formal definitions are given in Section 2). Examples of meanders can be viewed in Figure 1 and throughout this paper.

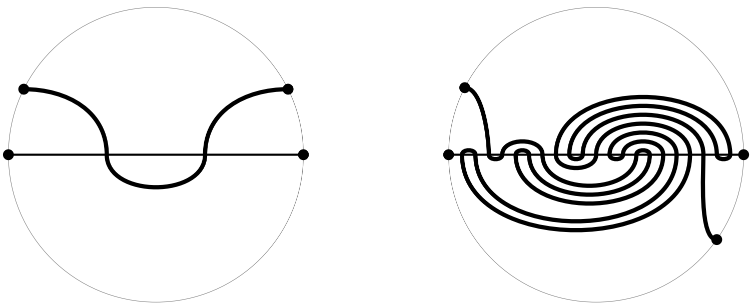

Figure 1: Examples of meanders.

V. Arnol'd was the first to use the term "meander" and to formulate the problem of counting them [Arn88]. A similar problem of counting closed meanders (under the name of planar permutation) was formulated by P. Rosenstiehl in [Ros84]. A detailed historical

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background to the subject of meanders can be found in [Zvo23]; however, this section briefly highlights some connections between meanders and other areas of mathematics and physics. Notable associations include the Temperley–Lieb algebras (see [DFGG97]), invariants of 3-manifolds (see [KS91]), models of statistical physics (see [DFGG00]), parabolic PDEs (see [FR91]), and moduli spaces of meromorphic quadratic differentials (see [DGZZ20]). The theory of meanders has been actively developed: various approaches to the calculation of meanders have been proposed (see, for example, [Jen00], [FE02], [BS10], [Lop22]) and the study of the asymptotic behavior of meander numbers ([DFGG00], [AP05]). In the recent article [DGZZ20] the exact asymptotics of the number of closed meanders with a fixed number of minimal arcs were obtained.

Despite the high interest in this area, the key questions remain open. The number of meanders with a given number of intersections is unknown, as is the asymptotic behavior of these numbers.

In this paper, we develop a new approach to the theory of meanders. We show that each meander can be canonically decomposed into prime factors of two different types: iterated snakes and irreducible meanders (Theorem 29). We also study each of these classes. It turns out that iterated snakes are fairly simple objects: we can explicitly write out their generating function (Theorem 36), and we can enumerate them very efficiently (Corollary 38). In addition, we have discovered the connection of iterated snakes to another open combinatorial problem — the enumeration of polyominoes (see Section 4.1). In contrast, irreducible meanders are mysterious and we have been able to find out only some of their basic properties.

The decomposition of meanders mirrors a phenomenon that occasionally occurs in the classification of low-dimensional topological objects, where typically, two primary classes of fundamental objects are identified: one elementary and the other more complex. All other objects are then constructed from these prime elements through specific operations, which can be described within the operadic framework. For instance, in knot theory, each knot can be classified as either a torus (representing the simpler prime objects) or hyperbolic (representing the complex prime objects) or can be constructed from them using a satellite operation (expressible in terms of operads, see [Bud12]). Similarly, in the theory of braid groups: up to conjugation each braid is either periodic, or pseudo-Anosov, or can be constructed from them using a cabling operation. This classification follows the famous Nielsen–Thurston classification (see, for example, [Thi22]); for details on braid operads see [Yau21].

The paper is structured as follows. In Section 2 we give basic definitions related to meanders; in Section 3 we discuss the factorization; in Section 4 we focus on iterated snakes, and, finally, in Section 5 we study irreducible meanders. Some computational results are given in Appendix A. More numerical data, as well as the code used to derive them, can be found in [Bel].

2 Basic definitions

Definition 1. A *singular meander* $(D, (p_1, p_2, p_3, p_4), (m, l))$ is a triple of

- Euclidean 2-dimensional disk D;
- four distinct points p_1, p_2, p_3, p_4 on the boundary ∂D such that there exists a connected component of $\partial D \setminus \{p_1, p_2\}$ containing $\{p_3, p_4\}$;
- the images m and l of smooth proper embeddings of the segment $[0; 1]$ into D such that $\partial m = \{p_1, p_3\}$, $\partial l = \{p_2, p_4\}$, and m and l intersect (not necessary transversely) in a non-zero finite number of points.

The intersection points of m and l are called *intersections of* M.

Remark 2. Usually, only meanders with transverse intersections are considered. However, it is convenient for us to extend the class of objects under consideration. In the rest of the paper we omit the word “singular”. If we wanted to emphasize the fact that a given meander has only transverse intersections, we would say “non-singular meander”.

Remark 3. What we call a “meander” is usually referred to in modern literature as an “open meander”, whereas a “meander” is a pair of closed curves in a disk. Our definition is the same as Arnol'd's original. Furthermore, it is not possible to apply our technique to closed meanders without considering open meanders.

Definition 4. We say that two meanders

$$M = (D, (p_1, p_2, p_3, p_4), (m, l))$$

and

$$M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))$$

are *equivalent* if there exists a homeomorphism $f : D \to D'$ such that $f(m) = m'$, $f(l) = l'$, and $f(p_i) = p'_i$ for each $i = 1, \ldots, 4$.

Remark 5. We now explain why we include the boundary points p_1, p_2, p_3, p_4 and impose the additional restriction — that there exists a connected component of $\partial D \setminus \{p_1, p_2\}$ containing $\{p_3, p_4\}$ — in the definition of meanders. If we do not regard the boundary points as part of the tuple, the meanders in Figure 2 would be equivalent (since they are related by a reflection across the vertical diameter of the disk and by isotopy). If we instead include the boundary points in the definition but do not impose the additional restriction, then the left meander in Figure 1 admits two non-equivalent interpretations, depending on whether the upper-left boundary point is labeled p_1 or p_3 (on both figures l is the horizontal diameter).

Remark 6. Throughout the figures we fix a drawing convention: we identify D with a Euclidean disk, draw l as a horizontal diameter with p_2 at the left end, and, after possibly reflecting across l, place p_1 above p_2. This uses that meanders are taken up to homeomorphism fixing the labeled boundary points and does not restrict generality. That is why we do not place l, m, p_1, p_2, p_3, p_4 in the figures. Examples of meanders with non-transverse intersections are given in Figure 3.

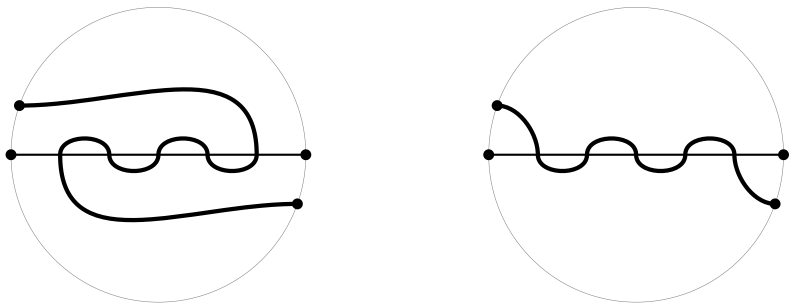


Figure 2: Meanders that would be equivalent if boundary points are not part of the data.

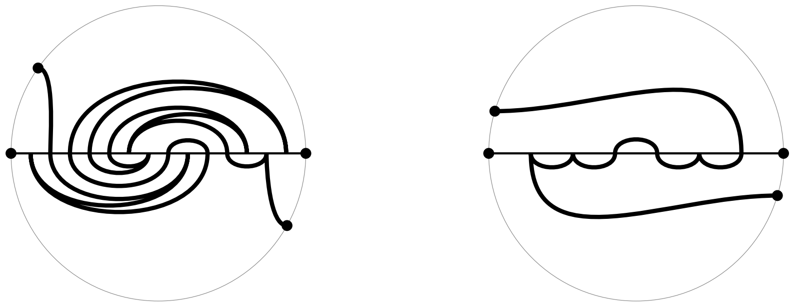


Figure 3: Examples of singular meanders.

Definition 7. Let $M = (D, (p_1, p_2, p_3, p_4), (m, l))$ be a meander, let $n_{\mathrm{t}}(M)$ be the number of transverse intersections of m and l, and let $n_{\mathrm{nt}}(M)$ be the number of non-transverse intersections of m and l. The *order of* M is the pair

$$(n, k) := \left(\max_{M' \in [M]} n_{\mathrm{t}}(M'), \min_{M' \in [M]} n_{\mathrm{nt}}(M') \right),$$

where $[M]$ is the set of all meanders equivalent to M. If the order of M is (n, k) then the *total order* of M is $n + k$. By $\mathfrak{M}_{n,k}$ we denote the set of all equivalence classes of meanders of order (n, k), and by $\mathcal{M}_{n,k}$ we denote the cardinality of $\mathfrak{M}_{n,k}$.

Remark 8. We present this definition to avoid non-transverse intersections that can be made transverse by a small isotopy — as, for example, the intersection of the graph $y = x^3$ with the x-axis $\{y = 0\}$.

Without loss of generality we always assume that if M is a meander of order (n, k), then $n_{\mathrm{t}}(M) = n$ and $n_{\mathrm{nt}}(M) = k$.

2.1 Meander permutation

It is convenient to work with meanders using their representation via permutations. To do this, we attach a permutation to each meander as follows. Let

$$M = (D, (p_1, p_2, p_3, p_4), (m, l))$$

be a meander of order (n, k). Consider a bijective map $\gamma : [0;\, n + k + 1] \to l$, such that

1. $\gamma(0) = p_2$,
2. $\gamma(t)$ is an intersection of M if and only if $t \in \{1, 2, \dots n + k\}$. We say that an intersection p has the *label* a if $\gamma(a) = p$.

If we write the labels in the order of movement from p_1 to p_3 along m, we get the *permutation of* M. Note that the labels do not depend on the choice of γ, so the permutation of M is well-defined. For example, the permutation of the second meander in Figure 3 is $(6, 5, 4, 3, 2, 1)$.

Remark 9. For non-singular meanders, the associated permutation uniquely determines the meander; by contrast, for meanders with non-transverse intersections a permutation alone does not suffice (in this case we need additional information on which intersections are transverse, and which are not).

2.2 Submeanders, and the insertion of meanders

The key ingredient in the meander factorization is the notion of submeanders (see Definition 10). For a given meander M and its submeander M', there is a canonical procedure for cutting M' from M. So, informally speaking, by choosing certain submeanders and cutting them from M we get a canonical procedure for decomposing a meander into simpler pieces.

Definition 10. We say that a meander $M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))$ is a *submeander* of a meander $M = (D, (p_1, p_2, p_3, p_4), (m, l))$ if

- $D' \subseteq D$.
- $m' = D' \cap m$.
- $l' = D' \cap l$.
- $p'_1 = \gamma_m(t_1)$, where $\gamma_m : [0;\, 1] \to D$ is any injective continuous map such that $\gamma_m([0;\, 1]) = m$, $\gamma_m(0) = p_1$, and $t_1 = \min\{t \in [0;\, 1] \mid \gamma_m(t) \in D'\}$.
- Let $S = \gamma_m^{-1}(l) \cap [0;\, t_1]$. If $S \neq \varnothing$, let $t_q = \max S$, and $q = \gamma_m(t_q)$; otherwise set $q := p_2$. Choose an injective continuous map $\gamma_l : [0;\, 1] \to D$ with $\gamma_l([0;\, 1]) = l$ such that $\gamma_l^{-1}(q) < t$ for all $t \in \gamma_l^{-1}(D')$. Then p'_2 is defined as $p'_2 = \gamma_l(t_2)$, where $t_2 = \min\{\, t \in [0;\, 1] \mid \gamma_l(t) \in D' \,\}$.

Definition 11. Let

$$M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))$$

and

$$M'' = (D'', (p''_1, p''_2, p''_3, p''_4), (m'', l''))$$

be two submeanders of a meander

$$M = (D, (p_1, p_2, p_3, p_4), (m, l)).$$

We say that M' and M'' are *equivalent with respect to* M if $D' \cap m \cap l = D'' \cap m \cap l$.

Remark 12. Note that two submeanders M' and M'' of M can be equivalent as meanders but not equivalent as submeanders with respect to M. See Figure 4: all highlighted submeanders are mutually equivalent as meanders, yet they are not equivalent as submeanders of M.

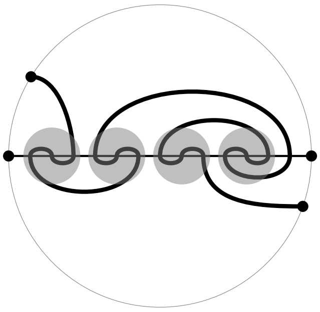


Figure 4: Examples of submeanders.

Definition 13. Let M be a meander, and let M_1 and M_2 be two of its submeanders. We say that $M_1 \leqslant M_2$ if there exists M_1' — a submeander of M that is equivalent to M_1 with respect to M, such that M_1' is also a submeander of M_2.
Thus there is a well-defined partial order on the set of all submeanders of M up to equivalence with respect to M; we denote this set by $\mathrm{Sub}(M)$.

Figure 5 shows examples of meanders and the Hasse diagram of their poset of submeanders.

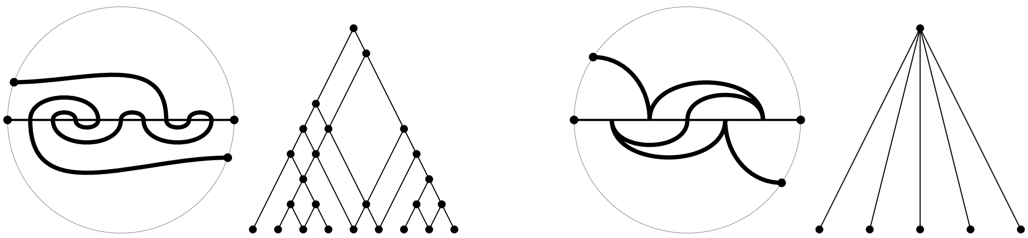


Figure 5: Examples of meanders and the Hasse diagrams of their posets of submeanders.

Definition 14. Let

$$M = (D, (p_1, p_2, p_3, p_4), (m, l))$$

and

$$M' = (D', (p_1', p_2', p_3', p_4'), (m', l'))$$

be two meanders of order (n, k) and (n', k') respectively, and let

$$M'' = (D'', (p_1'', p_2'', p_3'', p_4''), (m'', l''))$$

be a submeander of M of order (n'', k'') such that $n' \equiv n'' \mod 2$. Consider a map $f : \partial D'' \to \partial D'$ such that $f(p''_i) = p'_i$ for each $i = 1, \ldots, 4$. There is a well-defined meander

$$\tilde{M} = (\tilde{D}, (p_1, p_2, p_3, p_4), (\tilde{m}, \tilde{l}))$$

where

- $\tilde{D} = \big(D \setminus \mathrm{Int}(D'')\big) \cup_f D'$;
- $\tilde{m} = \big(m \setminus \mathrm{Int}(D'' \cap m)\big) \cup_f m'$;
- $\tilde{l} = \big(l \setminus \mathrm{Int}(D'' \cap l)\big) \cup_f l'$.

We say that $\tilde{M}$ is obtained by the *insertion of M' into M at M''*. If the total order of M' is one, we say that $\tilde{M}$ is obtained by the *cut of M' from M*.

Remark 15. Let M be a meander of total order n, and let M' be a submeander of M of total order $n' > 1$. If we cut M' from M, we get a meander M'' of total order $n - n' + 1$. An example of two consecutive cuts is shown in Figure 6.

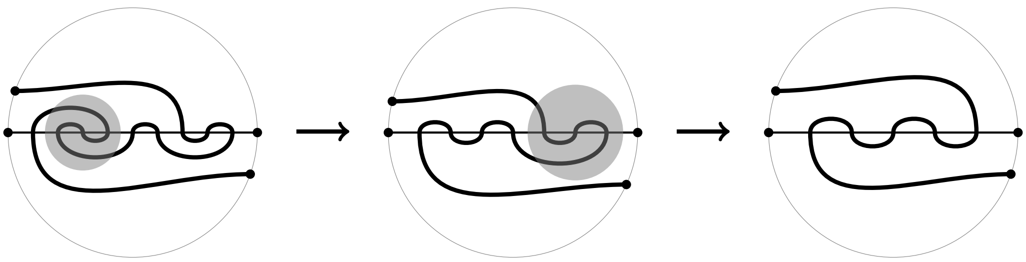


Figure 6: Example of two consecutive cuts.

2.3 The structure of a 2-colored operad

The insertion of one meander into another gives rise to the operations on the set of all equivalence classes of meanders. Let M be a meander of order (n, k), let $t_1 < t_2 < \cdots < t_n$ (resp. $s_1 < s_2 < \cdots < s_k$) be the labels of the transverse (resp. non-transverse) intersections of M. Then for a natural number i, let $M|_i$ (resp. $M|_{(i)}$) be a submeander of M with the only intersection with the label t_i (resp. s_i).

Now for an arbitrary meander M' of order $(2n'+1, k')$ we can define $M \circ_i M'$ to be a meander of order $(n + 2n', k + k')$ obtained by the insertion of M' into M at $M|_i$. Note that up to equivalence $M \circ_i M'$ is well-defined. Analogously, we can define $M \bullet_i M'$ to be the result of the insertion of a meander M' of order $(2n', k')$ into M at $M|_{(i)}$.

The straightforward check shows that these operations form a 2-colored operad on the set of equivalence classes of meanders (for the definition of a colored operad, see [LV12, Yau16], and for the examples of applications of operads in combinatorics see [Gir18]).

Theorem 16. *The set* $\mathfrak{M} = \bigcup_{n\geqslant 0,k\geqslant 0} \mathfrak{M}_{n,k}$ *together with the set of operations*

$$\circ_i : \mathfrak{M}_{n,k} \times \mathfrak{M}_{2n'+1,k'} \to \mathfrak{M}_{n+2n',k+k'} \qquad n \geqslant 1,\ k \geqslant 0,\ 1 \leqslant i \leqslant n,$$
$$\bullet_i : \mathfrak{M}_{n,k} \times \mathfrak{M}_{2n',k'} \to \mathfrak{M}_{n+2n',k+k'-1} \qquad n \geqslant 0,\ k \geqslant 1,\ 1 \leqslant i \leqslant k.$$

form a 2-colored operad.

3 Decomposition

In this section, we define prime meanders and show that each meander can be canonically decomposed into prime components.

3.1 Preliminary lemmas

Lemma 17. *Let*

$$M = (D, (p_1, p_2, p_3, p_4), (m, l))$$

be a meander of total order N *with permutation* $(\alpha_1, \alpha_2, \dots, \alpha_N)$*, and let*

$$A = \{\alpha_u, \alpha_{u+1}, \dots, \alpha_{u+v}\}$$

be some subset of its labels (here u *and* v *are natural numbers). Then there exists*

$$M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))$$

a submeander of M *containing exactly the intersections with labels from* A *if and only if*

$$\max_{\alpha\in A} \alpha - \min_{\alpha\in A} \alpha = v.$$

In other words, if and only if A *consists of consecutive numbers.*

Proof. Let M' be a submeander of M as above, and suppose

$$\max_{\alpha\in A} \alpha - \min_{\alpha\in A} \alpha \neq v.$$

Note that this difference cannot be less than v (as there are $v + 1$ different elements in A), hence it must be greater than v. In this case there exists an intersection with label $\tilde{\alpha}$ such that $\tilde{\alpha} \notin A$ and

$$\min_{\alpha\in A} \alpha < \tilde{\alpha} < \max_{\alpha\in A} \alpha.$$

Consequently, D' does not contain this point, resulting in $m' = D' \cap m$ being disconnected, which leads to a contradiction.

Conversely, assume A consists of labels as specified, with A being a set of consecutive labels. M' can be constructed explicitly. First of all, note that there exists $l' \subset l$ (resp. $m' \subset m$) — a connected subset of l (resp. m) containing precisely the intersections of M with the labels from A. All that remains is to select an arbitrary disk $D' \subset D$ such that $D \cap l = l'$ and $D \cap m = m'$. □

For a given meander M of order (n, k) and of total order greater than one, the cardinality of $\operatorname{Sub}(M)$ cannot be less than $n+k+1$, since there are always $n+k$ submeanders with a single intersection, and there is also a submeander equivalent to M. Conversely, the cardinality of $\operatorname{Sub}(M)$ cannot exceed $\frac{(n+k)(n+k+1)}{2}$, since each submeander can only contain intersections with consecutive labels (see Lemma 17).

Definition 18. Let M be a meander of order (n, k). M is said to be *irreducible* if its total order is more than two and $|\operatorname{Sub}(M)| = n+k+1$. M is said to be a *snake* if its total order is more than one and $|\operatorname{Sub}(M)| = \frac{(n+k)(n+k+1)}{2}$.
A meander is called *prime* if it is either a snake or an irreducible meander.

Prime meanders are the building blocks from which any meander can be constructed. The right meander in Figure 5 is irreducible, while the left one is non-prime. Examples of snakes are shown in Figure 7.

Remark 19. In [LZ92] a definition of an irreducible meander system was introduced. This definition is different from ours but they are similar. To be more precise, a *meander system* is a triple: $(D, \{m_1, \dots, m_r\}, l)$, where D is a Euclidean 2-dimensional disk, l is an image of smooth proper embedding of a segment into D, and $m_1, \dots, m_r$ (here $r \geqslant 1$) are pairwise disjoint images of smooth embeddings of a circle into D that intersects l only transversely. Two meander systems are called equivalent if they are homeomorphic as triples. A meander system $(D', \{m'_1, \dots, m'_{r'}\}, l')$ is a subsystem of $(D, \{m_1, \dots, m_r\}, l)$ if (i) $D' \subseteq D$, (ii) $\{m'_1, \dots, m'_{r'}\} = D' \cap \{m_1, \dots, m_r\}$, (iii) $l' = D' \cap l$. Finally, a meander system M is called *irreducible* if all its subsystems are equivalent to M.

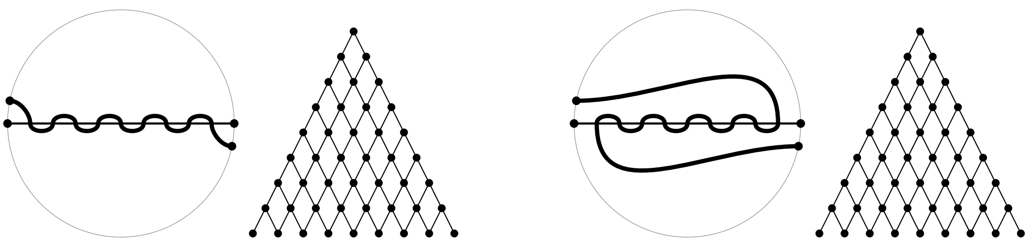


Figure 7: Examples of snakes.

Lemma 20. *If M is a non-prime meander of total order greater than one, then there exists prime meander M' that is a submeander of M.*

Proof. Let M be a non-prime meander of total order greater than one. Let us choose a submeander M' of M with the minimal total order greater than one (such submeander exists because M is non-prime). Let the total order of M' be k. If $k = 2$, then it is a snake (since all meanders of total order two are snakes); otherwise, the absence of any submeander of M' with a total order between one and k implies that M' is irreducible. □

Lemma 21. *A meander of total order n is a snake if and only if its permutation is either $(1, 2, \dots, n)$ or $(n, n-1, \dots, 1)$.*

Proof. This follows from Lemma 17. Indeed, if a meander with permutation $(\alpha_1, \ldots, \alpha_n)$ is a snake, then for each $1 \leqslant i < n$ we have $|\alpha_i - \alpha_{i+1}| = 1$. $\square$

Definition 22. A snake M of total order n is said to be *direct* if its permutation is $(1, 2, \ldots, n)$. Otherwise, M is said to be *inverse*.

Definition 23. Let M be a meander, and let M' be its submeander that is a snake. We say that M' is a *maximal snake in* M if for each snake M'' that is a submeander of M if $M' \leqslant M''$ then M' is equivalent to M''.

Lemma 24. *Let M be a meander and let M_1 and M_2 be two maximal snakes in M. If M_1 and M_2 are not equivalent with respect to M, then no submeander of M is both a submeander of M_1 and a submeander of M_2.*

Proof. The statement follows from Lemma 21. If both M_1 and M_2 are direct snakes, the statement is clear. Let M_1 be an inverse snake, and suppose that M_1 and M_2 have a common submeander. Then there exists M_2' — a submeander of M_2 such that (i) M_2' is equivalent to M_2, (ii) M_2' is a submeander of M_1. In this case, M_2 would not be a maximal snake, leading to a contradiction. $\square$

Lemma 25. *Let M be a meander, and let M_1 and M_2 be two irreducible meanders that are submeanders of M. If M_1 and M_2 are not equivalent with respect to M, then no submeander of M is both a submeander of M_1 and a submeander of M_2.*

Proof. Let M' be both a submeander of M_1 and a submeander of M_2. The total order of M' must be one (since M_1 and M_2 are irreducible and not equivalent with respect to M). Let $(\alpha_{i_1}, \ldots, \alpha_{i_r})$ be the permutation of M_1, and let (α') be the permutation of M'. Note that α' is either $\min\limits_{j=1,\ldots,r} \alpha_{i_j}$ or $\max\limits_{j=1,\ldots,r} \alpha_{i_j}$ (otherwise M' is not a submeander of M_2). But in this case

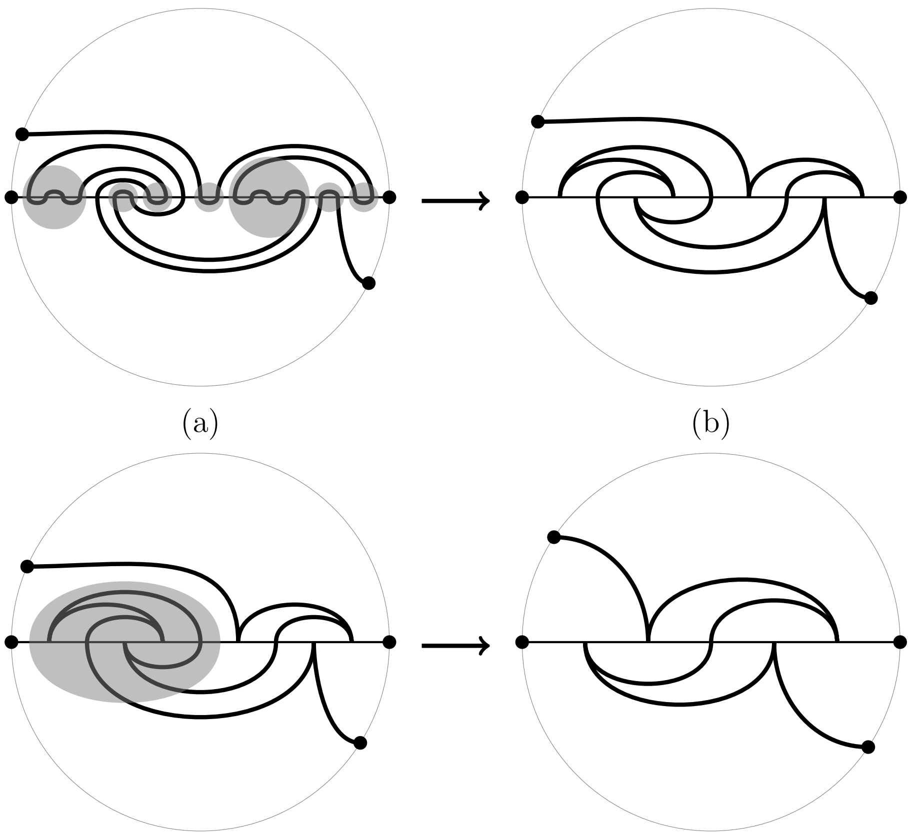
$$\max_{\substack{j=1,\ldots,r;\\ \alpha_{i_j} \neq \alpha'}} \alpha_{i_j} - \min_{\substack{j=1,\ldots,r;\\ \alpha_{i_j} \neq \alpha'}} \alpha_{i_j} = r - 2,$$

and from Lemma 17 it follows that M_1 is not irreducible (here we also use that the total order of M_1 is greater than two). $\square$

3.2 Description of the factorization

Now that we have established the foundational lemmas, we are ready to define the factorization process for an arbitrary meander. Let M be a meander of total order $N > 1$, and let $\mathrm{P}(M)$ be the set of all maximal snakes and irreducible submeanders in M that are not equivalent to each other with respect to M (due to Lemma 20 this set is not empty). Now, if we cut each $M' \in \mathrm{P}(M)$ from M, we obtain a meander M_1 of total order less than N (Lemma 24 and Lemma 25 ensure that M_1 is well-defined). If the total order of M_1 is greater than one, we can repeat this procedure. Thus, we obtain a finite sequence of meanders that ends with a meander of total order one.

Example 26. Let us consider an example of the decomposition. In Figure 8(a) we present a non-prime meander M with the highlighted set $\mathrm{P}(M)$. Figure 8(b) shows M_1 — the result of the cut. Figure 8(c) shows M_1 with the highlighted set $\mathrm{P}(M_1)$. Finally, Figure 8(d) shows M_2, which turns out to be an irreducible meander, so $\mathrm{P}(M_2) = \{M_2\}$, and M_3 has order $(1, 0)$ (so we do not provide a separate picture for it).



(a) (b) (c) (d)

Figure 8: Example of decomposition.

The language of 2-colored operads (see Section 2.3) provides a convenient framework for describing the construction of meanders using rooted trees. We represent $M \circ_i M'$ by a rooted tree with two vertices labeled M and M', joined by a directed edge labeled i that is oriented from M to M'. For the operation $M \bullet_i M'$ we use a dashed edge labeled i with the same orientation convention. An example of this construction, applied to the meander in Figure 8(a), is presented in Figure 9.

A meander may admit many such tree presentations. We single out a *canonical presentation* by imposing two constraints: (1) every vertex is labeled by a prime meander, and (2) there is no edge oriented from a snake to a direct snake (recall Definition 22).

Theorem 27. *Each open meander can be canonically constructed using snakes and irreducible meanders.*

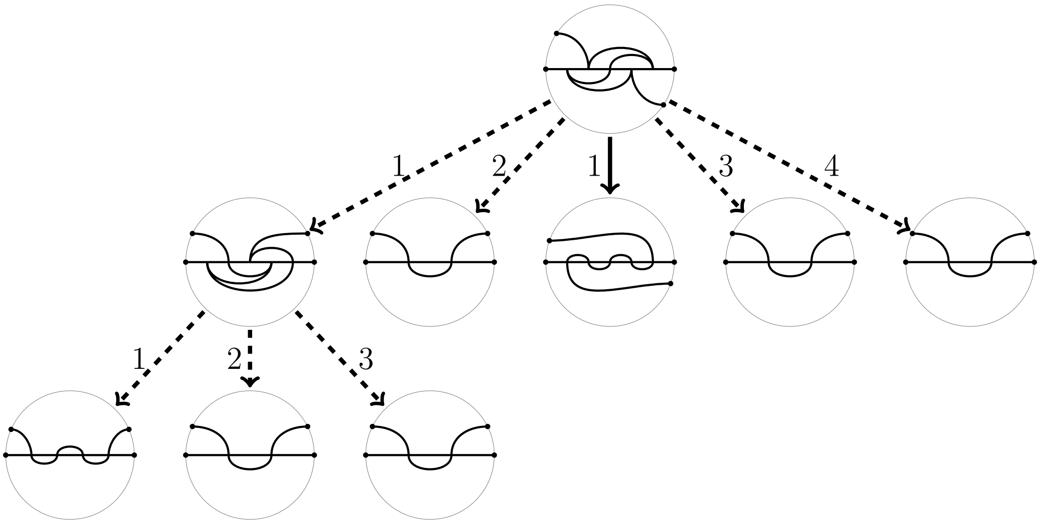


Figure 9: Construction of a meander via a rooted tree.

Proof. The existence of such a presentation is guaranteed by Lemma 20. Uniqueness is proved by induction on the depth of the tree. If the depth is zero (i. e. the meander is prime), there is nothing to prove. Assume M has a tree presentation of depth $d > 0$ satisfying (1)–(2). Note that the leaves of the tree correspond to submeanders in M. Moreover, we argue that this set of leaves is uniquely determined and corresponds precisely to the set of all irreducible submeanders and maximal snakes in M.

- If an irreducible submeander of M were not represented by a leaf, it would be part of an internal vertex. This would contradict constraint (1).
- If a leaf labeled with a snake represented a non-maximal snake, it would have to be a direct snake connected to another snake, as the only insertion that produces a snake is inserting a direct snake into a snake (this follows from Lemma 21). This would contradict constraint (2).

Therefore, the set of submeanders corresponding to the leaves is uniquely determined. If we cut these submeanders from M we obtain a meander M_1 and its tree presentation of depth $d - 1$ still satisfying (1)–(2). By the induction hypothesis, that presentation is unique, and therefore the original presentation of M is unique as well. □

We will consider another way of constructing meanders. For this purpose, we need to introduce a new class of meanders.

Definition 28. A meander is called an *iterated snake* if no irreducible meanders occur in its decomposition.

Iterated snakes form a simple class of meanders (we discuss this class in detail in Section 4). Examples of iterated snakes are shown in Figure 10. Using iterated snakes, the uniqueness of the construction can be stated as follows: each meander admits a

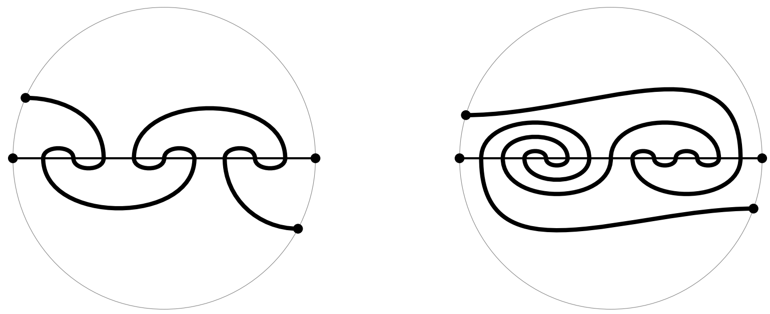


Figure 10: Example of iterated snakes.

unique tree presentation in which the vertices are labeled by either irreducible meanders or iterated snakes, with the additional condition that no edge is oriented from an iterated snake to another iterated snake.

Theorem 29. *Each open meander can be canonically constructed using iterated snakes and irreducible meanders.*

Remark 30. In fact, the decomposition of non-singular meanders can be expressed without passing to the singular ones. To do this, we need to insert meanders with an even number of intersections into *cups* — submeanders of total order two. However, this approach is much less convenient: the description of the decomposition becomes very cumbersome, and many constructions become less natural. In particular, the equation for the generating function of the meander numbers is not so easy to write (see Theorem 31).

3.3 Equation for the generating function.

The construction of meanders via trees allows us to express the generating function of all meanders in terms of the generating functions of iterated snakes and of irreducible meanders.

We use the following notation. Let $\{A_{n,k}\}_{n\geqslant 0,\, k\geqslant 0}$ be a bivariate sequence of numbers, and let $f(x,t) = \sum\limits_{n\geqslant 0, k\geqslant 0} A_{n,k}x^n t^k$ be its generating function. We decompose $f(x,t)$ into two parts (the "odd" part $f^{(1)}(x,t)$ and the "even" part $f^{(2)}(x,t)$) in the following way:

$$f(x,t) = \underbrace{\sum_{n\geqslant 0,k\geqslant 0} A_{2n+1,k}x^{2n+1}t^k}_{=:f^{(1)}(x,t)} + \underbrace{\sum_{n\geqslant 0,k\geqslant 0} A_{2n,k}x^{2n}t^k}_{=:f^{(2)}(x,t)}.$$

Let $f(x,t)$ and $g(x,t)$ be two bivariate generating functions with $g(0,0) = 0$. Then we use the following notation:

$$(f \boxdot g)(x,t) := f\left(g^{(1)}(x,t),\, g^{(2)}(x,t)\right).$$

This notation has a direct combinatorial meaning for meanders. The variable x marks transverse intersections, and t marks non-transverse ones. A meander with an odd number of transverse intersections can be inserted only at a transverse intersection, whereas one with an even number of transverse intersections can be inserted only at a non-transverse intersection. Thus insertion corresponds to the above composition of generating functions via $\boxdot$.

Theorem 31. *Let*

$$\phi(x,t) = \sum_{n,k} \mathcal{M}_{n,k} x^n t^k,$$
$$\phi_{(Ir)}(x,t) = \sum_{n,k} \mathcal{M}^{(Ir)}_{n,k} x^n t^k,$$
$$\phi_{(IS)}(x,t) = \sum_{n,k} \mathcal{M}^{(IS)}_{n,k} x^n t^k$$

be the generating functions for the numbers of equivalence classes of meanders, of irreducible meanders, and of iterated snakes, respectively. Then

$$\phi(x,t) = x + t + \left(\phi_{(Ir)} \boxdot \phi\right)(x,t) + \left(\phi_{(IS)} \boxdot \left(x + t + \left(\phi_{(Ir)} \boxdot \phi\right)\right)\right)(x,t). \qquad (1)$$

Proof. By Theorem 29, every meander is obtained in exactly one of the following ways: either it has total order 1 (contributing the term $x+t$), or else it is represented by a tree whose root is labeled with

- an irreducible meander, and its subtrees can represent arbitrary meanders (contributing $\phi_{(Ir)} \boxdot \phi$);
- an iterated snake, and its subtrees are not rooted at iterated snakes (contributing $\phi_{(IS)} \boxdot (x + t + \phi_{(Ir)} \boxdot \phi)$).

Insertions are encoded by $\boxdot$. Since the generating functions involved have zero constant term (there is no empty meander), these substitutions are well-defined. Collecting the three disjoint cases yields (1). □

Remark 32. In equation (1) the generating function $\phi_{(IS)}(x,t)$ is known (see Section 4), while for $\phi_{(Ir)}(x,t)$ only a few terms are known (see Section 5).

It follows from Theorem 31 that knowing the numbers $\left\{\mathcal{M}^{(Ir)}_{n,k}\right\}_{n,k}$ and $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n,k}$ we could easily compute the numbers $\{\mathcal{M}_{n,k}\}_{n,k}$ themselves. As it is shown in Section 4, $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n,k}$ can be computed quite easily even for large values of n and k. On the contrary, we do not know any efficient algorithm for finding the numbers $\left\{\mathcal{M}^{(Ir)}_{n,k}\right\}_{n,k}$ other than brute force (with one exception, see Theorem 42). Thus the problem of computing the numbers $\{\mathcal{M}_{n,k}\}_{n,k}$ is reduced to the problem of computing the numbers $\left\{\mathcal{M}^{(Ir)}_{n,k}\right\}_{n,k}$.

4 Iterated snakes

In this section, we discuss some properties of iterated snakes. In particular, we give an effective formula to calculate the number of equivalence classes of iterated snakes and discuss some results about the asymptotic behavior of these numbers.

Lemma 33. *Let $\mathcal{M}^{(S)}_{n,k}$ be the number of pairwise non-equivalent snakes of order (n,k). Then*

$$\mathcal{M}^{(S)}_{n,k} = \begin{cases} 0 & n+k<2, \\ \binom{n+k}{n} & n \text{ is even}, \\ 2\binom{n+k}{n} & n \text{ is odd}. \end{cases}$$

Proof. Let M be a snake of order (n,k). By Lemma 21 the permutation of M is either $(1,2,\ldots,n+k)$ or $(n+k,n+k-1,\ldots,1)$. Arbitrary n labels in the permutation can correspond to transverse intersections, and all the different ways of choosing such labels lead to different snakes. It remains to note that there are no inverse snakes of order $(2n,k)$ for $n \geqslant 0$. □

Corollary 34. *Let $\phi_{(S)}(x,t) = \sum_{n\geqslant 0, k\geqslant 0} \mathcal{M}^{(S)}_{n,k} x^n t^k$ be the generating function of the numbers of equivalence classes of snakes. Then*

$$\phi_{(S)}(x,t) = -\frac{3}{2(t+x-1)} - \frac{1}{2(x-t+1)} - t - 2x - 1.$$

Remark 35. If we consider meanders of total order one as snakes, the generating function and the corresponding numerical sequences take a somewhat more natural form. However, this requires additional caveats when discussing the factorization. Therefore, we prefer not to consider them as snakes.

Now, we can move on to the enumeration of iterated snakes.

Theorem 36. *Let $\mathcal{M}^{(IS)}_{n,k}$ be the number of pairwise non-equivalent iterated snakes of order (n,k), and let $\phi_{(IS)}(x,t)$ be the generating function of these numbers. Then*

$$\phi^{(1)}_{(IS)}(x,t) = \phi^{(1)}_{(S)}\left(x + \frac{\phi^{(1)}_{(IS)}(x,t)}{2},\ t\right), \tag{2}$$

$$\phi^{(2)}_{(IS)}(x,t) = \phi^{(2)}_{(S)}\left(x + \frac{\phi^{(1)}_{(IS)}(x,t)}{2},\ t\right). \tag{3}$$

Proof. The uniqueness of the decomposition requires forbidding insertions of direct snakes into other snakes (see Section 3.2). Therefore, each iterated snake is obtained by a sequence of insertions of inverse snakes into some snake. Note that inverse snakes have an odd number of transverse intersections, so the generating function of inverse snakes is $\frac{1}{2}\phi^{(1)}_{(S)}(x,t)$. □

Remark 37. We can use the equations (2)–(3) to explicitly find $\phi_{(IS)}(x,t)$, but the resulting formula is too cumbersome, so we do not include it in this paper.

We can use Theorem 36 to calculate the numbers $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n,k}$ directly. To write the resulting formula, we need to introduce some notation. Let n and k be non-negative integers. Then

- $\delta(n) := (n \bmod 2)$;
- $\mu \vdash (n,k)$ means that $\mu = ((a_1,b_1)^{s(a_1,b_1)}; (a_2,b_2)^{s(a_2,b_2)}; \ldots; (a_r,b_r)^{s(a_r,b_r)})$ is a partition of (n,k), (i. e. $\sum\limits_{(a,b)\in\mu} s_{(a,b)}a = n$ and $\sum\limits_{(a,b)\in\mu} s_{(a,b)}b = k$);
- if $\mu \vdash (n,k)$, then $|\mu| := \sum\limits_{(a,b)\in\mu} s_{(a,b)}$;
- if $\mu \vdash (n,k)$, then $\binom{n}{\mu} := \frac{n!}{(n-|\mu|)! \prod\limits_{(a,b)\in\mu}(s_{(a,b)}!)}$.

Corollary 38. *For $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n,k}$ the following recurrence relation holds:*

$$\mathcal{M}^{(IS)}_{n,k} = \sum_{r=1}^{\frac{n}{2}} \sum_{l=0}^{k} \mathcal{M}^{(S)}_{2r+\delta(n),l} \left(\sum_{\mu} \binom{2r+\delta(n)}{\mu} \prod_{(i_1,i_2)\in\mu} \left(\frac{\mathcal{M}^{(IS)}_{2i_1+1,i_2}}{2} \right)^{s(i_1,i_2)} \right), \qquad (4)$$

where the third summation goes through all partitions μ of $\left(\frac{n-2r-\delta(n)}{2}, k-l\right)$, such that $|\mu| \leqslant 2r+\delta(n)$.

We have used equations (2) and (3) to calculate the numbers $\left\{\mathcal{M}^{(IS)}_{n,k}\right\}_{n\leqslant 100,k\leqslant 30}$ and $\left\{\mathcal{M}^{(IS)}_{n,0}\right\}_{n\leqslant 500}$. The calculation was done with a C++ program (the code of the program and the results of the calculations are available at [Bel]).

4.1 Number sequences associated with iterated snakes

The numbers $\left\{\mathcal{M}^{(IS)}_{n,0}\right\}_{n\geqslant 2}$ are the same as the numbers of P-graphs with $2n$ edges defined in the work [Rea86][1] (see [OEI, A007165]). It is a well-studied number sequence, so we know the exact asymptotics of its even and odd parts (see [OEI, A100327] and [OEI, A003169] and references therein):

$$\mathcal{M}^{(IS)}_{2n+1,0} \sim \frac{\sqrt{4046+1122\sqrt{17}}}{136\sqrt{\pi}} \left(\frac{71+17\sqrt{17}}{16}\right)^n n^{-\frac{3}{2}},$$

$$\mathcal{M}^{(IS)}_{2n,0} \sim \frac{\sqrt{33\sqrt{17}-119}}{4\sqrt{34\pi}} \left(\frac{71+17\sqrt{17}}{16}\right)^n n^{-\frac{3}{2}}.$$

[1] P-graphs are nothing but a graph-theoretic reformulation of iterated snakes, so we omit the precise definition.

Remark 39. Note that the asymptotics of the numbers of non-singular iterated snakes with even and odd numbers of intersections are slightly different. We expect a similar phenomenon for irreducible meanders (this agrees with the results of numerical experiments: see Figure 14).

We found other interesting number sequences among the numbers of iterated snakes. The numbers $\left\{\mathcal{M}_{1,k}^{(IS)}\right\}_{k\geqslant 1}$ coincide[2] with the numbers [OEI, A007070] which form the sequence of numbers satisfying the following recurrence formula: $a_n = 4a_{n-1} - 2a_{n-2}$ where $a_0 = 1$ and $a_1 = 4$. It would be interesting to find a combinatorial explanation for why this recurrence relation holds for $\left\{\mathcal{M}_{1,k}^{(IS)}\right\}_{k\geqslant 1}$.

Another finding links the meander counting problem to a well-known open combinatorial problem — the counting of polyominoes, defined in [Gol54]. In [CFM+07] the authors introduce a subclass of polyominoes that can be enumerated via 2-compositions. The sum of the entries in the top rows of all 2-compositions of k ([OEI, A181292]) coincides with the numbers $\left\{\mathcal{M}_{2,k}^{(IS)}\right\}_{k\geqslant 0}$. As in the previous case, at the moment we do not know why these sequences match.

5 Irreducible meanders

In this section, we prove some elementary properties of irreducible meanders, in particular about their asymptotics.

As we said before, to calculate the numbers $\{\mathcal{M}_{n,k}\}_{n,k}$ one only needs to know $\left\{\mathcal{M}_{n,k}^{(Ir)}\right\}_{n,k}$ (because the numbers $\left\{\mathcal{M}_{n,k}^{(IS)}\right\}_{n,k}$ are easy to calculate). Unfortunately, we do not know any suitable way to calculate these numbers. We used a fairly straightforward brute-force algorithm (similar to the one described in [SL12]) to calculate $\left\{\mathcal{M}_{n,k}^{(Ir)}\right\}_{n,k}$ for $n+2k \leqslant 38$. The results of the calculations can be found in [Bel] and partially in Appendix A. In contrast to iterated snakes, no known numerical sequences were found among the numbers $\left\{\mathcal{M}_{n,k}^{(Ir)}\right\}_{n,k}$.

For what follows, it will be useful for us to transform singular meanders into non-singular ones. Let us describe this procedure. There exists a projection

$$c : \bigcup_{n,k\geqslant 0} \mathfrak{M}_{n,k} \to \bigcup_{n\geqslant 0} \mathfrak{M}_{n,0}$$

defined by the insertion of a meander of order $(2, 0)$ at each non-transverse intersection, so the meander of order (n, k) is mapped to the non-singular meander of order $(n+2k, 0)$. Note that this projection is surjective, and if M and M' are non-equivalent irreducible meanders, then $c(M)$ is not equivalent to $c(M')$ (due to the uniqueness of the factorization). A non-singular meander of order $(n, 0)$ is called *almost irreducible* if it is an image

[2] To be fully consistent with this sequence, a meander of order $(1, 0)$ must also be considered a snake (see Remark 35).

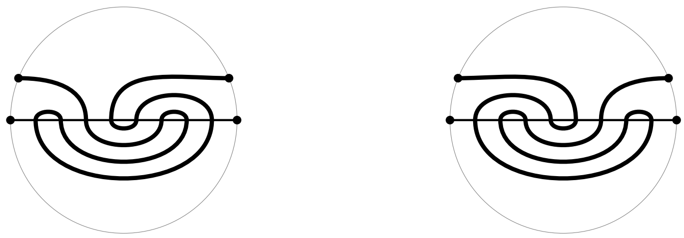
of an irreducible meander under the map c (examples of almost irreducible meanders are presented in Figure 11).

Proposition 40. $\mathcal{M}_{n,k}^{(Ir)} \equiv 0 \bmod 2$.

Proof. Let $M = (D,(p_1,p_2,p_3,p_4),(m,l))$ be an irreducible meander of order (n,k). If n is even, consider the meander $M' = (D,(p_3,p_4,p_1,p_2),(m,l))$. If n is odd, let $M' = (D,(p_3,p_2,p_1,p_4),(m,l))$. In both cases M' is irreducible. To conclude the proof we need to show that M and M' are not equivalent. Since there is a one-to-one correspondence between irreducible meanders of order (n,k) and almost irreducible meanders of total order $N = n+2k$ with precisely k cups, it suffices to show that $c(M)$ is not equivalent to $c(M')$.

Let $(a_1,a_2,\dots,a_N)$ be the permutation of $c(M)$. If n is odd, the permutation of $c(M')$ is $(a_N,a_{N-1},\dots,a_1)$, so M is not equivalent to M'.

Now let us consider the case when n is even. In this case the permutation of $c(M')$ is given by $(\sigma(a_N),\sigma(a_{N-1}),\dots,\sigma(a_1))$, where $\sigma(i) = N+1-i$; see the examples in Figure 11.



$(3,6,7,2,1,8,5,4)$ $(5,4,1,8,7,2,3,6)$

Figure 11: Examples of almost irreducible meanders $c(M)$ and $c(M')$.

Let $\tilde{M} = (D,(p_1,p_2,p_3,p_4),(\tilde{m},l))$ be a non-singular meander equivalent to $c(M)$. Since for non-singular meanders the permutation uniquely determines the equivalence class of a meander, $\tilde{M}$ is equivalent to $c(M')$ if and only if, for each $i = 1,\dots,N$, we have $a_i = N+1-a_{N+1-i}$. Assume that $\tilde{M}$ is equivalent to $c(M')$. In that case $a_1 \leqslant \frac{N}{2}$. Otherwise, $a_N = N+1-a_1 < \frac{N}{2}+1$, and the subarc of $\tilde{m}$ connecting p_1 to the point labeled a_1 would intersect the subarc of $\tilde{m}$ connecting p_3 to the point labeled a_N.

Let $r \geqslant 1$ be the maximal integer such that $a_i \leqslant \frac{N}{2}$ for all $1 \leqslant i \leqslant r$ (thus $a_{N+1-i} > \frac{N}{2}$ for $1 \leqslant i \leqslant r$). If $a_{r+1} \neq a_{N-r}$, then the subarc of $\tilde{m}$ connecting the points labeled a_r and a_{r+1} would intersect the subarc of $\tilde{m}$ connecting the points labeled a_{N-r+1} and a_{N-r} (since both of these subarcs lie on the same side of l). Thus $a_{r+1} = a_{N-r}$, and therefore the permutation of $\tilde{M}$ is

$$(a_1,\dots,a_{r-1},a_r,a_{N-r},a_{N-r+1},\dots,a_N),$$

so $r = \frac{N}{2}$. By Lemma 17, $\tilde{M}$ contains a submeander of total order $\frac{N}{2}$. Since an almost irreducible meander of total order N has only submeanders of total orders 1, 2, and N, and since there are no almost irreducible meanders of total order less than 8, the assumption that $c(M)$ is equivalent to $c(M')$ leads to a contradiction. □

Proposition 41. $\mathcal{M}_{n,k}^{(Ir)} = 0$, *if* $k < 3$. *In particular, there are no non-singular irreducible meanders.*

Proof. Closed meanders can be represented via a pair of words in Dyck language (see [LZ92]). A Dyck language consists of strings of balanced parentheses, where each opening parenthesis "(" is correctly matched and nested with a closing parenthesis ")". This is a well-known object in combinatorics, so we limit ourselves to this brief description (for details, see any book on combinatorics, for example [Sta99]). To work with (open) meanders, we extend the alphabet by adding an extra symbol — "—". The procedure of matching a non-singular meander M to a pair of words (A_M, B_M) in an extended Dyck language is shown in Figure 12.

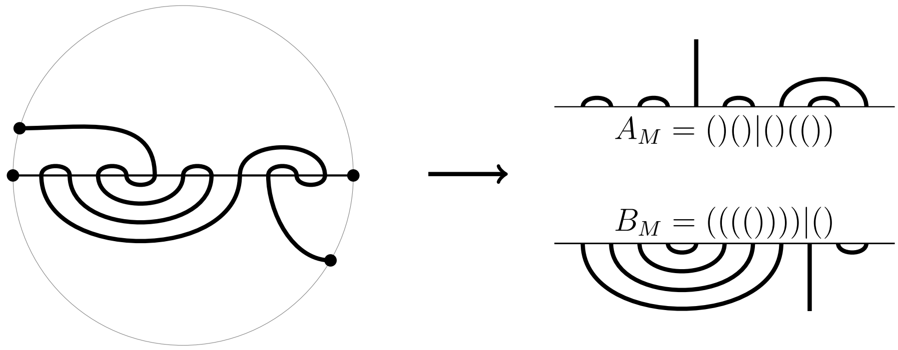


Figure 12: Correspondence between non-singular meander and a pair of words in the extended Dyck language.

A *cup* in a non-singular meander is a submeander of order $(2, 0)$. In terms of Dyck language it corresponds to a substring of the form $()$. The presentation of a non-singular meander as a pair of words in Dyck language shows that each non-singular meander of order greater than two has at least two cups. Furthermore, a non-singular meander M has precisely two cups only if the corresponding pair of words (A_M, B_M) is either $A_M = |(\dots()\dots)$ and $B_M = (\dots()\dots)|$ or $A_M = (\dots()\dots)|$ and $B_M = |(\dots()\dots)$. In both cases, it contains a submeander of order $(3, 0)$. It follows that each almost irreducible meander has at least three cups, and thus the number of non-transverse intersections in an irreducible meander is at least three. □

Theorem 42.

$$\mathcal{M}_{n,3}^{(Ir)} = \begin{cases} 0 & n \text{ is odd,} \\ \varphi\left(\frac{n}{2}+4\right) - 2 & n \text{ is even,} \end{cases}$$

where $\varphi(x)$ *is Euler's totient function.*

Proof. Let M be an almost irreducible meander of total order $n + 6$ with precisely three cups (each such meander corresponds to an irreducible meander of order $(n, 3)$). We can encode M with a pair of words (A_M, B_M) in extended Dyck language, as it was done in the proof of Proposition 41. Since M is almost irreducible, neither A_M nor B_M begins or ends with "|"; otherwise the associated permutation would begin or end with 1 or $n+6$, and hence, by Lemma 17, M would contain a submeander of total order $n+5$. It follows that

$$A_M = \underbrace{((\ldots(}_{n_1}\underbrace{))\ldots)}_{n_1}\,||\,\underbrace{((\ldots(}_{n_2}\underbrace{))\ldots)}_{n_2},$$
$$B_M = \underbrace{((\ldots(}_{n_1+n_2+1}\underbrace{))\ldots)}_{n_1+n_2+1},$$

where $n+6 = 2(n_1+n_2+1)$ with n_1 and n_2 being positive. In particular, we see that n must be even. Thus, we can encode almost irreducible meanders with three cups via a pair of positive integers (n_1, n_2). We show that such a pair of numbers corresponds to a meander if and only if n_1+1 and n_2+1 are coprime. The statement of the theorem follows from this immediately. Indeed,

$$\mathrm{GCD}(n_1+1, n_2+1) = \mathrm{GCD}\left(n_1+1, \frac{n}{2}+2-n_1+1\right) = \mathrm{GCD}\left(n_1+1, \frac{n}{2}+4\right).$$

Thus we obtain

$$\mathcal{M}_{n,3}^{(Ir)} = \sum_{\substack{n_1,n_2>0 \\ n+6=2(n_1+n_2+1) \\ \mathrm{GCD}(n_1+1,n_2+1)=1}} 1 = \sum_{\substack{n_1=1 \\ \mathrm{GCD}\left(n_1+1,\frac{n}{2}+4\right)=1}}^{\frac{n}{2}+1} 1 = \sum_{\substack{n_1=2 \\ \mathrm{GCD}\left(n_1,\frac{n}{2}+4\right)=1}}^{\frac{n}{2}+2} 1$$
$$= \varphi\left(\frac{n}{2}+4\right) - 2.$$

To demonstrate that a pair (n_1, n_2) corresponds to a meander if and only if n_1+1 and n_2+1 are coprime, we consider a meander with three cups as a curve in a disk with punctures. Such curves arose in the study of braid groups (see for example [Dyn02], [Mal04], [DW07]), and in particular such curves have been used to develop efficient algorithms for comparing braids. We use a simple special case of such an algorithm. For completeness, we mention that these are particular cases of a more general technique introduced by Agol, Hass, and Thurston [AHT06].

To a pair of numbers (n_1, n_2) we associate a pair of words in the extended Dyck language as above. Using such a pair we can construct a set of curves in a disk (as was done for meanders). Let us connect the two free points in order to obtain a set of closed curves (the number of connected components would not change), which we denote by $X_{(n_1,n_2)}$ (see the example in Figure 13). Without loss of generality, we can assume that $n_1 \geqslant n_2$. We consider several cases.

1. If $n_1 = n_2 \neq 0$, then $X_{(n_1,n_2)}$ has several connected components, and therefore (n_1, n_2) does not correspond to a meander.

2. If $n_2 = 0$, then $X_{(n_1,n_2)}$ is a single curve. (This case does not correspond to an almost irreducible meander with three cups, but we will need this case for later analysis).

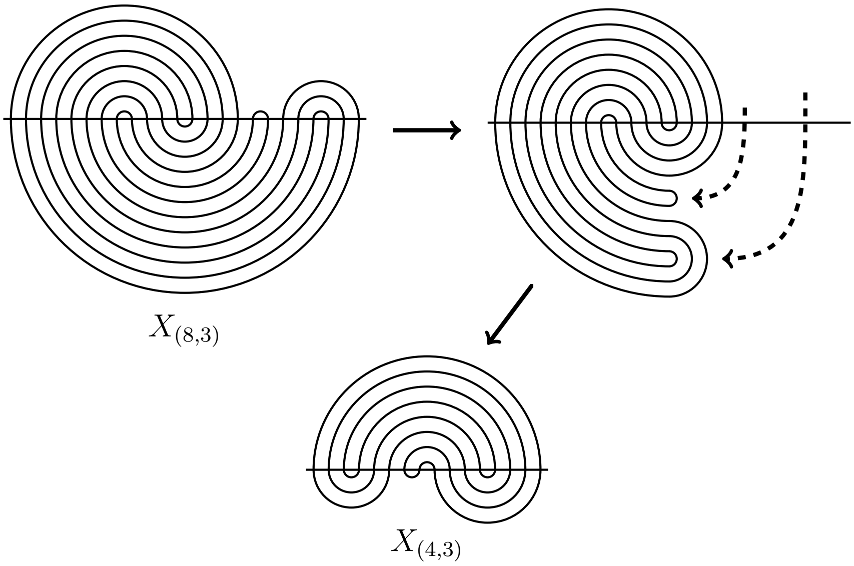


Figure 13: An example of simplification of $X_{(n_1,n_2)}$.

3. In general case we can simplify $X_{(n_1,n_2)}$ by pulling $2n_2+2$ arcs resulting in $X_{(n_2,n_1-n_2-1)}$, see Figure 13. Consequently, $X_{(n_1,n_2)}$ is a single curve if and only if $X_{(n_2,n_1-n_2-1)}$ is a single curve. That is, we almost get Euclid's algorithm: consider a sequence $\{n_i\}_{i=1,2,\dots}$ where for each $i > 2$ $n_i = n_{i-2} \mod (n_{i-1}+1)$. If $n_i = n_{i+1} \neq 0$ for some i, then $X_{(n_1,n_2)}$ has several connected components (according to the first case). Otherwise, $n_i = 0$ for some i and $X_{(n_1,n_2)}$ is a single curve (according to the second case). The only thing left to note is that $\{n_i\}_{i=1,2,\dots}$ stabilizes with zeroes if and only if n_1+1 and n_2+1 are coprime.

□

5.1 Asymptotics of the numbers of irreducible meanders

In this subsection, we show that the number of pairwise non-equivalent irreducible meanders with a fixed number of non-transverse intersections grows at most polynomially. By contrast, the growth rate of almost irreducible meanders is exponential.

Theorem 43. *For a fixed natural number k*

$$\sum_{n=1}^{N} \mathcal{M}_{n,k}^{(Ir)} = O(N^{2k-4}).$$

To prove this theorem we need the following lemma.

Lemma 44. *Let k be a natural number greater than two, and let $\mathcal{B}_{n,k}$ be the number of pairwise non-equivalent non-singular meanders with precisely n intersections and k cups. Then*

$$\mathcal{B}_{2n+1,k} \leqslant \mathcal{B}_{2n+2,k},$$
$$\mathcal{B}_{2n,k} \leqslant 2\mathcal{B}_{2n+1,k}.$$

Proof. Let M be a non-singular meander of total order $2n+1$ with precisely k cups, and let $(\alpha_1, \dots, \alpha_{2n+1})$ be the permutation of M. If $\alpha_{2n+1} \neq 2n+1$ then a non-singular meander M' with permutation $(\alpha_1, \dots, \alpha_{2n+1}, 2n+2)$ has the same number of cups. If $\alpha_{2n+1} = 2n+1$, then a non-singular meander M' with permutation $(1, \alpha_{2n+1}+1, \alpha_{2n}+1, \dots, \alpha_1+1)$ has the same number of cups. Clearly, for different M we obtain different M', and thus $\mathcal{B}_{2n+1,k} \leqslant \mathcal{B}_{2n+2,k}$.

Let M be a non-singular meander of total order $2n$ with precisely k cups, and let $(\alpha_1, \dots, \alpha_{2n})$ be the permutation of M. We need to consider several cases.

1. Case $\alpha_{2n} \neq 2n$. Consider a non-singular meander M' with permutation
$$(\alpha_1, \dots, \alpha_{2n}, 2n+1).$$
2. Case $\alpha_{2n} = 2n$, but $\alpha_1 \neq 1$. Consider a non-singular meander M' with permutation
$$(\alpha_{2n}+1, \alpha_{2n-1}+1, \dots, \alpha_1+1, 1).$$
3. Case $\alpha_{2n} = 2n$, $\alpha_1 = 1$, but $\alpha_{2n-1} \neq 2n-1$. Consider a non-singular meander M' with permutation
$$(1, \alpha_{2n-1}+1, \alpha_{2n-2}+1, \dots, \alpha_1+1, \alpha_{2n}+1).$$
4. Case $\alpha_{2n} = 2n$, $\alpha_1 = 1$, and $\alpha_{2n-1} = 2n-1$. Consider a non-singular meander M' with permutation
$$(\alpha_1+2, \alpha_2+2, \dots, \alpha_{2n-1}+2, 2, 1).$$

In all cases M' is a non-singular meander of total order $2n+1$ and with precisely k cups. The only possibilities where different M can lead to equivalent M' are cases 2 and 4. This means that each meander of total order $2n+1$ with precisely k cups corresponds to at most two different meanders of total order $2n$ with precisely k cups. It follows that $\mathcal{B}_{2n,k} \leqslant 2\mathcal{B}_{2n+1,k}$. □

Proof of Theorem 43. In [DGZZ20] it was proved that $\sum_{n=1}^{N} \mathcal{B}_{2n+1,k} = O(N^{2k-4})$. From Lemma 44 it follows that $\sum_{n=1}^{N} \mathcal{B}_{n,k} = O(N^{2k-4})$. It remains to note that $\mathcal{M}_{n,k}^{(Ir)} \leqslant \mathcal{B}_{n+2k,k}$. □

Corollary 45. *For each $k \geqslant 0$*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{n,k}^{(Ir)}}{\mathcal{M}_{n,k}} = 0.$$

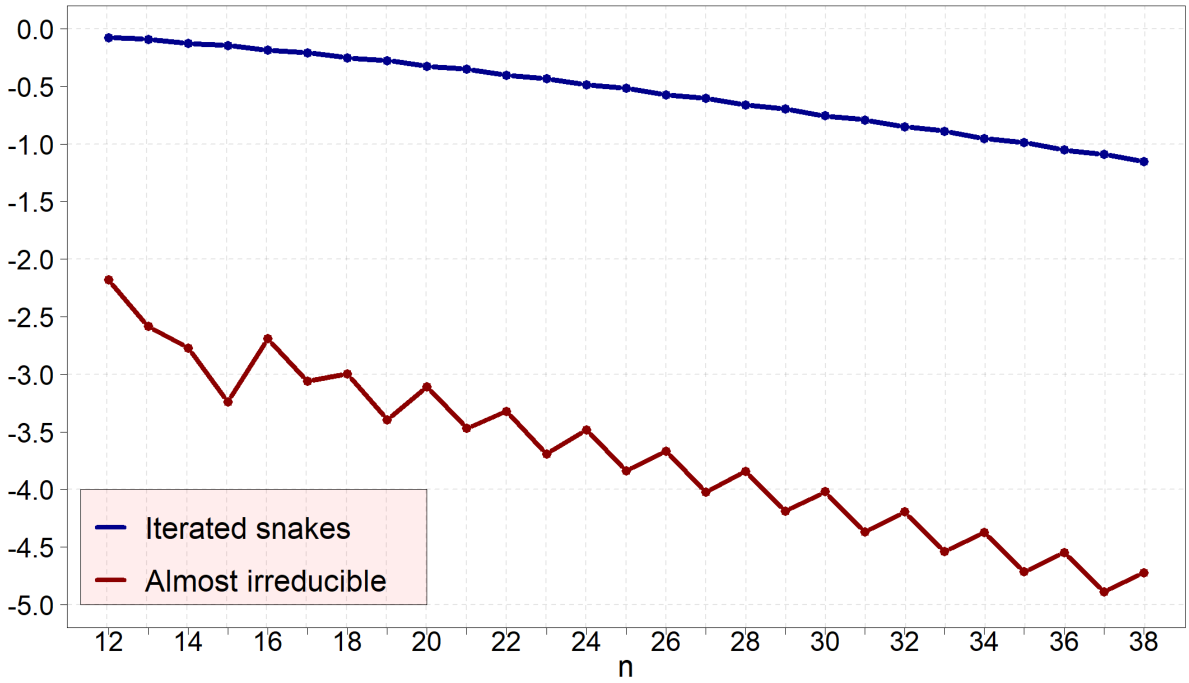


Figure 14: Base 10 logarithm of proportions of almost irreducible meanders and of non-singular iterated snakes among all non-singular meanders.

Proof. It is clear that $\mathcal{M}_{n,k} \geqslant \mathcal{M}_{n,0}$: indeed, starting from a non-singular meander with the permutation $(\alpha_1, \ldots, \alpha_n)$, we can construct a meander with the permutation $(1, 2, \ldots, k, k + \alpha_1, \ldots, k + \alpha_n)$, where intersections with the labels $1, 2, \ldots, k$ are non-transverse. But $\mathcal{M}_{n,0}$ grows exponentially with n (see [AP05]). $\square$

We denote the number of all pairwise non-equivalent almost irreducible meanders of total order n by $\mathcal{A}_n$. Note that $\mathcal{A}_n = \sum\limits_{r+2k=n} \mathcal{M}^{(Ir)}_{r,k}$. In Figure 14 one can see the logarithm of the proportion of almost irreducible meanders among all non-singular meanders of total order less than 39.

The following estimates of the growth rate of $\mathcal{A}_n$ were obtained jointly with A. Malyutin. The author is grateful for permission to include them in the present paper.

Theorem 46.

$$\limsup_{n\to\infty} \sqrt[n]{\mathcal{A}_n} < 3.313385 \tag{5}$$

Proof. Let M be an almost irreducible meander of total order n. Let us choose $\alpha \in (0; 1)$ such that αn is a natural number, and let $\mu = (a_1^{s_1}; a_2^{s_2}; \ldots; a_k^{s_k})$ be the partition of αn. We can choose $\sum_{i=1}^{k} s_i$ distinct intersections in M and at each of them insert a non-singular inverse snake of total order $2a_i + 1$. As a result, we obtain a non-singular meander of total order $n + \sum_{i=1}^{r} 2a_i = n + 2\alpha n$. In this way, we obtain $\sum\limits_{\mu \vdash \alpha n} \binom{n}{\mu} = \binom{n+\alpha n-1}{\alpha n}$ non-equivalent non-singular meanders of total order $n + 2\alpha n$ from a single almost irreducible meander of total order n. According to the uniqueness of the decomposition, different almost irreducible meanders also lead to different meanders. We have the following inequalities:

$$\begin{aligned}
\binom{n+\alpha n-1}{\alpha n}\mathcal{A}_n &\leqslant \mathcal{M}_{n+2\alpha n,0}, \\
\limsup_{n\to\infty}\sqrt[n]{\binom{n+\alpha n-1}{\alpha n}\mathcal{A}_n} &\leqslant \lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n+2\alpha n,0}} \\
\frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha}\limsup_{n\to\infty}\sqrt[n]{\mathcal{A}_n} &\leqslant \left(\lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n,0}}\right)^{1+2\alpha}, \\
\limsup_{n\to\infty}\sqrt[n]{\mathcal{A}_n} &\leqslant \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\left(\lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n,0}}\right)^{1+2\alpha}.
\end{aligned}$$

It is proved in [AP05] that $\lim_{n\to\infty}\sqrt[n]{\bar{\mathcal{M}}_n} \leqslant 12.901$, where $\bar{\mathcal{M}}_n$ is the number of pairwise non-equivalent closed meanders with precisely $2n$ intersections. It can be easily seen that $\mathcal{M}_{2n-1,0} = \bar{\mathcal{M}}_n$ and $\bar{\mathcal{M}}_n \leqslant \mathcal{M}_{2n,0} \leqslant n\bar{\mathcal{M}}_n$ (see, for example, [LC03] for details). From this it follows: $\lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n,0}} \leqslant \sqrt{12.901}$. Now for each $\alpha \in (0;\, 1)$ we have

$$\limsup_{n\to\infty}\sqrt[n]{\mathcal{A}_n} \leqslant \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\,(12.901)^{\frac{1+2\alpha}{2}}\,. \tag{6}$$

The function on the right side of equation (6) reaches the minimum at $\alpha \approx \frac{10}{119}$, where its value is approximately 3.313384. □

Corollary 47.

$$\lim_{n\to\infty}\frac{\mathcal{A}_n}{\mathcal{M}_{n,0}} = 0.$$

Proof. From [AP05] it follows that $\lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n,0}} > 3.37$. It remains to be noted that $\limsup_{n\to\infty}\sqrt[n]{\mathcal{A}_n} < \lim_{n\to\infty}\sqrt[n]{\mathcal{M}_{n,0}}$. □

Theorem 48.

$$\liminf_{n\to\infty}\sqrt[n]{\mathcal{A}_n} > 1.83669. \tag{7}$$

Proof. First, let us non-formally describe the main idea of the proof. Let M be an arbitrary non-singular meander of total order n, where n is odd (for even n the idea is the same). We can construct almost irreducible meanders of orders $2n+20$ and $2n+23$ from M by the following procedure. Consider a meander M_1 that is a concatenation of a non-singular meander with the permutation $(7, 6, 1, 2, 5, 8, 9, 4, 3)$ and M (see the example in Figure 15(a), where M is the meander with the permutation $(1, 2, 3, 4, 5)$). Next, we need to "double" M_1 (as in Figure 15(b)) to obtain a meander M_2. Finally, we can transform M_2 into an almost irreducible meander M_3 of total order $2(n+9)+2$ by adding two more

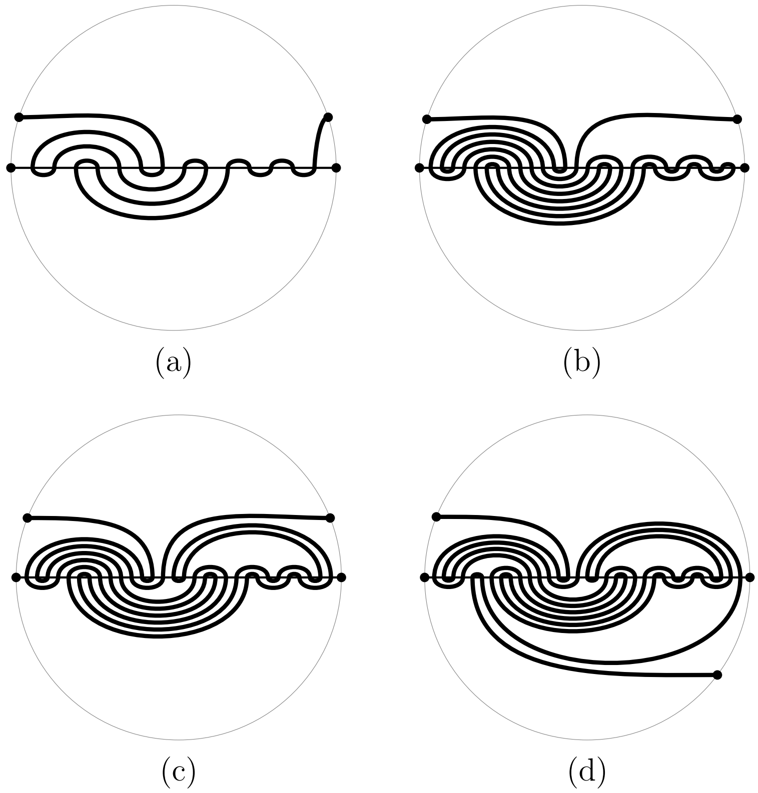


Figure 15: Constructing an almost irreducible meander for odd n.

intersections between points with labels 14 and 15 (see Figure 15(c)). We can also obtain an almost irreducible meander M_4 of odd total order in a similar way, see Figure 15(d).

Let us formalize this procedure. Let M be an arbitrary non-singular meander of total order n with permutation $(a_1, \ldots, a_n)$. If n is odd, consider a non-singular meander M' of total order $2n+20$ with permutation (recall that non-singular meanders are uniquely determined by their permutation):

$$\begin{aligned}(&13,\ 12,\ 1,\ 4,\ 9,\ 18,\ 19,\ 8,\ 5,\\ &20+2a_1,\ 20+2a_2-1,\ 20+2a_3,\ 20+2a_4-1,\ \ldots,\ 20+2a_n,\\ &15,\ 16,\\ &20+2a_n-1,\ 20+2a_{n-1},\ 20+2a_{n-2}-1,\ 20+2a_{n-3},\ \ldots,\ 20+2a_1-1,\\ &6,\ 7,\ 20,\ 17,\ 10,\ 3,\ 2,\ 11,\ 14).\end{aligned}$$

The only submeanders of M' are meanders of total order two (this follows from Lemma 17), and thus M' is almost irreducible. We also can consider non-singular meander M'' of total order $2n+23$ with permutation

$$\begin{aligned}(&15,\ 14,\ 1,\ 4,\ 11,\ 20,\ 21,\ 10,\ 7,\\ &22+2a_1,\ 22+2a_2-1,\ 22+2a_3,\ 22+2a_4-1,\ \dots,\ 22+2a_n,\\ &17,\ 18,\\ &22+2a_n-1,\ 22+2a_{n-1},\ 22+2a_{n-2}-1,\ 22+2a_{n-3},\ \dots,\ 22+2a_1-1,\\ &8,\ 9,\ 22,\ 19,\ 12,\ 3,\ 2,\ 13,\ 16,\\ &2n+23,\ 6,\ 5).\end{aligned}$$

The same argument shows that M'' is also almost irreducible.

If n is even, two non-singular meanders M' and M'' with permutations

$$\begin{aligned}(&15,\ 14,\ 1,\ 4,\ 11,\ 18,\ 19,\ 10,\ 7,\\ &20+2a_1,\ 20+2a_2-1,\ 20+2a_3,\ 20+2a_4-1,\ \dots,\ 20+2a_n-1,\\ &6,\ 5,\\ &20+2a_n,\ 20+2a_{n-1}-1,\ 20+2a_{n-2},\ 20+2a_{n-3}-1,\ \dots,\ 20+2a_1-1,\\ &8,\ 9,\ 20,\ 17,\ 12,\ 3,\ 2,\ 13,\ 16)\end{aligned}$$

and

$$\begin{aligned}(&17,\ 16,\ 1,\ 4,\ 13,\ 20,\ 21,\ 12,\ 9,\\ &22+2a_1,\ 22+2a_2-1,\ 22+2a_3,\ 22+2a_4-1,\ \dots,\ 22+2a_n-1,\\ &8,\ 7,\\ &22+2a_n,\ 22+2a_{n-1}-1,\ 22+2a_{n-2},\ 22+2a_{n-3}-1,\ \dots,\ 22+2a_1-1,\\ &10,\ 11,\ 22,\ 19,\ 14,\ 3,\ 2,\ 15,\ 18,\\ &2n+23,\ 6,\ 5)\end{aligned}$$

are almost irreducible of total order $2n+20$ and $2n+23$ respectively.

Thus we have the following inequalities

$$\liminf_{n\to\infty} \sqrt[n]{\mathcal{A}_n} \geqslant \lim_{n\to\infty} \sqrt[n]{\mathcal{M}_{\frac{n-23}{2},0}} = \sqrt{\lim_{n\to\infty} \sqrt[n]{\mathcal{M}_{n,0}}}.$$

The results of [AP05] imply that $\lim\limits_{n\to\infty} \sqrt[n]{\mathcal{M}_{n,0}} \geqslant \sqrt{11.38}$, and we finally get

$$\liminf_{n\to\infty} \sqrt[n]{\mathcal{A}_n} \geqslant \sqrt[4]{11.38} \approx 1.83669.$$

□

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A Table with meander numbers

Order n	$\mathcal{M}_{n,0}$	$\mathcal{M}_{n,0}^{(IS)}$	$\mathcal{A}_n$
1	1	0	0
2	1	1	0
3	2	2	0
4	3	3	0
5	8	8	0
6	14	14	0
7	42	42	0
8	81	79	2
9	262	252	2
10	538	494	0
11	1828	1636	0
12	3926	3294	26
13	13820	11188	36
14	30694	22952	52
15	110954	79386	64
16	252939	165127	516
17	933458	579020	816
18	2172830	1217270	2186
19	8152860	4314300	3296
20	19304190	9146746	15054
21	73424650	32697920	24946
22	176343390	69799476	84090
23	678390116	251284292	138352
24	1649008456	539464358	544652
25	6405031050	1953579240	934450
26	15730575554	4214095612	3377930
27	61606881612	15336931928	5831520
28	152663683494	33218794236	22075152
29	602188541928	121416356108	38959552
30	1503962954930	263908187100	143815358
31	5969806669034	968187827834	256128664
32	15012865733351	2110912146295	959463704
33	59923200729046	7769449728780	1732188588
34	151622652413194	16985386737830	6440145162
35	608188709574124	62696580696172	11727449592
36	1547365078534578	137394914285538	43825381338
37	6234277838531806	508451657412496	80571300722
38	15939972379349178	1116622717709012	300477174306

Table 1: Meander numbers.