Equating three degrees of graphs

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Abstract

It is proved that for every graph with at least 5 vertices, one can delete at most three vertices such that the subgraph obtained has at least three vertices with the same degree. This solves an open problem of Caro, Shapira and Yuster.

Mathematics Subject Classifications: 05C07

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1 Introduction

All graphs considered here are finite, simple and undirected. For a graph G, the *repetition number*, denoted by $\mathrm{rep}(G)$, is the maximum multiplicity of a vertex degree in G. Trivially, $\mathrm{rep}(G) \geqslant 2$ for any graph G with at least two vertices. There are also simple constructions showing that equality holds for infinitely many graphs. Repetition numbers of graphs and hypergraphs have been widely studied by various researchers, see [1, 2, 3, 4, 5, 7, 8, 9, 10].

Since there are infinitely many graphs having repetition number two, Caro, Shapira and Yuster [6] asked what is the smallest number of vertices one needs to delete from a graph in order to increase the repetition number of the remaining induced subgraph to 3 or higher. For any positive integer k, let $C(k)$ denote the least integer such that any n-vertex graph has an induced subgraph with at least $n - C(k)$ vertices whose repetition number is at least $\min\{k, n - C(k)\}$. Trivially, $C(1) = C(2) = 0$. Caro, Shapira and Yuster [6] established that $\Omega(k \log k) \leqslant C(k) \leqslant (8k)^k$ for any $k \geqslant 3$. Specially, for the first nontrivial case $C(3)$, the authors [6] proved that $C(3) \leqslant 6$, and the exact value was left as an open problem.

Problem 1 (Caro, Shapira and Yuster [6]). Determine the exact value of $C(3)$.

Recently, the upper bound of $C(3)$ was continuously improved by Kogan [11], and by Sun, Hou and Zeng [12]. In this paper, we consider Problem 1 and determine that $C(3) = 3$. We mention that the graph (see Figure 1) demonstrating that the lower bound

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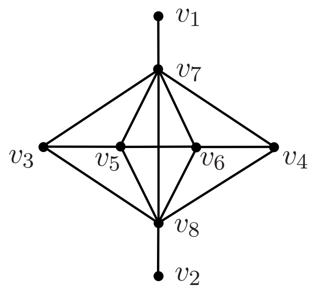

Figure 1: A graph demonstrating that $C(3) \geqslant 3$.

$C(3) \geqslant 3$ is constructed by Caro, Shapira and Yuster [6]. In addition, the path on 4 vertices shows that $|V(G)| \geqslant 5$ is necessary in the following theorem.

Theorem 2. *For any graph G with at least 5 vertices, one can delete at most three vertices such that the subgraph obtained has at least three vertices with the same degree. Consequently, $C(3) = 3$.*

Notation. Let G be a graph. For any $v \in V(G)$, denote $N_G(v)$ the set of *neighbors* of v in G and $d_G(v)$ the *degree* of v in G. Write $N_G[v] = N_G(v) \cup \{v\}$. For any $S \subseteq V(G)$, let $G[S]$ denote the induced subgraph of G on S. For each $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of G with one end in S and the other end in T. In addition, we usually write $[k] := \{1, \ldots, k\}$ for any integer $k \geqslant 2$.

2 Feasible sets and related lemmas

Let G be a graph, and let $S = \{x, y, z\}$ be a subset of $V(G)$ such that $d_G(x) \leqslant d_G(y) \leqslant d_G(z)$. We follow the definition of a feasible set in [12]. Call S *balanceable* if one of the following conditions holds:

(C1) $G[S]$ is an independent set;

(C2) $G[S]$ is a clique;

(C3) $G[S]$ contains only the edge xy;

(C4) $G[S]$ contains only the edges xy and xz.

Furthermore, call S *accessible* if one of the following conditions holds:

(C5) $G[S]$ contains only the edge xz, and $N_G(y) \backslash N_G(z) \neq \emptyset$;

(C6) $G[S]$ contains only the edges xy and yz, and $N_G(x) \backslash N_G[y] \neq \emptyset$;

(C7) $G[S]$ contains only the edge yz, and $N_G(x) \backslash N_G(y) \neq \emptyset$, $N_G(x) \backslash N_G(z) \neq \emptyset$;

(C8) $G[S]$ contains only the edges xz and yz, and $N_G(x) \backslash N_G[z] \neq \emptyset$, $N_G(y) \backslash N_G[z] \neq \emptyset$.

Usually, we call S *feasible* for short if it is balanceable or accessible. We also mention that the conditions (Ci) and (Cℓ) are related by complementation for each $\ell = i + 1$ and $i \in \{1, 3, 5, 7\}$.

We first present three lemmas given by Sun, Hou and Zeng [12], which are useful in our proof. For any set $S = \{x, y, z\}$ with $d_G(x) \leqslant d_G(y) \leqslant d_G(z)$, let $p(S) := d_G(z) - d_G(y)$ and $q(S) := d_G(y) - d_G(x)$.

Lemma 3 (Sun, Hou and Zeng [12]). *If G contains a feasible set S, then one can delete at most*

$$p(S) + q(S) + \max\{p(S), q(S)\}$$

vertices from G such that the subgraph obtained has at least three vertices with the same degree.

Remark. We mention that the above lemma actually implies that the existence of a feasible set S enables us to equate its three vertices by deleting some other vertices (see the proof of Lemma 2.2 or the comments before Lemma 2.2 in [12]). Actually, Lemma 3 is usually used in this form.

Lemma 4 (Sun, Hou and Zeng [12]). *Let $X = \{v_1, v_2, v_3, v_4\}$ be a subset of G such that $d_G(v_1) \leqslant d_G(v_2) \leqslant d_G(v_3) \leqslant d_G(v_4)$. If X does not contain balanceable sets, then $G[X]$ is an induced path such that v_1 and v_2 are terminal vertices.*

Lemma 5 (Sun, Hou and Zeng [12]). *Let G be a graph and $U = \{u_1, u_2, u_3, u_4, u_5\} \subseteq V(G)$. If $d_G(u_1) \leqslant \ldots \leqslant d_G(u_5)$, then there is a feasible set $S \subseteq U$ such that $u_3 \in S$.*

The following lemma plays a key role in our proof of Theorem 2.

Lemma 6. *Let G be a graph and let $U = \{v_1, v_2, v_3, v_4\} \subseteq V(G)$ with $d_G(v_1) = d_G(v_2) = d$ and $d_G(v_3) = d_G(v_4) = d + 2$ for any integer $d \geqslant 1$. If U contains a balanceable set, then one can delete at most 3 vertices from G such that the subgraph obtained has at least three vertices with the same degree.*

Proof. Suppose that $S \subseteq U$ is a balanceable set. Clearly, the degrees of the three vertices in S are either $d, d, d+2$ or $d, d+2, d+2$. We may assume that the three vertices in S have degrees $d, d, d+2$, respectively; otherwise, we may take the complement of G as our graph. Without loss of generality, let $S = \{v_1, v_2, v_3\}$. For convenience, we will drop the reference to the current graph G in the notations of $d_G(\cdot)$, $N_G(\cdot)$ and $e_G(\cdot\,, \cdot)$ since there is no danger of confusion in the following proof.

Claim 7. *The set $N(v_3) \backslash (N[v_1] \cup N[v_2])$ is empty.*

Proof. Suppose that there exists a vertex $v \in N(v_3) \backslash (N[v_1] \cup N[v_2])$. Clearly, deleting v reduces the degrees of v_1, v_2, v_3 from $(d, d, d+2)$ to $(d, d, d+1)$. Note that $\{v_1, v_2, v_3\}$ remains balanceable in the resulting graph as it is balanceable in G. It follows from Lemma 3 (and its remark) that we can delete at most two more vertices to equate the degrees of v_1, v_2, v_3. Overall, we delete at most 3 vertices, as desired. □

Claim 8. *The set $\{v_1, v_2, v_4\}$ is feasible.*

Proof. Suppose that $\{v_1, v_2, v_4\}$ is not a feasible set. It follows from the conditions (C1)-(C8) that at least one of $N(v_1) \setminus N[v_2] = \emptyset$, $N(v_1) \setminus N[v_4] = \emptyset$ and $N(v_2) \setminus N[v_4] = \emptyset$ holds.

If $N(v_1) \setminus N[v_2] = \emptyset$, then $|N[v_1] \cup N[v_2]| \leqslant d + 2$ as $d(v_1) = d(v_2) = d$. Note that $d(v_3) = d+2$ and $N(v_3) \setminus (N[v_1] \cup N[v_2]) = \emptyset$ by Claim 7. This forces that $N[v_1] \cup N[v_2] = N(v_3)$, meaning that $v_3 \in N[v_1] \cup N[v_2]$. This leads to a contradiction as $v_3 \notin N(v_3)$.

If $N(v_1) \setminus N[v_4] = \emptyset$, then we first assert that

$$v_1 v_2 \notin E(G). \tag{1}$$

Suppose that $v_1 v_2 \in E(G)$. It follows from the conditions (C2)-(C4) that $v_1 v_4 \notin E(G)$ and $v_2 v_4 \in E(G)$ as $\{v_1, v_2, v_4\}$ is not feasible. This further implies that $N(v_1) \setminus N[v_2] = \emptyset$ by the condition (C6). It follows that $N(v_3) \subseteq N[v_2]$ by Claim 7. This leads to a contradiction as $d(v_3) = d(v_2)+2$. Thus, we know that (1) holds. Recall that $\{v_1, v_2, v_3\}$ is a balanceable set. This together with (1) yields that $\{v_1, v_2, v_3\}$ is actually an independent set, i.e., $v_i \notin N(v_3)$ for each $i \in [2]$. Note that $N(v_3) \setminus (N[v_1] \cup N[v_2]) = \emptyset$ by Claim 7 and $d(v_1) = d(v_2) = d(v_3) - 2$. Thus, for each $i \in [2]$

$$|N(v_3) \setminus N[v_i]| = |N(v_3) \setminus N(v_i)| \geqslant 2 \text{ and } N(v_3) \setminus N(v_i) \subseteq N(v_{3-i}) \setminus N(v_i). \tag{2}$$

If $v_2 v_4 \in E(G)$, then it follows from (2) that $N(v_3) \setminus N(v_2) \subseteq N(v_1) \setminus N(v_2) \subseteq N(v_4)$ as $N(v_1) \setminus N[v_4] = \emptyset$ and $v_4 \in N(v_2)$. If $v_2 v_4 \notin E(G)$, then it follows from the condition (C5) that $N(v_2) \setminus N[v_4] = \emptyset$ as $\{v_1, v_2, v_4\}$ is not feasible. This together with (2) further implies that $N(v_3) \setminus N(v_1) \subseteq N(v_2) \setminus N(v_1) \subseteq N(v_4)$ as $v_4 \notin N(v_2)$. Thus, we conclude that there exists some $j \in [2]$ such that

$$N(v_3) \setminus N(v_j) \subseteq N(v_4).$$

Recall that $|N(v_3) \setminus N(v_i)| \geqslant 2$ for each $i \in [2]$ by (2). Consequently, we can always delete two vertices in $N(v_3) \setminus N(v_j)$ to equate the degrees of v_j, v_3 and v_4 from $(d, d+2, d+2)$ to (d, d, d) for some $j \in [2]$, as required.

Now, we may assume that $N(v_1) \setminus N[v_2] \neq \emptyset$, $N(v_1) \setminus N[v_4] \neq \emptyset$ and $N(v_2) \setminus N[v_4] = \emptyset$. Recall that $\{v_1, v_2, v_4\}$ is not a feasible set. In view of the conditions (C1)-(C8), it is easy to conclude that

$$v_1 v_2 \notin E(G) \text{ and } v_1 v_4 \in E(G). \tag{3}$$

Recall that $\{v_1, v_2, v_3\}$ is a balanceable set. This together with (3) yields that $\{v_1, v_2, v_3\}$ is actually an independent set, i.e., $v_i \notin N(v_3)$ for each $i \in [2]$. Note that $N(v_3) \setminus (N[v_1] \cup N[v_2]) = \emptyset$ by Claim 7 and $d(v_1) = d(v_2) = d(v_3) - 2$. Thus, for each $i \in [2]$, we also obtain (2). Since $v_1 v_4 \in E(G)$, we have $v_4 \notin N(v_2) \setminus N(v_1)$. As in (2), we have

$$N(v_3) \setminus N(v_1) \subseteq N(v_2) \setminus N(v_1) \subseteq N(v_4)$$

since $N(v_2) \setminus N[v_4] = \emptyset$. Thus, we conclude that

$$N(v_3) \setminus N(v_1) \subseteq N(v_4).$$

Recall that $|N(v_3) \setminus N(v_1)| \geqslant 2$ by (2). Consequently, we can always delete two vertices in $N(v_3) \setminus N(v_1)$ to equate the degrees of v_1, v_3 and v_4 from $(d, d+2, d+2)$ to (d, d, d), as required. Thus, we complete the proof of Claim 8. □

Claim 9. *The set* $N(v_4)\setminus(N[v_1] \cup N[v_2])$ *is empty.*

Proof. Suppose that there exists a vertex $v \in N(v_4)\setminus(N[v_1] \cup N[v_2])$. Clearly, deleting v reduces the degrees of v_1, v_2, v_4 from $(d, d, d+2)$ to $(d, d, d+1)$. Note that $\{v_1, v_2, v_4\}$ remains feasible in the resulting graph as it is feasible in G. It follows from Lemma 3 (and its remark) that we can delete at most two more vertices to equate the degrees of v_1, v_2, v_4. Overall, we delete at most 3 vertices, as desired. □

Combining Claims 7 and 9, we conclude that $(N(v_3) \cup N(v_4)) \subseteq (N[v_1] \cup N[v_2])$. Recall that $U = \{v_1, v_2, v_3, v_4\}$. Let

$$W := \big((N(v_1) \setminus N(v_2)) \cup (N(v_2) \setminus N(v_1))\big) \setminus U.$$

In what follows, we show that

$$e\left(\{v_3, v_4\}, W\right) - |W| \geqslant 4, \tag{4}$$

meaning that there exists two vertices u_1 and u_2 in $N(v_j) \setminus (N(v_{3-j}) \cup U)$ for some $j \in [2]$ such that $u_i v_3 \in E(G)$ and $u_i v_4 \in E(G)$ for each $i \in [2]$. It follows that we can always equate the degrees of v_{3-j}, v_3 and v_4 from $(d, d+2, d+2)$ to (d, d, d) by deleting u_1 and u_2 for some $j \in [2]$, as required.

Now, we prove (4). For two vertices $u, v \in V(G)$, let $\mathbf{1}_{uv} = 1$ if $uv \in E(G)$ and $\mathbf{1}_{uv} = 0$ if $uv \notin E(G)$. Note that

$$|W| = d(v_1) + d(v_2) - 2|\left(N(v_1) \cap N(v_2)\right) \setminus U| - 2 \cdot \mathbf{1}_{v_1 v_2} - e\left(\{v_1, v_2\}, \{v_3, v_4\}\right)$$

and

$$e(\{v_3, v_4\}, W) \geqslant d(v_3) + d(v_4) - 2 \cdot \mathbf{1}_{v_3 v_4} - e\left(\{v_3, v_4\}, \{v_1, v_2\}\right) - 2|\left(N(v_1) \cap N(v_2)\right) \setminus U|.$$

Thus, we have

$$e(\{v_3, v_4\}, W) - |W| \geqslant d(v_3) + d(v_4) - d(v_1) - d(v_2) + 2(\mathbf{1}_{v_1 v_2} - \mathbf{1}_{v_3 v_4}) = 4 + 2(\mathbf{1}_{v_1 v_2} - \mathbf{1}_{v_3 v_4}).$$

This implies (4) unless $v_1 v_2 \notin E(G)$ and $v_3 v_4 \in E(G)$. However, $v_1 v_2 \notin E(G)$ means that $\{v_1, v_2, v_3\}$ is an independent set as $\{v_1, v_2, v_3\}$ is a balanceable set, i.e., $v_3 \notin (N[v_1] \cup N[v_2])$. Note that $N(v_4) \subseteq (N[v_1] \cup N[v_2])$ by Claim 9. This leads to a contradiction as $v_3 \in N(v_4)$. Thus, (4) holds. This completes the proof of Lemma 6. □

3 Proof of Theorem 2

In this section, we prove Theorem 2.

Proof of Theorem 2. We aim to prove that $C(3) = 3$. First, the graph in Figure 1 shows that the lower bound $C(3) \geqslant 3$. It suffices to show that $C(3) \leqslant 3$. We prove this by way of contradiction and suppose that G is a smallest graph with at least five vertices for which one must delete more than 3 vertices to obtain an induced subgraph with at least three vertices with the same degree.

It is easy to check that Theorem 2 holds for $n = 5, 6$. So, assume $n \geqslant 7$. We may further assume that G has no isolated vertices; if it does, then there are at most two such vertices (otherwise $\mathrm{rep}(G) \geqslant 3$), and we may consider the graph without the isolated vertices (which has at least 5 vertices), contradicting the minimality of G.

Let

$$S = \{d \in [n-1] : \text{there exist exactly two vertices in } G \text{ with degree } d\}$$

and

$$T = \{d \in [n-1] : \text{there exists no vertex in } G \text{ with degree } d\}.$$

In what follows, we proceed our proof by showing the following series of claims. For some $x \notin [n-1]$, we use $x \in T$ (or $x \in S$) to denote that there exists $y \in T$ (or $y \in S$) such that $x \equiv y \pmod{n-1}$.

Claim 10. *If $d \in S$, then either $d-1 \in T$ or $d+1 \in T$.*

Proof. We first prove this for $2 \leqslant d \leqslant n-2$. Suppose that there exist 4 vertices v_1, v_2, v_3, v_4 in G with degrees $d-1$, d, d, $d+1$, respectively. If $X := \{v_1, v_2, v_3, v_4\}$ contains a balanceable set S_0, then we can delete at most 3 vertices to equate the degrees of the vertices in S_0 by Lemma 3, a contradiction. Otherwise, $G[X]$ is an induced path with v_1, v_2 as its terminal vertices by Lemma 4. Clearly, this path has two possible degree sequences $(d(v_1), d(v_3), d(v_4), d(v_2)) = (d-1, d, d+1, d)$ and $(d(v_1), d(v_4), d(v_3), d(v_2)) = (d-1, d+1, d, d)$. For the first case, deleting v_4 reduces the degrees of v_1, v_2, v_3 from $(d-1, d, d)$ to $(d-1, d-1, d-1)$, a contradiction; for the second case, deleting v_1 reduces the degrees of v_2, v_3, v_4 from $(d, d, d+1)$ to (d, d, d), a contradiction.

If $d = 1 \in S$, then we show that $n-1 \in T$ if $2 \notin T$. Suppose that $n-1 \notin T$. This means that there exist two vertices $u, v \in V(G)$ with $d(u) = n-1$ and $d(v) = 2$. Let $w \in N(v) \setminus \{u\}$ and $d(v_i) = 1$ for each $i \in [2]$. Since $d(u) = n-1$ and $d(v_i) = 1$ for each $i \in [2]$, we know that w and v_i are distinct vertices. Clearly, v, v_1, v_2 have the same degree 1 in the resulting graph by deleting w, a contradiction. If $d = n-1 \in S$, then every vertex in G has degree at least 2, i.e., $d+1 \in T$. ☐

Claim 11. *If $\{d, d+1\} \subseteq S$, then $\{d-1, d+2\} \subseteq T$.*

Proof. Note that $d \in S$ and $d+1 \notin T$. This implies that $d-1 \in T$ by Claim 10. Similarly, we have $d+2 \in T$ as $d+1 \in S$ and $d \notin T$. ☐

Claim 12. *If $\{d, d+2\} \subseteq S$, then $\{d-1, d+1, d+3\} \subseteq T$.*

Proof. We first prove this for $2 \leqslant d \leqslant n-4$. Let $U = \{v_1, v_2, v_3, v_4\} \subseteq V(G)$ with $d(v_1) = d(v_2) = d$ and $d(v_3) = d(v_4) = d+2$. If there exists a vertex $u \in V(G)$ with $d(u) = d+1$, then $U \cup \{u\}$ must contain a feasible set S_0 such that $u \in S_0$ in view of Lemma 5. It follows from Lemma 3 that we can always delete at most 3 vertices to equate the degrees of the vertices in S_0, a contradiction. By Lemma 6, U does not contain balanceable sets, implying that $G[U]$ is an induced path with v_1, v_2 as its terminal vertices. Hence, $v_1v_2 \notin E(G)$ and $v_3v_4 \in E(G)$. If there exists $v_0 \in V(G)$ with $d(v_0) = d-1$, then $\{v_0, v_1, v_2\}$ is feasible as $v_1v_2 \notin E(G)$ and $d(v_1) = d(v_2)$; if there exists $v_5 \in V(G)$ with $d(v_5) = d+3$, then $\{v_3, v_4, v_5\}$ is also feasible as $v_3v_4 \in E(G)$ and $d(v_3) = d(v_4)$. In both cases, we could delete at most 2 vertices to make their degrees equal by Lemma 3, a contradiction.

If $d = 1 \in S$, then it suffices to check that $n-1 \in T$ as the other cases are still valid by the proof above. However, we may also set $v_5 \in V(G)$ with $d(v_5) = n-1$ for the case $n-1 \in T$ and the desired result follows in a similar way as before.

If $d = n-3 \in S$, then we shall only need to check that $1 \in T$. This is clearly true as every vertex in G has degree at least 2 as $n-1 \in S$.

If $d = n-2 \in S$, then we have $1 \in S$ and we should prove that $\{n-3, n-1, 2\} \subseteq T$. Let $U = \{v_1, v_2, v_3, v_4\} \subseteq V(G)$ with $d(v_1) = d(v_2) = 1$ and $d(v_3) = d(v_4) = n-2$. Clearly, $G[U]$ is an induced path with v_1, v_2 as its terminal vertices, and $(\{v_3, v_4\}, V(G) \setminus U)$ is a complete bipartite subgraph. If there exists a vertex $v_0 \in V(G)$ with $d(v_0) = 2$, then v_0, v_1, v_2 both have degree 0 in the resulting graph by deleting v_3 and v_4, a contradiction. If there exists a vertex $v_5 \in V(G)$ with $d(v_5) = n-3$ or $d(v_5) = n-1$, then by Lemma 3 we can always delete at most 3 vertices to equate the degrees of v_3, v_4, v_5 as they form a triangle, a contradiction.

If $d = n-1 \in S$, then we have $2 \in S$ and we should prove that $\{n-2, 1, 3\} \subseteq T$. Since $n-1 \in S$, this implies that G contains two vertices of degree $n-1$, whence $1 \in T$. Let $U = \{v_1, v_2, v_3, v_4\} \subseteq V(G)$ with $d(v_1) = d(v_2) = 2$ and $d(v_3) = d(v_4) = n-1$. It is easy to see that $v_3v_4 \in E(G)$ and $(\{v_3, v_4\}, V(G) \setminus \{v_3, v_4\})$ is a complete bipartite subgraph. If there exists a vertex $v_0 \in V(G)$ with $d(v_0) = 3$, then v_0, v_1, v_2 both have degree 2 in the resulting graph by deleting some vertex $w \in N(v_0) \setminus \{v_3, v_4\}$, a contradiction. If there exists a vertex $v_5 \in V(G)$ with $d(v_5) = n-2$, then by Lemma 3 we can always delete at most 3 vertices to equate the degrees of v_3, v_4, v_5 as they form a triangle, a contradiction. □

Let $S_1 = \dot\bigcup_{i=1}^{\ell} S_{1i}$, where each S_{1i} is a maximal set of the form $\{d, d+2, \ldots, d+2j\} \subseteq S$ for some $d \in S$ and $j \geqslant 1$. Subject to this, we may further assume that ℓ is maximal. Let $T_{1i} = \{d-1, d+1, \ldots, d+2j+1\}$ for each S_{1i}. Clearly, $T_{1i} \cap T_{1k} = \emptyset$ for $i \neq k \in [\ell]$ by the maximality of S_{1i}, and $T_1 = \dot\bigcup_{i=1}^{\ell} T_{1i} \subseteq T$ by Claim 12. Thus, we conclude that $|T_1| \geqslant |S_1|$. Set $S_2 = S \setminus S_1$ and $T_2 = T \setminus T_1$. By the maximality of each S_{1i}, we have $T_1 \cap T_2 = \emptyset$. Moreover, for any two pairs $\{d_1, d_1+1\} \subseteq S_2$ and $\{d_2, d_2+1\} \subseteq S_2$, we have $\{d_1-1, d_1+2\} \cap \{d_2-1, d_2+2\} = \emptyset$ by the maximality of ℓ. Consequently, we have $|T_2| \geqslant |S_2|$ by Claims 10 and 11. This finally implies that $|T| \geqslant |S|$. Let

$Y = \{d \in [n-1] : \text{there exists exactly one vertex in } G \text{ with degree } d\}$. Since G has no isolated vertices and $\text{rep}(G) = 2$, we have $n = 2|S| + |Y|$ by the definitions of S and Y. Note also that

$$n - 1 = |S| + |Y| + |T| = |S| + (n - 2|S|) + |T|$$

as there are exactly $n-1$ possible degrees in an n-vertex graph without isolated vertices. That is $|T| = |S| - 1$. This leads to a contradiction and completes the proof of Theorem 2. □

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References

[1] P. Balister, B. Bollobás, J. Lehel and M. Morayne, Repeated degrees in random uniform hypergraphs. *SIAM J. Discrete Math.*, 27: 145–154, 2013.

[2] B. Bollobás, Degree multiplicities and independent sets in K_4-free graphs. *Discrete Math.*, 158: 27–35, 1996.

[3] B. Bollobás and A.D. Scott, Independent sets and repeated degrees. *Discrete Math.*, 170: 41–49, 1997.

[4] Y. Caro, A. Hansberg and R. Pepper, Regular independent sets. *Discrete Appl. Math.*, 203: 35–46, 2016.

[5] Y. Caro, J. Lauri and C. Zarb, Notes on spreads of degrees in graphs. *Bull. Inst. Combin. Appl.*, 85: 79–91, 2019.

[6] Y. Caro, A. Shapira and R. Yuster, Forcing k-repetitions in degree sequences. *Electron. J. Combin.*, 21:#P1.24, 2014.

[7] Y. Caro and D.B. West, Repetition number of graphs. *Electron. J. Combin.*, 16:#R7, 2009.

[8] Y. Caro and R. Yuster, Induced subgraphs with many repeated degrees. *Discrete Math.*, 343: 111828, 2020.

[9] G. Chen, J. Hutchinson, W. Piotrowski, W. Shreve and B. Wei, Degree sequences with repeated values. *Ars Combin.*, 59: 33–44, 2001.

[10] P. Erdős, R. Faudree, T.J. Reid, R. Schelp and W. Staton, Degree sequence and independence in K_4-free graphs. *Discrete Math.*, 141: 285–290, 1995.

[11] S. Kogan, A note on forcing 3-repetitions in degree sequences. *Discuss. Math. Graph Theory*, 43: 457–462, 2023.

[12] G. Sun, J. Hou and Q. Zeng, Large induced subgraphs with three repeated degrees. *Discrete Math.*, 346: 113344, 2023.