The sign character of the triagonal fermionic coinvariant ring

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Abstract

We determine the trigraded multiplicity of the sign character of the triagonal fermionic coinvariant ring $R_n^{(0,3)}$. As a corollary, this proves a conjecture of Bergeron (2020) that the multiplicity of the sign character of $R_n^{(0,3)}$ is $n^2 - n + 1$. We also give an explicit formula for double hook characters in the diagonal fermionic coinvariant ring $R_n^{(0,2)}$, and discuss methods towards calculating the sign character of $R_n^{(0,4)}$. Finally, we give a multigraded refinement of a conjecture of Bergeron (2020) that the multiplicity of the sign character of the $(1,3)$-bosonic-fermionic coinvariant ring $R_n^{(1,3)}$ is $\frac{1}{2}F_{3n}$, where F_n is a Fibonacci number.

Mathematics Subject Classifications: 05E10, 05A15, 05E05, 05E16, 20C30

1 Introduction

The diagonal coinvariant ring

$$R_n^{(2,0)} := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{\mathfrak{S}_n} \rangle \tag{1}$$

was introduced by Haiman in 1994 [12], and since then has been studied extensively. Its defining ideal is generated by all polynomials in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, with no constant term, which are invariant under the diagonal action of $\mathfrak{S}_n$:

$$\sigma \cdot p(x_1, \ldots, x_n, y_1, \ldots, y_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}). \tag{2}$$

Haiman found the dimension, bigraded Hilbert series, and bigraded Frobenius series of $R_n^{(2,0)}$ [13].

There has been much recent interest (see [2–4, 20]) in studying a more general class of coinvariant rings $R_n^{(k,j)}$ with k sets of n commuting (bosonic) variables $\boldsymbol{x}_n := \{x_1, \ldots, x_n\}$, $\boldsymbol{y}_n := \{y_1, \ldots, y_n\}$, $\boldsymbol{z}_n := \{z_1, \ldots, z_n\}$, etc., and j sets of n anticommuting (fermionic)

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variables $\boldsymbol{\theta}_n := \{\theta_1, \ldots, \theta_n\}$, $\boldsymbol{\xi}_n := \{\xi_1, \ldots, \xi_n\}$, $\boldsymbol{\rho}_n := \{\rho_1, \ldots, \rho_n\}$, etc. We define the (k, j)-bosonic-fermionic coinvariant ring by

$$R_n^{(k,j)} := \mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}] / \langle \mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{S}_n} \rangle, \tag{3}$$

where its defining ideal is generated by all polynomials in $\mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]$, without constant term, which are invariant under the diagonal action of the symmetric group $\mathfrak{S}_n$, given by permuting the indices of the variables. Commuting variables commute with all variables. Anticommuting variables anticommute with all anticommuting variables. That is, $\theta_i\theta_j = -\theta_j\theta_i$ for all i, j, and mixed products between different sets of fermionic variables likewise anticommute. Note that this implies that $\theta_i^2 = 0$.

Note that $R_n^{(k,j)}$ also can be defined in terms of symmetric and exterior algebras:

$$R_n^{(k,j)} = ((\mathrm{Sym}\,\mathbb{Q}^n)^{\otimes k} \otimes (\wedge \mathbb{Q}^n)^{\otimes j}) / \langle ((\mathrm{Sym}\,\mathbb{Q}^n)^{\otimes k} \otimes (\wedge \mathbb{Q}^n)^{\otimes j})_+^{\mathfrak{S}_n} \rangle. \tag{4}$$

We recall the definitions of the multigraded Hilbert and Frobenius series of $R_n^{(k,j)}$ (see for example [3]). For fixed integers $k, j \geqslant 0$, $R_n^{(k,j)}$ decomposes as a direct sum of multihomogeneous components, which are $\mathfrak{S}_n$-modules:

$$R_n^{(k,j)} = \bigoplus_{r_1, \ldots, r_k, s_1, \ldots, s_j \geqslant 0} (R_n^{(k,j)})_{r_1, \ldots, r_k, s_1, \ldots, s_j}. \tag{5}$$

We denote the multigraded Hilbert series by

$$\begin{aligned} &\mathrm{Hilb}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j) \\ &\quad := \sum_{r_1, \ldots, r_k, s_1, \ldots, s_j \geqslant 0} \dim\left((R_n^{(k,j)})_{r_1, \ldots, r_k, s_1, \ldots, s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}, \end{aligned} \tag{6}$$

and the multigraded Frobenius series by

$$\begin{aligned} &\mathrm{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j) \\ &\quad := \sum_{r_1, \ldots, r_k, s_1, \ldots, s_j \geqslant 0} F\,\mathrm{Char}\left((R_n^{(k,j)})_{r_1, \ldots, r_k, s_1, \ldots, s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}, \end{aligned} \tag{7}$$

where F denotes the Frobenius characteristic map and Char denotes the character. For simplicity, if $k \leqslant 2$, we will use q, t for q_1, q_2 and if $j \leqslant 4$, we will use u, v, w, z for u_1, u_2, u_3, u_4. Recall that $\langle \mathrm{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j), h_1^n \rangle = \mathrm{Hilb}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j)$. Furthermore, $R_n^{(k,j)}$ is a $\mathrm{GL}_k \times \mathrm{GL}_j \times \mathfrak{S}_n$-module (see [3, Section 2]), so its multigraded Frobenius character is a sum of products of three Schur functions, which are irreducible characters of polynomial representations of GL_k and GL_j, along with a Frobenius character. The main focus of this paper is on the case $(k, j) = (0, 3)$.

Observe that the Schur function $s_{(\ell-2)}(u,v,w)$ is a u,v,w-analogue of the binomial coefficient $\binom{\ell}{2}$; denote it by

$$\binom{\ell}{2}_{u,v,w} := s_{(\ell-2)}(u,v,w). \tag{8}$$

Recall the notation $[n]_{u,v} := u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1}$. It follows that as a polynomial in w with coefficients in u and v we have

$$\binom{\ell}{2}_{u,v,w} = [\ell-1]_{u,v} + [\ell-2]_{u,v}w + [\ell-3]_{u,v}w^2 + \cdots + [1]_{u,v}w^{\ell-2}, \tag{9}$$

and there are two similar expressions for $\binom{\ell}{2}_{u,v,w}$ as a polynomial in u or in v.

Our main result is the following.

Theorem 1.

$$\begin{aligned} \langle \mathrm{Frob}(R_n^{(0,3)}; u,v,w), s_{(1^n)}\rangle &= s_{(n-1)}(u,v,w) + s_{(n-2,1,1)}(u,v,w) \\ &= \binom{n+1}{2}_{u,v,w} + uvw\binom{n-1}{2}_{u,v,w}. \end{aligned} \tag{10}$$

The theorem immediately implies the following corollary, which was conjectured by Bergeron [3, Table 3].

Corollary 2.

$$\langle \mathrm{Frob}(R_n^{(0,3)}; 1,1,1), s_{(1^n)}\rangle = n^2 - n + 1. \tag{11}$$

Proof. Recall that $\binom{\ell}{2}_{u,v,w}|_{u=v=w=1} = \binom{\ell}{2}$. By evaluating Theorem 1 at $u=v=w=1$, the multiplicity is $\binom{n+1}{2} + \binom{n-1}{2} = n^2 - n + 1$. □

The organization of the paper is as follows. In Section 2, we detail the setting of $R_n^{(0,3)}$ from the perspective of both coinvariants and harmonics. In Section 3, we recall some results of Haglund–Sergel [11] and Kim–Rhoades [14], along with proving some preliminary results. In Section 4, building on the work of Haglund–Sergel and Kim–Rhoades, we give an upper bound on the multiplicity of the sign character of $R_n^{(0,3)}$ (Corollary 10). In Section 5, we construct two elements in the ring of triagonal fermionic harmonics T_n (Proposition 14), and study the GL_3-representations that they generate (Proposition 16), which constructs enough elements in T_n to show that the upper bound on the multiplicity of the sign character is achieved with equality, proving the main theorem.

In Section 6, we derive a formula for double hook characters of $R_n^{(0,2)}$ (Theorem 19). In Section 7, we discuss methods to analyze the sign character of $R_n^{(0,4)}$. In Section 8, we provide a q,u,v,w-refinement of a conjecture of Bergeron [3] on the sign character of $R_n^{(1,3)}$ (Conjecture 29).

2 Coinvariants and Harmonics

We now describe the setting in more detail, from the perspective of both coinvariants and, isomorphically, harmonics.

Specialize to $R_n^{(0,3)} := \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n} \rangle$, which we call the **triagonal fermionic coinvariant ring**. Its defining ideal $\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n} \rangle$ is the ideal generated by polynomials in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, without constant term, invariant under the diagonal action of $\mathfrak{S}_n$. A generating set for the ideal $\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n} \rangle$ is given by all the nonzero monomial symmetric functions in any of the three sets of variables:

$$\begin{aligned} &\{\theta_1 + \cdots + \theta_n, \xi_1 + \cdots + \xi_n, \rho_1 + \cdots + \rho_n, \theta_1\xi_1 + \cdots + \theta_n\xi_n, \\ &\theta_1\rho_1 + \cdots + \theta_n\rho_n, \xi_1\rho_1 + \cdots + \xi_n\rho_n, \theta_1\xi_1\rho_1 + \cdots + \theta_n\xi_n\rho_n\}. \end{aligned} \tag{12}$$

Consider the ring $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ in three sets of n anticommuting variables, which we arrange into the $3 \times n$ matrix

$$M := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \rho_1 & \rho_2 & \cdots & \rho_n \end{bmatrix}. \tag{13}$$

For each 3×3 invertible matrix A in GL_3, multiply M on the left by A to obtain the product $A \cdot M$. Then, in any polynomial $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, replace each variable θ_i, ξ_i, ρ_i by the corresponding entry of the matrix $A \cdot M$. This defines a left GL_3-action on $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$.

The ring $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is naturally trigraded. Denote by $(\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n])_{a,b,c}$ the homogeneous component spanned by all monomials that contain exactly a variables in $\boldsymbol{\theta}_n$, b variables in $\boldsymbol{\xi}_n$, and c variables in $\boldsymbol{\rho}_n$. Since GL_3 acts by linear combinations of the three rows, the GL_3-action preserves the total degree $d = a + b + c$. Hence each total degree d component $\bigoplus_{a+b+c=d} (\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n])_{a,b,c}$ is GL_3-invariant.

Each permutation $\sigma \in \mathfrak{S}_n$ corresponds to a permutation matrix P_σ, defined by $(P_\sigma)_{i,j} = 1$ if $i = \sigma(j)$ and 0 otherwise. Then the right multiplication $M \cdot P_\sigma$ has the effect on M of letting σ act on the indices of all variables: $\theta_i \mapsto \theta_{\sigma(i)}$, $\xi_i \mapsto \xi_{\sigma(i)}$, and $\rho_i \mapsto \rho_{\sigma(i)}$. Then, in any polynomial $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, replace each variable θ_i, ξ_i, ρ_i by the corresponding entry of the matrix $M \cdot P_\sigma$. This defines a right $\mathfrak{S}_n$-action on $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$.

Because left multiplication by A and right multiplication by P_σ commute, these GL_3 and $\mathfrak{S}_n$-actions commute, so $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is a $\mathrm{GL}_3 \times \mathfrak{S}_n$-module.

Recall the definition of derivative of anticommuting variables (where here each θ_i could be from any of the three sets of anticommuting variables) is (see for example [18, Section 1.5]):

$$\partial_{\theta_j} \theta_{i_1} \cdots \theta_{i_k} = \begin{cases} (-1)^{\ell-1} \theta_{i_1} \cdots \hat{\theta}_{i_\ell} \cdots \theta_{i_k} & \text{if } j = i_\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Define the space of **triagonal fermionic harmonics** by

$$T_n := \left\{ f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n] \;\middle|\; \sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell f = 0 \text{ for all } h + k + \ell > 0 \right\}. \tag{15}$$

Since any $\theta_i^2 = 0$, we need not check any second derivatives, hence

$$T_n = \left\{ f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n] \;\middle|\; \sum_{i=1}^{n} \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell f = 0 \text{ for all } h+k+\ell > 0 \text{ and } h,k,\ell \in \{0,1\} \right\}. \tag{16}$$

Looking at the infinitesimal action of the Lie algebra $\mathfrak{sl}_3$ on T_n allows us to study T_n as a GL_3-module.[1] Similarly to [12, Section 3.1], we define the operators

$$F^{\theta\to\xi} := \sum_{i=1}^{n} \xi_i \partial_{\theta_i}, \text{ and } F^{\xi\to\rho} := \sum_{i=1}^{n} \rho_i \partial_{\xi_i}, \tag{17}$$

and

$$E^{\theta\leftarrow\xi} := \sum_{i=1}^{n} \theta_i \partial_{\xi_i} \text{ and } E^{\xi\leftarrow\rho} := \sum_{i=1}^{n} \xi_i \partial_{\rho_i}. \tag{18}$$

Along with two H operators given by taking the commutators of the E and F operators, these generate the Lie algebra $\mathfrak{sl}_3$.

Furthermore T_n is a $\mathrm{GL}_3 \times \mathfrak{S}_n$-module. Write a trigraded component of T_n as $(T_n)_{a,b,c}$ and $R_n^{(0,3)}$ as $(R_n^{(0,3)})_{a,b,c}$. Then we have that $T_n \cong R_n^{(0,3)}$ as trigraded $\mathfrak{S}_n$-modules. Working with harmonics was advocated by Garsia: a benefit of working with harmonics over coinvariants is that one works with polynomials instead of equivalence classes.

A polynomial $p \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is called antisymmetric if $\sigma(p) = \mathrm{sgn}(\sigma) p$ for all $\sigma \in \mathfrak{S}_n$, where σ acts diagonally by permuting the variables. Let $(T_n)^\epsilon$ denote the antisymmetric subspace[2] of T_n, which consists of all elements $p \in T_n$ which are antisymmetric. This corresponds to the u,v,w-graded multiplicity of the sign character in $R_n^{(0,3)}$, that is, $\mathrm{Hilb}((T_n)^\epsilon; u,v,w) = \langle \mathrm{Frob}(R_n^{(0,3)}; u,v,w), s_{(1^n)} \rangle$.

3 Preliminary results

In this section, we recall or prove some preliminary results. Haglund and Sergel gave a formula for the graded Frobenius series of the fermionic coinvariants $R_n^{(0,1)} := \mathbb{Q}[\boldsymbol{\theta}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$.

Lemma 3 ([11, Lemma 4.10])**.** *For $n \geqslant 1$,*

$$\mathrm{Frob}(R_n^{(0,1)}; w) = \sum_{k=0}^{n-1} w^k s_{(n-k,1^k)}. \tag{19}$$

Kim and Rhoades gave a formula for the bigraded Frobenius series of the diagonal fermionic coinvariants $R_n^{(0,2)} := \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n} \rangle$.

[1] See for example [10, Section 15.5] for more on the relationship between SL_3 and GL_3-representations.
[2] This is sometimes called the alternating subspace.

Theorem 4 ([14, Theorem 6.1])**.** *For $n \geqslant 1$,*

$$\operatorname{Frob}(R_n^{(0,2)};u,v) = \sum_{0\leqslant i+j<n} u^i v^j \left(s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}\right), \quad (20)$$

where $*$ denotes the Kronecker product and $s_{(n+1,1^{-1})}$ is interpreted as 0.

In particular, they found bigraded multiplicities for the trivial, sign, and hook characters.

Proposition 5 ([14, Proposition 6.2])**.** *In $R_n^{(0,2)}$, the multiplicity of the trivial character is*

$$\langle \operatorname{Frob}(R_n^{(0,2)};u,v), s_{(n)}\rangle = 1, \quad (21)$$

the bigraded multiplicity of the sign character is

$$\langle \operatorname{Frob}(R_n^{(0,2)};u,v), s_{(1^n)}\rangle = [n]_{u,v}, \quad (22)$$

and for $0 < k < n-1$, the bigraded multiplicity of a hook character is

$$\langle \operatorname{Frob}(R_n^{(0,2)};u,v), s_{(n-k,1^k)}\rangle = [k+1]_{u,v} + uv[k]_{u,v}. \quad (23)$$

As a consequence of their result on the sign character, we conclude the following.

Corollary 6.

1. *If a nonzero harmonic polynomial in $T_n \cap \mathbb{Q}[\boldsymbol{\theta}_n]$ is antisymmetric, then it must be of degree exactly $n-1$.*
2. *If a nonzero harmonic polynomial in $T_n \cap \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]$ is antisymmetric, then it must be of degree exactly $n-1$.*

Proof. The first claim is implied by the second. To show the second, notice that Proposition 5 shows that the multiplicity of each component of the sign character is $n-1$. □

There is a useful condition for when antisymmetrizing a polynomial results in zero.

Lemma 7. *Let p be a polynomial in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$. If there exists a transposition $s \in \mathfrak{S}_n$ such that $s(p) = p$, then*

$$\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(p) = 0. \quad (24)$$

Proof. Consider the symmetric group $\mathfrak{S}_n$ as a set, and partition it $\mathfrak{S}_n = \mathfrak{S}_n' \sqcup \mathfrak{S}_n''$ into two equal-sized subsets such that for each $\sigma' \in \mathfrak{S}_n'$, we have that $\sigma' \cdot s = \sigma''$, for some $\sigma'' \in \mathfrak{S}_n''$. Since $\operatorname{sgn}(s) = -1$, then

$$\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(p) = \sum_{\sigma'\in\mathfrak{S}_n'} \operatorname{sgn}(\sigma')\sigma'(p) + \sum_{\sigma'\in\mathfrak{S}_n'} -\operatorname{sgn}(\sigma')\sigma'(p) = 0. \quad (25)$$

□

As a consequence, we establish that an antisymmetric polynomial must use enough distinct indices of variables to be nonzero.

Lemma 8. *If a polynomial in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is antisymmetric, and for the variables that appear, at most $n-2$ unique indices are used, then the polynomial is identically 0.*

Proof. Call the polynomial in question r. Without loss of generality, say that the indices used in r are $\{1, 2, \ldots, n-2\}$. Since r is antisymmetric, it can be obtained by applying an antisymmetrizing operator $\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ to some polynomial $p \in \mathbb{Q}[\boldsymbol{\theta}_{n-2}, \boldsymbol{\xi}_{n-2}, \boldsymbol{\rho}_{n-2}]$. Then p is invariant under the action of s_{n-1}. Apply Lemma 7 to conclude that $r = 0$. $\square$

4 An upper bound on degree

In this section, we prove an upper bound on the trigraded degree of the sign character. The following result is inspired by a similar result of Haglund–Sergel (on a different ring, $R_n^{(2,1)}$) [11, Theorem 4.11].

Proposition 9. *For $n \geqslant 1$,*

$$\langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle = \binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uv[n-1]_{u,v}. \tag{26}$$

Proof. As noted in [11, equation (4.9)], following from [5], for any $\mathfrak{S}_n$-modules A and B tensored under the diagonal action of $\mathfrak{S}_n$, we have that

$$\operatorname{Frob}(A \otimes B) = \sum_{\nu \vdash n} s_\nu \sum_{\lambda, \mu \vdash n} \langle \operatorname{Frob}(A), s_\lambda \rangle \langle \operatorname{Frob}(B), s_\mu \rangle \langle s_\lambda * s_\mu, s_\nu \rangle, \tag{27}$$

where $\langle s_\lambda * s_\mu, s_\nu \rangle = g(\lambda, \mu, \nu)$ are the Kronecker coefficients. In our case, we are interested in

$$\begin{aligned} &\langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle \\ &\quad = \sum_{\lambda, \mu \vdash n} \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_\lambda \rangle \langle \operatorname{Frob}(R_n^{(0,1)}; w), s_\mu \rangle g(\lambda, \mu, 1^n) \\ &\quad = \sum_{\lambda \vdash n} \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_{\lambda'} \rangle \langle \operatorname{Frob}(R_n^{(0,1)}; w), s_\lambda \rangle, \end{aligned} \tag{28}$$

since $g(\lambda, \mu, 1^n) = \delta_{\mu, \lambda'}$. By Lemma 3, which states that only hook Schur functions appear in $\operatorname{Frob}(R_n^{(0,1)}; w)$, we reduce to

$$\begin{aligned} \langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle &= \sum_{k=0}^{n-1} w^k \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_{(k+1, 1^{n-k-1})} \rangle \\ &= \sum_{k=0}^{n-1} w^k ([n-k]_{u,v} + uv[n-k-1]_{u,v}), \end{aligned} \tag{29}$$

where we applied Proposition 5. After a bit of algebra, we obtain the claimed formula. $\square$

Now the following result gives us an upper bound on the occurrences of the sign character in $R_n^{(0,3)}$. For multivariate polynomials A and B, the notation $A \leqslant B$ means that $B - A$ is a sum of monomials with only nonnegative coefficients.

Corollary 10. *For $n \geqslant 1$,*

$$\langle \mathrm{Frob}(R_n^{(0,3)}; u, v, w), s_{(1^n)} \rangle \leqslant \binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w}. \tag{30}$$

Proof. First note that any occurrence of the sign character in $R_n^{(0,3)}$ is predicated on it occurring in $R_n^{(0,2)} \otimes R_n^{(0,1)}$, since $R_n^{(0,3)}$ is a quotient of $R_n^{(0,2)} \otimes R_n^{(0,1)}$ under the diagonal action of $\mathfrak{S}_n$ (see [11]).

By permuting which sets of variables out of $\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n$ are assigned to $R_n^{(0,2)}$ and to $R_n^{(0,1)}$ in Proposition 9, we conclude that a sign character in $R_n^{(0,3)}$ must have trigraded multiplicity bounded above by

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uv[n-1]_{u,v}, \tag{31}$$

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uw[n-1]_{u,w}, \tag{32}$$

and

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + vw[n-1]_{v,w}. \tag{33}$$

Note that if a polynomial $A \geqslant 0$ satisfies $A \leqslant uv[n-1]_{u,v}$, $A \leqslant uw[n-1]_{u,w}$, and $A \leqslant vw[n-1]_{v,w}$, then $A = 0$. Hence by taking the bounds given by equations (31-33) together, we conclude that the sign character in $R_n^{(0,3)}$ must have trigraded multiplicity bounded above by

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w}. \tag{34}$$

□

5 Construction of basis elements

Now we will work towards the proof of the main theorem, by constructing two explicit elements in T_n, which are highest weight vectors for certain GL_3-representations. We ultimately show that the upper bound given in Corollary 10 is obtained with equality.

Definition 11. For $n \geqslant 1$, define the **primary theta-seed** by

$$\Delta_1(\boldsymbol{\theta}_n) := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \sigma(\theta_1 \theta_2 \cdots \theta_{n-1}). \tag{35}$$

Definition 12. For $n \geqslant 3$, define the **secondary theta-seed** by

$$\Delta_2(\boldsymbol{\theta}_n) := \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4 \cdots \theta_{n-1}). \tag{36}$$

When $n = 1$ or 3 in the definitions of the primary and secondary theta-seed, respectively, the empty product of θ_i's is interpreted as 1.

We now prove a technical lemma.

Lemma 13.

1. *The antisymmetrization operator* $\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ *commutes with the differential operator* $\sum_{i=1}^{n} \partial_{\theta_i}^{h}\partial_{\xi_i}^{k}\partial_{\rho_i}^{\ell}$.
2. *The antisymmetrization operator* $\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ *commutes with the operators* $F^{\theta\to\xi}$, $F^{\xi\to\rho}$, $E^{\theta\leftarrow\xi}$, *and* $E^{\xi\leftarrow\rho}$.

Proof. First we show (1). Let $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$. Observe that for any $\sigma \in \mathfrak{S}_n$,

$$\sum_{i=1}^{n} \partial_{\theta_i}^{h}\partial_{\xi_i}^{k}\partial_{\rho_i}^{\ell} = \sum_{i=1}^{n} \partial_{\sigma(\theta_i)}^{h}\partial_{\sigma(\xi_i)}^{k}\partial_{\sigma(\rho_i)}^{\ell}, \tag{37}$$

since addition is commutative. Thus,

$$\begin{aligned} \sum_{i=1}^{n} \partial_{\theta_i}^{h}\partial_{\xi_i}^{k}\partial_{\rho_i}^{\ell}\left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(f)\right) &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \partial_{\theta_i}^{h}\partial_{\xi_i}^{k}\partial_{\rho_i}^{\ell}(\sigma(f)) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \partial_{\sigma(\theta_i)}^{h}\partial_{\sigma(\xi_i)}^{k}\partial_{\sigma(\rho_i)}^{\ell}(\sigma(f)) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma\left(\sum_{i=1}^{n} \partial_{\theta_i}^{h}\partial_{\xi_i}^{k}\partial_{\rho_i}^{\ell} f\right), \end{aligned} \tag{38}$$

so the operators commute as claimed.

Next we show (2). For any $\sigma \in \mathfrak{S}_n$,

$$\sum_{i=1}^{n} \xi_i\partial_{\theta_i} = \sum_{i=1}^{n} \sigma(\xi_i)\partial_{\sigma(\theta_i)}, \tag{39}$$

since addition is commutative. Thus for $F^{\theta\to\xi} = \sum_{i=1}^{n} \xi_i\partial_{\theta_i}$,

$$\begin{aligned} \sum_{i=1}^{n} \xi_i\partial_{\theta_i}\left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(f)\right) &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \xi_i\partial_{\theta_i}\sigma(f) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \sigma(\xi_i)\partial_{\sigma(\theta_i)}\sigma(f) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma\left(\sum_{i=1}^{n} \xi_i\partial_{\theta_i} f\right), \end{aligned} \tag{40}$$

and similarly for the other operators. □

Now we show that $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$ are harmonic.

Proposition 14.

1. *The primary theta-seed $\Delta_1(\boldsymbol{\theta}_n)$ is in $(T_n)^\epsilon$.*

2. *The secondary theta-seed $\Delta_2(\boldsymbol{\theta}_n)$ is in $(T_n)^\epsilon$.*

Proof. We show (1). Consider the primary theta-seed

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2\cdots\theta_{n-1}). \tag{41}$$

By equation (16), to show it is in T_n, we only need to show that $\sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell \Delta_1(\boldsymbol{\theta}_n) = 0$ for $(h,k,\ell) \in \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$. Since only θ_i variables appear in $\Delta_1(\boldsymbol{\theta}_n)$, it only remains to check $(1,0,0)$. Consider $\theta_1\cdots\theta_{n-1}$, which is what we will antisymmetrize to get the primary theta-seed. By Lemma 13, the differential operator commutes with the antisymmetrization operator. We write

$$\sum_{i=1}^n \partial_{\theta_i}\theta_1\cdots\theta_{n-1} = \sum_{i=1}^n (-1)^{i-1}\theta_1\cdots\hat{\theta}_i\cdots\theta_{n-1}, \tag{42}$$

where each monomial on the right hand side is in only $n-2$ variables. By Lemma 8, this becomes 0 upon antisymmetrization. This shows $\Delta_1(\boldsymbol{\theta}_n)$ is in T_n, and since it is constructed via an antisymmetrization operator, it is in $(T_n)^\epsilon$.

We show (2). For $n \geqslant 3$, consider the secondary theta-seed

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}). \tag{43}$$

We must check that $\Delta_2(\boldsymbol{\theta}_n)$ is in T_n. We only need to show that $\sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell \Delta_2(\boldsymbol{\theta}_n) = 0$ for the following seven choices of (h,k,ℓ):

$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}. \tag{44}$$

Case 1: $(h,k,\ell) = (1,0,0)$. Let $f = (\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}$, which is what we will antisymmetrize to get the secondary theta-seed. For concision, write $\prod_{i=3}^{n-1}\theta_i$ for the ordered product $\theta_3\theta_4\cdots\theta_{n-1}$. We write that

$$\begin{aligned}\sum_{i=1}^n \partial_{\theta_i} f &= \partial_{\theta_1} f + \partial_{\theta_2} f + \sum_{i=3}^{n-1}\partial_{\theta_i} f \\ &= \xi_2\rho_2\prod_{i=3}^{n-1}\theta_i + (\xi_1\rho_2 + \xi_2\rho_1)\prod_{i=3}^{n-1}\theta_i \\ &\quad + \sum_{i=3}^{n-1}(-1)^i(\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\cdots\hat{\theta}_i\cdots\theta_{n-1}.\end{aligned} \tag{45}$$

Then upon antisymmetrization, both of the terms $\xi_2\rho_2 \prod_{i=3}^{n-1} \theta_i$ and $\sum_{i=3}^{n-1}(-1)^i(\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3 \cdots \hat{\theta}_i \cdots \theta_{n-1}$ will be 0 due to Lemma 8, since they only contain $n-2$ indices for variables. Since the simple transposition s_1 leaves the term $(\xi_1\rho_2 + \xi_2\rho_1) \prod_{i=3}^{n-1} \theta_i$ invariant, by Lemma 7, it becomes 0 upon antisymmetrization.

Case 2: $(h,k,\ell) = (0,1,0)$. Before antisymmetrization, we write that

$$\begin{aligned} \sum_{i=1}^{n} \partial_{\xi_i} f &= \partial_{\xi_1} f + \partial_{\xi_2} f \\ &= -\theta_2\rho_2 \prod_{i=3}^{n-1} \theta_i - (\theta_1\rho_2 + \theta_2\rho_1) \prod_{i=3}^{n-1} \theta_i. \end{aligned} \tag{46}$$

By Lemma 8, $\theta_2\rho_2 \prod_{i=3}^{n-1} \theta_i$ becomes 0 upon antisymmetrization. By Lemma 7, $(\theta_1\rho_2 + \theta_2\rho_1) \prod_{i=3}^{n-1} \theta_i$ becomes 0 upon antisymmetrization, since it is invariant under s_1.

Case 3: $(h,k,\ell) = (0,0,1)$. The same argument as in Case 2 applies.

Case 4: $(h,k,\ell) = (1,1,0)$. Before antisymmetrization, we write that

$$\begin{aligned} \sum_{i=1}^{n} \partial_{\theta_i}\partial_{\xi_i} f &= \partial_{\theta_1}\partial_{\xi_1} f + \partial_{\theta_2}\partial_{\xi_2} f + \sum_{i=3}^{n-1} \partial_{\theta_i}\partial_{\xi_i} f \\ &= 0 - \rho_1 \prod_{i=3}^{n-1} \theta_i + 0, \end{aligned} \tag{47}$$

which becomes 0 upon antisymmetrization by Lemma 8.

Case 5: $(h,k,\ell) = (1,0,1)$. The same argument as in Case 4 applies.

Case 6: $(h,k,\ell) = (0,1,1)$. The same argument as in Case 4 applies.

Case 7: $(h,k,\ell) = (1,1,1)$. Before antisymmetrization, we write that

$$\begin{aligned} \sum_{i=1}^{n} \partial_{\theta_i}\partial_{\xi_i}\partial_{\rho_i} f &= \partial_{\theta_1}\partial_{\xi_1}\partial_{\rho_1} f + \partial_{\theta_2}\partial_{\xi_2}\partial_{\rho_2} f + \sum_{i=3}^{n-1} \partial_{\theta_i}\partial_{\xi_i}\partial_{\rho_i} f \\ &= 0, \end{aligned} \tag{48}$$

which remains 0 upon antisymmetrization.

This completes the proof that $\Delta_2(\boldsymbol{\theta}_n)$ is in T_n, and since it is constructed via an antisymmetrization operator, it is in $(T_n)^\epsilon$. □

A priori, a harmonic polynomial in T_n could equal zero if certain term cancellations occur. However, the following result demonstrates that this does not happen for $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$.

Proposition 15.

1. *The primary theta-seed $\Delta_1(\boldsymbol{\theta}_n)$ is nonzero in T_n.*

2. *The secondary theta-seed $\Delta_2(\boldsymbol{\theta}_n)$ is nonzero in T_n.*

Proof. We show (1). Consider in T_n,

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2\cdots\theta_{n-1}). \tag{49}$$

When the antisymmetrization operator $\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma$ is applied to $\theta_1\theta_2\cdots\theta_{n-1}$, consider which σ will output the monomial $\theta_1\theta_2\cdots\theta_{n-1}$. Such a σ must fix n, but $1,\ldots,n-1$ can be permuted. For any σ in the symmetric group on letters $1,\ldots,n-1$, we have that $\operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2\cdots\theta_{n-1}) = \theta_1\theta_2\cdots\theta_{n-1}$ (see for example [6, Chapter III, Section 7.3, Proposition 5]). Hence antisymmetrization produces $(n-1)!$ copies of $\theta_1\theta_2\cdots\theta_{n-1}$, so $\Delta_1(\boldsymbol{\theta}_n)$ is nonzero in T_n, as desired.

Next we show (2). Consider in T_n,

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}). \tag{50}$$

When the operator $\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma$ is applied to $(\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}$, consider which σ will include the monomial $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ in its output. Since the ξ_i and ρ_i have the same index, it must come from an antisymmetrization of $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ (and not of $\theta_2\xi_1\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ nor $\theta_2\xi_2\rho_1\theta_3\theta_4\cdots\theta_{n-1}$). Thus such a σ must fix 2 and n, but $1,3,4,\ldots,n-1$ can be permuted. For any σ in the symmetric group on letters $1,3,4,\ldots,n-1$, it follows that $\operatorname{sgn}(\sigma)\sigma(\theta_1\theta_3\theta_4\cdots\theta_{n-1}) = \theta_1\theta_3\theta_4\cdots\theta_{n-1}$. This implies that $\operatorname{sgn}(\sigma)\sigma(\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}) = \theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$. Hence antisymmetrization produces $(n-2)!$ copies of $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$, so $\Delta_2(\boldsymbol{\theta}_n)$ is nonzero in T_n, as desired. □

Starting with $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$, the operators defined in equation (17) $F^{\theta\to\xi}$ and $F^{\xi\to\rho}$ create representations of GL_3.

Proposition 16.

1. *The primary theta-seed $\Delta_1(\boldsymbol{\theta}_n)$ is the highest weight vector for the representation of* GL_3 *with character $s_{(n-1)}(u,v,w)$.*

2. *The secondary theta-seed $\Delta_2(\boldsymbol{\theta}_n)$ is the highest weight vector for the representation of* GL_3 *with character $s_{(n-2,1,1)}(u,v,w)$.*

Proof. We show (1). Start with the primary theta-seed

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2\cdots\theta_{n-1}). \tag{51}$$

By Proposition 14, it is in $(T_n)^\epsilon$, and by Proposition 15, it is nonzero. Observe that $E^{\theta\leftarrow\xi}$ and $E^{\xi\leftarrow\rho}$ both kill $\Delta_1(\boldsymbol{\theta}_n)$. Thus $\Delta_1(\boldsymbol{\theta}_n)$ is a highest weight vector for the irreducible GL_3-representation with highest weight $(n-1,0,0)$, where the weight of any $p\in T_n$ is given by $(\deg_\theta p, \deg_\xi p, \deg_\rho p)$ (see for example [10, Section 14-15] for a reference on the Lie theory used). This GL_3-representation is in $(T_n)^\epsilon$ since Lemma 13 implies that $(T_n)^\epsilon$ is closed under $F^{\theta\to\xi}$ and $F^{\xi\to\rho}$. The GL_3-character of this representation is $s_{(n-1)}(u,v,w)$.

We show (2). Consider the secondary theta-seed

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4 \cdots \theta_{n-1}). \tag{52}$$

This is antisymmetric, as it is constructed using an antisymmetrization operator. By Corollary 6, there are no antisymmetric elements of degree n solely in any one or two sets of variables. This implies that $E^{\theta \leftarrow \xi}$ and $E^{\xi \leftarrow \rho}$ both kill $\Delta_2(\boldsymbol{\theta}_n)$. By Proposition 14, it is in $(T_n)^\epsilon$, and by Proposition 15, it is nonzero. Thus $\Delta_2(\boldsymbol{\theta}_n)$ is a highest weight vector for the irreducible GL_3-representation with highest weight $(n-2,1,1)$. Again, this GL_3-representation is in $(T_n)^\epsilon$ by Lemma 13. The GL_3-character of this representation is $s_{(n-2,1,1)}(u,v,w)$. □

Now we can prove the main theorem.

Theorem 1.

$$\begin{aligned} \langle \operatorname{Frob}(R_n^{(0,3)}; u,v,w), s_{(1^n)} \rangle &= s_{(n-1)}(u,v,w) + s_{(n-2,1,1)}(u,v,w) \\ &= \binom{n+1}{2}_{u,v,w} + uvw\binom{n-1}{2}_{u,v,w}. \end{aligned} \tag{53}$$

Proof. The representations constructed in Proposition 16 are sufficient to force the bound in Corollary 10 to be achieved with equality. A bit of algebra verifies that the formulation in terms of Schur functions is equivalent to the formulation in terms of u,v,w-binomial coefficients. □

Remark 17. While it is not necessarily true in general that the Lie algebra operators F and E will correspond to crystal operators, in the present case, it is true since every weight space for the representations studied in Proposition 16 is one-dimensional. The representation with highest weight $(n-1,0,0)$ has crystal structure isomorphic to a crystal of tableaux $\mathcal{B}_{(n-1)}$ and the representation with highest weight $(n-2,1,1)$ has crystal structure isomorphic to a crystal of tableaux $\mathcal{B}_{(n-2,1,1)}$ (see for example [7, Chapter 3]).

6 Double hook characters of the diagonal fermionic coinvariant ring

With Proposition 5, Kim and Rhoades gave explicit formulas for the trivial, sign, and hook characters of $R_n^{(0,2)}$. In this section, we extend the analysis to give an explicit formula for double hook characters of $R_n^{(0,2)}$, where a **double hook** is a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots)$ such that $\lambda_3 \leqslant 2$ and $\lambda_2 \geqslant 2$. All characters of $R_n^{(0,2)}$ indexed by shapes not contained in a double hook have multiplicity 0.

Rosas gave a combinatorial formula for the Kronecker coefficients $g(\lambda,(n-e,1^e),(n-f,1^f))$, for any shape λ. These Kronecker coefficients are only nonzero if λ is contained in a double hook shape. Here, we recall her formulas for double hook shapes, rows, and hook shapes.

Theorem 18 ([17, Theorem 3]). *Let $(n-e,1^e)$ and $(n-f,1^f)$ be hook shapes.*

1. *If λ is not contained in a double hook, then $g(\lambda,(n-e,1^e),(n-f,1^f))=0$.*
2. *Let $\lambda=(\lambda_1,\lambda_2,2^\ell,1^{n-2\ell-\lambda_1-\lambda_2})$ where $\lambda_1\geqslant\lambda_2\geqslant 2$ be a double hook. Then*

$$\begin{aligned} &g(\lambda,(n-e,1^e),(n-f,1^f)) \\ &\quad=\chi(\lambda_2-1\leqslant \tfrac{e+f-n+\lambda_1+\lambda_2}{2}\leqslant\lambda_1)\cdot\chi(|f-e|\leqslant n-2\ell-\lambda_1-\lambda_2) \\ &\quad+\chi(\lambda_2\leqslant \tfrac{e+f-n+\lambda_1+\lambda_2+1}{2}\leqslant\lambda_1)\cdot\chi(|f-e|\leqslant n-2\ell-\lambda_1-\lambda_2+1),\end{aligned}\tag{54}$$

where $\chi(P)$ is 1 if the proposition P is true and 0 if it is false.

3. *If $\lambda=(n)$, then $g(\lambda,(n-e,1^e),(n-f,1^f))=\chi(e=f)$.*
4. *If λ is a hook shape $(n-d,1^d)$, then*

$$g(\lambda,(n-e,1^e),(n-f,1^f))=\chi(|e-f|\leqslant d)\cdot\chi(d\leqslant e+f\leqslant 2n-d-2).\tag{55}$$

This implies that these coefficients are in $\{0,1,2\}$ [17, Corollary 4]. We are ready to prove the following result, establishing a formula for double hook characters in $R_n^{(0,2)}$.

Theorem 19. *Let $\lambda\vdash n$ be a double hook, that is, $\lambda=(\lambda_1,\lambda_2,2^\ell,1^{n-2\ell-\lambda_1-\lambda_2})$ where $\lambda_1\geqslant\lambda_2\geqslant 2$. Let $m:=n-2\ell-\lambda_1-\lambda_2+2$. Then if $\lambda_1=\lambda_2$, we have that*

$$\langle \mathrm{Frob}(R_n^{(0,2)};u,v),s_\lambda\rangle=(uv)^{\ell+\lambda_2-1}(uv[m-2]_{u,v}+[m]_{u,v}+[m-1]_{u,v}).\tag{56}$$

If $\lambda_1>\lambda_2$, we have that

$$\langle \mathrm{Frob}(R_n^{(0,2)};u,v),s_\lambda\rangle=(uv)^{\ell+\lambda_2-1}(uv[m-1]_{u,v}+uv[m-2]_{u,v}+[m]_{u,v}+[m-1]_{u,v}).\tag{57}$$

Proof. Recall from Theorem 4 that

$$\mathrm{Frob}(R_n^{(0,2)};u,v)=\sum_{0\leqslant i+j<n}u^iv^j\left(s_{(n-i,1^i)}*s_{(n-j,1^j)}-s_{(n-i+1,1^{i-1})}*s_{(n-j+1,1^{j-1})}\right).\tag{58}$$

Thus

$$\begin{aligned}&\langle \mathrm{Frob}(R_n^{(0,2)};u,v),s_\lambda\rangle\\ &=\sum_{0\leqslant i+j<n}u^iv^j\left(g(\lambda,(n-i,1^i),(n-j,1^j))-g(\lambda,(n-i+1,1^{i-1}),(n-j+1,1^{j-1}))\right).\end{aligned}\tag{59}$$

Since $\lambda=(\lambda_1,\lambda_2,2^\ell,1^{n-2\ell-\lambda_1-\lambda_2})$ is a double hook, by Theorem 18,

$$\begin{aligned}&g(\lambda,(n-i,1^i),(n-j,1^j))-g(\lambda,(n-i+1,1^{i-1}),(n-j+1,1^{j-1}))\\ &\quad=\left(\chi(\lambda_2-1\leqslant\tfrac{i+j-n+\lambda_1+\lambda_2}{2}\leqslant\lambda_1)-\chi(\lambda_2-1\leqslant\tfrac{i+j-n+\lambda_1+\lambda_2-2}{2}\leqslant\lambda_1)\right)\\ &\qquad\cdot\chi(|j-i|\leqslant n-2\ell-\lambda_1-\lambda_2)\\ &\quad+\left(\chi(\lambda_2\leqslant\tfrac{i+j-n+\lambda_1+\lambda_2+1}{2}\leqslant\lambda_1)-\chi(\lambda_2\leqslant\tfrac{i+j-n+\lambda_1+\lambda_2-1}{2}\leqslant\lambda_1)\right)\\ &\qquad\cdot\chi(|j-i|\leqslant n-2\ell-\lambda_1-\lambda_2+1).\end{aligned}\tag{60}$$

Define $d := \lambda_1 - \lambda_2$. We analyze a component of equation (60):

$$\chi(\lambda_2 - 1 \leqslant \tfrac{i+j-n+\lambda_1+\lambda_2}{2} \leqslant \lambda_1) - \chi(\lambda_2 - 1 \leqslant \tfrac{i+j-n+\lambda_1+\lambda_2-2}{2} \leqslant \lambda_1), \tag{61}$$

which is equivalent to

$$\chi(|i+j-(n-1)| \leqslant d+1) - \chi(|i+j-(n+1)| \leqslant d+1). \tag{62}$$

Since the bigraded component $(R_n^{(0,2)})_{i,j} = 0$ whenever $i+j \geqslant n$, we exclude these cases from our analysis. Thus $|i+j-(n-1)| \leqslant d+1$ is satisfied if $i+j = n-1$ or $n-2$ (if $d \geqslant 0$), if $i+j = n-3$ (if $d \geqslant 1$), if $i+j = n-4$ (if $d \geqslant 2$), etc. On the other hand, $|i+j-(n+1)| \leqslant d+1$ is satisfied if $i+j = n-1$ (if $d \geqslant 1$), if $i+j = n-2$ (if $d \geqslant 2$), etc. Putting these together, we get that equation (62) is 1 when $i+j \in \{n-1-d, n-2-d\}$ and 0 otherwise.

We analyze when condition $\chi(|j-i| \leqslant n-2\ell-\lambda_1-\lambda_2)$ is satisfied. At $i+j = n-1-d$, the condition is true exactly for integers j which satisfy

$$\ell + \lambda_2 \leqslant j \leqslant n-d-\ell-\lambda_2-1. \tag{63}$$

Summing up $u^{n-1-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2}[n-d-2\ell-2\lambda_2]_{u,v}. \tag{64}$$

At $i+j = n-2-d$, the condition is true exactly for integers j which satisfy

$$\ell + \lambda_2 - 1 \leqslant j \leqslant n-d-\ell-\lambda_2-1. \tag{65}$$

Summing up $u^{n-2-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2-1}[n-d-2\ell-2\lambda_2+1]_{u,v}. \tag{66}$$

Next we analyze another component of equation (60):

$$\chi(\lambda_2 \leqslant \tfrac{i+j-n+\lambda_1+\lambda_2+1}{2} \leqslant \lambda_1) - \chi(\lambda_2 \leqslant \tfrac{i+j-n+\lambda_1+\lambda_2-1}{2} \leqslant \lambda_1), \tag{67}$$

which is equivalent to

$$\chi(|i+j-(n-1)| \leqslant d) - \chi(|i+j-(n+1)| \leqslant d). \tag{68}$$

We have that $|i+j-(n-1)| \leqslant d$ is satisfied if $i+j = n-1$ (if $d \geqslant 0$), if $i+j = n-2$ (if $d \geqslant 1$), if $i+j = n-3$ (if $d \geqslant 2$), etc. On the other hand, $|i+j-(n+1)| \leqslant d$ is satisfied if $i+j = n-1$ (if $d \geqslant 2$), if $i+j = n-2$ (if $d \geqslant 3$), etc. Putting these together, when $d = 0$, we get that equation (68) is 1 when $i+j = n-1$ and 0 otherwise. When $d \geqslant 1$, we get that equation (68) is 1 when $i+j \in \{n-d, n-1-d\}$ and 0 otherwise.

We analyze when condition $\chi(|j-i| \leqslant n-2\ell-\lambda_1-\lambda_2+1)$ is satisfied. At $i+j = n-1-d$, the condition is true exactly for integers j which satisfy

$$\ell + \lambda_2 - 1 \leqslant j \leqslant n-d-\ell-\lambda_2. \tag{69}$$

Summing up $u^{n-1-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2-1}[n-d-2\ell-2\lambda_2+2]_{u,v}. \tag{70}$$

When $d \geqslant 1$, at $i+j=n-d$, the condition is true exactly for integers j which satisfy

$$\ell+\lambda_2 \leqslant j \leqslant n-d-\ell-\lambda_2. \tag{71}$$

Summing up $u^{n-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2}[n-d-2\ell-2\lambda_2+1]_{u,v}. \tag{72}$$

For concision, we use $m := n-2\ell-\lambda_1-\lambda_2+2 = n-d-2\ell-2\lambda_2+2$. When $d=0$, i.e., $\lambda_1=\lambda_2$, summing equations (64), (66), and (70) proves equation (56). When $d \geqslant 1$, i.e., $\lambda_1 > \lambda_2$, summing equations (64), (66), (70), and (72) proves equation (57).

□

7 Four sets of fermions

A natural further question is to determine $\langle \mathrm{Frob}(R_n^{(0,j)}; u_1, \ldots, u_j), s_{(1^n)}\rangle$ for $j \geqslant 4$, of which the simplest next case is $\langle \mathrm{Frob}(R_n^{(0,4)}; u, v, w, z), s_{(1^n)}\rangle$. If the hook characters in $R_n^{(0,3)}$ are all known, then a similar process as in Proposition 9 using $\mathrm{Frob}(R_n^{(0,3)} \otimes R_n^{(0,1)}; u, v, w, z)$ can be attempted. Using Theorem 18, we can first bound $\langle \mathrm{Frob}(R_n^{(0,3)}; u, v, w), s_{(n-d,1^d)}\rangle$.

Example 20. Let $n=3$. Suppose we want to calculate $\langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(2,1)}\rangle$. Consider that any $s_{(2,1)}$ in $\mathrm{Frob}(R_3^{(0,3)}; u, v, w)$ must appear in $\mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,1)}; u, v, w)$. We compute that

$$\begin{aligned}&\langle \mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,1)}; u, v, w), s_{(2,1)}\rangle \\ &\qquad = \sum_{\lambda\vdash 3}\sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_\lambda\rangle \langle \mathrm{Frob}(R_3^{(0,1)}; w), s_{(3-d,1^d)}\rangle g(\lambda, (3-d, 1^d), (2,1)).\end{aligned} \tag{73}$$

Using Rosas' formula for Kronecker coefficients of two hooks, we determine that the only pairs of λ and $(3-d, 1^d)$ which do not have multiplicity of 0 are $(3), (2,1)$; $(2,1), (3)$; $(2,1), (2,1)$; $(2,1), (1^3)$; and $(1^3), (2,1)$, which all have multiplicity of 1. Using Proposition 5 and Lemma 3, we obtain

$$w + ([2]_{u,v}+uv) + ([2]_{u,v}+uv)w + ([2]_{u,v}+uv)w^2 + [3]_{u,v}w. \tag{74}$$

By permuting which sets of variables are assigned to $R_n^{(0,2)}$ and $R_n^{(0,1)}$, we can similarly obtain

$$v + ([2]_{u,w}+uw) + ([2]_{u,w}+uw)v + ([2]_{u,w}+uw)v^2 + [3]_{u,w}v, \tag{75}$$

$$u + ([2]_{w,v}+wv) + ([2]_{w,v}+wv)u + ([2]_{w,v}+wv)u^2 + [3]_{w,v}u. \tag{76}$$

Summing the monomials which appear in all three equations, we obtain

$$u + v + w + uv + uw + vw + 2uvw. \tag{77}$$

Now a computer calculation finds that

$$\langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(2,1)} \rangle = u + v + w + uv + uw + vw, \tag{78}$$

which shows that the bound is not tight in this case due to the monomial $2uvw$. It will require additional work to determine how to cut out the extra monomials in general.

Example 21. Let $n = 3$. Suppose we know all of the hook characters in $R_3^{(0,3)}$; in this case $\mathrm{Frob}(R_3^{(0,3)}; u, v, w) = (uvw + u^2 + uv + v^2 + uw + vw + w^2)s_{(1^3)} + (uv + uw + vw + u + v + w)s_{(2,1)} + s_{(3)}$. Consider that any sign character in $R_3^{(0,4)}$ must appear in $R_3^{(0,3)} \otimes R_3^{(0,1)}$. We compute that

$$\begin{aligned}\langle \mathrm{Frob}(R_3^{(0,3)} \otimes R_3^{(0,1)}; u, v, w, z), s_{(1^3)} \rangle &= \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{\lambda'} \rangle \langle \mathrm{Frob}(R_3^{(0,1)}; z), s_\lambda \rangle \\ &= \sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(d+1,1^{2-d})} \rangle z^d,\end{aligned} \tag{79}$$

using Lemma 3. This simplifies to

$$u^2 + v^2 + w^2 + z^2 + uv + uw + uz + vw + vz + wz + uvw + uvz + uwz + vwz, \tag{80}$$

which is unchanged under changing which sets of variables are assigned to $R_n^{(0,3)}$ and $R_n^{(0,1)}$. In this example, equation (80) is equal to $\langle \mathrm{Frob}(R_3^{(0,4)}; u, v, w, z), s_{(1^3)} \rangle$ determined by computer calculation.

Example 22. The previous example also gives tight bounds for $n = 4$, with graded multiplicity $s_{(3)}(u, v, w, z) + s_{(2,1,1)}(u, v, w, z) + s_{(2,1,1,1)}(u, v, w, z)$, and for $n = 5$, with graded multiplicity $s_{(4)}(u, v, w, z) + s_{(3,1,1)}(u, v, w, z) + s_{(3,1,1,1)}(u, v, w, z) + s_{(2,2,1,1)}(u, v, w, z) + s_{(2,2,2,1)}(u, v, w, z)$.

Another approach is to use $\mathrm{Frob}(R_n^{(0,2)} \otimes R_n^{(0,2)}; u, v, w, z)$ to bound the sign character in $\mathrm{Frob}(R_n^{(0,4)}; u, v, w, z)$ directly. As Kim and Rhoades note, only Schur functions of shapes contained within double hooks occur in $\mathrm{Frob}(R_n^{(0,2)}; u, v)$, so the shapes which index sign characters are limited.

Example 23. Let $n = 3$. Consider that any sign character in $R_3^{(0,4)}$ must appear in $R_3^{(0,2)} \otimes R_3^{(0,2)}$. We compute that

$$\langle \mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,2)}; u, v, w, z), s_{(1^3)} \rangle = \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{\lambda'} \rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_\lambda \rangle. \tag{81}$$

In this case, there are only three partitions of 3: $(1^3), (2,1), (3)$. Using Proposition 5, we get

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(3)}\rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(1^3)}\rangle = [3]_{w,z}, \tag{82}$$

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(2,1)}\rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(2,1)}\rangle = ([2]_{u,v} + uv)([2]_{w,z} + wz), \tag{83}$$

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(1^3)}\rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(3)}\rangle = [3]_{u,v}. \tag{84}$$

Putting these together, we obtain

$$[3]_{w,z} + ([2]_{u,v} + uv)([2]_{w,z} + wz) + [3]_{u,v}. \tag{85}$$

By changing which sets of variables are assigned to each $R_n^{(0,2)}$, we can similarly obtain

$$[3]_{v,z} + ([2]_{u,w} + uw)([2]_{v,z} + vz) + [3]_{u,w}, \tag{86}$$

$$[3]_{u,z} + ([2]_{w,v} + wv)([2]_{u,z} + uz) + [3]_{w,v}. \tag{87}$$

In this case, all three are equal to each other, and expand to give

$$uvwz + uvw + uvz + uwz + vwz + u^2 + uv + v^2 + uw + vw + w^2 + uz + vz + wz + z^2. \tag{88}$$

Now a computer calculation finds that

$$\begin{aligned} &\langle \mathrm{Frob}(R_3^{(0,4)}; u, v, w, z), s_{(1^3)}\rangle \\ &\qquad = uvw + uvz + uwz + vwz + u^2 + uv + v^2 + uw + vw + w^2 + uz + vz + wz + z^2, \end{aligned} \tag{89}$$

which shows that the bound is not tight in this case due to the monomial $uvwz$. It will require additional work to determine how to cut out the extra monomials in general.

8 One set of bosons and three sets of fermions

In this section, we study the sign character in $R_n^{(1,3)}$, the $(1,3)$-bosonic-fermionic coinvariant ring. We first recall the "Theta conjecture" of D'Adderio, Iraci, and Vanden Wyngaerd, which expresses the multigraded Frobenius series of $R_n^{(2,2)}$ in terms of certain Theta operators and the nabla operator.

Conjecture 24 ([9, Conjecture 8.2])**.** For all $n \geqslant 1$,

$$\mathrm{Frob}(R_n^{(2,2)}; q, t; u, v) = \sum_{k+\ell<n} u^k v^\ell \Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}. \tag{90}$$

The Theta conjecture specialized at $t = 0$ is the following.

Conjecture 25 (D'Adderio, Iraci, and Vanden Wyngaerd)**.** For all $n \geqslant 1$,

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{k+\ell<n} u^k v^\ell \left(\Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}\right)|_{t=0}. \tag{91}$$

We recall the following result on the conjectural hook characters in $R_n^{(1,2)}$.

Theorem 26 ([15, Theorem 8.4])**.** *If the Theta conjecture specialized at $t = 0$ (Conjecture 25) is true, then*

$$\begin{aligned}&\langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{(d+1,1^{n-d-1})}\rangle \\ &\quad = \sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.\end{aligned} \tag{92}$$

Now we compute an example.

Example 27. Let $n = 3$. Consider that any sign character in $R_3^{(1,3)}$ must appear in $R_3^{(1,2)} \otimes R_3^{(0,1)}$. We compute that

$$\begin{aligned}&\langle \mathrm{Frob}(R_3^{(1,2)} \otimes R_3^{(0,1)}; q; u, v, w), s_{(1^3)}\rangle \\ &\quad = \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(1,2)}; q; u, v), s_{\lambda'}\rangle \langle \mathrm{Frob}(R_3^{(0,1)}; w), s_\lambda\rangle \\ &\quad = \sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(1,2)}; q; u, v), s_{(d+1,1^{2-d})}\rangle w^d \\ &\quad = (q^3 + vq[2]_q + uq[2]_q + uv[2]_q + v^2 + u^2) + (uv + q[2]_q + v[2]_q + u[2]_q)w + w^2.\end{aligned} \tag{93}$$

In this example, this is equal to $\langle \mathrm{Frob}(R_3^{(1,3)}; q; u, v, w), s_{(1^3)}\rangle$ determined by computer calculation.

Proposition 28. *If the Theta conjecture specialized at $t = 0$ (Conjecture 25) is true, then we have the following upper-bound:*

$$\begin{aligned}&\langle \mathrm{Frob}(R_n^{(1,3)}; q; u, v, w), s_{(1^n)}\rangle \\ &\quad \leqslant \sum_{k,\ell,d \geqslant 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.\end{aligned} \tag{94}$$

Proof. Any sign character in $R_n^{(1,3)}$ must appear in $R_n^{(1,2)} \otimes R_n^{(0,1)}$. Assume the Theta conjecture specialized at $t = 0$ (Conjecture 25) is true. By Theorem 26, we compute that

$$\begin{aligned}
&\langle \mathrm{Frob}(R_n^{(1,2)} \otimes R_n^{(0,1)}; q; u, v, w), s_{(1^n)}\rangle \\
&\quad = \sum_{\lambda \vdash n} \langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{\lambda'}\rangle \langle \mathrm{Frob}(R_n^{(0,1)}; w), s_\lambda\rangle \\
&\quad = \sum_{d=0}^{n-1} w^d \langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{(d+1,1^{n-d-1})}\rangle \\
&\quad = \sum_{d=0}^{n-1} w^d \sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q \\
&\quad = \sum_{k,\ell,d \geqslant 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q,
\end{aligned} \tag{95}$$

where the last line follows by considering when the q-binomial coefficients must be 0 (specifically, one could take $k + \ell < n$, $\ell + d < n$, and $d + k < n$). □

Based on data for $n \leqslant 5$, we propose the following conjecture.

Conjecture 29.

$$\begin{aligned}
&\langle \mathrm{Frob}(R^{(1,3)}; q; u, v, w), s_{(1^n)}\rangle \\
&\quad = \sum_{k,\ell,d \geqslant 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.
\end{aligned} \tag{96}$$

Notice that upon specializing q, u, v, w all to 1, the conjecture becomes

$$\langle \mathrm{Frob}(R^{(1,3)}; 1; 1, 1, 1), s_{(1^n)}\rangle = \sum_{k,\ell,d \geqslant 0} \binom{n-1-d}{\ell} \binom{n-1-k}{d} \binom{n-1-\ell}{k}. \tag{97}$$

Let F_n be the nth Fibonacci number, defined by the initial conditions $F_0 = 0$, $F_1 = 1$, and recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. Inspired by a conjecture of Zabrocki on $\mathrm{Frob}(R_n^{(2,1)}; q, t; u)$ [19], Bergeron made a general conjecture on how the Frobenius series of bosonic-fermionic coinvariant rings could be derived from the Frobenius series of purely bosonic coinvariant rings [3, Conjecture 1]. While collecting computational evidence, Bergeron conjectured the following.

Conjecture 30 ([3, Table 3])**.** For all $n \geqslant 1$,

$$\langle \mathrm{Frob}(R_n^{(1,3)}; 1; 1, 1, 1), s_{(1^n)}\rangle = \frac{1}{2} F_{3n}. \tag{98}$$

Beginning at $n = 1$, the sequence $\frac{1}{2}F_{3n}$ is $1, 4, 17, 72, 305, 1292, 5473, 23184, \ldots$ (see [16, Sequence A001076]). The two proposed enumerations are connected by the following result.

Proposition 31 ([8]; see [1] for a bijective proof)**.** *For* $n \geqslant 1$*,*

$$\sum_{k,\ell,d\geqslant 0} \binom{n-1-d}{\ell}\binom{n-1-k}{d}\binom{n-1-\ell}{k} = \frac{1}{2}F_{3n}. \tag{99}$$

Remark 32. If the Theta conjecture specialized at $t = 0$ (Conjecture 25) is true, and the conjecture of Bergeron (Conjecture 30) that $\langle \mathrm{Frob}(R^{(1,3)}; 1; 1, 1, 1), s_{(1^n)} \rangle = \frac{1}{2}F_{3n}$ is true, then by Proposition 31, the sign character has multiplicity large enough to force equality in the upper bound given in Proposition 28. This would prove Conjecture 29.

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