Clique number of xor-powers of Kneser graphs

Zoltán Füredi[a] András Imolay[b] Ádám Schweitzer[c]

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Abstract

Let $f_\ell(n,k)$ denote the clique number of the xor-product of ℓ isomorphic Kneser graphs $KG(n,k)$. Alon and Lubetzky investigated the case of complete graphs as a coding theory problem and showed $f_\ell(n,1) \leqslant \ell n+1$. Imolay, Kocsis, and Schweitzer proved that $f_2(n,k) \leqslant \lfloor \frac{n}{k} \rfloor + c(k)$.

Here, the order of magnitude of $c(k)$ is determined to be $\Theta\left(k\binom{2k}{k}\right)$. By explicit constructions and by an algebraic proof, it is shown that $\ell n - 2\ell - 1 \leqslant f_\ell(n,1) \leqslant \ell n - \ell + 1$ (for all $n \geqslant 1$ and $\ell \geqslant 3$). Finally, it is proved that the order of magnitude of f lies between $\Omega\left(n^{\lfloor \log_2(\ell+1) \rfloor}\right)$ and $O\left(n^{\lfloor \frac{\ell+1}{2} \rfloor}\right)$ (as ℓ, k are given and $n \to \infty$).

We conjecture that the lower bound gives the correct exponent.

Mathematics Subject Classifications: 05D05, 05C69, 05C76

1 Introduction

1.1 Kneser graphs

A Kneser graph $G := KG(A,k)$ has a *base* set A, the vertex set of G consists of all subsets of k elements of A. We denote this as $V(G) := \binom{A}{k}$, and a pair $\{X,Y\}$ forms an edge of G when $X \cap Y = \emptyset$. We also use $KG(n,k)$ for a Kneser graph with an n-element base set. A complete subgraph in the Kneser graph corresponds to a family of mutually disjoint k-element sets in the base set A. So, the size of the largest clique $\omega(KG(n,k)) = \lfloor n/k \rfloor$.

The parameters of Kneser graphs are widely studied in combinatorics. Lovász [13] determined the chromatic number of Kneser graphs and Erdős, Ko and Rado [7] determined their independence number. Brešar and Valencia-Pabon [5] examined the independence

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[a] Alfréd Rényi Institute for Mathematics, Reáltanoda u. 13–15, H-1364 Budapest, Hungary (`furedi.zoltan@renyi.hu`).

[b] Eötvös Loránd University, H-1117 Budapest, Pázmány Péter sétány 1/C (`andras.imolay@ttk.elte.hu`).

[c] KTH Royal Institute of Technology, Kungliga Tekniska högskolan, 100 44 Stockholm (`adasch@kth.se`).

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number of Kneser graphs of different graph products. In this article, we study the clique number of the xor-products.

1.2 The xor-product

Given two graphs $G=(V(G),E(G))$ and $H=(V(H),E(H))$, their *xor-product* $G\cdot H$ is a graph with the vertex set $V(G)\times V(H)$ and two vertices (g,h) and (g',h') are connected in $G\cdot H$ if and only if either $gg'\in E(G)$ and $hh'\notin E(H)$ or $gg'\notin E(G)$ and $hh'\in E(H)$. The xor-product is not as well understood as other graph products, but there are a number of highly non-trivial results about it, e.g., by Alon and Lubetzky [2] and Thomason [15]. They were also motivated to compare it to the Shannon capacity of graphs, see Alon and Lubetzky [3], Lovász [14]. Let $f_\ell(n,k)$ denote the clique number of the xor-product of ℓ isomorphic Kneser graphs $KG(n,k)$.

Taking a clique $C\subset V(G)$ and a vertex $b\in V(H)$ the set $C\times\{b\}$ forms a clique in $G\cdot H$ so we obtain $\omega(G\cdot H)\geqslant\max\{\omega(G),\omega(H)\}$. Hence

$$\omega\left(KG(n,k)\cdot KG(n,k)\right)\geqslant\lfloor n/k\rfloor.$$

Imolay, Kocsis, and Schweitzer [11] showed that the function $f_2(n,k)-\lfloor n/k\rfloor$ is bounded for any given k. Define

$$c(k):=\sup_{n\geqslant k}\left\{f_2(n,k)-\left\lfloor\frac{n}{k}\right\rfloor\right\}.$$

Our first result determines the order of magnitude of $c(k)$ as $k\to\infty$.

Theorem 1. *For all $k\geqslant 1$ and $n\geqslant\frac{1}{2}\left(\binom{2k}{k}-2\right)k^2$,*

$$f_2(n,k)\geqslant\left\lfloor\frac{n}{k}\right\rfloor+\binom{2k}{k}\frac{k}{2}-k. \tag{1.1}$$

On the other hand, as $k\to\infty$ we have

$$c(k)\leqslant(1+o(1))k\binom{2k}{k}. \tag{1.2}$$

The proof of Theorem 1 is presented in Section 2. It might be easier to determine $f_2(n,k)$ for large n. In [11] it was proved that $f_2(n,2)=\lfloor n/2\rfloor+4$ for sufficiently large n. Let us define

$$c_\infty(k):=\limsup_{n\to\infty}\left\{f_2(n,k)-\left\lfloor\frac{n}{k}\right\rfloor\right\}.$$

We have $c_\infty(1)=c(1)=0$, $c_\infty(2)=4$, and in general the true order of magnitudes

$$\binom{2k}{k}\frac{k}{2}-k\leqslant c_\infty(k)\leqslant c(k)\leqslant(1+o(1))k\binom{2k}{k}.$$

We conjecture that here the equality holds for $c_\infty(k)$ for all $k\geqslant 1$ and the best construction is the one from Section 2.1.

Conjecture 2. For all k, if n is large enough then

$$f_2(n,k) = \left\lfloor \frac{n}{k} \right\rfloor + \binom{2k}{k}\frac{k}{2} - k.$$

Maybe more is true and $c_\infty(k) = c(k)$ for all k.

1.3 Multiproducts of the complete graphs

The rest of our article tackles the question of higher powers of Kneser graphs. We investigate $f_\ell(n,k)$ the clique number of the ℓ-th xor-power (the xor-product of ℓ isomorphic copies) of the Kneser graph $KG(n,k)$.

Even the case $k = 1$ is not trivial when $\ell \geqslant 3$. Note that $KG(n,1)$ is the complete graph on n vertices. The function $f_\ell(n,1)$ was considered by Alon and Lubetzky in [2]. They proved an upper bound $f_\ell(n,1) \leqslant \ell n + 1$. Here we give tighter bounds.

Theorem 3. *For all $\ell \geqslant 3$ and $n \geqslant 1$*

$$\ell n - 2\ell - 1 \leqslant f_\ell(n,1) \leqslant \ell n - \ell + 1.$$

1.4 Higher powers of Kneser graphs

We give bounds for the magnitude of $f_\ell(n,k)$ for large n. In particular, we show that it is not necessarily linear in n.

Theorem 4. *We have*

$$f_\ell(n,k) \leqslant 2^{\lfloor \frac{\ell}{2} \rfloor} \cdot \left\lfloor \frac{\ell}{2} \right\rfloor ! \cdot n^{\lfloor \frac{\ell+1}{2} \rfloor}. \tag{1.3}$$

On the other hand, if $k \geqslant \lfloor \log_2(\ell+1) \rfloor$, then

$$f_\ell(n,k) \geqslant \left(\left\lfloor \frac{n}{k} \right\rfloor \right)^{\lfloor \log_2(\ell+1) \rfloor}. \tag{1.4}$$

This settles the exact magnitude for the cases $\ell \leqslant 4$. We conjecture that the lower bound has the correct order of magnitude.

Conjecture 5. For any fixed ℓ and k if k is large enough then

$$f_\ell(n,k) = \Theta(n^{\lfloor \log_2(\ell+1) \rfloor}).$$

1.5 Semi-intersecting families

The vertex set of a Kneser graph $KG(A,k)$ is a k-uniform hypergraph. Given ℓ Kneser graphs $KG(A_i,k)$ $(1 \leqslant i \leqslant \ell)$ with pairwise disjoint n-element base sets $A_1, \dots, A_\ell$ the vertices of their xor-product are naturally correspond to those $k\ell$-element subsets S of $A_1 \cup A_2 \cup \ldots \cup A_\ell$ where $|S \cap A_i| = k$ for each i. The set pair $\{S, S'\}$ corresponds to an edge in the xor-product $KG(A_1,k) \cdot \ldots \cdot KG(A_\ell,k)$ if $S \cap S' \cap A_i = \emptyset$ in an odd number of cases for $1 \leqslant i \leqslant \ell$.

Definition 6. A family of sets $\mathcal{S}$ on the pairwise disjoint base sets $A_1 \cup A_2 \cup \ldots \cup A_\ell$ is called an *ℓ-semi-intersecting family* with parameters n and k if

- $|A_1| = |A_2| = \ldots = |A_\ell| = n$,
- $|S \cap A_1| = |S \cap A_2| = \ldots = |S \cap A_\ell| = k$ for each $S \in \mathcal{S}$, and
- for distinct S, $T \in \mathcal{S}$, we have $S \cap T \cap A_i = \emptyset$ for an odd number of i's, $1 \leqslant i \leqslant \ell$.

There is a one-to-one correspondence between ℓ-semi-intersecting families and cliques in $KG(n,k)^\ell$. Hence $f_\ell(n,k) = \omega\left(KG(n,k)^\ell\right)$ is the maximum size of an ℓ-semi-intersecting family with parameters n and k.

We prefer to work with this equivalent hypergraph reformulation. Similar questions in extremal combinatorics with two part set systems were studied extensively, see, e.g., [10] and [12].

2 Determining the order of magnitude of $c(k)$

In this section we prove the bounds stated in Theorem 1. In this case $\ell = 2$, we use simply semi-intersecting instead of 2-semi-intersecting, and we denote the base sets A_1, A_2 by A and B.

2.1 Lower bound construction

Here we give a construction yielding (1.1).

Proof. For easier notation introduce $m := \frac{1}{2}\binom{2k}{k}$. Choose a subset of $K \subset A$ with $|K| = 2k$. Label the k-element subsets of K by

$$H_1, H_2, \ldots, H_m, G_1, G_2, \ldots, G_m$$

such that H_i and G_i are disjoint $(1 \leqslant i \leqslant m)$. Let $L_2, L_3, \ldots, L_m$ be pairwise disjoint k^2-element subsets of B. This is possible as $n \geqslant \frac{1}{2}\left(\binom{2k}{k} - 2\right)k^2$. Let us arrange the elements of L_i to a $k \times k$ rectangular point lattice. There are $n - (m-1)k^2$ elements of $B \setminus \bigcup L_i$, so we can select additional pairwise disjoint k-element subsets $F_1, F_2, \ldots, F_d$ from them, where $d = \left\lfloor \frac{n}{k} \right\rfloor - (m-1)k$.

Define a semi-intersecting family $\mathcal{S}$ as follows. We let $S \subset A \cup B$ in $\mathcal{S}$ if one of the following holds.

- $S \cap A = H_i$ and $S \cap B$ corresponds to a row of the lattice in L_i,
- $S \cap A = G_i$ and $S \cap B$ corresponds to a column of the lattice in L_i,
- $S \cap A = H_1$ and $S \cap B = F_i$ for some $1 \leqslant i \leqslant d$.

This $\mathcal{S}$ is a semi-intersecting family with parameters n and k, because if $S, T \in \mathcal{S}$ with $S \neq T$, and they are disjoint in A then $\{S \cap A, T \cap A\} = \{H_i, G_i\}$ for some $2 \leqslant i \leqslant m$. On the other hand, this is the only case when S and T intersect in B, as they intersect only if $S \cap B$ is a row (or column) of some L_i and $T \cap B$ is a column (or row) of the same L_i.

It remains to count the members of $\mathcal{S}$. There are $(m-1)k$ sets from both the first and second bullet point, and d from the third one. Hence

$$|\mathcal{S}| = 2(m-1)k + d = \left(\binom{2k}{k} - 2\right)k + \left\lfloor \frac{n}{k} \right\rfloor - \frac{\binom{2k}{k} - 2}{2}k = \left\lfloor \frac{n}{k} \right\rfloor + \binom{2k}{k}\frac{k}{2} - k.$$

□

2.2 Cross intersecting matchings

A set of hypergraphs $\mathcal{A}_1, \ldots, \mathcal{A}_t$ is called a k-uniform *cross intersecting* matching of size t and of type $(d_1, \ldots, d_t)$ if $t, k \geqslant 2$, $d_i \geqslant 2$ for $1 \leqslant i \leqslant t$, each $\mathcal{A}_i$ consists of d_i pairwise disjoint k-element sets and $X \cap Y \neq \emptyset$ for $X \in \mathcal{A}_i$, $Y \in \mathcal{A}_j$ whenever $1 \leqslant i \neq j \leqslant t$. Obviously, $d_i \leqslant k$ for every i. The classical set-pair theorem of Bollobás [4] implies that $t \leqslant \frac{1}{2}\binom{2k}{k}$ and here equality holds only if $d_1 = \ldots = d_t = 2$ and each $\mathcal{A}_i$ consists of a complementary pair of k-sets of a base set L, $|L| = 2k$. There are many generalizations of Bollobás's theorem, see, e.g., Alon [1] where the exterior algebra method was introduced. Even the case of 2-independent d-partitions is highly non-trivial, i.e., when $\cup\mathcal{A}_i$ is the same dk-element set for all i. For this case Gargano, Körner, and Vaccaro [9] showed that for any fixed d one can have $t = \Omega(4^{k(1-o(1))})$ using their Sperner capacity method in information theory. Here we show an upper bound.

Lemma 7. *There exists a sequence $\gamma(2), \gamma(3), \ldots$ with $\gamma(k) \to 0$ exponentially as $k \to \infty$, such that for every k-uniform cross intersecting matching of type $(d_1, \ldots, d_t)$,*

$$\sum_i d_i(d_i - 1) \leqslant (1 + \gamma(k))\binom{2k}{k}. \tag{2.1}$$

We *conjecture* that the true value of γ is zero, for all k.

Proof. Define $A := \cup_i(\cup\mathcal{A}_i)$, $n := |A|$. Given any ordering π of A and two non-empty subsets $X, X' \subset A$, we say that $X <_\pi X'$ if $\pi(x) < \pi(x')$ for all $x \in X$ and $x' \in X'$. We call π of *type* i if there are two sets $X, X' \in \mathcal{A}_i$ with $X <_\pi X'$. Every permutation π can have only at most one type. Indeed, if $X_i, X_i' \in \mathcal{A}_i$ with $X_i <_\pi X_i'$ and $X_j, X_j' \in \mathcal{A}_j$ with $X_j <_\pi X_j'$ then the cross intersection property implies that there are elements $u \in X_i \cap X_j'$ and $v \in X_i' \cap X_j$. From $X_i <_\pi X_i'$ we get $\pi(u) < \pi(v)$ and from $X_j <_\pi X_j'$ we get $\pi(v) < \pi(u)$, a contradiction.

Consider a uniform probability distribution on the $n!$ possible orderings of A. Let E_i denote the event that the random variable π is of type i. We have $\sum_i \Pr(E_i) \leqslant 1$ because for $i \neq j$ the events E_i and E_j cannot occur simultaneously. Fix i, our goal is to approximate $\Pr(E_i)$. To simplify the presentation we leave out the index i from d_i in the following calculation. From now on, let $X_1, X_2, \dots, X_{d_i}$ denote the members of $\mathcal{A}_i$.

Let $O_{\alpha,\beta}$ be the event that $X_\alpha < X_\beta$. There are $d(d-1)$ such events, because $1 \leqslant \alpha, \beta \leqslant d$ and $\alpha \neq \beta$. Since $E_i = \bigcup_{\alpha\neq\beta} O_{\alpha,\beta}$ the inclusion-exclusion principle yields

$$\Pr(E_i) \geqslant \sum_{\alpha\neq\beta} \Pr(O_{\alpha,\beta}) - \frac{1}{2} \sum_{\alpha_1\neq\beta_1,\alpha_2\neq\beta_2,(\alpha_1,\beta_1)\neq(\alpha_2,\beta_2)} \Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2}).$$

In the first sum, $\Pr(O_{\alpha,\beta}) = \frac{1}{\binom{2k}{k}}$, because this is the probability that π arranges the elements of $X_\alpha \cup X_\beta$ such that $X_\alpha < X_\beta$. Now we calculate $\Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2})$ for all possible $\alpha_1, \beta_1, \alpha_2, \beta_2$. We categorize them into six groups.

— $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$. In this case

$$\Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2}) = 0.$$

— The numbers $\alpha_1, \beta_1, \alpha_2$ and β_2 are all distinct. In this case O_{α_1,β_1} and O_{α_2,β_2} are independent events, hence

$$\Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2}) = \frac{1}{\binom{2k}{k}\binom{2k}{k}}.$$

Here there are $d(d-1)(d-2)(d-3)$ possibilities for $\alpha_1, \beta_1, \alpha_2, \beta_2$.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_2 = \beta_1$. In this case $X_{\alpha_1} < X_{\beta_1} = X_{\alpha_2} < X_{\beta_2}$, therefore

$$\Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2}) = \frac{1}{\binom{3k}{2k}\binom{2k}{k}}.$$

There are $d(d-1)(d-2)$ such possibilities.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_1 = \beta_2$. This can be calculated in the same way as the previous case.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\beta_1 = \beta_2$. This means that $X_{\alpha_1} < X_{\beta_1}$ and $X_{\alpha_2} < X_{\beta_1}$ are both true. Hence

$$\Pr(O_{\alpha_1,\beta_1} \cap O_{\alpha_2,\beta_2}) = \frac{1}{\binom{3k}{k}}.$$

As in the previous cases, there are $d(d-1)(d-2)$ possibilities for this.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_1 = \alpha_2$. This is the same calculation as the previous case.

Combining the calculations above and using $2 \leqslant d \leqslant k$ we arrive at the following inequalities.

$$\begin{aligned}
\Pr(E_i) &\geqslant \frac{d(d-1)}{\binom{2k}{k}} - \frac{1}{2}\frac{d(d-1)(d-2)(d-3)}{\binom{2k}{k}\binom{2k}{k}} - \frac{d(d-1)(d-2)}{\binom{3k}{2k}\binom{2k}{k}} - \frac{d(d-1)(d-2)}{\binom{3k}{k}} \\
&= \frac{d(d-1)}{\binom{2k}{k}}\left(1 - \frac{1}{2}\frac{(d-2)(d-3)}{\binom{2k}{k}} - \frac{(d-2)}{\binom{3k}{2k}} - \frac{(d-2)\binom{2k}{k}}{\binom{3k}{k}}\right) \\
&\geqslant \frac{d(d-1)}{\binom{2k}{k}}\left(1 - \frac{(k-2)(k-3)}{2\binom{2k}{k}} - \frac{(k-2)}{\binom{3k}{2k}} - \frac{(k-2)\binom{2k}{k}}{\binom{3k}{k}}\right) := \frac{d(d-1)}{\binom{2k}{k}} \cdot \frac{1}{1+\gamma(k)}.
\end{aligned}$$

Here $\gamma(2) = 0$, $\gamma(3) = 1/3$, $\gamma(4) < 0.44$, $\gamma(5) < 0.36$ and then it exponentially converges to 0 as $k \to \infty$.

Summing these lower bounds for all i we get (2.1). $\square$

2.3 Proof of the upper bound for $c(k)$

We prove (1.2) in the following form. For all $n, k \geqslant 2$,

$$f_2(n,k) \leqslant \left\lfloor \frac{n}{k} \right\rfloor + (1+\gamma(k))k\binom{2k}{k},$$

where we define $\gamma(2) = \frac{1}{3}$ and $\gamma(k)$ comes from the proof of Lemma 7 for $k \geqslant 3$.

Consider a semi-intersecting family $\mathcal{S}$ with parameters n and k and base sets A and B. Since $(1+\gamma(k))k\binom{2k}{k} \geqslant 2k^3$ we may suppose $|\mathcal{S}| > 2k^3$. For $x \in A \cup B$, we use $\deg_{\mathcal{S}}(x)$ to denote the degree of x in $\mathcal{S}$, i.e., the number of sets in $\mathcal{S}$ containing x.

Lemma 8. *If $|\mathcal{S}| > 2k^3$ then either all degrees in A are at most k, or all degrees in B are at most k.*

Proof. Assume that there is an $a \in A$ with degree more than k. Let $S_1, S_2, \ldots, S_{k+1} \in \mathcal{S}$ be some (distinct) sets containing a and define $X := \bigcup_{1 \leqslant i \leqslant k+1}(S_i \cap A)$. Note that $|X| \leqslant k^2$ as we take the union of $k+1$ sets with k elements, all containing a. The sets $B \cap S_i$ are pairwise disjoint for $1 \leqslant i \leqslant k+1$ so no $T \in \mathcal{S}$ can intersect each of them in B. Hence $T \cap X \neq \emptyset$ for all $T \in \mathcal{S}$.

We claim that the degree of each $y \in B$ is at most $|X|$, i.e., $\deg_{\mathcal{S}}(y) \leqslant k^2$. Indeed, a set $T \in \mathcal{S}$ with $y \in T$ contains a pair $\{x, y\}$ with $x \in X$ and every such pair can appear in at most one member of $\mathcal{S}$.

Similarly, $b \in B$ and $\deg_{\mathcal{S}}(b) > k$ imply $\deg_{\mathcal{S}}(x) \leqslant k^2$ for every $x \in A$.

Fix any member $T \in \mathcal{S}$. Since $\mathcal{S}$ is intersecting we obtain $|\mathcal{S}| \leqslant \sum_{z \in T} \deg_{\mathcal{S}}(z) \leqslant 2k \cdot k^2$, and we are done. $\square$

From now on, we may suppose that $\deg_{\mathcal{S}}(y) \leqslant k$ for all $y \in B$. Starting with $\mathcal{S}_0 := \mathcal{S}$ we define a series of families $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \ldots \supset \mathcal{S}_q$ as follows. If the families $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{i-1}$ have already been created, and the members of $\mathcal{S}_{i-1}$ were pairwise disjoint in B then we let $q := i-1$ and $\mathcal{S}_q := \mathcal{S}_{i-1}$. Note that, $|\mathcal{S}_q| \leqslant \left\lfloor \frac{n}{k} \right\rfloor$.

Otherwise, define $\mathcal{S}_i$ as follows. Take a vertex $p_i \in B$ with maximum degree in $\mathcal{S}_{i-1}$, let $\mathcal{Z}_i \subset \mathcal{S}_{i-1}$ be the family of sets containing p_i, and let $d_i := |\mathcal{Z}_i|$. We have $2 \leqslant d_i \leqslant k$. Denote by $\mathcal{M}_i \subset \mathcal{S}_{i-1} \setminus \mathcal{Z}_i$ the family of sets which intersect at least one set of $\mathcal{Z}_i$ in B. Finally, let $\mathcal{S}_i := \mathcal{S}_{i-1} \setminus (\mathcal{Z}_i \cup \mathcal{M}_i)$.

Now we give an upper bound for $|\mathcal{S}_{i-1} \setminus \mathcal{S}_i|$. Since $p_i \in \bigcap_{Z \in \mathcal{Z}_i} Z$ we have $\left|B \cap \bigcup_{Z \in \mathcal{Z}_i} Z\right| \leqslant 1 + (k-1)d_i$. An element of $B \cap \left(\bigcup_{Z \in \mathcal{Z}_i} Z\right) \setminus \{p_i\}$ can meet at most $d_i - 1$ members of $\mathcal{M}_i$, so we get $|\mathcal{M}_i| \leqslant (k-1)d_i(d_i-1)$. Hence $|\mathcal{S}_{i-1} \setminus \mathcal{S}_i| = |\mathcal{Z}_i \cup \mathcal{M}_i| \leqslant d_i + (k-1)d_i(d_i-1)$ and

$$|\mathcal{S}| = |\mathcal{S}_q| + \sum_{1 \leqslant i \leqslant q} |\mathcal{S}_{i-1} \setminus \mathcal{S}_i| \leqslant \left\lfloor \frac{n}{k} \right\rfloor + \sum_{1 \leqslant i \leqslant q} (d_i + (k-1)d_i(d_i-1)).$$

We get

$$|\mathcal{S}| - \left\lfloor \frac{n}{k} \right\rfloor \leqslant \sum_{1\leqslant i\leqslant q} d_i + (k-1)\sum_{1\leqslant i\leqslant q} d_i(d_i-1) \leqslant k \sum_{1\leqslant i\leqslant q} d_i(d_i-1).$$

We need to bound $\sum_{1\leqslant i\leqslant q} d_i(d_i-1)$.

Observe that the sets in the family $\mathcal{Z}_i$ are pairwise disjoint in A as they have a common element in B. Define $\mathcal{A}_i$ as $\{Z\cap A : Z\in \mathcal{Z}_i\}$. If $S\in\mathcal{S}_i$ and $Z\in\mathcal{Z}_i$ then $S\cap Z\cap B=\emptyset$, as any set from $\mathcal{S}_{i-1}$ that intersects Z in B is in $\mathcal{Z}_i\cup\mathcal{M}_i$ by definition. Hence $S\cap Z\cap A\neq\emptyset$. In particular, any $Z\in\mathcal{Z}_i$ and $Z'\in\mathcal{Z}_j$ intersect in A if $i<j$, i.e., $X\cap X'\neq\emptyset$ if $X\in\mathcal{A}_i$, $X'\in\mathcal{A}_j$, and $i\neq j$.

If $q\geqslant 2$ then $\mathcal{A}_1,\ldots,\mathcal{A}_q$ form a k-uniform cross intersecting matching and then Lemma 7 completes the proof. In case of $q\leqslant 1$ we have $|\mathcal{S}|\leqslant \left\lfloor \frac{n}{k}\right\rfloor + 2k^2(k-1)$ and we are done. $\square$

3 Multiproducts of the complete graphs

3.1 Algebraic upper bound for the product of complete graphs

Complete graphs are also Kneser graphs with $k=1$. We prove the upper bound $f_\ell(n,1)\leqslant n\ell-\ell+1$ in Theorem 3 in the following stronger form. Suppose that $\ell\geqslant 2$, $n_1,\ldots,n_\ell$ are positive integers and $A_1,\ldots,A_\ell$ are disjoint sets of sizes $n_1,\ldots,n_\ell$, $V:=A_1\cup A_2\cup\ldots\cup A_\ell$.

Theorem 9. *Let G be the xor-product of the complete graphs $K_{n_1},\ldots,K_{n_\ell}$. Then $\omega(G)\leqslant |V|-\ell+1$.*

The vertices of G corresponds to ℓ-element sets T with $|T\cap A_i|=1$ for each i. A clique in G corresponds to an ℓ-semi-intersecting family $\mathcal{S}$ of ℓ-element subsets of V, i.e., for distinct $S,T\in\mathcal{S}$ we have $|S\setminus T|=\ell-|S\cap T|$ is odd, so $(\ell+1+|S\cap T|)$ is even. Let $\mathbf{F}_2$ be the 2-element field. For every subset $X\subseteq V$ let $\widehat{X}\in\mathbf{F}_2^V$ denote the characteristic vector of X. Thus $\widehat{\emptyset}$ is the $|V|$ dimensional zero-vector. Let $\mathcal{A}$ denote the family $\{A_1,\ldots,A_\ell\}$.

Lemma 10. *Suppose that the cardinality $|\mathcal{S}|$ is odd and suppose that the vectors $\{\widehat{X}: X\in \mathcal{S}\cup\mathcal{A}\}$ have a non-trivial linear dependency, $\sum_{X\in\mathcal{S}\cup\mathcal{A}}\alpha(X)\widehat{X}=\widehat{\emptyset}$ for some $\alpha(X)\in\mathbf{F}_2$, not all coefficients are zero. Then this dependency is unique and $\alpha(X)=1$ for each $X\in\mathcal{S}\cup\mathcal{A}$.*

Proof of Lemma 10. The scalar product $\langle\widehat{X},\widehat{Y}\rangle=|X\cap Y|$. So for any $Y\subseteq V$,

$$0=\langle\widehat{\emptyset},\widehat{Y}\rangle=\sum_{X\in\mathcal{S}\cup\mathcal{A}}\alpha(X)|X\cap Y|.$$

Here every integer is taken modulo 2. Substituting to Y a single element $v\in A_i$, then a fixed member $A_i \in \mathcal{A}$, and finally a member $T \in \mathcal{S}$ we get

$$0 = \left(\sum_{S: v \in S \in \mathcal{S}} \alpha(S) \right) + \alpha(A_i), \tag{3.1}$$

$$0 = \left(\sum_{S \in \mathcal{S}} \alpha(S) \right) + \alpha(A_i)|A_i|, \tag{3.2}$$

$$0 = \left(\sum_{S \in \mathcal{S}} |S \cap T| \alpha(S) \right) + \left(\sum_j \alpha(A_j) \right). \tag{3.3}$$

Add $(\ell+1)$ times (3.2) to (3.3). For given T and A_i we get

$$0 = \left(\sum_{S \in \mathcal{S}} (\ell + 1 + |S \cap T|) \alpha(S) \right) + (\ell+1)\alpha(A_i)|A_i| + \left(\sum_j \alpha(A_j) \right). \tag{3.4}$$

For distinct $S, T \in \mathcal{S}$ $(\ell+1+|S \cap T|)$ is even and for $S = T$ we have $\ell+1+|S \cap T| = 2\ell+1 = 1$ (in $\mathbf{F}_2$). So the first term in (3.4) is exactly $\alpha(T)$. The second and the third terms of (3.4) are independent from T, so we obtain that all $\alpha(T)$ are equal.

If $\alpha(T) = 0$ for each $T \in \mathcal{S}$ then (3.1) gives that $\alpha(A_i) = 0$ for all i, a contradiction. Therefore each $\alpha(T) = 1$, so the first term in (3.2) is $|\mathcal{S}|$. By our assumptions this is odd, so $\alpha(A_i)|A_i|$ should be odd. In particular $\alpha(A_i) = 1$ for all i. □

Proof of Theorem 9. If $|\mathcal{S}|$ is odd, then by Lemma 10 the vectors $\{\widehat{X} : X \in \mathcal{S} \cup \mathcal{A}\}$ are either linearly independent in $\mathbf{F}_2^V$ or has a unique linear dependency. So they generate a subspace of dimension at least $|\mathcal{S}| + |\mathcal{A}| - 1$. This is at most $|V|$ and we are done.

If $|\mathcal{S}|$ is even then we can assume that $|\mathcal{S}| \geqslant 2$. Take two distinct members $T_1, T_2 \in \mathcal{S}$. Then $|\mathcal{S} \setminus \{T_i\}|$ is odd (for $i = 1, 2$). If either of the set of vectors $\{\widehat{X} : X \in (\mathcal{S} \setminus \{T_i\}) \cup \mathcal{A}\}$ is independent, we get $|\mathcal{S}| + |\mathcal{A}| - 1 \leqslant |V|$ as desired. If both are dependent, then again by Lemma 10 they have unique linear dependencies, namely $\sum \{\widehat{X} : X \in (\mathcal{S} \setminus \{T_i\}) \cup \mathcal{A}\} = \widehat{\emptyset}$. Adding up these equations we get $\widehat{T_1} + \widehat{T_2} = \widehat{\emptyset}$. This contradiction completes the proof. □

3.2 An explicit construction for the case of complete graphs

We prove the lower bound $f_\ell(n, 1) \geqslant \ell n - 2\ell - 1$ in Theorem 3 in the following stronger form. Suppose that $\ell \geqslant 3$, $A_1, \ldots, A_\ell$ are disjoint sets of sizes $n_1, \ldots, n_\ell$, $V := A_1 \cup A_2 \cup \ldots \cup A_\ell$.

Theorem 11. *Let G be the xor-product of the complete graphs $K_{n_1}, \ldots, K_{n_\ell}$. Suppose that $\ell \geqslant 3$ and $n_i \geqslant 2$ for each $i \in [\ell]$. Then $\omega(G) \geqslant |V| - 2\ell - 1$.*

Proof. We show a construction. For a given partition $A_1, \ldots, A_\ell$ $(\ell \geqslant 3)$ we call the family of sets $\mathcal{B} := \{B_1, \ldots, B_\ell\}$ an *ℓ-core* if

(i) each B_i is an $(\ell-1)$-set with $B_i \cap A_i = \emptyset$ but $|B_i \cap A_j| = 1$ for $i \neq j$ and

(ii) $|B_i \cap B_j| \not\equiv \ell \pmod 2$ for all $1 \leqslant i, j \leqslant \ell$.

This intersection condition can be reformulated as $|B_i \cap B_j| + \ell$ is always odd. Let $U(\mathcal{B})$ denote $\cup B_i$. A core $\mathcal{B}$ generates an ℓ-uniform family $\mathcal{S}(\mathcal{B})$ by enlarging each core set by extra elements as follows.

$$\mathcal{S}(\mathcal{B}) := \{B_i \cup \{x\} : i \in [\ell],\ x \in A_i \setminus U(\mathcal{B})\}.$$

We claim that $\mathcal{S}(\mathcal{B})$ is an ℓ-semi-intersecting family with $k = 1$ of size $|V| - |U|$. Indeed, by definition each $S \in \mathcal{S}(\mathcal{B})$ intersects every A_i in exactly one element. Let $S, T \in \mathcal{S}(\mathcal{B})$ with $S \neq T$. Say $S = B_i \cup \{x\}$ and $T = B_j \cup \{y\}$. Then $S \cap T = B_i \cap B_j$. So $S \cap T \cap A_\alpha = \emptyset$ in $\ell - |B_i \cap B_j|$ cases of α. Here $\ell - |B_i \cap B_j|$ is odd by (ii), so $\mathcal{S}(\mathcal{B})$ is an ℓ-semi-intersecting family.

The next step in the proof of Theorem 11 is to find a core $\mathcal{B}$ with small $|U(\mathcal{B})|$. We need a couple of more definitions. The *type* of $\mathcal{B}$ is the tuple

$$(|U(\mathcal{B}) \cap A_1|, |U(\mathcal{B}) \cap A_2|, \ldots, |U(\mathcal{B}) \cap A_\ell|).$$

The set $U(\mathcal{B}) \cap A_i$ is called the ith *class* of $\mathcal{B}$.

Lemma 12. *Suppose that $p, q \geqslant 3$, $\mathcal{B}'_p$ is a p-core of type $(x_1, \ldots, x_p)$ and $\mathcal{B}''_q$ is a q-core of type $(y_1, \ldots, y_q)$. Then there exists a $(p+q-1)$-core $\mathcal{B}$ of type $(x_1, \ldots, x_{p-1}, x_p + y_1, y_2, \ldots, y_q)$.*

Proof of Lemma 12. We may assume that $U(\mathcal{B}'_p)$ and $U(\mathcal{B}''_q)$ are disjoint. We have $|\mathcal{B}'_p| = p$, $|\mathcal{B}''_q| = q$. The classes of $\mathcal{B}'_p$ are denoted by $A'_1, \ldots, A'_p$ ($|A'_i| = x_i$), its hyperedges are $B'_1, \ldots, B'_p$. The classes of $\mathcal{B}''_q$ are denoted by $A''_1, \ldots, A''_q$ ($|A''_j| = y_j$), its hyperedges are $B''_1, \ldots, B''_q$. We define the core $\mathcal{B} = \mathcal{B}_{p+q-1}$ as follows. Its classes are $A_1, \ldots, A_{p+q-1}$ where $A_i := A'_i$ for $1 \leqslant i \leqslant p-1$, $A_p := A'_p \cup A''_1$, and $A_j := A''_{j-p+1}$ for $p+1 \leqslant j \leqslant p+q-1$. The hyperedges $B_1, \ldots, B_{p+q-1}$ of $\mathcal{B}_{p+q-1}$ are defined as unions of the form $B'_\alpha \cup B''_\beta$ as follows: $B_p := B'_p \cup B''_1$ and in general $B_i := B'_i \cup B''_1$ for $1 \leqslant i \leqslant p$ and $B_j := B'_p \cup B''_{j-p+1}$ for $p+1 \leqslant j \leqslant p+q-1$.

We claim that $\mathcal{B}$ is a $(p+q-1)$-core. $B_\alpha \cap A_\alpha = \emptyset$ and $|B_\alpha \cap A_\beta| = 1$ for $\alpha \neq \beta$ follows from the definition of $\mathcal{B}$. Consider $|B_\alpha \cap B_\beta|$. We have to show that $|B_\alpha \cap B_\beta| + (p+q-1)$ is odd. Write B_α in the form $B'_e \cup B''_f$ and let $B_\beta = B'_g \cup B''_h$ where $B'_e, B'_g \in \mathcal{B}'$ and $B''_f, B''_h \in \mathcal{B}''$. We have $B_\alpha \cap B_\beta = (B'_e \cup B''_f) \cap (B'_g \cup B''_h)$ which is the disjoint union of $B'_e \cap B'_g$ and $B''_f \cap B''_h$. Since $|B'_e \cap B'_g| + p$ and $|B''_f \cap B''_h| + q$ are both odd their sum is even, so

$$|B_\alpha \cap B_\beta| + p + q - 1 = |B'_e \cap B'_g| + |B''_f \cap B''_h| + p + q - 1$$

is odd. So $\mathcal{B}$ is a $(p+q-1)$-core, completing the proof of Lemma 12. $\square$

The procedure described in the proof of Lemma 12 will be referred to as the *fusion* of $\mathcal{B}_p$ and $\mathcal{B}_q$. Using this construction, we prove by induction that there exists an ℓ-core $\mathcal{B}$ with $|U(\mathcal{B})| \leqslant 2\ell + 1$ if $\max_i n_i \geqslant 3$. Note that we can assume that $\max_i n_i \geqslant 3$ as if each $n_i = 2$ then $|V| = 2\ell$, so the lower bound from Theorem 11 obviously holds. First, we define an ℓ-core for $\ell = 3, 4, 5$.

There is a 3-core $\mathcal{B}_3$ of type $(2,2,2)$ with three sets $B_\alpha := \{a_{\alpha-1,\alpha}, a_{\alpha+1,\alpha}\}$ (indices are taken modulo 3) where these a's are six distinct vertices with $a_{i,j} \in A_i$ ($i, j \in \{1,2,3\}$, $i \neq j$).

There is a 4-core $\mathcal{B}_4$ of type $(2,2,2,1)$ on 7 vertices $\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,3}, a_{3,1}, a_{3,2}, a_4\}$ where $\{a_{1,2}, a_{1,3}\} \subset A_1$, $\{a_{2,1}, a_{2,3}\} \subset A_2$, $\{a_{3,1}, a_{3,2}\} \subset A_3$, and $a_4 \in A_4$. The core sets are $B_1 := \{a_{2,1}, a_{3,1}, a_4\}$, $B_2 := \{a_{1,2}, a_{3,2}, a_4\}$, $B_3 := \{a_{1,3}, a_{2,3}, a_4\}$, and $B_4 := \{a_{1,3}, a_{2,1}, a_{3,2}\}$.

There is a 5-core $\mathcal{B}_5$ of type $(2,2,2,1,1)$ on 8 vertices $\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,3}, a_{3,1}, a_{3,2}, a_4, a_5\}$ where $\{a_{1,2}, a_{1,3}\} \subset A_1$, $\{a_{2,1}, a_{2,3}\} \subset A_2$, $\{a_{3,1}, a_{3,2}\} \subset A_3$, $a_4 \in A_4$, and $a_5 \in A_5$. The core sets are $B_1 := \{a_{2,1}, a_{3,1}, a_4, a_5\}$, $B_2 := \{a_{1,2}, a_{3,2}, a_4, a_5\}$, $B_3 := \{a_{1,3}, a_{2,3}, a_4, a_5\}$, $B_4 := \{a_{1,3}, a_{2,1}, a_{3,2}, a_5\}$, and $B_5 := \{a_{1,2}, a_{2,3}, a_{3,1}, a_4\}$.

Now we give the construction for any $\ell \geqslant 3$. Fusing m copies of $\mathcal{B}_5$ of type $(1,2,2,2,1)$ we obtain a $(4m+1)$-core $\mathcal{B}_{4m+1}$ of type $(1,2,\dots,2,1)$ for each $m \geqslant 1$. The fusion of a $\mathcal{B}_4$ of type $(2,2,2,1)$ and a $\mathcal{B}_{4m+1}$ defines a $(4m+4)$-core $\mathcal{B}_{4m+4}$ of type $(2,2\dots,2,1)$ for each $m \geqslant 0$. Fusing this with a $\mathcal{B}_4$ of type $(1,2,2,2)$ yields a $(4m+7)$-core of type $(2,\dots,2)$ (for each $m \geqslant 0$). Finally, fusing $\mathcal{B}_3$ of type $(2,2,2)$ and a $4m+4$-core of type $(1,2,\dots,2)$ ($m \geqslant 0$) one gets a $(4m+6)$-core of type $(2,2,3,2,\dots,2)$, which finishes the proof. □

3.3 Remarks

Note that in the proof of Theorem 9 we have adapted a method of Deza, Frankl and Singhi [6]. Their 'even town theorem' can be applied to prove $f_\ell(n,1) \leqslant \ell n - \ell + 1$ when ℓ and n are both even. If ℓ is even and n is odd one can still apply the even town theorem to get $\ell n + 1$, the same upper bound as in Alon and Lubetzky [2]. In the case ℓ is odd the bound $f_\ell(n,1) \leqslant \ell n$ follows by a theorem of Frankl and Wilson [8].

Suppose that there exists a finite projective plane of order $\ell - 1$, i.e., an ℓ-uniform, ℓ-regular set system $\mathcal{L}$ of size $\ell^2 - \ell + 1$ such that $|L \cap L'| = 1$ for all pairwise intersections. Also suppose that ℓ is even. Take any vertex v, we have ℓ lines containing it, $L_1, \dots, L_\ell$. Set $A_i := L_i \setminus \{v\}$, and let $\mathcal{S} := \mathcal{P} \setminus \{L_1, \dots, L_\ell\}$. This $\mathcal{S}$ is an ℓ-semi-intersecting family of size $(\ell-1)^2$ on $\ell \times (\ell - 1)$ vertices. Since such finite planes exist whenever $\ell - 1$ is a power of an odd prime, we got infinitely many cases when the lower bound is tight in Theorem 9. This example was also mentioned by Alon and Lubetzky [2, 3]. They were more interested from coding theory point of view, i.e., when n is fixed and $\ell \to \infty$.

Finding the exact value of $f_\ell(n,1)$ is still open.

4 Higher powers, the general case

In this section we study the order of magnitude of $f_\ell(n,k)$.

4.1 Proof of the upper bound (1.3) by induction on ℓ

Suppose that $\ell \geqslant 2$ and let $\mathcal{S}$ be an ℓ-semi-intersecting family with parameters n and k and base sets $A_1, \dots, A_\ell$. Take any $v \in A_i$. Define $\mathcal{S}[v] := \{S \setminus A_i : v \in S \in \mathcal{S}\}$. Then $\mathcal{S}[v]$ does not contain multiple hyperedges, it is an $(\ell-1)$-semi-intersecting family. Hence

$|\mathcal{S}[v]| = \deg_{\mathcal{S}}(v) \leqslant f_{\ell-1}(n,k)$. Take this inequality for each $v \in A_i$ and suppose that $|\mathcal{S}|$ has maximum size. We get

$$f_\ell(n,k) = |\mathcal{S}| = \frac{1}{k}\sum_{v \in A_i} \deg_{\mathcal{S}}(v) \leqslant \frac{n}{k} f_{\ell-1}(n,k). \tag{4.1}$$

If ℓ is even then $\mathcal{S}$ is intersecting. Taking the degrees of any given $T \in \mathcal{S}$ we obtain

$$f_\ell(n,k) = |\mathcal{S}| \leqslant 1 + \sum_{v \in T}(\deg_{\mathcal{S}}(v) - 1) \leqslant 1 + k\ell\,(f_{\ell-1}(n,k) - 1) \leqslant k\ell f_{\ell-1}(n,k). \tag{4.2}$$

We have $f_1(n,k) \leqslant n/k$. Apply (4.2), we get $f_2(n,k) \leqslant 2kf_1(n,k) \leqslant 2n$. Then (4.1) gives $f_3(n,k) \leqslant (n/k)f_2(n,k) \leqslant 2n^2/k$. Apply again (4.2), we get $f_4(n,k) \leqslant 4kf_3(n,k) \leqslant 2 \cdot 4 \cdot n^2$. Continuing this way, we get for each even ℓ that $f_\ell(n,k) \leqslant 2 \cdot 4 \cdot \ldots \cdot \ell \cdot n^{\ell/2}$ and $f_\ell(n,k) \leqslant 2 \cdot 4 \cdot \ldots \cdot (\ell-1) \cdot n^{(\ell+1)/2}/k$ when ℓ is odd. □

4.2 Construction showing the lower bound (1.4)

Note that $f_\ell(n,k)$ is monotonous in n and also increases monotonously in ℓ, since an ℓ-semi-intersecting family $\mathcal{S}$ can be extended to an $(\ell+1)$-semi-intersecting family by adding $A_{\ell+1}$ to the base sets and a fixed k-element $S_{\ell+1} \subset A_{\ell+1}$ to all $S \in \mathcal{S}$. So it is enough to prove the theorem for $\ell = 2^t - 1$ where $t \geqslant 2$ is an integer, and we also suppose that $k \geqslant t$. Let $m := \left\lfloor \frac{n}{k} \right\rfloor$.

Take ℓ disjoint sets $A_1, \ldots, A_\ell$ of sizes $|A_\alpha| = mk$. We are going to define an ℓ-semi-intersecting family $\mathcal{S}$ of size m^t with parameters mk and k with these base sets. Let H be 0-1 matrix of size $\ell \times t$ with $2^t - 1$ pairwise distinct nonzero rows. Note that this matrix is unique up to a permutation of its rows. Let C be any $\ell \times t$ matrix with non-negative integer entries and row sums equal to k (i.e. $\sum_{1 \leqslant \beta \leqslant t} C_{\alpha,\beta} = k$) and so that $C_{\alpha,\beta} = 0$ if and only if $H_{\alpha,\beta} = 0$. Such a matrix exists as $k \geqslant t$. Partition each A_α into subsets $A^p_{\alpha,\beta}$ where $1 \leqslant p \leqslant m$ and $|A^p_{\alpha,\beta}| = C_{\alpha,\beta}$. In particular, let $A^p_{\alpha,\beta} = \emptyset$ if $C_{\alpha,\beta} = 0$. Make another partition of $\cup A_\alpha$ into mt sets by joining some of the $A^p_{\alpha,\beta}$ as follows. For each β and p where $1 \leqslant \beta \leqslant t$ and $1 \leqslant p \leqslant m$ define

$$S^p_\beta := \bigcup_{1 \leqslant \alpha \leqslant \ell} A^p_{\alpha,\beta}.$$

For each of the m^t functions $\varphi : [t] \to [m]$ define $S_\varphi := \cup_{1 \leqslant \beta \leqslant t} S^{\varphi(\beta)}_\beta$. Finally, let $\mathcal{S} := \{S_\varphi : \varphi : [t] \to [m]\}$.

Let us prove that $\mathcal{S}$ is an ℓ-semi-intersecting family. Each $\mathcal{S}_\varphi \in \mathcal{S}$ intersects every A_α in k elements, as

$$|S_\varphi \cap A_\alpha| = \left| \bigcup_{1 \leqslant \beta \leqslant t} (A_\alpha \cap S^{\varphi(\beta)}_\beta) \right| = \left| \bigcup_{1 \leqslant \beta \leqslant t} A^{\varphi(\beta)}_{\alpha,\beta} \right| = \sum_{1 \leqslant \beta \leqslant t} C_{\alpha,\beta} = k.$$

Let $D_\beta \subset [\ell]$ denote the set of indices of nonzero elements of the column β of H, i.e., $D_\beta := \{\alpha : H_{\alpha,\beta} = 1\}$. For any set $X \subset \cup A_\alpha$ let $\pi(X) \subset [\ell]$ denote its projection,

$\pi(X) := \{\alpha \in [\ell] : A_\alpha \cap X \neq \emptyset\}$. We have $\pi(S^p_\beta) = D_\beta$ for all p. Even more, each such set has the type $(C_{1,\beta}, \ldots, C_{\ell,\beta})$, i.e., $|S^p_\beta \cap A_\alpha|$ is exactly $C_{\alpha,\beta}$. Note that for any $Q \subseteq [t]$ we have $|\cup_{q\in Q} D_q| = 2^t - 2^{t-|Q|}$, an even number except in the case $Q = [t]$.

Given two functions φ, σ we claim that $|\pi(S_\varphi \cap S_\sigma)|$ is even except in the case $\varphi = \sigma$. This implies that $\mathcal{S}$ is ℓ-semi-intersecting, as claimed. We have $S_\varphi \cap S_\sigma = \left(\cup S^{\varphi(\beta)}_\beta\right) \cap \left(\cup S^{\sigma(\beta)}_\beta\right)$. Since the sets S^p_β form a partition of $\cup A_\alpha$ we have that $S_\varphi \cap S_\sigma = \cup\{S^p_\beta : \varphi(\beta) = \sigma(\beta)\}$. Hence $\pi(S_\varphi \cap S_\sigma) = \cup\{D_\beta : \varphi(\beta) = \sigma(\beta)\}$. This set has even cardinality whenever $\varphi \neq \sigma$. So we find that S_φ and S_σ are disjoint in an odd number of base sets A_α which finishes the proof. □

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