Hankel Determinants for Convolution Powers of Narayana Polynomials

Guo-Niu Han[a]

Submitted: Dec 14, 2025; Accepted: Jan 28, 2026; Published: May 22, 2026

Abstract

We prove and generalize a conjecture of Johann Cigler on the Hankel determinants of convolution powers of Narayana polynomials. Our method follows a "guess-and-prove" strategy, relying on established techniques involving Hankel continued fractions. While the final forms of our theorems are given by simple closed expressions, the proofs require us to formulate and manage extremely large and intricate explicit expressions at intermediate stages. Most of the technically involved and lengthy formal verifications are carried out using a symbolic computation program, whose code is available on the author's personal webpage for independent verification. We emphasize that our program delivers rigorous symbolic proofs, rather than merely verifying the initial terms.

Mathematics Subject Classifications: 05A15, 11A55, 11B39, 11C20, 15A15

1 Introduction

The purpose of this paper is to establish and extend a conjecture of Johann Cigler concerning the Hankel determinants of convolution powers of Narayana polynomials [7]. Let $\gamma_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th *Catalan number* (We avoid the traditional notation C_n because of possible confusion with some other object introduced later). Hankel determinants formed from the Catalan sequence have been investigated in [1, 12, 21, 25, 5, 3, 19]. In 2002, Cvetković, Rajković, and Ivković [11] determined the Hankel determinants of the sequence whose entries are sums of consecutive Catalan numbers:

$$\det\left(\gamma_{i+j} + \gamma_{i+j+1}\right)_{i,j=0}^{n-1} = F_{2n+1}, \tag{1}$$

where F_n denotes the n-th *Fibonacci number*. Since then, this identity has been extended in various directions [2, 22, 13, 4, 23, 26, 9, 24]. Motivated by (1), Cigler [7] studied the

[a]I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, F-67084 Strasbourg, France (`guoniu.han@unistra.fr`).

Hankel determinants associated with convolution powers of Narayana polynomials. Recall that the *Narayana polynomials* are given by

$$\gamma_n(t) = \sum_{k=0}^{n} \binom{n}{k}\binom{n-1}{k}\frac{1}{k+1}t^k, \tag{2}$$

which specialize to the Catalan numbers when $t = 1$. The initial terms of the sequence $(\gamma_n(t))_{n\geqslant 0}$ are

$$1, 1, 1+t, 1+3t+t^2, 1+6t+6t^2+t^3, 1+10t+20t^2+10t^3+t^4, \ldots$$

The generating function of the Narayana polynomials

$$\gamma(t,q) = \sum_{n\geqslant 0} \gamma_n(t)q^n$$

satisfies the quadratic relation

$$-1 + (1-q+tq)\,\gamma(t,q) - tq\,\gamma(t,q)^2 = 0. \tag{3}$$

The *convolution powers of the Narayana polynomials* $\gamma_n^{(\tau)}(t)$ are defined via the following generating functions, depending on the parity of τ:

$$\sum \gamma_n^{(2\tau)}(t)q^n = G(t,q)^{\tau},$$
$$\sum \gamma_n^{(2\tau+1)}(t)q^n = \gamma(t,q)\,G(t,q)^{\tau},$$

where $G(t,q) = \frac{\gamma(t,q)-1}{q}$ denotes the generating function of the shifted Narayana polynomial sequence. Observe that

$$\gamma_n^{(\tau)}(1) = \frac{\tau}{2n+\tau}\binom{2n+\tau}{n}$$

gives the τ-fold convolution power of the Catalan numbers. In [7], Cigler investigated the Hankel determinants associated with the convolution powers of Narayana polynomials $\gamma_n^{(\tau)}(t)$ and derived explicit expressions for the cases $\tau = 3$ and $\tau = 4$. He also proposed a conjecture for $\tau = 6$. We restate these results below. Let

$$\Delta_{\varepsilon}^{(m)} = \det\left(\gamma_{i+j-\varepsilon}^{(m)}(t)\right)_{i,j=0}^{N-1}. \tag{4}$$

Theorem 1 ([7], Theorem 5.2)**.** *We have*

$$\Delta_0^{(3)} = t^{\binom{N}{2}} \sum_{k=0}^{\lfloor N/2\rfloor} (-1)^k \binom{N-k}{k} t^{-k}. \tag{5}$$

Theorem 2 ([7], Theorem 7.4)**.** *We have*

$$\Delta_0^{(4)} = \begin{cases} (-1)^n t^{2n(n-1)}[n+1]_{t^2}, & \text{if } N = 2n, \\ (-1)^n t^{2n^2}[n+1]_{t^2}, & \text{if } N = 2n+1, \end{cases} \tag{6}$$

where $[n]_q$ *is the standard notation for q-number:*

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

Conjecture 3 ([7], Conjecture 7.6)**.** We have

$$\Delta_0^{(6)} = \begin{cases} (-1)^n t^{9n(n-1)/2}[n+1]_{t^3}^2, & \text{if } N = 3n, \\ (-1)^n t^{3n(3n-1)/2}[n+1]_{t^3}^2, & \text{if } N = 3n+1, \\ (-1)^{n+1} 3t^{3n(3n+1)/2}[3]_t r_n(t), & \text{if } N = 3n+2, \end{cases} \tag{7}$$

where

$$r_n(t) = 1 + 3t^3 + 6t^6 + \cdots + \binom{n+1}{2} t^{3(n-1)} + \binom{n+2}{2} t^{3n} + \binom{n+1}{2} t^{3(n+1)} + \cdots + t^{6n}.$$

For further references on these Hankel determinants, we refer the reader to [26, 9, 10, 14, 16].

In this paper, we establish generalizations of Theorem 2 and Conjecture 3. Rather than working with $G(t,q)^m$, we focus on the sequence generated by $(\gamma(t,q)-1)^m$, and derive closed-form expressions for the Hankel determinants of this sequence as well as for its first, second, and third shifted versions:

$$(\gamma(t,q)-1)^m, \frac{(\gamma(t,q)-1)^m}{q}, \frac{(\gamma(t,q)-1)^m}{q^2}, \frac{(\gamma(t,q)-1)^m}{q^3}.$$

It is straightforward to see that the second shifted sequence coincides with $\gamma_n^{(4)}(t)$ when $m = 2$, and that the third shifted sequence coincides with $\gamma_n^{(6)}(t)$ when $m = 3$. We also set $\gamma_i^{(2m)}(t) = 0$ for $i < 0$. Our main theorems are presented below, where we write $\xi_1 = (-1)^{nm(m-1)/2}$ for short.

Theorem 4. *For* $m \geqslant 1$ *we have*

$$\Delta_m^{(2m)} = \begin{cases} 1, & \text{if } N = 0, \\ -\xi_1 t^{m(n-1)(mn-2)/2}[n-1]_{t^m}, & \text{if } N = mn, \\ (-1)^m \xi_1 t^{m^2 n(n-1)/2}[n]_{t^m}, & \text{if } N = mn+1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5. *For $m \geqslant 1$ we have*

$$\Delta_{m-1}^{(2m)} = \begin{cases} \xi_1 t^{m^2 n(n-1)/2}, & \text{if } N = mn, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6. *For $m \geqslant 2$ we have*

$$\Delta_{m-2}^{(2m)} = \begin{cases} \xi_1 t^{m^2 n(n-1)/2}[n+1]_{t^m}, & \text{if } N = mn, \\ (-1)^{m-1}\xi_1 t^{m(n-1)(mn-2)/2}[n]_{t^m}, & \text{if } N = mn-1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 7. *For $m \geqslant 3$ we have*

$$\Delta_{m-3}^{(2m)} = \begin{cases} \xi_1 t^{m^2 n(n-1)/2}[n+1]_{t^m}^2, & \text{if } N = mn, \\ (-1)^{m-1}\xi_1 t^{m(n-1)(mn-2)/2} R(m;t,n-1), & \text{if } N = mn-1, \\ -\xi_1 t^{m(n-1)(mn-4)/2}[n]_{t^m}^2, & \text{if } N = mn-2, \\ 0, & \text{otherwise.} \end{cases}$$

where

$$R(m;t,n) = m[m]_t \left(\sum_{i=0}^{n} \binom{i+2}{2} t^{mi} + \sum_{i=n+1}^{2n} \binom{2n-i+2}{2} t^{mi} \right). \tag{8}$$

We see that Theorem 6 implies Theorem 7.4 of Cigler (6) when $m=2$, and Theorem 7 confirmes his conjecture 7.6 (7) when $m=3$.

Our method is essentially a "guess-and-prove" approach that relies on established techniques involving Hankel continued fractions. While the final forms of our theorems are quite simple and closed, the proof process itself forced us to guess and manipulate extremely large and intricate explicit expressions. For instance, the complete expression for C_{3j} given in Lemma 19 is

$$\begin{aligned} C_{3j} = &-q^2\Big(1 + \frac{[j]_{t^m}[j+1]_{t^m} q}{R(m;t,j-1)}\Big) \sum_{d=0}^{m-2} \frac{(-q)^d m}{m-d} \sum_{i=0}^{d} \binom{m-1-d+i}{i} \binom{m-1-i}{d-i} t^i \\ &- \frac{R(m;t,j)}{[j+1]_{t^m}^2}(-q)^{m+1} + \Big(m[m]_t \frac{[j]_{t^m}[j+1]_{t^m}}{R(m;t,j-1)} - \frac{(1+t^{m(j+1)})}{[j+1]_{t^m}}\Big)(-q)^{m+2} \\ &- \frac{t^{mj}}{R(m;t,j-1)}(-q)^{m+3}, \end{aligned}$$

where $R(m;t,n)$ is defined in (8). Observe that in this formula, the terms $R(m;t,n)$ appear both in numerators and in denominators.

To establish our main theorems, we use the Hankel continued fraction approach developed in [17]. The underlying idea is recalled in Section 3. Our strategy proceeds as follows:

1. Derive a quadratic equation for each of the series

$$\frac{(C(t,q)-1)^m}{q^2} \quad \text{and} \quad \frac{(C(t,q)-1)^m}{q^3}.$$

To this end, we obtain a general formula in Section 5, Corollary 17.

2. Apply Algorithm `NextABC` to the initial coefficients appearing in the quadratic equations derived above. This produces the initial terms of a sequence of six-tuples (23)

$$(A_{n+1}, B_{n+1}, C_{n+1}; k_n, a_n, D_n).$$

Based on these initial terms, we formulate a conjectural closed-form description of the sequence; see Lemmas 18 and 19.

3. Prove that the conjectured formulas are correct, i.e., that they satisfy the relations specified in Algorithm `NextABC`, in Sections 7 and 8. Because the proofs are very long and technically involved, we reproduce only the more accessible part in this paper. The more intricate computational verifications are carried out by computer; the programs are available on my homepage at

`https://irma.math.unistra.fr/~guoniu/narayana.html`

We emphasize that our program delivers rigorous symbolic proofs, rather than merely verifying the initial terms.

4. Construct the Hankel continued fraction from the resulting sequence of quantities; see Section 3, Lemmas 12-15.

5. Finally, compute the Hankel determinants from the Hankel continued fractions. This is carried out in Section 4.

2 Notations and properties

Since the expressions we obtain are rather lengthy, we first introduce some notation to simplify the exposition.

Definition 8. For $d = 0, 1, 2, \ldots, m$, define

$$\rho(m;t,d) = \sum_{i=0}^{d} \frac{m(m-d)}{(m-i)(m-d+i)} \binom{m-d+i}{i} \binom{m-i}{d-i} t^i,$$

$$S(m;t,n) = \frac{m(1-t^m)}{1-t} \left(\sum_{i=0}^{n} (i+1)^2 t^{mi} + \sum_{i=n+1}^{2n} (2n-i+1)^2 t^{mi} \right),$$

$$\begin{aligned} \beta(m;t,q) &= \sum_{d=0}^{m} \rho(m;t,d)(-q)^d \\ &= 1 - m(t+1)q + (\binom{m}{2} + m(m-2)t + \binom{m}{2} t^2) q^2 + \cdots \\ &\quad + \frac{m(1-t^m)}{1-t}(-q)^{m-1} + (1+t^m)(-q)^m. \end{aligned}$$

We also set

$$\alpha(m;t,q) = \sum_{d=0}^{m-2} \rho(m;t,d)(-q)^d,$$

so that

$$\beta(m;t,q) - \alpha(m;t,q) = \frac{m(1-t^m)}{1-t}(-q)^{m-1} + (1+t^m)(-q)^m.$$

For brevity, we write $\rho(d) = \rho(m;t,d), R(n) = R(m;t,n), S(n) = S(m;t,n), \beta(q) = \beta(m;t,q), \alpha(q) = \alpha(m;t,q)$.

Remark. When $d = m$, we obtain $\rho(m;t,m) = 1 + t^m$. For $d = 0, 1, 2, \ldots, m-1$, the function $\rho(m;t,d)$ can be expressed as

$$\rho(m;t,d) = \frac{m}{m-d} \sum_{i=0}^{d} \binom{m-1-d+i}{i} \binom{m-1-i}{d-i} t^i.$$

Recall that the *Lucas polynomials* are given by $L_0(x,s) = 2$ and, for $m \geqslant 1$,

$$L_m(x,s) = \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} \frac{m}{m-i} s^i x^{m-2i}.$$

They satisfy

$$L_m(x,s) = xL_{m-1}(x,s) + sL_{m-2}(x,s) \tag{9}$$

with initial values $L_0(x,s) = 2$ and $L_1(x,s) = x$.

Lemma 9. *We have*

$$\beta(m;t,q) = L_m(1-q-tq, -tq^2).$$

Proof. Consider the right-hand side of the claimed identity:

$$\begin{aligned}
\text{RHS} &= \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} \frac{m}{m-i} (1-q-tq)^{m-2i}(-tq^2)^i \\
&= \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} \frac{m}{m-i} \sum_{\ell=0}^{m-2i} \binom{m-2i}{\ell} (-(1+t)q)^\ell (-tq^2)^i.
\end{aligned}$$

For $0 \leqslant d \leqslant m$ and $0 \leqslant j \leqslant d$, we extract the coefficient of $q^d t^j$ from the above expression. First,

$$\begin{aligned}
[q^d]\,\text{RHS} &= \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{m-i}{i} \frac{m}{m-i} \binom{m-2i}{d-2i} (-(1+t))^{d-2i}(-t)^i \\
&= (-1)^d \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{m-i}{i} \frac{m}{m-i} \binom{m-2i}{d-2i} \sum_{\ell=0}^{d-2i} \binom{d-2i}{\ell} t^\ell (-t)^i.
\end{aligned}$$

To obtain $[q^d t^j]\,\mathrm{RHS}$, set $\ell = j - i$ in the last sum. The condition $0 \leqslant \ell \leqslant d - 2i$ becomes $0 \leqslant j - i \leqslant d - 2i$, i.e., $i \leqslant j \leqslant d - i$. Hence $i \leqslant j$ and $i \leqslant d - j$, and in particular $i \leqslant \lfloor d/2 \rfloor$. Therefore,

$$[q^d t^j]\,\mathrm{RHS} = (-1)^d \sum_{i=0}^{\min\{j,d-j\}} \binom{m-i}{i} \frac{m}{m-i} \binom{m-2i}{d-2i} \binom{d-2i}{j-i} (-1)^i. \tag{10}$$

We now check that these coefficients coincide with the corresponding coefficients in $\beta(m; t, q)$. For $d \leqslant m - 1$, we have

$$\begin{aligned}
[q^d t^j]\,\mathrm{RHS} &= (-1)^d \frac{m(m-1+j-d)!}{j!(m-d)!} \sum_{i=0}^{\min\{j,d-j\}} \binom{j}{i} \binom{m-i-1}{d-i-j} (-1)^i \\
&= (-1)^d \frac{m(m-1+j-d)!}{j!(m-d)!} \binom{m-1-j}{d-j} \\
&= [q^d t^j] \rho(m; t, d)(-q)^d,
\end{aligned}$$

where the sum over i is evaluated using "The Methode of Coefficients"; see [15].

If $d = m$, then from (10) we obtain $[q^m t^0]\,\mathrm{RHS} = [q^m t^m]\,\mathrm{RHS} = (-1)^m$. For $d = m$ and $1 \leqslant j \leqslant m - 1$, (10) specializes to

$$\begin{aligned}
[q^m t^j]\,\mathrm{RHS} &= (-1)^m \frac{m}{j} \sum_{i=0}^{\min\{j,m-j\}} \binom{j}{i} \binom{m-i-1}{m-i-j} (-1)^i \\
&= (-1)^m \frac{m}{j} \binom{m-1-j}{m-j} = 0.
\end{aligned}$$

□

Moreover, the quantities $S(j)$ and $R(j)$ are connected through the following relationships:

$$\begin{aligned}
\frac{m(1-t^m)}{1-t} &= \frac{2R(j) - S(j)}{[j+1]_{t^m}^2} = \frac{-2t^m R(j) + S(j+1)}{[j+2]_{t^m}^2} \\
&= \frac{-2t^m R(j-1) + S(j)}{[j+1]_{t^m}^2} = \frac{R(j) - t^m R(j-1)}{[j+1]_{t^m}^2}, \\
S(j) &= R(j) + t^m R(j-1).
\end{aligned}$$

Throughout this paper, we adopt the following rule, referred to as the "*index convention*" [18]. It stipulates that, in any expression defined by cases, each formula applies only to those integer indices that have not already appeared as special values. For instant, in Lemma 12, the formula for v_{2j+1} is applicable for $j \geqslant 1$, but not for $j = 0$, because v_1 has already been explicitly specified earlier.

3 Hankel continued fractions

The Jacobi continued fraction is a useful tool for evaluating Hankel determinants when all of them are nonzero. Since, in our case, some Hankel determinants vanish, we instead have to use the so-called "Hankel continued fractions" developed in [17]. For further references, see [20, 6, 8, 18]. We now briefly recall the definition and the basic properties of Hankel continued fractions from [17].

Definition 10. For each positive integer δ, a *super continued fraction* associated with δ, called *super δ-fraction* for short, is defined to be a continued fraction of the following form

$$F(q) = \frac{v_0 q^{k_0}}{1 + u_1(q)q} - \frac{v_1 q^{k_0+k_1+\delta}}{1 + u_2(q)q} - \frac{v_2 q^{k_1+k_2+\delta}}{1 + u_3(q)q} - \cdots \tag{11}$$

where $v_j \neq 0$ are constants, k_j are nonnegative integers and $u_j(q)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, 0 is of degree -1.

When $\delta = 1$ (resp. $\delta = 2$) and all $k_j = 0$, the super δ-fraction (11) is the traditional S-fraction (resp. J-fraction). A super 2-fraction is called *Hankel continued fraction*.

Theorem 11. *(i) Let δ be a positive integer. Each super δ-fraction defines a power series, and conversely, for each power series $F(q)$, the super δ-fraction expansion of $F(q)$ exists and is unique. (ii) Let $F(q)$ be a power series such that its H-fraction is given by* (11) *with $\delta = 2$. Then, all non-vanishing Hankel determinants of $F(q)$ are given by*

$$H_{s_j}(F(q)) = (-1)^{\epsilon_j} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}}, \tag{12}$$

where $\epsilon_j = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geqslant 0$.

By applying Theorem 11, the proof of our main results reduces to explicitly determining the corresponding Hankel continued fractions. This is exactly what we now proceed to do. The H-fraction will be given in the standard form

$$\mathcal{H}(q; (k_j), (v_j), (u_j)) := \frac{v_0 q^{k_0}}{1 + u_1(q)q} - \frac{v_1 q^{k_0+k_1+2}}{1 + u_2(q)q} - \frac{v_2 q^{k_1+k_2+2}}{1 + u_3(q)q} - \cdots \tag{13}$$

with explict values for k_j, v_j, u_j.

We are now prepared to write down the explicit H-fractions of

$$(\gamma(t,q) - 1)^m, \quad \frac{(\gamma(t,q) - 1)^m}{q}, \quad \frac{(\gamma(t,q) - 1)^m}{q^2}, \quad \frac{(\gamma(t,q) - 1)^m}{q^3}.$$

Lemma 12. *For $m \geqslant 1$, the power series $(\gamma(t,q) - 1)^m$ has the following H-fraction expansion:*

$$(\gamma(t,q) - 1)^m = \mathcal{H}((k_j), (v_j), (u_j)),$$

where $k_0 = m, k_{2j+1} = m-2, k_{2j+2} = 0, v_0 = 1, v_1 = t^m$, *and*

$$\begin{aligned}
v_{2j+1} &= -(-t)^m [j]_{t^m}/[j+1]_{t^m},\\
v_{2j+2} &= (-1)^{m+1}[j+2]_{t^m}/[j+1]_{t^m};\\
1 + u_1(q)q &= \beta(m;t,q),\\
1 + u_{2j+2}(q)q &= \beta(m;t,q) - (-q)^m(1+t^m),\\
u_{2j+1}(q) &= 0.
\end{aligned}$$

Lemma 13. *For* $m \geqslant 1$, *the power series* $(\gamma(t,q)-1)^m/q$ *has the following* H*-fraction expansion:*

$$\frac{(\gamma(t,q)-1)^m}{q} = \mathcal{H}((k_j),(v_j),(u_j)),$$

where $k_j = m-1, v_0 = 1, v_j = t^m, 1 + u_j(q)q = \beta(m;t,q)$.

Lemma 14. *For* $m \geqslant 2$, *the power series* $(\gamma(t,q)-1)^m/q^2$ *has the following* H*-fraction expansion:*

$$\frac{(\gamma(t,q)-1)^m}{q^2} = \mathcal{H}((k_j),(v_j),(u_j)),$$

where $k_{2j} = m-2, k_{2j+1} = 0, v_0 = 1$, *and*

$$\begin{aligned}
v_{2j} &= -(-t)^m [j]_{t^m}/[j+1]_{t^m},\\
v_{2j+1} &= (-1)^{m+1}[j+2]_{t^m}/[j+1]_{t^m},\\
1 + u_{2j+1}(q)q &= \beta(m;t,q) - (-q)^m(1+t^m),\\
u_{2j}(q) &= 0.
\end{aligned}$$

Lemma 15. *For* $m \geqslant 3$, *the power series* $(\gamma(t,q)-1)^m/q^3$ *has the following* H*-fraction expansion:*

$$\frac{(\gamma(t,q)-1)^m}{q^3} = \mathcal{H}((k_j),(v_j),(u_j)),$$

where $k_{3j} = m-3, k_{3j+1} = 0, k_{3j+2} = 0, v_0 = -1$, *and*

$$\begin{aligned}
v_{3j} &= (-t)^{m+1} R(j-1)/[j+1]^2_{t^m},\\
v_{3j+1} &= (-1)^m R(j)/[j+1]^2_{t^m},\\
v_{3j+2} &= -[j+1]^2_{t^m}[j+2]^2_{t^m}/R(j)^2,\\
u_{3j}(q) &= -[j]_{t^m}[j+1]_{t^m}/R(j-1),\\
1 + u_{3j+1}(q)q &= \alpha(q),\\
u_{3j+2}(q) &= [j+1]_{t^m}[j+2]_{t^m}/R(j).
\end{aligned}$$

4 Proofs of the main theorems

In this section we prove the four main theorems using the four explicit H-fraction given in the previous section.

Proof of Theorem 4. We apply the second part of Theorem 11 to the H-fraction from Lemma 12 that corresponds to $(\gamma(t,q)-1)^m$. In this setting, we obtain $s_0=0$,

$$s_{2j+1} = k_0 + k_1 + \cdots + k_{2j} + 2j + 1 = m(j+1)+1,$$
$$s_{2j+2} = s_{2j+1} + k_{2j+1} + 1 = m(j+1) + 1 + (m-2) + 1 = m(j+2),$$

and $\epsilon_0 = 0$,

$$\begin{aligned}\epsilon_{2j+1} &= m(m+1)/2 + \sum_{i=1}^{2j} k_i(k_i+1)/2 \\ &= m(m+1)/2 + j(m-2)(m-1)/2, \\ \epsilon_{2j+2} &= \epsilon_{2j+1} + (m-2)(m-1)/2 \\ &= m(m+1)/2 + (j+1)(m-2)(m-1)/2.\end{aligned}$$

Since $v_0 = 1$ and $v_{2i+2}v_{2i+3} = t^m$, we have $H_0 = 1, H_{s_1} = (-1)^{m(m+1)/2}$,

$$\begin{aligned}H_{s_{2j+1}}(F(q)) &= (-1)^{\epsilon_{2j+1}} v_1^{s_{2j+1}-s_1} v_2^{s_{2j+1}-s_2} \cdots v_{2j}^{s_{2j+1}-s_{2j}}, \\ &= (-1)^{\epsilon_{2j+1}} v_1^{s_{2j+1}-s_1} v_{2j}^{s_{2j+1}-s_{2j}} \prod_{i=0}^{j-2} v_{2i+2}^{s_{2j+1}-s_{2i+2}} v_{2i+3}^{s_{2j+1}-s_{2i+3}} \\ &= (-1)^{(j+1)(m-2)(m-1)/2+m+(m+1)(j-1)} t^{m^2 j} \\ &\quad \times \frac{[j+1]_{t^m}}{[j]_{t^m}} \prod_{i=0}^{j-2} t^{m(mj-m+1)-im^2-m} \prod_{i=0}^{j-2} \frac{[i+2]_{t^m}}{[i+1]_{t^m}} \\ &= (-1)^{(j+1)m(m-1)/2+m} t^{m^2 j(j+1)/2} [j+1]_{t^m},\end{aligned}$$

and

$$\begin{aligned}H_{s_{2j+2}}(F(q)) &= (-1)^{\epsilon_{2j+2}} v_1^{s_{2j+2}-s_1} v_2^{s_{2j+2}-s_2} \cdots v_{2j+1}^{s_{2j+2}-s_{2j+1}}, \\ &= (-1)^{\epsilon_{2j+2}} v_1^{s_{2j+2}-s_1} \prod_{i=0}^{j-1} v_{2i+2}^{s_{2j+2}-s_{2i+2}} v_{2i+3}^{s_{2j+2}-s_{2i+3}} \\ &= (-1)^{m(m+1)/2+(j+1)(m-2)(m-1)/2} t^{m(mj+m-1)} \\ &\quad \times \prod_{i=0}^{j-1} (v_{2i+2}v_{2i+3})^{m(j-i)} \prod_{i=0}^{j-1} v_{2i+3}^{-1} \\ &= (-1)^{m(m+1)/2+(j+1)(m-2)(m-1)/2} t^{m(mj+m-1)} \\ &\quad \times \prod_{i=0}^{j-1} (t^m)^{m(j-i)} \prod_{i=0}^{j-1} \frac{(-1)^{m+1}[i+2]_{t^m}}{t^m[i+1]_{t^m}} \\ &= (-1)^{(j+2)m(m-1)/2+1} t^{m(j+1)(mj+2m-2)/2} [j+1]_{t^m}.\end{aligned}$$

□

Proof of Theorem 5. Apply the second part of Theorem 11 to the H-fraction from Lemma 13 corresponding to $(\gamma(t,q)-1)^m/q$. Then we obtain

$$s_0 = 0, \qquad s_j = k_0 + k_1 + \cdots + k_{j-1} + j = j(m-1) + j = mj,$$

and

$$\epsilon_j = \sum_{i=0}^{j-1} k_i(k_i+1)/2 = jm(m-1)/2.$$

Since $v_0 = 1$, we have $H_0 = 1$,

$$\begin{aligned} H_{s_j}(F(q)) &= (-1)^{\epsilon_j} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}} \\ &= (-1)^{jm(m-1)/2} v_1^{jm-m} v_2^{jm-2m} \cdots v_{j-1}^{jm-(j-1)m} \\ &= (-1)^{jm(m-1)/2} (t^m)^{jm-m+jm-2m+\cdots+jm-(j-1)m} \\ &= (-1)^{jm(m-1)/2} t^{m^2 j(j-1)/2}. \end{aligned}$$

□

Proof of Theorem 6. We apply the second part of Theorem 11 to the H-fraction from Lemma 14, corresponding to $(\gamma(t,q)-1)^m/q^2$. This yields $s_0 = 0$,

$$\begin{aligned} s_{2j+1} &= k_0 + k_1 + \cdots + k_{2j} + 2j + 1 \\ &= (j+1)(m-2) + 2j + 1 = (j+1)m - 1, \\ s_{2j+2} &= k_0 + k_1 + \cdots + k_{2j+1} + 2j + 2 \\ &= (k_0 + k_1 + \cdots + k_{2j} + 2j + 1) + 1 = (j+1)m, \end{aligned}$$

and $\epsilon_0 = 0$,

$$\begin{aligned} \epsilon_{2j+1} &= \sum_{i=0}^{2j} k_i(k_i+1)/2 = (j+1)(m-2)(m-1)/2, \\ \epsilon_{2j+2} &= (j+1)(m-2)(m-1)/2. \end{aligned}$$

Since $v_0 = 1$ and $v_{2i+1}v_{2i+2} = t^m$, we have $H_0 = 1$,

$$\begin{aligned} H_{s_{2j+1}}(F(q)) &= (-1)^{\epsilon_{2j+1}} v_1^{s_{2j+1}-s_1} v_2^{s_{2j+1}-s_2} \cdots v_{2j}^{s_{2j+1}-s_{2j}}, \\ &= \xi_2 \prod_{i=0}^{j-1} v_{2i+1}^{(j-i)m} v_{2i+2}^{(j-i)m-1} \\ &= \xi_2 \prod_{i=0}^{j-1} (v_{2i+1}v_{2i+2})^{(j-i)m} \prod_{i=0}^{j-1} v_{2i+2}^{-1} \\ &= \xi_2 \prod_{i=0}^{j-1} (t^m)^{(j-i)m} \Big/ \prod_{i=0}^{j-1} \frac{(-1)^{m+1} t^m [i+1]_{t^m}}{[i+2]_{t^m}} \\ &= (-1)^{(m+1)(mj+m-2)/2 + m(j+1)} t^{mj(mj+m-2)/2} [j+1]_{t^m}, \end{aligned}$$

where $\xi_2 = (-1)^{(j+1)(m-2)(m-1)/2}$, and

$$\begin{aligned}
H_{s_{2j+2}}(F(q)) &= (-1)^{\epsilon_{2j+2}} v_1^{s_{2j+2}-s_1} v_2^{s_{2j+2}-s_2} \cdots v_{2j+1}^{s_{2j+2}-s_{2j+1}}, \\
&= (-1)^{\epsilon_{2j+2}} \prod_{i=0}^{j-1} v_{2i+1}^{s_{2j+2}-s_{2i+1}} v_{2i+2}^{s_{2j+2}-s_{2i+2}} \times v_{2j+1}^{s_{2j+2}-s_{2j+1}} \\
&= \xi_2 v_{2j+1} \prod_{i=0}^{j-1} (v_{2i+1} v_{2i+2})^{(j-i)m+1} \prod_{i=0}^{j-1} v_{2i+2}^{-1} \\
&= \xi_2 v_{2j+1} \prod_{i=0}^{j-1} (t^m)^{(j-i)m+1} / \prod_{i=0}^{j-1} \frac{(-1)^{m+1} t^m [i+1]_{t^m}}{[i+2]_{t^m}} \\
&= (-1)^{(j+1)m(m+1)/2+m(j+1)} t^{m^2 j(j+1)/2} [j+2]_{t^m}.
\end{aligned}$$

□

Proof of Theorem 7. We apply the second part of Theorem 11 to the H-fraction from Lemma 15 corresponding to $(\gamma(t,q)-1)^m/q^3$. In this setting, we have $s_0 = 0$,

$$\begin{aligned}
s_{3j+1} &= k_0 + k_1 + \cdots + k_{3j} + 3j + 1 \\
&= (j+1)(m-3) + 3j + 1 = (j+1)m - 2, \\
s_{3j+2} &= s_{3j+1} + k_{3j+1} + 1 = (j+1)m - 1, \\
s_{3j+3} &= s_{3j+2} + k_{3j+2} + 1 = (j+1)m,
\end{aligned}$$

and $\epsilon_0 = 0$,

$$\begin{aligned}
\epsilon_{3j+1} &= \epsilon_{3j+2} = \epsilon_{3j+3} \\
&= \sum_{i=0}^{3j} k_i(k_i+1)/2 = (j+1)(m-3)(m-2)/2.
\end{aligned}$$

We can verify that

$$\begin{aligned}
v_{3i+1} v_{3i+2} v_{3i+3} &= t^m, \\
v_{3i+2} v_{3i+3}^2 &= -t^{2m} [i+1]_{t^m}^2 / [i+2]_{t^m}^2, \\
v_{3j+1}^2 v_{3j+2} &= -[j+2]_{t^m}^2 / [j+1]_{t^m}^2.
\end{aligned}$$

Since $v_0 = 1$, we have $H_0 = 1$,

$$\begin{aligned}
H_{s_{3j+1}}(F(q)) &= (-1)^{\epsilon_{3j+1}} v_1^{s_{3j+1}-s_1} v_2^{s_{3j+1}-s_2} \cdots v_{3j}^{s_{3j+1}-s_{3j}}, \\
&= (-1)^{\epsilon_{3j+1}} \prod_{i=0}^{j-1} v_{3i+1}^{s_{3j+1}-s_{3i+1}} v_{3i+2}^{s_{3j+1}-s_{3i+2}} v_{3i+3}^{s_{3j+1}-s_{3i+3}} \\
&= \xi_3 \prod_{i=0}^{j-1} v_{3i+1}^{(j-i)m} v_{3i+2}^{(j-i)m-1} v_{3i+3}^{(j-i)m-2}
\end{aligned}$$

$$
\begin{aligned}
&= \xi_3 \prod_{i=0}^{j-1} \left(v_{3i+1}v_{3i+2}v_{3i+3})\right)^{(j-i)m} \prod_{i=0}^{j-1} v_{3i+2}^{-1}v_{3i+3}^{-2} \\
&= \xi_3 \prod_{i=0}^{j-1} (t^m)^{(j-i)m} \Big/ \prod_{i=0}^{j-1} \frac{-t^{2m}[i+1]_{t^m}^2}{[i+2]_{t^m}^2} \\
&= (-1)^{1+(j+1)m(m-1)/2} t^{mj(mj+m-4)/2} [j+1]_{t^m}^2,
\end{aligned}
$$

where $\xi_3 = (-1)^{(j+1)(m-3)(m-2)/2}$, and

$$
\begin{aligned}
H_{s_{3j+2}}(F(q)) &= (-1)^{\epsilon_{3j+2}} v_1^{s_{3j+2}-s_1} v_2^{s_{3j+2}-s_2} \cdots v_{3j+1}^{s_{3j+2}-s_{3j+1}}, \\
&= (-1)^{\epsilon_{3j+2}} \prod_{i=0}^{j-1} v_{3i+1}^{s_{3j+2}-s_{3i+1}} v_{3i+2}^{s_{3j+2}-s_{3i+2}} v_{3i+3}^{s_{3j+2}-s_{3i+3}} \times v_{3j+1}^{s_{3j+2}-s_{3j+1}} \\
&= \xi_3 v_{3j+1} \prod_{i=0}^{j-1} v_{3i+1}^{(j-i)m+1} v_{3i+2}^{(j-i)m} v_{3i+3}^{(j-i)m-1} \\
&= \xi_3 v_{3j+1} \prod_{i=0}^{j-1} \left(v_{3i+1}v_{3i+2}v_{3i+3}\right)^{(j-i)m+1} \prod_{i=0}^{j-1} v_{3i+2}^{-1}v_{3i+3}^{-2} \\
&= \xi_3 v_{3j+1} \prod_{i=0}^{j-1} (t^m)^{(j-i)m+1} \Big/ \prod_{i=0}^{j-1} \frac{-t^{2m}[i+1]_{t^m}^2}{[i+2]_{t^m}^2} \\
&= (-1)^{m-1+(j+1)m(m-1)/2} R(j) t^{mj(mj+m-2)/2},
\end{aligned}
$$

and

$$
\begin{aligned}
H_{s_{3j+3}}(F(q)) &= (-1)^{\epsilon_{3j+3}} v_1^{s_{3j+3}-s_1} v_2^{s_{3j+3}-s_2} \cdots v_{3j+2}^{s_{3j+3}-s_{3j+2}}, \\
&= (-1)^{\epsilon_{3j+3}} \prod_{i=0}^{j-1} v_{3i+1}^{s_{3j+3}-s_{3i+1}} v_{3i+2}^{s_{3j+3}-s_{3i+2}} v_{3i+3}^{s_{3j+3}-s_{3i+3}} \\
&\quad \times v_{3j+1}^{s_{3j+3}-s_{3j+1}} v_{3j+2}^{s_{3j+3}-s_{3j+2}} \\
&= \xi_3 v_{3j+1}^2 v_{3j+2} \prod_{i=0}^{j-1} \left(v_{3i+1}v_{3i+2}v_{3i+3}\right)^{(j-i)m+2} \prod_{i=0}^{j-1} v_{3i+2}^{-1}v_{3i+3}^{-2} \\
&= \xi_3 \frac{[j+2]_{t^m}^2}{[j+1]_{t^m}^2} \prod_{i=0}^{j-1} (t^m)^{(j-i)m+2} \Big/ \prod_{i=0}^{j-1} \frac{-t^{2m}[i+1]_{t^m}^2}{[i+2]_{t^m}^2} \\
&= (-1)^{(j+1)m(m-1)/2} t^{m^2 j(j+1)/2} [j+2]_{t^m}^2. \qquad \square
\end{aligned}
$$

5 Basic transformations for quadratic power series

Lemma 16. *Let A, B, C, U be polynomials. Suppose F is a quadratic power series satisfying*

$$0 = A + BF + CF^2.$$

Then the series UF, $F+U$, and F^n are also quadratic power series, and they satisfy

$$\begin{aligned} 0 &= AU^2 + BU(UF) + C(UF)^2, \\ 0 &= (A - BU + CU^2) + (B - 2CU)(F+U) + C(F+U)^2, \\ 0 &= A^n + (-1)^{n+1} L_n(B, -AC) F^n + C^n F^{2n}. \end{aligned}$$

Proof. The initial two situations, UF and $F+U$, are straightforward, so we focus on the case $G = F^n$.

Begin with the special case $A = C = 1$, and assume

$$0 = 1 + BF + F^2, \tag{14}$$
$$0 = 1 + B_n F^n + F^{2n}. \tag{15}$$

Clearly, $B_0 = -2$ and $B_1 = B$. From (14) and (15) we derive

$$\begin{aligned} BB_n &= \left(\frac{1}{F} + F\right)\left(\frac{1}{F^n} + F^n\right) \\ &= \left(\frac{1}{F^{n+1}} + F^{n+1}\right) + \left(F^{n-1} + \frac{1}{F^{n-1}}\right) \\ &= -B_{n+1} - B_{n-1}. \end{aligned}$$

Comparing this with (9), we obtain

$$\begin{aligned} B_n &= (-1)^{n+1} L_n(B, -1) \\ &= (-1)^{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k B^{n-2k}. \end{aligned} \tag{16}$$

Now consider the general case, where

$$\begin{aligned} 0 &= A + BF + CF^2, \\ 0 &= A_n + B_n F^n + C_n F^{2n}. \end{aligned} \tag{17}$$

Set $F = \sqrt{A/C}\, G$. Then the first relation becomes

$$\begin{aligned} 0 &= A + B\sqrt{\frac{A}{C}}\, G + C\left(\sqrt{\frac{A}{C}}\, G\right)^2, \\ 0 &= 1 + \frac{B}{\sqrt{AC}}\, G + G^2. \end{aligned}$$

By (16) it follows that

$$0 = 1 + \bar{B}_n G^n + G^{2n}, \tag{18}$$

where $\bar{B}_0 = -2$ and

$$\bar{B}_n = (-1)^{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k \left(\frac{B}{\sqrt{AC}}\right)^{n-2k}.$$

On the other hand, from (18) we get

$$
\begin{aligned}
0 &= 1 + \bar{B}_n \left(\sqrt{\frac{C}{A}}\right)^n F^n + \left(\frac{C}{A}\right)^n F^{2n} \\
&= A^n + \bar{B}_n(\sqrt{AC})^n F^n + C^n F^{2n}.
\end{aligned}
$$

Comparing this with (17), we identify $A_n = A^n$, $C_n = C^n$, and

$$
\begin{aligned}
B_n &= \bar{B}_n(\sqrt{AC})^n \\
&= (-1)^{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k \left(\frac{B}{\sqrt{AC}}\right)^{n-2k} (\sqrt{AC})^n \\
&= (-1)^{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k B^{n-2k} (AC)^k \\
&= (-1)^{n+1} L_n(B, -AC).
\end{aligned}
$$

□

Corollary 17. *The power series*

$$
F(q) = \frac{(\gamma(t,q)-1)^m}{q^{m_0}}
$$

satisfies the following quadartic equation

$$
0 = -q^{m-m_0} + \beta(m;t,q)F - t^m q^{m+m_0} F^2.
$$

Proof. Invoking Lemma 16 in the case $F+U$ and applying it to (3), we obtain

$$
-q + (1-q-tq)\left(\gamma(t,q)-1\right) - tq\left(\gamma(t,q)-1\right)^2 = 0.
$$

Next, applying Lemma 16 once more, now in the case F^n, and using Lemma 9, we deduce

$$
-q^m + \beta(m;t,q)(\gamma(t,q)-1)^m - (tq)^m \left(\gamma(t,q)-1\right)^{2m} = 0.
$$

The statement of the corollary follows by one further application of the same lemma, this time in the case FU. □

6 Hankel continued fraction of quadratic series

To derive the Hankel continued fraction of a quadratic series, we recall the main idea presented in [17, 19]. Suppose $F(q)$ is a power series that satisfies the quadratic equation

$$
A(q) + B(q)F(q) + C(q)F(q)^2 = 0.
$$

Then $F(q)$ can be expressed in the form

$$
F(q) = \frac{-aq^k}{D(q) - q^{k+\delta}G(q)},
$$

where $G(q)$ is another power series that satisfies a transformed quadratic equation

$$A^*(q) + B^*(q)G(q) + C^*(q)G(q)^2 = 0.$$

Moreover, the quantities A^*, B^*, C^*, k, a, and D can be computed explicitly by means of the following algorithm.

Algorithm `[NextABC]` (for $\delta = 2$)

Prototype: $(A^*, B^*, C^*; k, a, D) = \texttt{NextABC}(A, B, C)$

Input: $A(q), B(q), C(q) \in \mathbb{Q}[q]$ three polynomials such that $B(0) = 1$, $C(0) = 0$, $C(q) \neq 0, A(q) \neq 0$;

Output: $A^*(q), B^*(q), C^*(q) \in \mathbb{Q}[q]$, $k \in \mathbb{N}^+$, $a \neq 0 \in \mathbb{Q}$, $D(q) \in \mathbb{Q}[q]$ a polynomial of degree less than or equal to $k+1$ such that $D(0) = 1$.

Step 1 [Define k, a]. Since $A(q) \neq 0$, let $A(q) = aq^k + O(q^{k+1})$ with $a \neq 0$.

Step 2 [Define D]. Define $D(q)$ by

$$\frac{aq^k B}{A} - \frac{aq^k C}{B} = D(q) + O(q^{k+2}), \tag{19}$$

where $D(q)$ is a polynomial of degree less than or equal to $k+1$ such that $D(0) = 1$.

Step 3. [Define A^*, B^*, C^*]. Let

$$A^*(q) = \left(-D^2 A/a + BDq^k - Caq^{2k}\right)/q^{2k+2}; \tag{20}$$

$$B^*(q) = 2AD/(aq^k) - B; \tag{21}$$

$$C^*(q) = -Aq^2/a. \tag{22}$$

Remark. In Step 2, if $C = q^2 C'$, then

$$\frac{aq^k B}{A} = D(q) + O(q^{k+2}).$$

Similarly, if $A = aq^k$, then

$$B = D(q) + O(q^{k+2}).$$

By repeatedly applying Algorithm `NextABC`, we generate a sequence of six-tuples

$$(A_{n+1}, B_{n+1}, C_{n+1}; k_n, a_n, D_n), \tag{23}$$

satisfying, for each n,

$$(A_{n+1}, B_{n+1}, C_{n+1}; k_n, a_n, D_n) = \texttt{NextABC}(A_n, B_n, C_n). \tag{24}$$

This yields an H-fraction expansion of $F(q)$ of the form

$$F(q) = \mathcal{H}((k_j), (-a_j), (u_j)),$$

where u_j is determined by the relation $1 + u_j(q)q = D_{j-1}$.

We will employ this method to establish the H-fraction representations stated in Section 3. To do this, we first conjecture the sequence (23) from computational data obtained experimentally, and then confirm its validity by verifying the relations (24). In other words, we must check the three steps of the algorithm.

7 Proof of Lemma 14

For $m \geqslant 2$, let

$$F(q) = \frac{(\gamma(t,q)-1)^m}{q^2}.$$

By Corollary 17, we have

$$0 = -q^{m-2} + \beta(m,t,q)F - t^m q^{m+2} F^2.$$

Thus, we run Algorithm `NextABC` starting from the initialization

$$A_0 = -q^{m-2}, \quad B_0 = \beta(m;t,q), \quad C_0 = -t^m q^{m+2}.$$

Lemma 18. *If we use A_n, B_n, C_n as the input to Algorithm* `NextABC`*, then the outputs are*

$$A_{n+1}, B_{n+1}, C_{n+1}, k_n, a_n, D_n,$$

where $A_n, B_n, C_n, k_n, a_n, D_n$ for $n \geqslant 0$ are defined as follows:

$$\begin{aligned}
A_0 &= -q^{m-2}, \qquad A_{2j} = \frac{(-t)^m q^{m-2} J_0}{J_1}, \\
A_{2j+1} &= \frac{(-1)^m J_2}{J_1}\beta(m;t,q) + (\frac{1-2J_2}{J_1} - \frac{t^{m(j+1)}J_2}{J_1^2})q^m, \\
B_{2j} &= \beta(m;t,q) + 2(\frac{1}{J_1} - 1)(-q)^m, \\
B_{2j+1} &= \beta(m;t,q) - 2(\frac{t^{m(j+1)}}{J_1} + 1)(-q)^m, \\
C_0 &= -t^m q^{m+2}, \qquad C_{2j+1} = -q^m, \\
C_{2j} &= -\beta(m;t,q)q^2 - (\frac{1}{J_1} - \frac{t^{mj}}{J_0} - 2)(-q)^{m+2} \\
k_{2j} &= m-2, \qquad k_{2j+1} = 0, \\
a_0 &= -1, \qquad a_{2j} = \frac{(-t)^m J_0}{J_1}, \qquad a_{2j+1} = \frac{(-1)^m J_2}{J_1}, \\
D_{2j} &= \beta(m;t,q) - (-q)^m(1+t^m), \qquad D_{2j+1} = 1,
\end{aligned}$$

where, for brevity, we denote

$$J_0 = [j]_{t^m}, \qquad J_1 = [j+1]_{t^m}, \qquad J_2 = [j+2]_{t^m}.$$

Proof. The values of k_j and a_j are straightforward to determine. For D_j, we distinguish two cases.

In the even case, we have $k_{2j} = m-2$. Since $C_n = O(q^2)$, it follows that

$$\frac{a_{2j} q^k B_{2j}}{A_{2j}} = D_{2j}(q) + O(q^m),$$

which implies

$$B_{2j} = D_{2j}(q) + O(q^m).$$

Thus, D_{2j} is simply the polynomial obtained from B_{2j} by truncating its q-expansion at order q^{m-1}.

In the odd case, we have $k_{2j+1} = 0$, and

$$\frac{a_{2j+1}B_{2j+1}}{A_{2j+1}} = D_{2j+1} + O(q^2).$$

Hence, $D_{2j+1} = 1$.

The remaining three relations, (20), (21), and (22), are verified using a computer algebra system. □

8 Proof of Lemma 15

For $m \geqslant 3$, let

$$F(q) = \frac{(\gamma(t,q) - 1)^m}{q^3}.$$

By Corollary 17, we have

$$0 = -q^{m-3} + \beta(m,t,q)F - t^m q^{m+3} F^2$$

So we run Algorithm `NextABC` starting from the initial values

$$A_0 = -q^{m-3}, \quad B_0 = \beta(m;t,q), \quad C_0 = -t^m q^{m+3}.$$

Lemma 19. *If we input A_n, B_n, C_n into Algorithm* `NextABC`*, then the algorithm produces*

$$A_{n+1}, B_{n+1}, C_{n+1}, k_n, a_n, D_n,$$

where, for $n \geqslant 0$, the sequences $A_n, B_n, C_n, k_n, a_n, D_n$ are defined as follows:

$$
\begin{aligned}
A_0 &= -q^{m-3}, \qquad A_{3j} = (-t)^m q^{m-3} R(j-1) J_1^{-2}, \\
A_{3j+1} &= \alpha(q) \left(u(j) + (-1)^m J_2 J_1^{-1} q\right) \\
&\quad - \left((-1)^{m+1} u(j) + w_3(j) q + t^{mj} q^2\right) t^m q^{m-1} J_1^{-2} R(j-1), \\
A_{3j+2} &= \frac{-v(j)^2 C_{3j+3}}{q^2}, \\
B_{3j} &= \alpha(q) + w_2(j), \qquad B_{3j+1} = \alpha(q) - w_2(j), \\
B_{3j+2} &= 2(1 + qv(j)) w_1(j) - \alpha(q) + w_2(j), \\
C_0 &= -t^m q^{m+3}, \\
C_{3j} &= -\alpha(q) q^2 (1 + v(j-1)q) - u(j) q^{m+1} \\
&\quad + w_3(j)(-q)^{m+2} - t^{mj} R(j-1)^{-1} (-q)^{m+3}, \\
C_{3j+1} &= -q^{m-1}, \qquad C_{3j+2} = -q^2 w_1(j), \\
k_{3j} &= m-3, \qquad k_{3j+1} = k_{3j+2} = 0, \\
a_0 &= -1, \quad a_{3j} = (-t)^m R(j-1) J_1^{-2}, \\
a_{3j+1} &= u(j), \qquad a_{3j+2} = v(j)^2, \\
D_{3j} &= \alpha(q), \qquad D_{3j+1} = 1 + qv(j), \qquad D_{3j+2} = 1 - qv(j),
\end{aligned}
$$

where, for brevity, we define

$$
\begin{aligned}
u(j) &= \frac{(-1)^{m+1} R(j)}{J_1^2}, \qquad v(j) = \frac{J_1 J_2}{R(j)}, \qquad w_1(j) = \frac{A_{3j+1}}{a_{3j+1}}, \\
w_2(j) &= (-q)^{m-1} S(j) J_1^{-2} + (-q)^m (1 + t^{(j+1)m}) J_1^{-1}, \\
w_3(j) &= m[m]_t v(j-1) - (1 + t^{m(j+1)}) J_1^{-1}.
\end{aligned}
$$

Proof. Turn $(3j)$. From A_0 we derive the formulas for k_0 and a_0. Using (19), we obtain $B_0 = D_0 + O(q^{m-1})$, which validates the expression of D_0. For $j \geqslant 1$, A_{3j} yields the expressions for k_{3j} and a_{3j}. Again, by (19), we have $B_{3j} = D_{3j} + O(q^{m-1})$, which confirms the expression of D_{3j}.

Turn $(3j+1)$. From the explicit forms of A_{3j+1} and B_{3j+1}, we obtain $k_{3j+1} = 0$ and $a_{3j+1} = u(j)$. Define

$$
w_1(j) = \frac{A_{3j+1}}{a_{3j+1}} = \alpha(q)\big(1 - v(j)q\big) + O(q^{m-1}).
$$

Consequently,

$$
\frac{B_{3j+1}}{w_1(j)} = \frac{1}{1 - v(j)q} + O(q^{m-1}).
$$

Since $m \geqslant 3$, we have $C_{3j+1} = O(q^2)$, which implies $D_{3j+1} = 1 + qv(j)$.

Turn $(3j+2)$. We obtain $k_{3j+2} = 0$ and $a_{3j+2} = v(j)^2$. Because $C_{3j+2} = O(q^2)$, we get

$$\begin{aligned} B_{3j+2}(q) &= 2D_{3j+1}w_1(j) - B_{3j+1} \\ &= 2(1+v(j)q)\alpha(q)(1-v(j)q) - \alpha(q) + O(q^2) \\ &= 1 - m(t+1)q + O(q^2). \end{aligned}$$

Thus,

$$\frac{a_{3j+2}B_{3j+2}}{A_{3j+2}} = 1 - v(j)q + O(q^2).$$

Hence $D_{3j+2} = 1 - qv(j)$. The remaining three relations (20), (21), and (22) are verified using a computer algebra system. □

9 Proof of Lemmas 12 and 13

In this section, we obtain Lemma 12 as a consequence of Lemma 14, and we establish Lemma 13 by a direct calculation.

Proof of Lemma 12. For $m \geqslant 0$, define $F(q) = (\gamma(t,q) - 1)^m$. By Corollary 17, we have

$$0 = -q^m + \beta(m;t,q)\,F - t^m q^m F^2.$$

Hence

$$F = \frac{q^m}{\beta(m;t,q) - t^m q^m (\gamma(t,q)-1)^m} = \frac{q^m}{\beta(m;t,q) - t^m q^{m+2} \frac{(\gamma(t,q)-1)^m}{q^2}}.$$

Since the H-fraction of $(\gamma(t,q)-1)^m/q^2$ is determined in Lemma 14, this directly yields the H-fraction expansion of F. □

Proof of Lemma 13. Set

$$F = \frac{(\gamma(t,q)-1)^m}{q}.$$

By Corollary 17, we obtain

$$F = \frac{q^{m-1}}{\beta(m;t,q) - t^m q^{m+1} F}.$$

This identity provides the H-fraction representation of F. □

References

[1] Martin Aigner. Catalan-like numbers and determinants. *J. Combin. Theory Ser. A*, 87(1):33–51, 1999.

[2] Marc Chamberland and Christopher French. Generalized Catalan numbers and generalized Hankel transformations. *J. Integer Seq.*, 10(1):Article 07.1.1, 7, 2007.

[3] Xiang-Ke Chang, Xing-Biao Hu, and Ying-Nan Zhang. A direct method for evaluating some nice Hankel determinants and proofs of several conjectures. *Linear Algebra Appl.*, 438(5):2523–2541, 2013.

[4] J. Cigler and C. Krattenthaler. Hankel determinants of linear combinations of moments of orthogonal polynomials. *Int. J. Number Theory*, 17(2):341–369, 2021.

[5] Johann Cigler. Some nice Hankel determinants. *arXiv*, 1109.1449, 2011.

[6] Johann Cigler. A special class of Hankel determinants. *arXiv*, 1302.4235, 2013.

[7] Johann Cigler. Catalan numbers, Hankel determinants and Fibonacci polynomials. *arXiv*, 1801.05608, 2018.

[8] Johann Cigler. Some observations about determinants which are connected with Catalan numbers and related topics. *arXiv*, 1902.10468, 2019.

[9] Johann Cigler. Some experimental observations about Hankel determinants of convolution powers of Catalan numbers. *arXiv*, 2308.07642, 2023.

[10] Johann Cigler. Hankel determinants of convolution powers of Catalan numbers revisited. *arXiv*, 2403.11244, 2024.

[11] Aleksandar Cvetković, Predrag Rajković, and Milo ˇ s Ivković. Catalan numbers, and Hankel transform, and Fibonacci numbers. *J. Integer Seq.*, 5(1):Article 02.1.3, 8, 2002.

[12] Myriam de Sainte-Catherine and Gérard Viennot. Enumeration of certain Young tableaux with bounded height. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 58–67. Springer, Berlin, 1986.

[13] Ömer Eğecioğlu. A Catalan-Hankel determinant evaluation. In *Proceedings of the Fortieth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 195, pages 49–63, 2009.

[14] Ömer Eğecioğlu, Timothy Redmond, and Charles Ryavec. Almost product evaluation of Hankel determinants. *Electron. J. Combin.*, 15(1):#R6, 58, 2008.

[15] Ira M. Gessel. The Methode of Coefficients. *slide available on the author's personal homepage*, 2008.

[16] Ira M. Gessel and Guoce Xin. The generating function of ternary trees and continued fractions. *Electron. J. Combin.*, 13(1):#R53, 48, 2006.

[17] Guo-Niu Han. Hankel continued fraction and its applications. *Adv. Math.*, 303:295–321, 2016.

[18] Guo-Niu Han. Hankel continued fractions and Hankel determinants of the Euler numbers. *Trans. Amer. Math. Soc.*, 373(6):4255–4283, 2020.

[19] Guo-Niu Han and Emmanuel Pedon. Hankel continued fractions and hankel determinants for q-deformed metallic numbers. *arxiv*, 2502.05993, 2025.

[20] C. Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.*, 42:Art. B42q, 67 pp., 1999. `http://www.mat.univie.ac.at/~slc/`.

[21] C. Krattenthaler. Advanced determinant calculus: a complement. *Linear Algebra Appl.*, 411:68–166, 2005.

[22] C. Krattenthaler. Determinants of (generalised) Catalan numbers. *J. Statist. Plann. Inference*, 140(8):2260–2270, 2010.

[23] C. Krattenthaler. Hankel determinants of linear combinations of moments of orthogonal polynomials, II. *Ramanujan J.*, 61(2):597–627, 2023.

[24] Predrag M. Rajković, Marko D. Petković, and Paul Barry. The Hankel transform of the sum of consecutive generalized Catalan numbers. *Integral Transforms Spec. Funct.*, 18(3-4):285–296, 2007.

[25] Ulrich Tamm. Some aspects of Hankel matrices in coding theory and combinatorics. *Electron. J. Combin.*, 8(1):#A1, 31, 2001.

[26] Ying Wang and Guoce Xin. Hankel determinants for convolution powers of Catalan numbers. *Discrete Math.*, 342(9):2694–2716, 2019.