A Note on Large Degenerate Induced Subgraphs in Sparse Graphs

Alexander Clow Sean Kim Ladislav Stacho

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Abstract

Given a graph G and a non-negative integer d let $\alpha_d(G)$ be the order of a largest induced d-degenerate subgraph of G. We prove that for any pair of non-negative integers $k > d$, if G is a k-degenerate graph, then $\alpha_d(G) \geqslant \max\{\frac{(d+1)n}{k+d+1}, n-\alpha_{k-d-1}(G)\}$. For k-degenerate graphs this improves a more general lower bound of Alon, Kahn, and Seymour. By modifying our arguments we also obtain an improved lower bounds on $\alpha_d(G)$ for graphs of bounded genus. This extends earlier work on degenerate subgraphs of planar graphs.

Mathematics Subject Classifications: 05C40

1 Introduction

A graph G is d-degenerate if every subgraph H of G has minimum degree at most d. The degeneracy of a graph G is the least d such that G is d-degenerate. Thus, a graph is 0-degenerate if and only if it has no edges, and 1-degenerate if and only if it is a forest. Given a fixed integer d and a graph G, we let $\alpha_d(G)$ be the order of a largest induced d-degenerate subgraph of G. Hence, by setting $d = 0$ one can recover the independence number of the graph G. For more background and definitions in graph theory we refer the reader to [24].

Alon, Kahn, and Seymour [4] began the study of $\alpha_d(G)$ when they proved that in any graph G,

$$\alpha_d(G) \geqslant \sum_{v\in V(G)} \min\left\{1, \frac{d+1}{\deg(v)+1}\right\}.$$

Notice that this bound is tight for any disjoint union of cliques. As remarked in [4], when $d = 0$ this returns a classic lower bound for independence number proven by Caro [8] and independently by Wei [23]. As a corollary of this result, graphs with average degree $D \geqslant 2d$ have

$$\alpha_d(G) \geqslant \frac{(d+1)n}{D+1}.$$

Department of Mathematics, Simon Fraser University, Canada (alexander_clow@sfu.ca, sean_kim_8@sfu.ca, lstacho@sfu.ca).

Observe that every k-degenerate graph has average degree at less than $2k$, but for all $\varepsilon > 0$ there exists a k-degenerate graph with average degree greater than $2k - \varepsilon$. So for any $d < k$, Alon, Kahn, and Seymour proved that in any k-degenerate graph $\alpha_d(G) \geqslant \frac{(d+1)n}{2k+1}$.

There is a long history of studying α_d for planar graphs. As corollary of the Four Colour Theorem, for all planar graphs G, $\alpha_0(G) \geqslant \frac{n}{4}$. This bound is tight as witnessed by K_4. Proving this lower bound without the Four Colour Theorem remains open, although there has been some progress, see [3, 10].

Similarly, K_4 demonstrates that there are planar graphs with $\alpha_1(G) = \frac{n}{2}$. Albertson and Berman [2] and Akiyama and Watanabe [1] independently conjectured that $\alpha_1(G) \geqslant \frac{n}{2}$ for all planar graphs. This conjecture has seen some significant attention since being proposed. Indeed, a corollary of Borodin's proof [6] that all planar graphs are acyclically 5-colourable is that for all planar graphs G, $\alpha_1(G) \geqslant \frac{2n}{5}$. This remains the best general lower bound on α_1 for planar graphs. The conjecture is known to be true for triangle-free planar graphs, see [21]. The current best lower bound on α_1 for triangle-free planar graphs G is from Dross, Montassier, and Pinlou [11] who proved that $\alpha_1(G) \geqslant \frac{6n+7}{11}$.

More recently, studying α_d for $2 \leqslant d \leqslant 4$ in planar graphs has garnered some attention. The octahedron witnesses that there exists planar graphs with $\alpha_2(G) = \frac{2n}{3}$. Gu, Kierstead, Oum, Qi, and Zhu [13] conjecture that every planar graph has $\alpha_2(G) \geqslant \frac{2n}{3}$. Notably, Dvořák and Kelly [12] proved that if G is a triangle-free planar graph then the stronger bound $\alpha_2(G) \geqslant \frac{4n}{5}$ holds. When $d = 3$ the octahedron and icosahedron witness that there exists planar graphs with $\alpha_3(G) = \frac{5n}{6}$. Gu, Kierstead, Oum, Qi, and Zhu [13] showed that all planar graphs have $\alpha_3(G) \geqslant \frac{3n}{4}$. For $d = 4$ the icosahedron witnesses that there exists planar graphs with $\alpha_4(G) = \frac{11n}{12}$. The best bound here is from Lukot'ka, Mazák, and Zhu [16] who showed that for all planar graphs $\alpha_4(G) \geqslant \frac{8n}{9}$.

In this paper, we consider this problem for k-degenerate graphs. Thus, our results are for a more general class of graphs than planar graphs, but more specific than the average degree cases studied in [4]. Our primary contribution is the following theorem.

Theorem 1. *Let $k > d$ be non-negative integers. If G is a k-degenerate graph, then*

$$\alpha_d(G) \geqslant \max\left\{\frac{(d+1)n}{k+d+1}, n - \alpha_{k-d-1}(G)\right\}.$$

Furthermore, $V(G)$ can be partitioned into sets X and Y such that $G[X]$ is d-degenerate and $G[Y]$ is $(k-d-1)$-degenerate.

Intuitively, Theorem 1 follows from taking a k-degeneracy ordering, then greedy adding vertices from this ordering to a d degenerate induced subgraph. If our degeneracy ordering satisfies that each vertex has at most d neighbours that come before it in the ordering, then this greedy strategy leads to a partition of G into a d-degenerate subgraph and a $(k-d-1)$-degenerate subgraph. If the ordering instead satisfies that each vertex has at most d neighbours that come later it in the ordering, then we can prove the resulting d-degenerate subgraph is large by counting the number of forward edges between our d-degenerate subgraph and the rest of the graph. This number of edges is bounded by the choice of degeneracy ordering, and our choice of greedy selection procedure.

Since Theorem 1 provides linear lower bound on α_d, for pairs of integers $k > d$, we define

$$\alpha_d(k) := \inf_{G \text{ with degeneracy } k} \frac{\alpha_d(G)}{|V(G)|},$$

and note that $\alpha_d(k)$ is bounded away from 0.

Trivially, as witnessed by the clique K_{k+1}, for all $k > d$, $\alpha_d(k) \leqslant \frac{d+1}{k+1}$. Together with Theorem 1 this implies $\frac{2}{k+2} \leqslant \alpha_1(k) \leqslant \frac{2}{k+1}$ for all k. Meaning that for large k our bound in Theorem 1 is close to optimal. On the other hand as d grows relative to k, our lower bound that $\alpha_d(G) \geqslant \frac{(d+1)n}{k+d+1}$ tends to Alon, Kahn, and Seymour's bound for graphs with average degree at most $2k$. In these cases, the other term in our lower bound can be helpful. For example, Theorem 1 implies that if G is k-degenerate and the independence number of G is $\frac{n}{k+1}$, the minimum for maximally k-degenerate graphs, then $\alpha_d(G) \geqslant \frac{(d-1)n}{k+1}$. This nearly matches the $\alpha_d(k)$ upper bound witnessed by cliques.

Notice that for certain values of k and d, for example $k = 3$ and $d = 1$, $\alpha_d(k) = \frac{d+1}{k+1}$, while for other values of k and d, for example $k = 2$ and $d = 1$, $\alpha_d(k) < \frac{d+1}{k+1}$. To see that $\alpha_1(3) = \frac{1}{2}$ notice that Theorem 1 implies $\alpha_1(G) \geqslant n - \alpha_1(G)$ for any 3-degenerate graph G. Meanwhile, K_4 with one edge subdivided is an example of 2-degenerate graph that witnesses $\alpha_1(2) \leqslant \frac{3}{5} < \frac{2}{3}$.

Given the literature on planar graphs, studying this problem for graphs of bounded genus is natural. For preliminaries on graphs on more general surfaces than the plane we recommend [19]. By adapting our proof of Theorem 1 we obtain the following bound for graphs of bounded genus.

Theorem 2. *Let g be an integer. If G is a graph with genus at most g, then*

$$\alpha_d(G) \geqslant \begin{cases} \frac{n-12g+1}{4} & \text{if } d = 1 \\ \frac{dn-4g+4}{d+1} & \text{if } 2 \leqslant d \leqslant 5 \end{cases}$$

Recalling that for planar graphs, the best lower bound on α_1 comes from acyclic colouring, we note that this approach an acyclic colouring argument does not improve Theorem 2 in the $d = 1$ case for graphs with larger genus. This is because there are graphs with genus g and acyclic chromatic number $\Omega(g^{\frac{4}{7}-o(1)})$, see [5]. Even if we restrict to locally planar graphs, which are the classic example of graphs of large genus that behave like planar graphs, one does not achieve an improvement to Theorem 2. This is because the best upper bound for the acyclic chromatic number of locally planar graphs is 7 [14].

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we prove Theorem 2. While Section 4 discusses our computational approach to find improved upper bounds for $\alpha_d(k)$ for small values of k and d. We conclude with a discussion of future work.

2 Degenerate Graphs

This section will be dedicated to proving Theorem 1. Before launching into the proof of Theorem 1, we note that an equivalent condition for a graph to be d-degenerate is the existence of linear ordering of its vertices where each vertex has at most d neighbours that come later it in the ordering. See [15], Proposition 1 for a proof that the existence of such a vertex ordering is equivalent to a graph being d-degenerate.

Proof of Theorem 1. Let G be a k-degenerate graph, and let $v_1, \dots, v_n$ be a linear ordering of $V(G)$ such that for all i, $|N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leqslant k$. We will prove $\alpha_d(G) \geqslant \frac{(d+1)n}{k+d+1}$

and $\alpha_d(G) \geqslant n - \alpha_{k-d-1}(G)$ using separate, but similar, arguments. We begin by demonstrating that $\alpha_d(G) \geqslant \frac{(d+1)n}{k+d+1}$.

We colour the vertices of G blue or red in order starting from v_1, using the following procedure: let B be the set of vertices already coloured blue (initially empty), and let R be the set of vertices already coloured red (initially empty). Let the blue-degree of a vertex v_i, written $\text{degb}(v_i)$, be defined by $\text{degb}(v_i) := |B \cap N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$, and let the red-degree of a vertex v_i, written $\text{degr}(v_i)$, be defined by $\text{degr}(v_i) := |R \cap N(v_i) \cap \{v_{i+1}, \dots, v_n\}|$. We colour v_1 blue. At step $i > 1$, we will assume that vertices $v_1, \dots, v_{i-1}$ are already coloured, and that no vertex $v_i, \dots, v_n$ has been coloured. We will then colour vertex v_i blue if $\text{degb}(v_i) \leqslant d$, or red if $\text{degb}(v_i) > d$.

Note that this procedure will colour all vertices of G. By our colouring rule, the reader can easily observe that $G[B]$ is a d-degenerate graph, and the original ordering (with red coloured vertices removed) witnesses its d-degeneracy. Furthermore, by our choice of colouring procedure each vertex $v_j \in R$ satisfies $\text{degb}(v_j) \geqslant d + 1$. Hence, $\sum_{u \in R} \text{degb}(u) \geqslant (d+1)|R|$. Next, observe that

$$\sum_{u \in R} \text{degb}(u) = \sum_{v \in B} \text{degr}(v)$$

since both sides of this equation count the number of edges $v_i v_j \in E(G)$ where $i < j$, $v_i \in B$, and $v_j \in R$. Finally, we note that for all vertices v_i, $\text{degr}(v_i) \leqslant |N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leqslant k$ by our choice of vertex order $v_1, \dots, v_n$. Thus, $\sum_{v \in B} \text{degr}(v) \leqslant k|B|$.

We conclude that $(d+1)|R| \leqslant k|B|$. This implies $|R| \leqslant \frac{k}{d+1}|B|$. Recalling that B and R partition $V(G)$, since all vertices receive a colour, we note that $n = |R| + |B|$. Hence,

$$\begin{aligned} n &= |R| + |B| \\ &\leqslant \left(\frac{k}{d+1}|B|\right) + |B| \\ &= \frac{k+d+1}{d+1}|B|. \end{aligned}$$

It follows that $\frac{(d+1)n}{k+d+1} \leqslant |B|$. Recalling that $G[B]$ is d-degenerate this implies $\alpha_d(G) \geqslant \frac{(d+1)n}{k+d+1}$ as desired.

Now, we turn our attention to proving $\alpha_d(G) \geqslant n - \alpha_{k-d-1}(G)$. Let $u_1, \dots, u_n$ be a linear ordering of $V(G)$ such that for all i, $|N(u_i) \cap \{u_1, \dots, u_{i-1}\}| \leqslant k$. Such an ordering exists, since one can be obtained by reversing the indices of the order $v_1, \dots, v_n$, that is $u_i = v_{n-i+1}$.

Perform the same procedure as before to colour vertices of G either blue or red, this time respecting the vertex order $u_1, \dots, u_n$ (rather than $v_1, \dots, v_n$). Let B' and R' be the resulting sets of blue and red vertices, respectively. Observe that $G[B']$ is d-degenerate by the same argument as for $G[B]$.

We claim that $G[R']$ is $(k-d-1)$-degenerate. Notice that if true, $\alpha_d(G) + \alpha_{k-d-1}(G) \geqslant n$, since B' and R' partition $V(G)$. To see that $G[R']$ is $(k-d-1)$-degenerate, we note that, by the definition of our colouring procedure each vertex $u_i \in R'$ has $\text{degb}(u_i) \geqslant d+1$. Since, $|N(u_i) \cap \{u_1, \dots, u_{i-1}\}| \leqslant k$ by our choice of vertex ordering, this implies each vertex u_i has at most $k - d - 1$ red neighbours in $\{u_1, \dots, u_{i-1}\}$. Hence, $u_1, \dots, u_n$ with the blue vertices removed from it is a $(k - d - 1)$-degenerate ordering of $G[R']$. This concludes the proof. □

3 Graphs on Surfaces

In this section we will prove Theorem 2. We begin considering graphs on surfaces by noting a well known corollary of Euler's formula. For completeness a proof is included.

Lemma 3. *If G is a triangle-free graph with genus at most g, then $|E(G)| \leqslant 2n+4g-4$.*

Proof. Let G be a graph of Euler genus at most g. Let Π be a minimum genus embedding of G. By the handshaking lemma, the sum of the size of all faces in Π is equal to twice the number of edges in G. Since G is triangle-free, this implies $F \leqslant \frac{1}{2}|E(G)|$, where F denotes the number of faces in Π. Then Euler's formula implies $2-2g \leqslant n - \frac{1}{2}|E(G)|$. Isolating $|E(G)|$ implies the desired result. □

Next, we consider another useful corollary of Euler's formula. This lemma also appears in [7, 9]. For completeness a proof is included here. Observe that this implies that if g is a constant, then $\alpha_6(G) = (1-o(1))n$ for any graphs G with genus at most g.

Lemma 4. *Let $k \geqslant 1$ be an integer. If G is a graph of genus at most g and minimum degree at least $k+6$, then G has fewer than $\frac{12g}{k}$ vertices.*

Proof. Let G be a graph of Euler genus at most g and $\delta(G) \geqslant k+6$. By Euler's formula and the handshaking lemma, $n - \frac{1}{3}|E| > -2g$. Rearranging this,

$$\sum_{v \in V(G)} (\deg(v) - 6) < 12g.$$

Since, each vertex has degree at least $k+6$, the number of terms in this sum is less than $\frac{12g}{k}$, completing the proof. □

We are now prepared to prove Theorem 2.

Proof of Theorem 2. Let G be a graph with genus at most g. The outline of the proof is consistent across different choices of $2 \leqslant d \leqslant 5$. Unfortunately, the same argument does not apply to the $d=1$ case. Hence, we consider these two cases separately.

Suppose $d \geqslant 2$. Let $v_1, \dots, v_n$ be a fixed but arbitrary linear ordering of $V(G)$. Colour vertices blue or red, using the same procedure as in the proof of Theorem 1 with respect to the vertex order $v_1, \dots, v_n$ and our choice of d. Also, let blue-degree and red-degree be defined per the proof of Theorem 1. Let B and R be the resulting set of blue and red vertices, respectively.

By the definition of our vertex colouring procedure, for all $u \in R$, $\text{degb}(u) \geqslant d+1$. Furthermore, like in the proof of Theorem 1, $G[B]$ is d-degenerate. Let X be the set of all edges v_iv_j such that $i<j$, $v_i \in B$, and $v_j \in R$, and let H be the subgraph of G consisting of all edges in X and their incident vertices. Then $R \subseteq V(H)$, H is a bipartite graph, and $X \geqslant (d+1)|R|$.

Since G has genus at most g and H is a subgraph of G, H has genus at most g. Since H is a bipartite graph with genus at most g, and $|V(H)| \leqslant n$, observe that Lemma 3 implies that $|E(H)| = |X| \leqslant 2n+4g-4$.

We have shown that $(d+1)|R| \leqslant 2n+4g-4$. Hence, $|R| \leqslant \frac{2n+4g-4}{d+1}$. Since B and R partition $V(G)$, $n = |B|+|R|$. Combining these facts we obtain $|B| \geqslant n - \frac{2n+4g-4}{d+1}$. Since $G[B]$ is d-degenerate, and $d \geqslant 2$ this completes the proof of this case $d \geqslant 2$.

Now suppose $d = 1$. Let $v_1, \ldots, v_n$ be a linear ordering of the vertices of $V(G)$ such that for all i, v_i is a vertex of minimum degree in $G[\{v_1, \ldots, v_i\}]$. Then, Lemma 4 implies for all $i \geqslant 12g$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leqslant 6$. Letting $H = G - \{v_1, \ldots, v_{12g-1}\}$, this implies H is 6-degenerate.

Since H is 6-degenerate, Theorem 1 implies $\alpha_1(H) \geqslant \frac{2}{8}|V(H)|$. Given H is an induced subgraph of G, we note that $\alpha_1(G) \geqslant \frac{2|V(H)|}{8}$. Thus, $\alpha_1(G) \geqslant \frac{n-12g+1}{4}$ as required. This completes the proof. □

4 Future Work

The most natural open problem resulting from our work is to determine exact values for $\alpha_d(k)$. After conducting extensive computational work, we conjecture the following.

Conjecture 5. $\alpha_1(2) = \frac{3}{5}$.

Conjecture 6. For all $k \geqslant 3$ and $k > d$, $\alpha_d(k) = \frac{d+1}{k+1}$.

Our code is available at [22]. As a short summary, using `gtools`, a package inside of `nauty` [17] we exhaustively generate small graphs with degree sequences that were of interest. When this did not produce examples better than K_4 with an edge subdivided, or cliques, we wrote a genetic algorithm to search for larger graphs that might disprove Conjecture 5 or Conjecture 6. Again this did not produce better examples, although in several cases it could find large graphs that match the bounds in Conjecture 5 and Conjecture 6.

We computed $\alpha_1(G)$ using an integer programming approach from [18]. For $d \geqslant 2$, our only means to calculate $\alpha_d(G)$ was to check the degeneracy of all induced subgraphs. This computing bottleneck limited our ability to run our genetic algorithm at scale for cases where $d > 1$. Note that there is a Monte Carlo algorithm given in [20] for computing $\alpha_d(G)$. However, for graphs with a small number of vertices (say < 50), which is where we found the most success for our genetic algorithm, we did not find that random algorithms provided a substantial advantage. Our code was run on the FIR supercomputer at Simon Fraser University.

This leads us to the following problem that might assist future computation work regarding α_d.

Problem 7. For each $d \geqslant 2$, determine an integer program that computes $\alpha_d(G)$ for an input graph G, or prove no such program exists.

Given Theorem 2, it is also natural to ask about induced subgraphs of graphs on surfaces. We are particularly interested in induced forests in graphs on surfaces. This leads us to conjecture the following.

Conjecture 8. For all non-negative integers g, there exists a constant $f(g)$ such that, if G is a graph with genus g, then $\alpha_1(G) \geqslant \frac{n}{2} - f(g)$.

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