Lower Bound on the Maximum Denominator of Fractional Chromatic Numbers

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Abstract

What is the least integer $\mathrm{sd}(n)$ such that every graph on n vertices has fractional chromatic number p/q, where p and q are positive integers and $q \leqslant \mathrm{sd}(n)$? An upper bound on the determinants of Hadamard matrices implies that $\mathrm{sd}(n) \leqslant 2^{-n}(n+1)^{(n+1)/2}$. The only known lower bound on $\mathrm{sd}(n)$ that is exponential in n (asymptotically, roughly $1.346^n/\sqrt{\log n}$) was obtained using an iterated Mycielski construction [D. C. Fisher, J. Graph Theory 20 (1995), 403–409]. We improve on this bound by constructing a family of graphs which shows that $\mathrm{sd}(n) \geqslant 2^{n/2}$.

Mathematics Subject Classifications: 05C15, 05C72

1 Introduction

The fractional chromatic number $\chi_f(G)$ of a graph G can be defined as the optimal value of the relaxation of the integer program that defines the usual chromatic number $\chi(G)$ of G. This directly implies that the fractional chromatic number is a rational number, attained by a rational solution. From this point of view, determining $\chi_f(G)$ can be done in time polynomial in the size of the program thanks to the ellipsoid method. However, the size of this program is generally exponential in the size of the original graph, and determining the fractional chromatic number of a graph actually turns out to be NP-hard. More precisely, it is NP-hard to approximate the fractional chromatic number of an n-vertex graph within a factor of $n^{1-\varepsilon}$, for every $\varepsilon > 0$ [2].

Yet the fractional chromatic number is an ubiquitous notion, which appeared independently under various guises. In particular, it was also defined in terms of $(a:b)$-colourings

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of graphs (where each vertex is assigned a subset of size b of $\{1,\dots,a\}$ such that adjacent vertices are assigned disjoint sets). More precisely, if G is a graph then $\chi_f(G) \leqslant \frac{p}{q}$ if and only if there exists a positive integer t such that G admits a $(pt : qt)$-colouring. While there is no *a priori* knowledge on an upper bound on the least such integer t in terms of the number of vertices of the graph itself, one shows that the value of the integer t can be taken to be the least common denominator of the values in any rational optimal solution of the linear program defining $\chi_f(G)$.

Such considerations raise several questions about the sizes of the various numbers involved in fractional colourings, and we focus on how large can be the least integers expressing the fractional chromatic number of a graph as an irreducible fraction, in terms of the size of the graph itself. This seems particularly relevant for the least denominator q such that $\chi_f(G) = \frac{p}{q}$ for some integer p. Note that this denominator is 1 if and only if $\chi_f(G) = \chi(G)$. For convenience, let us call $\mathrm{sd}(n)$ the least positive integer d such that for every n-vertex graph G there exist positive integers p and q such that $\chi_f(G) = \frac{p}{q}$ and $q \leqslant d$.

Chvátal, Garey and Johnson [3] seem to have been the first to provide an upper bound on $\mathrm{sd}(n)$. Their approach is an application of Hadamard's inequality for determinants that shows, using Cramer's rules, that $\mathrm{sd}(n) \leqslant n^{n/2}$. (In fact, the argument provides $\mathrm{sd}(n) \leqslant \alpha(G)^{n/2}$ where $\alpha(G)$ is the independence number of G.) Using the specialisation of Hadamard's inequality to matrices with all entries in $\{0,1\}$, Alon, Tuza and Voigt [1] observed that the upper bound strengthens to $2^{-n}(n+1)^{(n+1)/2}$.

Erdős had raised the question of whether one has $\chi_f(G) = \frac{p}{q}$, with p, q positive integers and $q \leqslant n$, for every n-vertex graph G. Chvátal, Garey and Johnson [3] provided a negative answer by exhibiting a construction (based on successive joins of odd cycles with suitable lengths) providing a lower bound on $\mathrm{sd}(n)$ that is super-polynomial in n. Yet this lower bound is not exponential in n and it was only in 1995 that Fisher [4], using a then-recent work of Larsen, Propp and Ullman [5], managed to obtain a lower bound on $\mathrm{sd}(n)$ that grows exponentially with n. Indeed Larsen, Propp and Ullman [5] had proved that if $M(G)$ is the Mycielskian of a graph G then $\chi_f(M(G)) = \chi(G) + \frac{1}{\chi(G)}$. This relation allowed Fisher [4] to discover that iterating the Mycielski operation starting with the complete graph on 3 vertices yields a sequence $(G_n)_{n\geqslant 0}$ of graphs showing that $\mathrm{sd}(n) \geqslant \frac{\lambda^n}{f(n)}$ where $f(n) = O(\sqrt{\log n})$ and λ is a constant approximately equal to 1.346193.

The gap between the known lower and upper bounds on $\mathrm{sd}(n)$ is thus enormous. We present here a new family of graphs that allows us to prove that $\sup_{n\in\mathbf{N}} \mathrm{sd}(n)^{1/n} \geqslant \sqrt{2}$, by demonstrating the following statement.

Theorem 1. *For every integer N and every positive real number ε there exists an n-vertex graph G such that $n > N$ and $\chi_f(G) = \frac{p}{q}$ with $q > (\sqrt{2}-\varepsilon)^n$ and p, q are relatively prime integers.*

2 Definitions and construction of tower graphs

This section gathers the notation used throughout and present the construction of the tower graphs, which allow us to establish Theorem 1.

2.1 Generalities

Let $\mathbf{Q}$ and $\mathbf{R}_{\geqslant 0}$ denote the set of rational numbers and the set of nonnegative reals, respectively. For a (finite) graph G, we write $V(G)$ for its vertex set. For a function $\overline{\omega}\colon V(G) \to \mathbf{R}_{\geqslant 0}$ and a subset V' of $V(G)$ we write $\overline{\omega}(V')$ to mean $\sum_{v\in V'} \overline{\omega}(v)$. Conversely for a (real-valued) function ω defined over a collection $\mathcal{C}$ of subsets of $V(G)$ we write $\omega[v]$ to mean $\sum_{\substack{C\in\mathcal{C}\\ v\in C}} \omega(C)$, for every vertex $v \in V(G)$. We let $\mathcal{I}(G)$ be the collection of all independent sets of G.

In accordance with the discussion in Section 1, a *fractional colouring* of a graph G is defined as a function $\omega : \mathcal{I}(G) \to \mathbf{R}_{\geqslant 0}$ such that $\omega[v] \geqslant 1$ for each vertex $v \in V(G)$. The infimum of all real numbers c such that G admits a fractional colouring ω with $\sum_{I\in\mathcal{I}(G)} \omega(I) \leqslant c$ is a rational number and thus a minimum, attained by a fractional colouring of G taking non-negative rational values. The *fractional chromatic number* $\chi_f(G)$ of a graph G is this minimum.

The dual of the linear program defining the fractional chromatic number of a graph G yields an invariant called the fractional clique number of G. Formally, a *fractional clique* of a graph G is a function $\overline{\omega}\colon V(G) \to \mathbf{R}_{\geqslant 0}$ such that $\overline{\omega}(I) \leqslant 1$ for every independent set $I \in \mathcal{I}(G)$. The supremum of all real numbers c such that G admits a fractional clique $\overline{\omega}$ with $\overline{\omega}(V(G)) \geqslant c$ is the *fractional clique number* $\omega_f(G)$. The strong LP-duality theorem ensures that $\omega_f(G) = \chi_f(G)$ and therefore $\omega_f(G)$ is a rational number and a maximum, attained by a fractional clique taking non-negative rational values. For more information on fractional colouring and fractional graph theory in general the reader is referred to the book by Scheinerman and Ullman [8].

Finally we let K_2 be the complete graph with 2 vertices, and if G is a graph then $G\times K_2$ is the graph with vertex set $\{(v, r) : v \in V(G) \text{ and } r \in \{0, 1\}\}$ and an edge between (v, r) and (u, s) if u and v are adjacent in G and $r \neq s$. Moreover, for every positive integer ℓ, the graph G^ℓ has the same vertices as the graph G and two distinct vertices are adjacent in G^ℓ if and only if their distance in G is at most ℓ.

2.2 Construction of the (k, ℓ)-tower graph

We define a family of simple graphs for each pair (k, ℓ) of positive integers such that $k \geqslant \ell$. Figure 1 illustrates the following definition when $(k, \ell) = (5, 3)$.

Definition 2. Let $k \in \mathbf{N}$ and $\ell \in \mathbf{N}$ such that $k \geqslant \ell$. We first construct the graph $(P_{k+\ell})^\ell \times K_2$, where $P_{k+\ell} = p_{1-\ell}p_{2-\ell}\dots p_k$ is a path on $k+\ell$ vertices. In a natural way the vertices of the obtained graph are called $p_{i,j}$ with $1-\ell \leqslant i \leqslant k$ and $j \in \{0, 1\}$. Next for each $i \in \{1-\ell, \dots, 0\}$ we identify the vertices $p_{i,0}$ and $p_{i,1}$ into a single vertex named $x_{i+\ell}$, which yields the vertices $x_1, \dots, x_\ell$. Last we join the vertices $p_{k,0}$ and $p_{k,1}$, which are the vertices farthest from the vertex x_ℓ, by an edge named e_k. We call the resulting graph the (k,ℓ)-*tower graph* $G_{k,\ell}$. We define X to be the subset $\{x_1,\dots,x_\ell\}$ of $V(G_{k,\ell})$ and for positive integers a,b, we let $P(a,b)$ be the set consisting of all the vertices $p_{i,0}$ and $p_{i,1}$ with $a \leqslant i \leqslant \min(b,k)$. Note that $V(G_{k,\ell}) = X \cup P(1,k)$. Furthermore, $P(a,b)$ is empty if $a > b$.

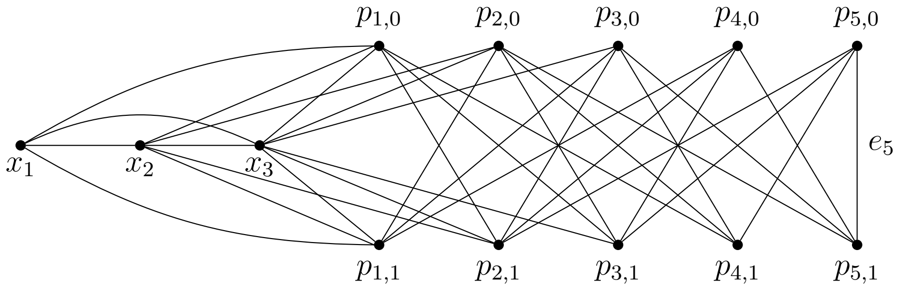

Figure 1: The tower graph $G_{5,3}$ with contracted vertices $\{x_1,x_2,x_3\}$ on the left side and the edge e_5 joining the vertices $p_{5,0}$ and $p_{5,1}$ on the right side.

We note that $G_{k,1}$ is isomorphic to the odd cycle C_{2k+1}, and $G_{k,\ell}$ has $\ell+2k$ vertices. The following observation will be useful. It conveys the fact that up to the presence of the edge e_k, the introduced subgraph is isomorphic to a complete bipartite graph minus a perfect matching. The proof directly follows from Definition 2.

Assertion 3. *Let k and ℓ be two positive integers with $k \geqslant \ell$. For positive integers a,b such that $1 \leqslant a \leqslant b \leqslant \min(k, a+\ell)$, let H be the subgraph of $G_{k,\ell}$ induced by $P(a,b)$. Then for each $i,j \in \{a,\dots,b\}$, the vertices $p_{i,0}$ and $p_{j,1}$ are adjacent in H if and only if $i \neq j$.*

2.3 Fibonacci numbers of arbitrary order

Fibonacci numbers of higher orders allow us to express the fractional chromatic numbers of tower graphs. We give the definition of the generalised sequences of Fibonacci numbers and introduce some convenient notation.

Definition 4. For every positive integer ℓ, the sequence $(F_i^\ell)_{i\geqslant 0}$ of ℓ-*Fibonacci numbers* is recursively defined by

- $F_i^\ell = 0$ if $i \in \{0,\dots,\ell-2\}$;
- $F_{\ell-1}^\ell = 1$; and
- $F_i^\ell = \sum_{j=i-\ell}^{i-1} F_j^\ell$ if $i \geqslant \ell$.

For all non-negative integers a,b,ℓ with $b,\ell \geqslant 1$, we set $S^\ell(a,b) = \sum_{j=b-1}^{a+b-2} F_j^\ell$, where $S^\ell(0,b) = 0$.

Thus, $S^\ell(a,b)$ is the sum of a consecutive ℓ-Fibonacci numbers starting with F^ℓ_{b-1}. In particular,

$$S^\ell(\ell,b) = F^\ell_{b+\ell-1}.$$

It will come handy to note that

$$S^\ell(c,a+b) + S^\ell(a,b) = S^\ell(a+c,b) \tag{1}$$

for all non-negative integers a,b,c,ℓ with $b,\ell \geqslant 1$, which we will sometimes use (notably with $c=a$) without reference in what follows.

3 Fractional chromatic number of tower graphs

Our goal is to determine the fractional chromatic number of the graph $G_{k,\ell}$ for all integers $k \geqslant \ell \geqslant 1$. To this end we prove matching upper and lower bounds in the next two sections, the former one obtained by giving an explicit fractional colouring of $G_{k,\ell}$, and the latter one deduced by the strong duality theorem from an explicit fractional clique of $G_{k,\ell}$. These two bounds (Lemma 6 and Corollary 11) and the LP-duality yield the following statement.

Theorem 5. *For all integers $k \geqslant \ell \geqslant 1$,*

$$\chi_f(G_{k,\ell}) = (\ell+1) + \frac{1}{S^\ell(k,\ell)}.$$

3.1 Lower bound on the fractional chromatic number

We define a particular fractional clique of $G_{k,\ell}$, which provides a lower bound on its fractional chromatic number by LP-duality.

Let k and ℓ be integers such that $k \geqslant \ell \geqslant 1$. We define $\overline{\omega}\colon V(G_{k,\ell}) \to \mathbf{N}$ such that for all positive integers i,j such that $i \leqslant k$ and $j \leqslant \ell$,

$$\overline{\omega}(p_{i,0}) = \overline{\omega}(p_{i,1}) = F^\ell_{k-i+\ell-1} \tag{2}$$
$$\overline{\omega}(x_j) = S^\ell(j, k+\ell-j). \tag{3}$$

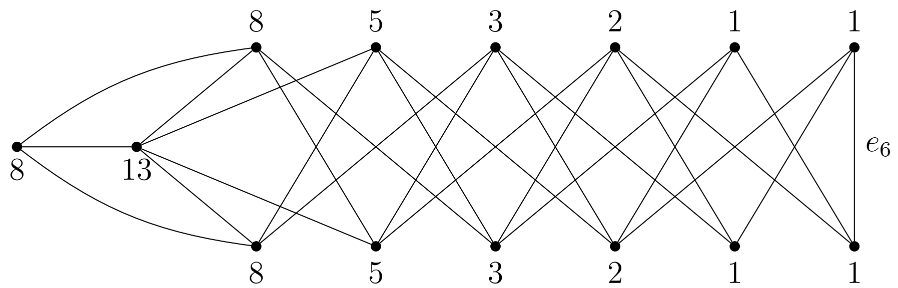

Figure 2: The function $\overline{\omega}$ over $V(G_{6,2})$, which gives an optimal fractional clique when divided by $S^2(6,2) = 20$.

Our goal is to prove the following.

Lemma 6. *The weight assignment* $\frac{1}{S^\ell(k,\ell)}\cdot\overline{\omega}\colon V(G_{k,\ell}) \to \mathbf{Q}^+$ *is a fractional clique of* $G_{k,\ell}$. *In particular,* $\chi_f(G_{k,\ell}) \geqslant (\ell+1) + \frac{1}{S^\ell(k,\ell)}$.

Let us first compute the total sum of the weights assigned to the vertices, which will establish the second part of Lemma 6 provided the first part is true.

Assertion 7. *For all integers* k *and* ℓ *such that* $k \geqslant \ell \geqslant 1$,

$$\overline{\omega}(V(G_{k,\ell})) = (\ell+1)S^\ell(k,\ell) + 1. \tag{4}$$

Proof. For each $r \in \{0,1\}$, the sum of $\overline{\omega}(p_{i,r})$ over $i \in \{1,\dots,k\}$ equals $S^\ell(k,\ell)$. It follows that

$$\overline{\omega}(V(G_{k,\ell})) = 2S^\ell(k,\ell) + \sum_{j=1}^{\ell} S^\ell(j, k+\ell-j).$$

Calling $f(k,\ell)$ the right side of the above equality for convenience, it thus suffices to prove that

$$f(k,\ell) = (\ell+1)S^\ell(k,\ell) + 1. \tag{5}$$

Observe that since $k \geqslant \ell \geqslant 1$,

$$\begin{aligned} f(k+1,\ell) - f(k,\ell) &= 2\left(S^\ell(k+1,\ell) - S^\ell(k,\ell)\right) \\ &\quad + \sum_{j=1}^{\ell}\left(S^\ell(j,(k+1)+\ell-j) - S^\ell(j,k+\ell-j)\right) \\ &= 2F^\ell_{k+\ell-1} + \sum_{j=1}^{\ell}\left(F^\ell_{k+\ell-1} - F^\ell_{k+\ell-j-1}\right) \\ &= (\ell+2)F^\ell_{k+\ell-1} - S^\ell(\ell,k) = (\ell+1)F^\ell_{k+\ell-1}. \end{aligned}$$

Since on the other hand $F^\ell_{k+\ell-1}$ equals $S^\ell(k+1,\ell) - S^\ell(k,\ell)$ by definition, it suffices to establish (5) when $k=\ell$ and the full statement will then hold (by induction on k).

Let us therefore consider the case that $k = \ell \geqslant 1$. We have

$$f(\ell,\ell) = 2S^\ell(\ell,\ell) + \sum_{j=1}^{\ell} S^\ell(j, 2\ell-j). \tag{6}$$

Consider the $j=\ell$ term in the summation in (6): we write it out as a sum of ℓ-Fibonacci numbers $F^\ell_{\ell-1} + \cdots + F^\ell_{2\ell-2}$, and distribute them one by one, bar the first, to each of the other terms in the summation, obtaining

$$f(\ell,\ell) = 2S^\ell(\ell,\ell) + \sum_{j=1}^{\ell-1}\left(S^\ell(j,2\ell-j) + F^\ell_{2\ell-1-j}\right) + F^\ell_{\ell-1}. \tag{7}$$

In the term in the brackets on the right side of this equation, we can expand $F^\ell_{2\ell-1-j}$ as a sum of ℓ-Fibonacci numbers and then use (1):

$$\begin{aligned} S^\ell(j,2\ell-j)+F^\ell_{2\ell-1-j} &= S^\ell(j,2\ell-j)+S^\ell(\ell,\ell-j) \\ &= S^\ell(\ell+j,\ell-j)=S^\ell(\ell,\ell), \end{aligned}$$

where the last equality follows from the fact that $F^\ell_i=0$ for $i<\ell-1$. Plugging this back into (7) and noting that $F^\ell_{\ell-1}=1$, we infer (5) for $k=\ell$ as desired. □

To prove that $\frac{1}{S^\ell(k,\ell)}\cdot\overline{\omega}$ is a fractional clique of $G_{k,\ell}$ we use the following lemma. Recall the definition of the set $P(a,b)$ given in Definition 2.

Lemma 8. *Let k and ℓ be positive integers such that $k\geqslant\ell$. If J is an independent set of $G_{k,\ell}$ contained in $P(j,k)$ for some $j\in\{1,\dots,k+1\}$, then $\overline{\omega}(J)\leqslant S^\ell(k-j+1,\ell)$.*

Proof. We proceed by induction on $k-j$. The base case, $j=k+1$, is trivial. Suppose that $j\leqslant k$ and let $J'=J\setminus P(j,j)$. Furthermore, let c be the number of vertices of $P(j,j)$ contained in J. (Thus, $c\in\{0,1,2\}$.) We have

$$\overline{\omega}(J)=\overline{\omega}(J')+c\cdot F^\ell_{k-j+\ell-1}. \tag{8}$$

If $c\leqslant 1$, then the statement of the lemma follows by applying the induction hypothesis to J', since

$$\overline{\omega}(J)\leqslant S^\ell(k-j,\ell)+F^\ell_{k-j+\ell-1}=S^\ell(k-j+1,\ell).$$

Suppose then that $c=2$, i.e., $P(j,j)\subseteq J$. Assertion 3 implies that J contains no vertex in $P(j+1,j+\ell)$ and therefore $J'\subseteq P(j+\ell+1,k)$. Assume first that $j\leqslant k-\ell$, in which case we may apply the induction hypothesis to J' with $j+\ell+1$ in place of j, yielding that $\overline{\omega}(J')\leqslant S^\ell(k-j-\ell,\ell)$. Expressing the ℓ-Fibonacci number in the right side of (8) as the sum of ℓ smaller ones, we have $F^\ell_{k-j+\ell-1}=S^\ell(\ell,k-j)$. Substituting into (8),

$$\overline{\omega}(J)\leqslant S^\ell(k-j-\ell,\ell)+S^\ell(\ell,k-j)+F^\ell_{k-j+\ell-1}=S^\ell(k-j+1,\ell).$$

It remains to consider the case that $j>k-\ell$. Here, J' is empty, so $J=P(j,j)$. Expressing $F^\ell_{k-j+\ell-1}$ as before, and using the fact that the $\ell-k+j$ least terms of $S^\ell(\ell,k-j)$ are zero in this case, we find that $S^\ell(\ell,k-j)=S^\ell(k-j,\ell)$. Hence,

$$\overline{\omega}(J)=S^\ell(k-j,\ell)+F^\ell_{k-j+\ell-1}=S^\ell(k-j+1,\ell),$$

which concludes the proof. □

With Lemma 8 at hand, we can now establish Lemma 6 easily.

Proof of Lemma 6. As mentioned earlier, by Assertion 7 we only need to prove the first statement.

Let I be an independent set of $G_{k,\ell}$. Our goal is to prove that $\overline{\omega}(I)\leqslant S^\ell(k,\ell)$. Note that I contains at most one vertex of X. If $I\cap X$ is empty, then $I\subseteq P(1,k)$ and Lemma 8 implies that $\overline{\omega}(I)\leqslant S^\ell(k,\ell)$ as needed.

We may therefore assume that $I\cap X=\{x_i\}$, where $1\leqslant i\leqslant\ell$. Since $I\setminus X$ is contained in $P(i+1,k)$, Lemma 8 implies that $\overline{\omega}(I\setminus X)\leqslant S^\ell(k-i,\ell)$. Considering $\overline{\omega}(x_i)=S^\ell(i,k+\ell-i)$, we deduce that $\overline{\omega}(I)\leqslant S^\ell(k-i,\ell)+S^\ell(i,k+\ell-i)=S^\ell(k,\ell)$ by (1). □

3.2 Upper bound on the fractional chromatic number of tower graphs

Our goal now is to provide a fractional colouring of $G_{k,\ell}$ showing that $\chi_f(G) \leqslant (\ell+1) + \frac{1}{S^\ell(k,\ell)}$ for all positive integers ℓ and $k \geqslant \ell$. To this end we first introduce the independent sets of $G_{k,\ell}$ to which a positive weight will be assigned. We use the following notation: for a positive integer j, we let $[j]$ and $[\![j]\!]$ be the integers congruent to j modulo $\ell+1$ such that $1 \leqslant [j] \leqslant \ell+1$ and $k-\ell \leqslant [\![j]\!] \leqslant k$.

The independent sets we use come in two kinds. First, for $j \in \{1,\dots,k\}$ and $r \in \{0,1\}$ the set $A_{j,r} \subseteq V(G_{k,\ell})$ consists of the following vertices:

- all vertices $p_{i,r}$ with $j \leqslant i \leqslant k$,
- all vertices $p_{i,0}$ and $p_{i,1}$ with $1 \leqslant i < j$ and $i \equiv j \pmod{\ell+1}$,
- the vertex $x_{[j]-1}$ if $[j] \neq 1$.

Second, for $j \in \{k-\ell,\dots,k-1\}$ the set $B_j \subseteq V(G_{k,\ell})$ consists of the following vertices:

- all vertices $p_{i,0}$ and $p_{i,1}$ with $1 \leqslant i \leqslant j$ and $i \equiv j \pmod{\ell+1}$,
- the vertex $x_{[j]-1}$ if $[j] \neq 1$.

For convenience, we define B_k as the empty set.

We let $\mathcal{A}_{k,\ell}$ be the collection of the above sets $A_{j,r}$ and B_j. It is straightforward to check that each of them is independent in $G_{k,\ell}$.

We define a function ω that assigns to each independent set I of $G_{k,\ell}$ an integer $\omega(I)$. For the sets in $\mathcal{A}_{k,l}$, we define

$$\begin{aligned}\omega(A_{j,r}) &= F^\ell_{j+\ell-2} \quad \text{for } j \in \{1,\dots,k\} \text{ and } r \in \{0,1\},\\ \omega(B_j) &= F^\ell_{(j+1)+\ell-2} + \cdots + F^\ell_{k+\ell-2}\\ &= S^\ell(k-j, j+\ell) \quad \text{for } j \in \{k-\ell,\dots,k\}.\end{aligned}$$

For any other independent set I, we set $\omega(I)=0$. (Note that $\omega(B_k)=0$.)

Lemma 9. *For any vertex v of $G_{k,\ell}$, the total weight $w[v]$ of the independent sets containing v equals $S^\ell(k,\ell)$.*

Proof. Consider first the case where $v \in X$, that is, there exists $i \in \{2,\dots,\ell+1\}$ such that $v = x_{i-1}$. The vertex x_{i-1} is contained in the following independent sets from $\mathcal{A}_{k,\ell}$:

- $A_{j,0}$ and $A_{j,1}$, where $j \equiv i \pmod{\ell+1}$ and $i \leqslant j \leqslant k$,
- $B_{[\![i]\!]}$ (which is empty and has zero weight if $[\![i]\!] = k$).

The total weight of these sets is

$$\omega[x_{i-1}] = \Bigg(\sum_{\substack{i \leqslant j \leqslant k \\ j \equiv i \pmod{\ell+1}}} 2 \cdot F^{\ell}_{j+\ell-2} \Bigg) + \Big(F^{\ell}_{(\llbracket i \rrbracket+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2} \Big) \tag{9}$$

regardless of whether $\llbracket i \rrbracket = k$ or not. Since

$$F^{\ell}_{j+\ell-2} = F^{\ell}_{j-2} + F^{\ell}_{j-1} + \cdots + F^{\ell}_{j+\ell-3},$$

we can rewrite (9) as

$$\begin{aligned} \omega[x_{i-1}] &= \sum_{\substack{i \leqslant j \leqslant k \\ j \equiv i \pmod{\ell+1}}} \Big(F^{\ell}_{j-2} + \cdots + F^{\ell}_{j+\ell-3} + F^{\ell}_{j+\ell-2} \Big) \\ &\quad + \Big(F^{\ell}_{(\llbracket i \rrbracket+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2} \Big) \\ &= \sum_{j=1}^{k} F^{\ell}_{j+\ell-2}, \end{aligned}$$

where in the last equality we used the fact that the largest integer congruent to i modulo $\ell + 1$ is $\llbracket i \rrbracket$, as well as the fact that the first $\ell - 1$ of the ℓ-Fibonacci numbers, from F^{ℓ}_0 to $F^{\ell}_{\ell-2}$, are equal to zero. We conclude that $\omega[x_{i-1}] = S^{\ell}(k, \ell)$ as required.

It remains to consider the case where $v \notin X$. By symmetry of the vertices $p_{i,0}$ and $p_{i,1}$ in $G_{k,\ell}$ and the independent sets in $\mathcal{A}_{k,\ell}$, we can assume that $v = p_{i,0}$ with $1 \leqslant i \leqslant k$. The vertex $p_{i,0}$ is contained in the following sets from $\mathcal{A}_{k,\ell}$:

- $A_{j,0}$, where $1 \leqslant j \leqslant i$,
- $A_{j,0}$ and $A_{j,1}$, where $j \equiv i \pmod{\ell + 1}$ and $i < j$,
- $B_{\llbracket i \rrbracket}$.

Summing the weights of these sets in the given order, we obtain

$$\begin{aligned} w[p_{i,0}] &= \sum_{j=1}^{i} F^{\ell}_{j+\ell-2} \\ &\quad + \sum_{\substack{i < j \leqslant k \\ j \equiv i \pmod{\ell+1}}} 2 \cdot F^{\ell}_{j+\ell-2} \\ &\quad + \Big(F^{\ell}_{(\llbracket i \rrbracket+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2} \Big). \end{aligned}$$

As in the case of (9), we can rewrite this as

$$w[p_{i,0}] = \Big(F^{\ell}_{\ell-1} + \cdots + F^{\ell}_{i+\ell-2} \Big)$$

$$
\begin{aligned}
&+ \sum_{\substack{i<j\leqslant k \\ j\equiv i \pmod{\ell+1}}} \left(F^{\ell}_{j-2} + \cdots + F^{\ell}_{j+\ell-2}\right) \\
&+ \left(F^{\ell}_{([\![i]\!]+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2}\right) \\
&= \sum_{j=1}^{k} F^{\ell}_{j+\ell-2} = S^{\ell}(k,\ell),
\end{aligned}
$$

concluding the proof. □

Lemma 10. *The total weight of all the sets in $\mathcal{A}_{k,\ell}$ equals $(\ell+1)\cdot S^{\ell}(k,\ell)+1$.*

Proof. Each of the sets in $\mathcal{A}_{k,\ell}$ contains at most one vertex from X. At the same time, the total weight of all independent sets containing a fixed vertex of X is $S^{\ell}(k,\ell)$ by Lemma 9. Since $|X|=\ell$, all we need to prove is that the total weight of the sets in $\mathcal{A}_{k,\ell}$ containing no vertex of X equals $S^{\ell}(k,\ell)+1$. Let $\mathcal{N}$ be the family of such sets.

By the definition of $\mathcal{A}_{k,\ell}$, the family $\mathcal{N}$ consists of the following sets:

- $A_{j,0}$ and $A_{j,1}$, where $1 \leqslant j \leqslant k$ and $j \equiv 1 \pmod{\ell+1}$,
- $B_{[\![1]\!]}$.

Using similar considerations as in the proof of Lemma 9, we express the sum of the weights of these sets as

$$
\begin{aligned}
\omega(\mathcal{N}) &= \sum_{\substack{1\leqslant j\leqslant k \\ j\equiv 1 \pmod{\ell+1}}} 2\cdot F^{\ell}_{j+\ell-2} \\
&+ \left(F^{\ell}_{([\![1]\!]+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2}\right) \\
&= 1 + F^{\ell}_{\ell-1} + \sum_{\substack{1<j\leqslant k \\ j\equiv 1 \pmod{\ell+1}}} \left(F^{\ell}_{j-2} + \cdots + F^{\ell}_{j+\ell-2}\right) \\
&+ \left(F^{\ell}_{([\![1]\!]+1)+\ell-2} + \cdots + F^{\ell}_{k+\ell-2}\right) \\
&= 1 + \sum_{j=1}^{k} F^{\ell}_{j+\ell-2} = S^{\ell}(k,\ell)+1.
\end{aligned}
$$

□

Lemmas 9 and 10 directly imply the following.

Corollary 11. *The weight assignment $\frac{1}{S^{\ell}(k,\ell)}\cdot\omega$ is a fractional colouring of $G_{k,\ell}$. In particular,*

$$\chi_f(G_{k,\ell}) \leqslant (\ell+1) + \frac{1}{S^{\ell}(k,\ell)}.$$

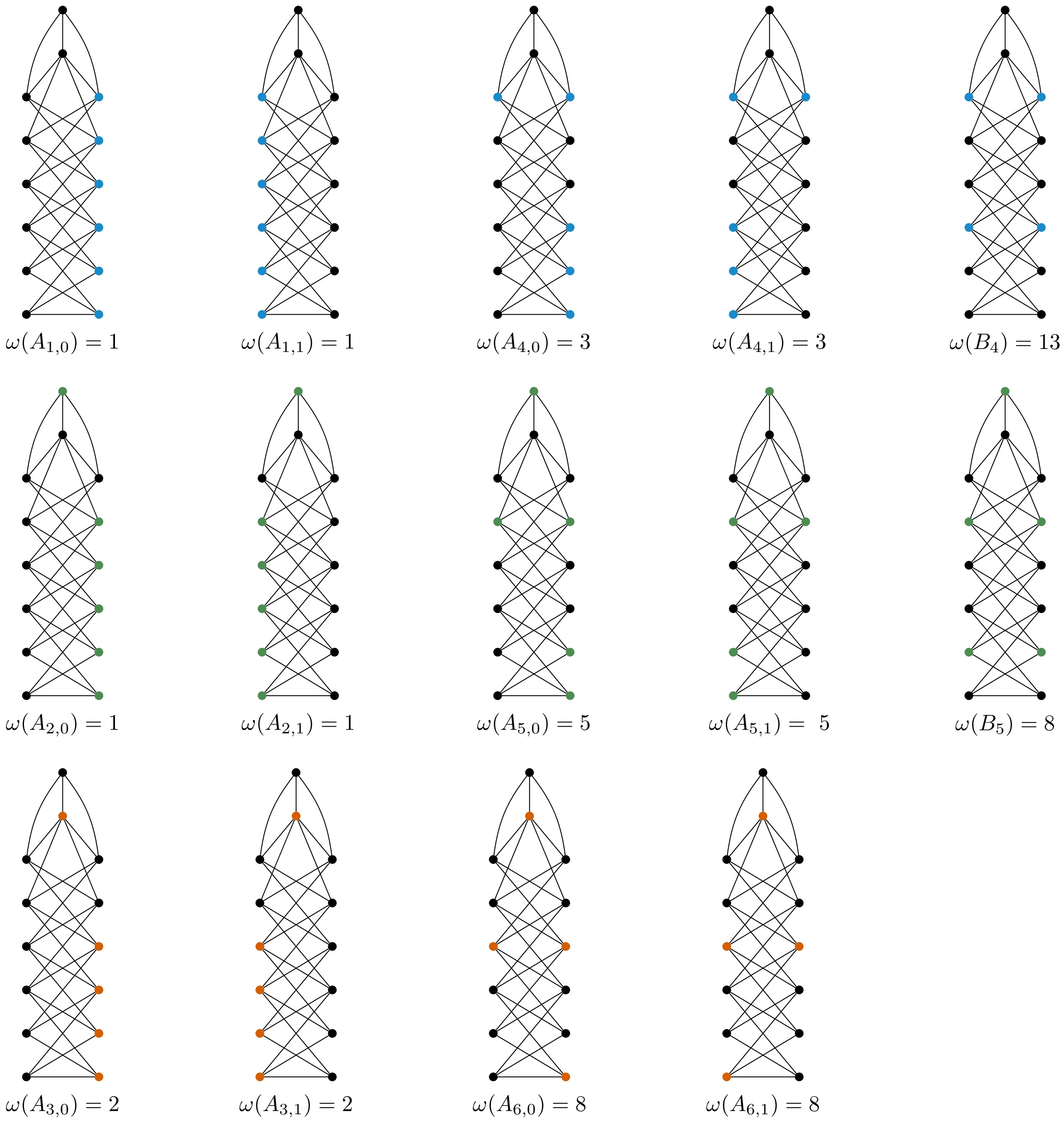

Figure 3: The values taken by ω over $\mathcal{A}_{6,2}$, implying that $\chi_f(G_{6,2}) \leqslant \frac{61}{20}$. The coloured vertices are those in the independent set given with its ω-value below the graph.

4 Bounding $\mathbf{sd}(n)$

Let ℓ be a positive integer and let r_ℓ be the largest root of the polynomial $P(x) = x^\ell - \sum_{i=0}^{\ell-1} x^i$. It has been proved [9, Lemma 3.6] that $2 - 2^{-\ell+1} < r_\ell < 2$ and that F_k^ℓ is the integer closest to $r_\ell^k \cdot \frac{1-1/r_\ell}{r_\ell+\ell(r_\ell-2)}$ for all $k \geqslant \ell-1$. (See also [6,7].) In particular $\frac{F_k^\ell}{F_{k-1}^\ell} = (1+o(1))r_\ell^{k-1}$ if $k \geqslant \ell$, where the $o(1)$ term tends to 0 as $\ell \to \infty$. It follows that $S^\ell(k,\ell) = (1+o(1))r_\ell^{k+\ell-2}$ where $o(1) \to 0$ as $\ell \to \infty$.

Consequently, by considering the graph $G_{t\ell,\ell}$ for a fixed positive integer $t \geqslant 4$, Theorem 5 implies that

$$\mathrm{sd}((2t+1)\ell) \geqslant (1+o(1))r_\ell^{(t+1)\ell-2}.$$

By letting ℓ tend to infinity, we infer that the supremum of $\mathrm{sd}(n)^{\frac{1}{n}}$ over all positive integers n is at least $\sqrt{2}$, which proves Theorem 1.

5 Conclusion: the tower construction

In analogy with the Mycielski construction, one may consider tower graphs as the particular case of a generic construction applied to a particular seed graph, in this case the graph K_2. (This has been independently suggested by a reviewer of this paper.) We thus provide the following generic definition, and suggest it could be interesting to investigate how the fractional chromatic number behaves with respect to it.

The *tensor product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two vertices (g,h) and (g',h') are adjacent if $\{g,g'\}$ is an edge in G and $\{h,h'\}$ is an edge in H. (This generalises the definition provided in Section 2.1.)

Definition 12. Let $k \in \mathbf{N}$ and $\ell \in \mathbf{N}$ such that $k \geqslant \ell$. Given a graph H, we first construct the graph $(P_{k+\ell})^\ell \times H$, where $P_{k+\ell} = p_{1-\ell}p_{2-\ell}\dots p_k$ is a path on $k+\ell$ vertices. In a natural way the vertices of the obtained graph are called $p_{i,h}$ with $1-\ell \leqslant i \leqslant k$ and $h \in V(H)$. Next for each $i \in \{1-\ell,\dots,0\}$, we identify the vertices in $\{p_{i,h} : h \in V(H)\}$ into a single vertex named $x_{i+\ell}$, which yields the vertices $x_1,\dots,x_\ell$. Last, we add the edge $\{p_{k,h}, p_{k,h'}\}$ for each edge $\{h,h'\}$ of H, so $\{p_{k,h} : h \in V(H)\}$ induces a copy of H. We call the resulting graph $\mathcal{T}_{k,\ell}(H)$, the *(k,ℓ)-tower graph on H*.

Note that $\mathcal{T}_{k,\ell}(K_2)$ is the tower graph $G_{k,\ell}$ defined and studied in this work.

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