The α-representation for Tait coloring and the characteristic polynomial of a matroid

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Abstract

Consider a finite field $\mathbb{F}_q$, $q = p^d$, where p is an odd prime. Let $M = (E, r)$ be a regular matroid; denote by $\mathcal{B}$ the family of its bases, $\bar{s}(M; \alpha) = \sum_{B\in\mathcal{B}} \prod_{e\notin B} \alpha_e$, where $\alpha_e \in \mathbb{F}_q$, $\alpha_e \neq 0$. Let a subset $A \equiv A(\alpha)$ in E have maximum cardinality and satisfy the condition $\bar{s}(M|A; \alpha) \neq 0$, while $r^*(\alpha) = |A| - r(E)$. Let us represent the value of the characteristic polynomial of the matroid M at the point q as a linear combination of Legendre symbols with respect to $\bar{s}(M|A; \alpha)$, whose coefficients are equal in modulus to $1/q^{r^*(\alpha)/2}$. This representation generalizes the formula for the flow polynomial of a graph which was obtained by us earlier. The latter formula is an analog of the so-called α-representation of vacuum Feynman amplitudes over finite fields, which inspired the Kontsevich conjecture (1997). The α-representation technique is also applicable to expressing the number of Tait colorings for a cubic biconnected planar graph in terms of principal minors of the face matrix of this graph.

Mathematics Subject Classifications: 05B35, 05C31

1 Introduction

1.1 Historical background and the goal of the paper

Since the very appearance of the theory of matroids, its significant problems have been related to finite fields; it suffices to mention the critical problem stated by H. Crapo and G.-C. Rota. The problem consists in finding the critical exponent of an $\mathbb{F}_q$-representable matroid M of rank n, defined as the minimum number of hyperplanes in $\mathbb{F}_q^n$ whose intersection has no elements in common with M [4]. Nevertheless, in calculations of values of characteristic polynomials of matroids, one rarely uses number-theoretic ingredients directly.

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The idea of this paper has a long history. Let the symbol $\mathcal{T}(G)$ stand for the set of spanning trees of a connected graph G. Consider sums

$$s(G;\alpha) = \sum_{T\in\mathcal{T}(G)} \prod_{e\in E(T)} \alpha_e, \tag{1}$$

where α_e are elements of the finite field $\mathbb{F}_q$. In December 1997, while giving a talk at the Gelfand seminar at Rutgers University, Maxim Kontsevich proposed a conjecture that the number of nonzero values of (1) for $\alpha \in \mathbb{F}_q^E$ is a polynomial in q. This conjecture was inspired by the study of analogous sums (with complex-valued α_e) in quantum field theory. Although the conjecture was never published, it aroused the interest of experts in combinatorics (see [3, 18, 19]). The conjecture admits a natural generalization to matroids, where the set of all possible bases of a matroid represents an analog of $\mathcal{T}(G)$. However, the increased interest in the Kontsevich conjecture and its generalization negatively affected the development of such techniques, since the conjecture was subsequently refuted [2]. Nevertheless, by appropriately adapting the α-representation techniques used in quantum theory for the field of real numbers to the case of finite fields[1], we obtain a new representation for the flow polynomial of a graph [8].

The goal of this paper is to generalize the results obtained in [8] to the case of arbitrary regular matroids and to apply the α-representation technique to derive explicit formulas for cases of particular importance in graph theory. In particular, we consider the number of Tait colorings for a cubic biconnected planar graph. The fact that the number of Tait colorings for any such graph is non-zero is equivalent to the assertion of the Four Color Theorem.

In obtaining the α-representation for the characteristic polynomial of an arbitrary regular matroid, the fact that the representation matrix is unimodular plays an important role. In the non-regular case, we obtain a more complicated formula which depends on the representation matrix of the matroid over the field $\mathbb{F}_q$.

The expression for the number of Tait colorings for a cubic planar graph G turns out to be rather simple. It is related to the face matrix of this graph, which (as far as we know) has not been studied previously. In this matrix, rows and columns correspond to the faces F_i of the graph G. The (i, j)-th element of this matrix equals the sum of the values of variables $a(v)$, where v is a vertex belonging to both faces F_i and F_j, while the $a(v)$ are non-zero elements in the field $\mathbb{F}_3$ associated with the graph vertices (we treat these variables, which take on values ± 1, as spins). Note that the diagonal elements of this matrix are not inherently zero, as they represent the sum of $a(v)$ over all vertices v of the corresponding face; however, the sum of all entries in any row is zero in $\mathbb{F}_3$. The number of Tait colorings depends on the sum of Legendre symbols of principal non-zero minors of this matrix of maximum possible rank, taken over all possible a.

The appearance of the Legendre symbol in the α-representation is due to a specific technique: replacing the linear form in the exponent of the Fourier transform of the delta function with a quadratic form (see Eq. (13), subsection 3.1). This leads to quadratic

[1]See Remark 20 (subsection 3.1) for a brief history of the term "α-representation".

Gauss sums, well-known in number theory, which are expressed in terms of the Legendre symbol.

The non-triviality of their evaluation does not preclude the possibility of deriving the α-representation for combinatorial quantities other than those considered in this paper. This is always possible whenever the quantity of interest is defined as the number of nowhere-zero solutions to a system of linear homogeneous equations over a finite field $\mathbb{F}_q$ of odd characteristic. The reader is left to explore other applications; in the Conclusion, we will only discuss the development of the topic involving Tait colorings addressed in this article.

The remainder of the paper is structured as follows. In the next subsections, we state the main results and provide illustrative examples. Section 2 is devoted to auxiliary assertions used in the proofs of the main results. In particular, we state (and later prove) a more general theorem which is valid for all matroids representable over the field $\mathbb{F}_q$. An important role in this research is played by expressions for the characteristic function of the dual matroid in terms of representation matrices of the initial matroid (presented in subsection 2.5) and properties of the generalized Laplace–Kirchhoff matrix of an arbitrary matroid. In subsection 2.6, we recall Heawood's theorem on the number of Tait colorings, which is used below. In subsection 2.7, we introduce an important technical tool: multidimensional Gauss sums. Section 3 is specifically devoted to the α-representation. We first prove the theorem on the number of Tait colorings. Readers who are interested only in understanding this result may restrict themselves to the corresponding parts of subsections 1.2 and 1.3 and, after familiarizing themselves with subsections 2.6 and 2.7, proceed to this proof in Section 3. We use it to clarify the relatively simple principle behind the proof of the main result concerning the α-representation of matroids. We believe that after thorough preparation, the reader will easily understand this idea. In the Conclusion, we summarize the results obtained and discuss the further development of this work.

1.2 Notation and statements of the main results

Let us formally state the main results of this paper. We define a *matroid* M as a pair (E, r), where $r \equiv r_M$ is the *rank function* of M, and E is the *ground set* of M. Recall [14] that r is defined on the power set 2^E of E, takes nonnegative integer values in $\mathbb{N}_0$, and possesses the following properties:
(1) If $A \subseteq E$, then $0 \leqslant r(A) \leqslant |A|$.
(2) If $B \subseteq A \subseteq E$, then $r(B) \leqslant r(A)$.
(3) If A and B are subsets of E, then $r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B)$.

We denote a singleton $\{e\} \subseteq E$ as e; if $r(e) = 0$, then e is said to be a *loop*. By definition, $r(M) \equiv r(E)$; a set $B \subseteq E$ is called a *basis* if $r(B) = |B| = r(M)$. Let $\mathcal{B} \equiv \mathcal{B}_M$ denote the set of all bases of the matroid.

Let $A \subseteq E$. The restriction of $M = (E, r)$ to A is the matroid $M|A$ with the ground set A and a rank function coinciding with r on 2^A.

Recall that the characteristic polynomial χ_M of the matroid M is given by the formula

$$\chi_M(x) = \sum_{A: A\subseteq E} (-1)^{|A|} x^{r(E)-r(A)}.$$

As is easily seen, if M contains loops, then $\chi_M(x) \equiv 0$.

A matroid is said to be *representable* over a field $\mathbb{F}$, if there exists a matrix $\mathtt{M}$ such that its elements belong to the field $\mathbb{F}$, numbers of its columns are elements of the set E, and $r(A)$ equals the rank of the set of columns of the matrix $\mathtt{M}$ with indices in A. A matroid is said to be *regular*, if it is representable over any field $\mathbb{F}$ (see, for example, [14, 15] for more details).

We consider *the field* $\mathbb{F}_q$ of an odd characteristic, i.e., $q = p^d$, where p is an odd prime, and $d \in \mathbb{N}$. Denote by η *the multiplicative quadratic character* of the field $\mathbb{F}_q$: $\eta(0) = 0$, in other cases, $\eta(x) = 1$ or $\eta(x) = -1$ depending on whether x is a square in the field $\mathbb{F}_q$ or not. For $d = 1$ we have $\eta(x) = \left(\frac{x}{p}\right)$, where $\left(\frac{x}{p}\right)$ is *the Legendre symbol* of the residue field modulo prime p. Let us define a function $g(q, m)$, where q is the number of elements in the mentioned field, $m \in \mathbb{N}$, by the formula

$$g(q,m) = \begin{cases} 1/q^{m/2}, & \text{if } p \bmod 4 = 1,\ m \bmod 2 = 0, \\ 1/(-q)^{m/2}, & \text{if } p \bmod 4 = 3,\ m \bmod 2 = 0, \\ 0, & \text{if } m \bmod 2 = 1. \end{cases}$$

Let us associate each element $e \in E$ of the matroid with a nonzero value $\alpha_e \in \mathbb{F}_q$. Let $\alpha = (\alpha_e)_{e\in E}$ and define

$$s(M) \equiv s(M;\alpha) = \sum_{B\in\mathcal{B}} \prod_{e\in B} \alpha_e, \qquad \bar{s}(M) \equiv \bar{s}(M;\alpha) = \sum_{B\in\mathcal{B}} \prod_{e\notin B} \alpha_e.$$

Here the product over the empty set equals one. In subsection 3.1, we discuss the historical background of the choice of the denotation α. Let us put

$$A^*(M;\alpha) \equiv A^* = \operatorname*{argmax}_A \{|A| : s(M|A) \neq 0\}, \quad r^*(M;\alpha) = |A^*| - r(M).$$

Note that with the number of nowhere-zero α the condition $s(M|A;\alpha) \neq 0$ is equivalent to the inequality $\bar{s}(M|A;\alpha) \neq 0$. The set $A^*(\alpha)$ is not defined uniquely, but the value of the function η in the next theorem is independent of its choice.

Theorem 1. *Put $q = p^d$, where p is an odd prime number. The following formula is valid for any regular matroid without loops:*

$$\chi_M(q) = \sum g(q, r^*(M;\alpha))\ \eta(\bar{s}(M|A^*)); \tag{2}$$

where the sum is taken over all nowhere-zero vectors α in $\mathbb{F}_q^E$.

Note that one can seek for the subset A^* through a narrower set, see Corollary 9 in subection 2.4 for more details.

Let us now consider the result of another application of the same technique. Given a *simple biconnected planar cubic* graph $G = (V, E)$ (in this case, the biconnectivity of the graph is equivalent to the absence of bridges in it), we assume that the number of edges in the graph equals $3n$, while the number of vertices and faces is $2n$ and $n + 2$, correspondingly, $n = 2, 3, \dots$. *The Tait coloring* is a coloring of edges in E in 3 colors such that all edges with a common vertex are colored differently. The existence of such coloring for any graph G in the class under consideration is equivalent to the assertion of the Four Color Theorem. We denote *the number of various Tait colorings* for the graph G by the symbol $\chi'_G(3)$.

Denote by $a(v)$ the variable (spin) of vertex v ($v \in V(G)$), which takes on values ± 1 in the field $\mathbb{F}_3$. Let $F_1, \dots, F_{n+2}$ be all faces of the graph G. We understand the matrix of faces of the graph G as an $(n+2)\times(n+2)$-matrix $\texttt{FM}(a)$, whose i, j-th element equals the sum of values $a(v)$, where v belongs to both faces F_i and F_j. Observe that the diagonal entries of this matrix are not inherently zero. They are defined as the sum of $a(v)$ over all vertices v belonging to the corresponding face. However, the sum of all entries in any row is zero in $\mathbb{F}_3$.

Evidently, the matrix $\texttt{FM}(a)$ is symmetric and degenerate (since the graph is cubic, the sum of elements in any row of this matrix equals zero in the field $\mathbb{F}_3$). Denote the rank of the matrix $\texttt{FM}(a)$ by the symbol $\operatorname{rank}(\texttt{FM}(a))$. Let $\texttt{FM}'(a)$ be an arbitrary principal minor $\texttt{FM}(a)$ of the order $\operatorname{rank}(\texttt{FM}(a))$ such that $\det \texttt{FM}'(a)$ differs from zero. Let $\operatorname{sign}(\texttt{FM}(a)) = +1$, if $\det \texttt{FM}'(a) = 1$, and $\operatorname{sign}(\texttt{FM}(a)) = -1$ otherwise. In other words, $\operatorname{sign}(\texttt{FM}(a)) = \left(\frac{\det \texttt{FM}'(a)}{3}\right)$.

Theorem 2. *The following formula holds:*

$$\chi'_G(3) = 3 \sum (-1/3)^{\operatorname{rank}(\texttt{FM}(a))/2} \operatorname{sign}(\texttt{FM}(a)); \tag{3}$$

here the sum is taken over all nowhere-zero values of spins $a = (a(v))_{v\in V(G)}$ such that the rank of the matrix $\texttt{FM}(a)$ is even.

1.3 Examples of calculations using the formulas given in theorems 1 and 2

Let us illustrate theorems 1 and 2 by applying them to the calculation of the number of proper vertex colorings in 3 colors (Example 1) and Tait colorings (Example 2) for the graph shown in Fig. 1. Recall that a vertex coloring is said to be proper if adjacent vertices have different colors. The ratio between the number of such colorings for a connected graph and the number of colors q (in the case of a connected graph) equals the value of the characteristic polynomial of the cyclic matroid M_G at the point q. The ground set E of the cyclic matroid M_G is the set of edges of the graph G. The restriction $M_G|A$ represents the cyclic matroid of the subgraph induced by the set of edges A; this subgraph is not necessarily connected. Recall that for a connected graph G, each spanning

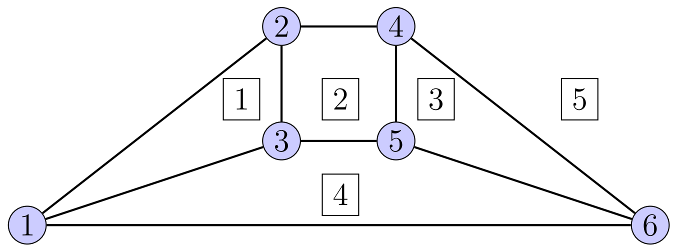

Figure 1: We will calculate the number of proper colorings of vertices and edges of this graph in 3 colors with the help of theorems 1 and 2, correspondingly.

tree is a basis of the matroid M_G. In the case of a disconnected graph, a basis of a cyclic matroid is the union of spanning trees of all connected components.

Example 1. In order to apply Theorem 1, let us make use (see, for example, [18] and references therein) of the fact that the generating function of spanning trees $s(M_G;\alpha)$ is the cofactor of any element in the weighted Laplace–Kirchhoff matrix of the graph G (a generalized version of the matrix tree theorem). For the graph under consideration, this matrix takes the form

$$L = \begin{pmatrix} \ell_{11}(\alpha) & -\alpha_{12} & -\alpha_{13} & 0 & 0 & -\alpha_{16} \\ -\alpha_{12} & \ell_{22}(\alpha) & -\alpha_{23} & -\alpha_{24} & 0 & 0 \\ -\alpha_{13} & -\alpha_{23} & \ell_{33}(\alpha) & 0 & -\alpha_{35} & 0 \\ 0 & -\alpha_{24} & 0 & \ell_{44}(\alpha) & -\alpha_{45} & -\alpha_{46} \\ 0 & 0 & -\alpha_{35} & -\alpha_{45} & \ell_{55}(\alpha) & -\alpha_{56} \\ -\alpha_{16} & 0 & 0 & -\alpha_{46} & -\alpha_{56} & \ell_{66}(\alpha) \end{pmatrix},$$

where each diagonal entry represents the sum of all the remaining entries in the corresponding row with the opposite sign. This fact allows us to evaluate (with the help of a computer program) 512 cases (with $\alpha_e \in \{1, -1\} \subseteq \mathbb{F}_3$) of the Legendre symbol $\left(\frac{\bar{s}(M_G;\alpha)}{3}\right) = \left(\frac{s(M_G;\alpha)\prod_e \alpha_e}{3}\right)$. In 164 cases, the cofactor equals zero. In 186 cases, $\left(\frac{\bar{s}(M_G;\alpha)}{3}\right) = +1$, and in 162 cases, $\left(\frac{\bar{s}(M_G;\alpha)}{3}\right) = -1$. In addition, the number of basis elements (edges of the spanning tree) is 5; consequently, $r^*(M_G;\alpha) = |E| - 5 = 4$. The contribution of the $186 + 162$ cases to the right-hand side of formula (2) equals $\frac{186}{9} - \frac{162}{9}$.

To handle the remaining 164 cases, we need to determine the values of $s(M_G|A;\alpha)$ with $|A| = 8$ (there are, evidently, 9 such subsets A). This coincides with the value $s(M_G;\alpha)|_{\alpha_e=0}$, where $\{e\} = E - A$. Among the 164 cases, there are only 16 where all 9 values are equal to zero (the "zero cases"). In the remaining cases, $r^*(M_G;\alpha) = |A| - 5 = 3$; consequently, these cases do not affect the right-hand side of formula (2).

In all 16 "zero cases", we have $s_0(\alpha) \equiv s(M_G;\alpha)|_{\alpha_{12}=\alpha_{13}=0} \neq 0$, i.e., for $A = E - \{(1,2),(1,3)\}$, we have $\bar{s}(M_G|A;\alpha) = s_0(\alpha)\prod_{e\in A}\alpha_e \neq 0$. Here $r^*(M_G;\alpha) = 2$. Applying the latter formula to calculate $\bar{s}(M_G|A;\alpha)$, we find that $\left(\frac{\bar{s}(M_G|A;\alpha)}{3}\right) = 1$ in 6 cases and $\left(\frac{\bar{s}(M_G|A;\alpha)}{3}\right) = -1$ in 10 cases. As a result, we conclude that for the graph G

shown in Fig. 1,

$$\chi_{M_G}(3) = \frac{186 - 162}{9} - \frac{6 - 10}{3} = 4;$$

consequently, there are 12 proper vertex colorings of this graph in 3 colors. Certainly, this fact can be easily verified by standard techniques developed for the calculation of chromatic polynomials.

Example 2. Let us now calculate $\chi'_G(3)$ according to Theorem 2. For brevity, let $(a(1), a(2), a(3), a(4), a(5), a(6)) \equiv (b, c, d, x, y, z)$. Then the face matrix for our graph takes the form

$$\begin{pmatrix} b+c+d & c+d & 0 & b+d & b+c \\ c+d & c+d+x+y & x+y & d+y & c+x \\ 0 & x+y & x+y+z & y+z & x+z \\ b+d & d+y & y+z & b+d+y+z & b+z \\ b+c & c+x & x+z & b+z & b+c+x+z \end{pmatrix}.$$

All principal minors of the fourth order are identical and equal to

$$\sum_{\{i_1,i_2,i_3,i_4\} \notin F_2,F_4,F_5} a(i_1)a(i_2)a(i_3)a(i_4); \tag{4}$$

here the sum is taken over all quadruples of distinct spins, except for those with indices $\{2,3,4,5\}$, $\{1,3,5,6\}$, and $\{1,2,4,6\}$. Note that when deriving the formulas, we have taken into account the fact that $a(i)$ are non-zero elements of the field $\mathbb{F}_3$ and, consequently, $a(i)^2 = 1$. Calculating the sum according to formula (3) over the 64 cases of values of $a \in \{-1,1\}^V$, we find that expression (4) yields a non-zero contribution in 30 cases; namely, in 18 cases this minor equals $-1 \pmod 3$, and in 12 cases it equals 1. In another 8 cases, this minor equals zero, as do all principal minors of the third order; furthermore, several principal minors of the second order are non-zero, but in all such cases, they equal $-1 \pmod 3$. Therefore, the sum in expression (3) equals

$$\frac{-18 + 12}{9} - \frac{-8}{3} = 2,$$

i.e., $\chi'_G(3) = 6$.

2 Auxiliary assertions necessary for proving theorems 1 and 2

2.1 Remark on the set A^*

In Theorem 1, we consider the set A^* whose cardinality is maximum among all subsets A of the set E such that $s(M|A) \neq 0$. We can significantly reduce the number of subsets to be enumerated. Here, we prove one simple property of the set A^*, and later in Corollary 9, we reduce the search space even more substantially.

Lemma 3. *In the notation of Theorem 1, the following equality holds: $r(A^*) = r(M)$.*

Proof. Assume that $s(M|A) \neq 0$ for a certain set A with rank $r(A) < r(M)$. Let I stand for a basis of the matroid $M|A$, i.e., $|I| = r(A)$. According to the so-called independence augmentation property (see [14, subsection 1.1]), there exists an element e of a basis of the matroid M such that $r(I \cup e) = r(A) + 1$. Consequently, $r(A \cup e) > r(A)$.

Let B be an arbitrary basis of the matroid $M|(A \cup e)$. It necessarily contains e; otherwise, for sets A and B, we would obtain a contradiction with the second axiom of the rank function of a matroid (see subsection 1.2). But then $s(M|(A \cup e)) = s(A) \times \alpha_e \neq 0$. Therefore, the set A under consideration does not coincide with $\operatorname{argmax}\{|A| : s(M|A) \neq 0\}$, i.e., $r(A^*) = r(M)$. □

2.2 Another statement of Theorem 1

In the paper [8], the flow polynomial is expressed as a linear combination of Legendre symbols of the values of the generating function of spanning trees of a connected graph. The flow polynomial is the characteristic polynomial of the bond matroid of the graph. However, spanning trees are not bases of the bond matroid; they represent bases of the cyclic matroid dual to the bond matroid. Note that in Theorem 1, the sum is taken over the bases of the matroid whose characteristic polynomial is the one being sought. Here, we state another assertion whose matrix variant implies (as a particular case) the main result of the paper [8].

Recall that *the dual matroid* $M^\perp = (E, r^\perp)$ of a matroid $M = (E, r)$ is defined by the rank function $r^\perp$, which is defined on the same ground set as r and satisfies the relation

$$r^\perp(A) = r(E - A) + |A| - r(E), \quad \text{for any } A \subseteq E. \tag{5}$$

It is easily verified that *there is a one-to-one correspondence between the bases of the original matroid and those of its dual*: if B is a basis of M, then $E - B$ is a basis of $M^\perp$, and vice versa. Hence $(M^\perp)^\perp = M$. It is also well known (see, for example, [14]) that the dual of a matroid representable over a field $\mathbb{F}$ is also representable over the same field. Consequently, the dual of a regular matroid is also regular. If $e \in E$ is a loop of the matroid $M^\perp$, then e is said to be a *coloop* of the matroid M.

Let $A \subseteq E$ and $C = E - A$. *The contraction* of M onto A is the matroid $M.A \equiv M/C$ with the ground set A and the rank function satisfying the relation

$$r_{M.A}(B) = r_M(C \cup B) - r_M(C) \quad \text{for any } B \subseteq A.$$

Assume that

$$A^*_\perp(\alpha) \equiv A^*_\perp = \operatorname*{argmax}_A \{|A| : s(M.A; \alpha) \neq 0\}.$$

Although the set $A^*_\perp$ is not uniquely defined, the contribution of each term in the sum (6) is independent of the choice of this set.

Let $\mathbb{F}_q^*$ denote the *set of non-zero elements* of the field $\mathbb{F}_q$ (which form a multiplicative group).

Theorem 4. *Let $q = p^d$, where p is an odd prime. For any regular matroid without coloops, the following formula holds:*

$$\chi_{M^\perp}(q) = \sum\nolimits_{\alpha \in (\mathbb{F}_q^*)^E} g(q, r(M.A^*_\perp(\alpha)))\, \eta(s(M.A^*_\perp;\alpha)). \tag{6}$$

Lemma 5. *Theorem 4 is equivalent to Theorem 1.*

Proof. As is well known, $(M|A)^\perp = M^\perp.A$ and $M^\perp|A = M.A$. In addition, as was mentioned above, each basis B of the matroid $M|A$ corresponds to the basis of $(M|A)^\perp$ given by $A - B$; therefore, $\bar{s}(M|A;\alpha) = s((M|A)^\perp;\alpha)$. Thus, $A^*(M^\perp;\alpha) = A^*_\perp(\alpha)$.

Consequently, to establish a one-to-one correspondence between the terms in sums (2) and (6) (where the matroid $M^\perp$ plays the role of M in the first formula), it suffices to show that $r^*(M^\perp;\alpha) = r(M.A^*_\perp(\alpha))$.

Let us make use of Lemma 3. The following assertion holds: $r_{M^\perp}(A^*_\perp) = r_{M^\perp}(E)$, i.e., $A^*_\perp$ contains a basis of the matroid $M^\perp$. Consequently, the complement of this set, $E - A^*_\perp$, is a subset of a basis of the matroid M, and thus $r(E - A^*_\perp) = |E| - |A^*_\perp|$.

We conclude that

$$r^*(M^\perp;\alpha) = |A^*_\perp| - r^\perp(E) = r(E) - |E| + |A^*_\perp| = r(E) - r(E - A^*_\perp) = r(M.A^*_\perp).$$

□

2.3 A more general version of Theorem 1

Theorem 1 does not hold for non-regular matroids. For example, the uniform matroid $U_{2,4}$, which is representable over all fields except for $\mathbb{F}_2$ (see [14, Proposition 6.5.2]), satisfies $\chi_{U_{2,4}}(q) = (q-1)(q-3)$. At the same time, the right-hand side of formula (2) equals 0 for $q = 3$, but is not equal to 8 for $q = 5$.

Note that the matroid $U_{2,4}$ cannot be defined by a unimodular matrix over any field $\mathbb{F}_q$ because it is not regular (see [14, Theorem 6.6.3]). This matroid is representable by the matrix

$$\mathtt{M}\,(U_{2,4}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}. \tag{7}$$

Evidently, the determinant of the submatrix formed by the last two columns over any field whose characteristic exceeds 3 is not equal to ± 1.

Let us now state a generalization of Theorem 4 that holds for all matroids representable over the field $\mathbb{F}_q$. The statement of this generalization involves the representation matrix of the matroid.

Let $\mathtt{M}$ be a representation matrix for the matroid M over the field $\mathbb{F}_q$; let the symbol E denote the set of indices of its columns, and let the symbol V denote the set of indices of its rows. Without loss of generality, we assume that the number of rows is equal to the rank of this matrix. Denote by $\mathtt{M}\,|_B$ the non-degenerate square submatrix of $\mathtt{M}$ formed by the columns whose indices belong to the set B (and all rows of $\mathtt{M}$). Put

$$s'(\mathtt{M};\alpha) = \sum_{B \in \mathcal{B}(\mathtt{M})} \det{}^2(\mathtt{M}\,|_B) \prod_{e \in B} \alpha_e,$$

where $\mathcal{B}(\mathtt{M})$ is the collection of all subsets of columns of this matrix such that $\det(\mathtt{M}|_B) \neq 0$.

Let $W \subseteq V$. Denote by $\mathtt{M}/W$ the matrix obtained from $\mathtt{M}$ by deleting the rows whose indices belong to the set W; this matrix corresponds to a certain matroid. Evidently, the rank of this matroid is $|V| - |W|$.

Let $W^*(\mathtt{M}, \alpha)$ denote an arbitrary subset W of V of minimum cardinality for which the sum $s'(\mathtt{M}/W; \alpha)$ is non-zero. Denote by $r^*(\mathtt{M}; \alpha)$ the difference $|V| - |W^*(\mathtt{M}, \alpha)|$.

Theorem 6. *We use the following notation: $\mathbb{F}_q$ is a finite field of odd characteristic, M is an $\mathbb{F}_q$-linear matroid, and $\mathtt{M}$ is its representation matrix. The following formula holds:*

$$\chi_{M^\perp}(q) = \sum\nolimits_{\alpha \in (\mathbb{F}_q^*)^E} g(q, r^*(\mathtt{M}; \alpha))\, \eta(s'(\mathtt{M}/W^*(\mathtt{M}, \alpha); \alpha)). \tag{8}$$

Furthermore, not only the sum itself, but each individual term depends neither on the choice of the representation matrix $\mathtt{M}$ of the matroid, nor on the choice of the subset W^* of rows of this matrix.

Remark 7. In [8], we considered a particular case of Theorem 6 for a cyclic matroid M. We did not state the independence of each term in the sum (8) on the representation matrix of the matroid, but considered a fixed (canonical) matrix $\mathtt{M}$ representing the incidence matrix of a connected graph (with the row corresponding to one of the vertices deleted). Since this matrix is unimodular, we conclude that $\det^2(\mathtt{M}|_B) \equiv 1$, and the sum taken over the bases $\mathcal{B}(\mathtt{M}/W)$ in this case represents the sum over the spanning trees of the corresponding graph.

Example 3. Let us illustrate the relation (8) in the case of the matroid $U_{2,4}$. Note that the matroid $U_{2,4}$ is self-dual, while the left-hand side of identity (8) is $\chi_{U_{2,4}^\perp}(q) = (q-1)(q-3)$. If $U_{2,4}$ is defined by matrix (7), then

$$s'(\mathtt{M}(U_{2,4}); \alpha) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + 4\alpha_3\alpha_4.$$

It can be shown that $r^*(\mathtt{M}; \alpha) = 0$ only in $q-1$ cases (namely, when $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, and $\alpha_1 = -2\alpha_3$). Therefore, for $U_{2,4}$ with representation matrix (7), Theorem 6 is equivalent to the identity

$$(q-1)(q-4) = g(q, 2) \sum_{\alpha \in (\mathbb{F}_q^*)^4} \eta(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + 4\alpha_3\alpha_4).$$

We see that identity (8) implies several new relations for the Legendre symbol.

2.4 Theorem 4 as a corollary of Theorem 6 and some refinements regarding the choice of A^* and $A^*_\perp$

Lemma 8. *Theorem 6 implies Theorem 4.*

Proof. Let M be a regular matroid as stated in the assumptions of Theorem 4. According to Lemma 3 and Proposition 5, $E - A^*_\perp(\alpha)$ is a subset of a certain basis of the matroid M. Denote *this basis* by B^*. Consider (for a given fixed α) the representation matrix $\mathtt{M}$ of the matroid M, whose columns corresponding to the basis B^* form the identity matrix. Note that due to the regularity of the matroid M, we can choose $\mathtt{M}$ as a unimodular matrix. Therefore, the value of the function s' for the matrix $\mathtt{M}$ equals the value of the function s for the matroid corresponding to this matrix.

Let A' be the part of the basis B^* corresponding to columns whose identity entries are located in rows whose indices belong to the set W. Then, in accordance with [14, Proposition 3.2.6], by deleting the rows W and columns A' from the matrix $\mathtt{M}$, we obtain the representation matrix of the matroid M/A'. If we delete only the rows and do not delete the columns, we obtain a matrix with $|A'|$ zero columns, whose matroid differs from M/A' in having $|A'|$ additional loops. In both cases, the function s remains the same.

Therefore, to establish a one-to-one correspondence between the terms in the sums (6) and (8), it suffices to ensure that in the identity submatrix of the matrix $\mathtt{M}$ composed of columns of B^*, the non-zero elements in rows W^* are located exactly in the columns of $E - A^*_\perp$. The latter property is evident because, by construction, $(E - A^*_\perp) \subseteq B^*$ and $A^*_\perp$ has maximum cardinality among all such subsets A with non-zero $s(M.A)$. We choose the subset W^* using the same principle (naturally replacing the maximum with the minimum and s with s'). □

Note that the number of all subsets W of the rows of the matrix, in general, is smaller than the number of all possible subsets of the set E; in the proof given above, we fix some basis and make use of the fact that each term in the sum (8) is independent of the representation matrix of the matroid.

Corollary 9. *If Theorem 6 holds, then theorems 1 and 4 remain valid, provided that the search for the subsets A^* and $A^*_\perp$ is performed by enumerating fewer cases. Namely, let B' be some fixed basis of the matroid M as stated in the assumptions of theorems 1 and 4. Then one can define the sets A^* and $A^*_\perp$, respectively, by the formulas:*

$$A^* = \operatorname{argmax}_A\{|A| : B' \subseteq A,\ s(M|A;\alpha) \neq 0\},$$
$$A^*_\perp = \operatorname{argmax}_A\{|A| : (E - A) \subseteq B',\ s(M.A;\alpha) \neq 0\}.$$

In particular, in Example 1, for 164 cases of values of α, we find that $s(M|A;\alpha)$ equals zero for every choice of $A \subseteq E$, $|A| = 8$, among the 9 possible ones. If we fix the edges B' of some spanning tree and consider only sets A such that $B' \subseteq A$, it suffices to verify the mentioned equality for only 4 values instead of 9. Naturally, if we consider subsets A of the set E of smaller cardinality, the reduction of the enumeration process is even more significant.

2.5 Flows and an analogue of the Laplace–Kirchhoff matrix

In the paper [8], when considering a particular case of Theorem 6, we make use of the properties of the Laplace–Kirchhoff matrix of a graph and the fact that the value of the flow polynomial at point q is, by definition, equal to the number of nowhere-zero solutions in the field $\mathbb{F}_q$ of a system of linear homogeneous equations with the incidence matrix of the graph. Let us now state analogues of these properties for an arbitrary matroid representable over the field $\mathbb{F}_q$. We shall use these properties in the proof of Theorem 6 in the next section.

Let us first recall the classical Crapo–Rota result ([4, Theorem 1 in subsection 16.4]) on the geometric interpretation of the value $\chi_M(q)$ for a matroid M such that the columns of its representation matrix define $|E|$ points in the space $\mathbb{F}_q^V$. As it turns out, $\chi_M(q)$ is equal to the number of linear functionals on this space which take on non-zero values at all these points.

J. P. S. Kung [9, Section 1] defines *a flow* as a vector $\mathbf{f} = (f(e) : e \in E, f(e) \in \mathbb{F}_q)$ such that $\mathtt{M}\,\mathbf{f}^T = 0$. From the matrix $\mathtt{M}$ that defines the matroid M, he constructed another matrix (orthogonal to $\mathtt{M}$) which defines the dual matroid $M^\perp$. He then established a bijection between the set of nowhere-zero flows for the original matrix $\mathtt{M}$ and the linear functionals which take on non-zero values at the points defining the columns of the new matrix. Thus, J. P. S. Kung obtained the following assertion ([9, Theorem 1.1, particular case], see also [10]) as a corollary of the Crapo–Rota result.

Proposition 10. *The number of nowhere-zero flows for the matrix* $\mathtt{M}$ *is equal to* $\chi_{M^\perp}(q)$.

Certainly, this assertion is well known in the case of a cyclic matroid M. In this particular case, it is also well known that the principal minor of the weighted Laplace–Kirchhoff matrix, which can be expressed in terms of $\mathtt{M}$, is the generating function of spanning trees $s(M;\alpha)$. Let us now consider the case of an arbitrary representable matroid.

Let $\mathtt{M}$ be a representation matrix of an arbitrarily chosen matroid M. Recall that, by our assumption, the number of rows of this matrix is equal to $r(M)$. Consider the following square matrix $L(\mathtt{M};\alpha)$ of order $r(M)$:

$$L(\mathtt{M};\alpha) = \mathtt{M}\Lambda\mathtt{M}^T, \tag{9}$$

where Λ is the $|E| \times |E|$-diagonal matrix whose diagonal entries are α_e, $e \in E$, and T denotes the transpose. One can easily prove the following assertion by applying the Binet–Cauchy formula.

Proposition 11 ([15], Exercise 11 (c)). *Let an F_q-linear matroid M be defined by a matrix* $\mathtt{M}$. *Then*

$$\det(L(\mathtt{M};\alpha)) = s'(\mathtt{M},\alpha).$$

2.6 The Heawood Theorem

To prove Theorem 2, we (similarly to Proposition 10) need to represent the desired value (in this case, $\chi'_G(3)$) as the number of nowhere-zero solutions to a system of linear equations over a finite field (here, over the field $\mathbb{F}_3$). This representation of the number of Tait colorings for a cubic graph was obtained by P. J. Heawood [5].

Proposition 12 (Heawood, 1898). *Let us associate each vertex v of an arbitrary planar biconnected cubic graph G with a variable (spin) $\sigma(v)$, which takes values in the set $\{-1, 1\} \equiv \mathbb{F}_3^*$. Then the normed number of Tait colorings, $\chi'_G(3)/3$, is equal to the number of all possible sets of spins $(\sigma(v), v \in V(G))$ such that for any face F of this graph, the sum of the spins of all vertices of face F is zero.*

In P. J. Heawood's original paper, this assertion is stated in terms of the graph dual to G (see also [1] for another proof of the same theorem). However, it is often more convenient to consider a cubic graph (cf. [13, Theorem 9.3.4]).

2.7 Multidimensional Gauss sums over the field $\mathbb{F}_q$

In the application of the α-representation to the case of a real field, explicit formulas for calculating Gaussian integrals with the imaginary unit in the exponent play an important role. In the case of a finite field, we refer to multidimensional Gauss sums. Here, we also require these formulas (established in [8]).

The function $\exp(ix)$ is a homomorphism from the additive group of real numbers to the group of complex numbers of modulus one; all such homomorphisms (additive characters) are parameterized by a parameter $k \in \mathbb{R}$ and take the form $\exp(ikx)$. In the case of a finite field $\mathbb{F}_q$, $q = p^d$, an analogous role is played by the function

$$h(x) = \exp\left(2\pi i \ \mathrm{Tr}(x)/p\right), \quad \text{where } \mathrm{Tr}(x) = x + x^p + x^{p^2} + \ldots + x^{p^{d-1}}; \tag{10}$$

all additive characters take the form $h_k(x) = h(kx)$, $k \in \mathbb{F}_q$ ([12, Theorem 5.7]).

Definition 13. Let C be an arbitrary symmetric $n \times n$ matrix with elements in $\mathbb{F}_q$, where q is odd, and let $\mathbf{x}\, C\, \mathbf{x}^T$ be a quadratic form with this matrix ($\mathbf{x}$ is a row vector of the corresponding dimension). We define a multidimensional Gauss sum as the expression given by the formula: $\mathrm{Gau}_q(C) = \sum_{\mathbf{x} \in \mathbb{F}_q^n} h(\mathbf{x}\, C\, \mathbf{x}^T)$.

In the one-dimensional case with $C = 1$, following [6], we use the notation $g_1(q)$ for the Gauss sum: $g_1(q) = \sum_{x \in \mathbb{F}_q} h(x^2)$. For example,

$$g_1(3) = 1 + 2\exp(2\pi i/3), \qquad g_1(5) = 1 + 2\exp(2\pi i/5) + 2\exp(2\pi 4i/5).$$

As is well known (see [12, Theorem 5.15]), for a field with odd characteristic, it satisfies the formula

$$g_1(q) = \begin{cases} (-1)^{d-1}\sqrt{q}, & \text{if } p \bmod 4 = 1, \\ (-1)^{d-1} i^d \sqrt{q}, & \text{if } p \bmod 4 = 3. \end{cases}$$

Note that for even m, the function $g(q, m)$ defined earlier coincides with $\left[\frac{g_1(q)}{q}\right]^m$. Below, we provide an explicit formula for $\mathrm{Gau}_q(C)$, but first, let us make a useful remark.

Remark 14. If matrices C and A are congruent, i.e., $A = PCP^T$ for some non-degenerate $n \times n$ matrix P, then $\mathrm{Gau}_q(C) = \mathrm{Gau}_q(A)$.

Remark 14 holds because, under the substitution $\mathbf{x}' = P\,\mathbf{x}$, the sum $\mathrm{Gau}_q(A)$ is transformed into the sum $\mathrm{Gau}_q(C)$.

Lemma 15 (Lemma 8 in [8]). *Let $q = p^d$ with an odd prime p, and let* $\operatorname{rank} C = r$. *Then*

$$\frac{\mathrm{Gau}_q(C)}{q^n} = \eta(\det C_r)\left[\frac{g_1(q)}{q}\right]^r, \tag{11}$$

where $\det C_r$ *is an arbitrary non-zero principal minor of order r.*

In view of Remark 14, the proof of Lemma 15 reduces to the diagonal case (see [16, Chapter IV] for the technique of reducing a quadratic form over a finite field to diagonal form). The diagonal case factorizes into the one-dimensional case, which in turn reduces to the calculation of the sum $g_1(q)$.

Corollary 16. *For any symmetric matrix of rank r, the value $\eta(\det C_r)$, where $\det C_r$ is a non-zero principal minor of C of order r, is independent of the choice of C_r.*

Remark 17. In the case of the zero matrix C, the formula obtained in Lemma 15 remains valid, provided that in this case $\eta(\det C_r) = 1$.

Corollary 18. *Assume that all entries of a symmetric matrix C are linear functions with respect to some collection of variables $\alpha \in (F_q^*)^k$, and let $r(C(\alpha))$ be the rank of this matrix. Then* $\sum_{\substack{\alpha:\ \alpha\in(F_q^*)^k,\\ r(C(\alpha)) \bmod 2=1}} \mathrm{Gau}_q(C(\alpha)) = 0$.

Proof. Let γ be an arbitrary element of the field $\mathbb{F}_q$ such that $\eta(\gamma) = -1$. Let us replace the collection α in the sum under consideration with $\gamma\alpha$. Note that in the formula for $\mathrm{Gau}_q(C(\gamma\alpha))$, we then obtain the value $\gamma^r \det C_r$ instead of $\det C_r$. For odd r, we obtain the equality $\eta(\gamma^r \det C_r) = -\eta(\det C_r)$. Consequently, the sum under consideration equals its own negative, and thus it must be zero. ☐

3 The α-representation

3.1 The Fourier transform over the field $\mathbb{F}_q$ and its properties

To prove theorems 2 and 6, we use several simple properties of the Fourier transform over the field $\mathbb{F}_q$.

Definition 19. Consider complex-valued functions $f(k)$ whose argument k belongs to the field $\mathbb{F}_q$. We define the Fourier transform of such a function as the function $\widehat{f}(x) = \sum_{k\in\mathbb{F}_q} f(k)h(kx)/q$, where h satisfies formula (10).

Note that our definition of the Fourier transform is usually used for the inverse discrete Fourier transform. It is more convenient for us to use the terminology proposed in the paper [8]. Using the classical definition would require the introduction of a conjugation sign and an additional factor q in the right-hand sides of the corresponding formulas.

Let the symbol $\mathbf{1}(k)$ denote the function $f(k)$ that is identically equal to one, and let $\delta(x)$ denote the delta function (the Kronecker symbol), i.e., $\delta(0) = 1$ and $\delta(x) = 0$ for all

$x \in \mathbb{F}_q^*$. It is well known that $\sum_{k\in\mathbb{F}_q} h(kx) = 0$ for any non-zero x ([12, Theorem 5.4]). Hence

$$\widehat{\mathbf{1}}(x) = \delta(x). \tag{12}$$

Relation (12) implies the formula

$$\sum\nolimits_{x\in\mathbb{F}_q^*} h(kx) = q\delta(k) - 1 = q\delta(k^2) - 1 = \sum\nolimits_{y\in\mathbb{F}_q^*} h(k^2y); \tag{13}$$

we use it in what follows.

Remark 20. In the case of a finite field, we can treat the function $f(k) = 1 - \delta(k)$ as the norm of an element of a finite field and regard the sum $\sum_{x\in\mathbb{F}_q^*} h(kx)$ as the Fourier transform of the norm raised to a certain power. In the case of a real field, an analogue of this sum is the Fourier transform of the generalized function $|k|^\gamma$. In quantum field theory, such functions are known as propagators of Feynman amplitudes (in the massless case). The parametric representation for integrals of propagators of Feynman amplitudes as integrals of characters with a quadratic argument was proposed by R. Feynman. The paper [17, p. 691] by K. Symanzik gave rise to their systematic application; the symbol α stands there for an analogue of the variable y used by us. In quantum field theory, this representation of a propagator is called the α-representation. Following this tradition, we recall that Theorem 4 is a direct generalization of the main result obtained in the paper [8]; it has a real analogue in the theory of Feynman amplitudes.

3.2 Proof of Theorem 2

To clarify the idea of using the α-representation (which is quite evident in the discrete case), it is helpful to consider a more specific case first.

Proof of Theorem 2. According to the Heawood theorem (Proposition 12), $\chi'_G(3)$ equals three times the value of the sum

$$S = \sum_{\sigma\in\{-1,1\}^{V(G)}} \prod_{F\in\mathcal{F}} \delta\left(\sum_{v\in F}\sigma(v)\right), \tag{14}$$

where $\mathcal{F}$ is the *set of all faces of the graph G*.

Let us transform the right-hand side of formula (14), using the fact that each δ-function is the Fourier transform of the identity function (see (12)). Rewriting the product of exponentials as the exponential of a sum and changing the order of summation, we obtain the relation

$$S = \sum_{\mathbf{k}\in\mathbb{F}_3^{\mathcal{F}}} \sum_{\sigma\in\{-1,1\}^{V(G)}} \exp\left(\frac{2\pi i}{3}\sum_{F\in\mathcal{F}} k_F \sum_{v\in F}\sigma(v)\right)/3^{|\mathcal{F}|}.$$

We can represent the sum in the exponent in another way, namely,

$$\sum_{F\in\mathcal{F}} k_F \sum_{v\in F}\sigma(v) = \sum_{v\in V(G)}\sigma(v)\sum_{F:v\in F} k_F.$$

This allows us to apply formula (13). We obtain the relation

$$S = \sum_{\mathbf{k}\in\mathbb{F}_3^{\mathcal{F}}} \sum_{a\in\{-1,1\}^{V(G)}} \exp\left(\frac{2\pi i}{3} \sum_{v\in V(G)} a(v) \left(\sum_{F:v\in F} k_F\right)^2\right) /3^{|\mathcal{F}|}.$$

By changing the order of the outer sums and collecting terms in the exponent, we obtain the following relation:

$$S = \sum_{a\in\{-1,1\}^{V(G)}} \sum_{\mathbf{k}\in\mathbb{F}_3^{\mathcal{F}}} \frac{\exp(2\pi i\ \mathbf{k}\,\mathtt{FM}\,(a)\,\mathbf{k}^T\ /3)}{3^{|\mathcal{F}|}} = \sum_{a\in\{-1,1\}^{V(G)}} \frac{\mathrm{Gau}_3(\ \mathtt{FM}\,(a))}{3^{|\mathcal{F}|}},$$

where $\mathbf{k}$ is the row vector $(k_F, F\in\mathcal{F})$. Let us now make use of Corollary 18. The right-hand side of formula (11) with $q=3$ and an even rank of the argument of the function Gau_3 coincides with the term in the sum (3). □

3.3 Proof of Theorem 6

The proof of Theorem 6 proposed by us in many respects is analogous to the proof of Theorem 2. Note also the similarity of final results.

In accordance with formula (9) and Proposition 11 the value $s'(\,\mathtt{M}\,/W;\alpha)$ represents the determinant of the submatrix of the matrix $L(\,\mathtt{M}\,;\alpha)$ obtained from the latter by deleting rows and columns, whose numbers belong to the set W. Correspondingly, by Lemma 15 each term in sum (8) multiplied by $q^{r(M)}$ equals (with even rank of the matrix $L(\,\mathtt{M}\,;\alpha)$) the Gauss sum $\mathrm{Gau}_q(L(\,\mathtt{M}\,;\alpha))$.

Recall that numbers of rows of the matrix $\mathtt{M}$ belong to a certain set V, $r(M)=|V|$.

Proof of Theorem 6. Proposition 5 implies the formula

$$\chi_{M^\perp}(q) = \sum_{\mathbf{x}\in(\mathbb{F}_q^*)^E} \prod_{j\in V} \delta(\,\mathtt{M}_j\,\mathbf{x}^T),$$

where $\mathtt{M}_j$ is the j-th row of the matrix $\mathtt{M}$ and $\mathbf{x}=(x_e, e\in E)$.

Similarly to the proof of Theorem 2, we use formula (12) and, by transforming the product of characters h and changing the order of summation, obtain the relation

$$\chi_{M^\perp}(q) = \sum_{\mathbf{k}\in\mathbb{F}_q^V} \sum_{\mathbf{x}\in(\mathbb{F}_q^*)^E} h\left(\sum_{j\in V} k_j\,\mathtt{M}_j\,\mathbf{x}^T\right) /q^{|V|}.$$

Let us rewrite the sum in the argument of the function h by collecting terms for each x_e, namely,

$$\sum_{j\in V} k_j\,\mathtt{M}_j\,\mathbf{x}^T = \sum_{e\in E} x_e \sum_{j\in V} k_j\,\mathtt{M}_{j,e}.$$

This allows us to apply formula (13), thus yielding the relation

$$\chi_{M^\perp}(q) = \sum_{\mathbf{k}\in\mathbb{F}_q^V} \sum_{\alpha\in(\mathbb{F}_q^*)^E} h\left(\sum_{e\in E}\alpha_e \left(\sum_{j\in V} k_j \, \mathtt{M}_{j,e}\right)^2\right) /q^{|V|}.$$

Changing the order of the outer sums again and transforming the argument of the function h, we conclude that

$$\chi_{M^\perp}(q) = \sum_{\alpha\in(\mathbb{F}_q^*)^E} \sum_{\mathbf{k}\in\mathbb{F}_q^V} \frac{h(\mathbf{k}\, L(\mathtt{M};\alpha)\,\mathbf{k}^T)}{q^{|V|}} = \sum_{\alpha\in(\mathbb{F}_q^*)^E} \mathrm{Gau}_q(L(\mathtt{M};\alpha))/q^{|V|}.$$

In accordance with Corollary 18, we can neglect terms where the matrix $L(\mathtt{M};\alpha)$ has an odd rank.

It can be easily shown that each term in the latter sum is independent of the choice of the representation matrix of the matroid M. Indeed, let $\mathtt{M}'$ be another representation matrix of the same matroid. Then $\mathtt{M}' = P\,\mathtt{M}$ for some non-degenerate matrix P. Then the matrices $L(\mathtt{M};\alpha) = \mathtt{M}\,\Lambda\,\mathtt{M}^T$ and $L(\mathtt{M}';\alpha)$ are congruent; consequently, by Remark 14,

$$\mathrm{Gau}_q(L(\mathtt{M};\alpha)) = \mathrm{Gau}_q(L(\mathtt{M}';\alpha)).$$

□

4 Conclusion

This paper is devoted to extending the scope of the α-representation technique initially used in [8] for combinatorial problems to general structures in enumerative combinatorics. To this end, we first express the number of nowhere-zero solutions of a system of homogeneous equations over a finite field $\mathbb{F}_q$ as a sum of analogues of multidimensional Gaussian integrals over $\mathbb{F}_q$. The latter admit an explicit representation in terms of the Legendre symbols of the principal minors of the matrix associated with the quadratic form. As a result, we obtain a linear combination of these Legendre symbols with rational coefficients, the magnitudes of which are powers of $1/q$.

We have considered several versions of these expressions for the characteristic polynomial of a matroid. Note that in the preprint [11], we propose another (more complex) proof of a simplified version of Theorem 6; there, we use results obtained by C. Chevalley regarding the number of points at which a quadratic form vanishes over the field $\mathbb{F}_q$.

We have also obtained an α-representation for the number of Tait colorings of planar cubic graphs. In Example 2, we observed the equality of all principal minors of the fourth order, which is a standard property of the Laplace–Kirchhoff matrix. In [7], the face matrix was represented as a weighted Laplace–Kirchhoff matrix for the dual graph, and the cases of edge weights included in the sum were explicitly described. This sum is expressed over the spanning trees of the dual graph, which is a maximal planar graph. Note that the sum over the spanning trees of the dual graph corresponds to the sum over the complements of the spanning trees of the original graph (see (5)). This provides grounds for the hope that an α-representation exists for all cubic graphs, not just planar ones. Certainly, this research was inspired by its connection to the Four Color Theorem; we aim to find an algebraic and number-theoretic explanation for why the number of proper 3-edge-colorings of snarks is zero.

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