The α-spectral Turán type problems for graphs

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Abstract

For $0 \leqslant \alpha < 1$, the α-spectral radius of a graph G is defined as the largest eigenvalue of $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $D(G)$ and $A(G)$ are the diagonal matrix of degrees and adjacency matrix of G, respectively. A graph is called color-critical if it contains an edge whose deletion reduces its chromatic number. The celebrated Erdős-Stone-Simonovits theorem asserts that $\mathrm{ex}(n, \mathcal{F}) = \big(1 - \frac{1}{\chi(\mathcal{F})-1} + o(1)\big)\frac{n^2}{2}$, where $\chi(\mathcal{F})$ is the chromatic number of $\mathcal{F}$. Nikiforov and Zheng et al. established the adjacency spectral and signless Laplacian spectral versions of this theorem, respectively. In this paper, we present the α-spectral version of this theorem, which unifies the aforementioned results. Furthermore, we characterize the α-spectral extremal graphs for color-critical graphs, thereby extending the existing results on adjacency spectral and signless Laplacian spectral extremal graphs for such graphs.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual, we use $|G|$ and $e(G)$ to represent the number of vertices and edges of a graph G, respectively. For $v \in V(G)$, the set of neighbors of v in G are denoted by $N_G(v)$, and $d_G(v) = |N_G(v)|$ denotes its degree in G. If there is no ambiguity, we simplify $N_G(v)$ and $d_G(v)$ as $N(v)$ and $d(v)$, respectively. We write $\Delta(G)$ for the maximum degree of G and $\delta(G)$ for the minimum degree of G. Let $T_{n,r}$ denote the r-partite Turán graph of order n. The k-vertex independent set is denoted by I_k. For two graphs G and H, the disjoint union of G and H is denoted by $G \cup H$. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding all possible edges between G and H. Denote $S_{n,k} = K_k \vee I_{n-k}$ and $S^+_{n,k} = K_k \vee (K_2 \cup I_{n-k-2})$. Let M_{k+1} be a matching of size $k+1$

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and P_k be a path of order k. A cycle on k vertices is denoted by C_k. A graph is called *color-critical* if it contains an edge whose deletion reduces its chromatic number.

Let $\mathcal{F}$ be a family of graphs. A graph is called $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The classic *Turán type problem* is to determine the maximum number of edges over all $\mathcal{F}$-free graphs of order n. The *Turán number* of $\mathcal{F}$ is defined as follows:

$$\mathrm{ex}(n, \mathcal{F}) = \max\{e(G) |\ G \text{ is an } n\text{-vertex } \mathcal{F}\text{-free graph}\}.$$

Let $\chi(\mathcal{F}) = \min_{F \in \mathcal{F}} \chi(F)$. When $\mathcal{F} = \{F\}$, we simplify the notation to $\mathrm{ex}(n, F)$ instead of $\mathrm{ex}(n, \{F\})$. The same simplification applies to all other relevant functions. The *Turán density* of $\mathcal{F}$ is defined as

$$\pi(\mathcal{F}) := \lim_{n \to \infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{2}},$$

for which the existence is guaranteed by an averaging argument due to Katona, Nemetz and Simonovits in [11].

Let G be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. The *signless Laplacian matrix* of a graph G is defined as $Q(G) := D(G) + A(G)$. In 2017, Nikiforov [18] proposed to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1 - \alpha)A(G), \quad 0 \leqslant \alpha < 1.$$

Obviously, $A(G) = A_0(G)$ and $Q(G) = 2A_{1/2}(G)$. The largest eigenvalue of $A_\alpha(G)$ is called the *α-spectral radius* of G, denoted by $\lambda_\alpha(G)$. The $\lambda_0(G)$ (resp. $2\lambda_{1/2}(G)$) of G is usually referred to as the spectral radius (resp. signless Laplacian spectral radius) of G, denoted by $\lambda(G)$ (resp. $q(G)$). In 2017, Nikiforov [18] proposed the following problem.

Proposition 1. *Given a family of graphs $\mathcal{F}$, what is the maximum $\lambda_\alpha(G)$ of a graph G of order n, with no subgraph isomorphic to any $F \in \mathcal{F}$?*

Let $\mathrm{ex}_\alpha(n, \mathcal{F})$ denote the maximum α-spectral radius among all $\mathcal{F}$-free graphs of order n. An $\mathcal{F}$-free graph of order n that achieves $\mathrm{ex}_\alpha(n, \mathcal{F})$ is called an *α-spectral extremal graph* for $\mathcal{F}$, and we let $\mathrm{Ex}_\alpha(n, \mathcal{F})$ denote the family of all such graphs. For specific spectral parameters, $\mathrm{ex}_\lambda(n, \mathcal{F})$ stands for the maximum spectral radius of $\mathcal{F}$-free graphs of order n, while $\mathrm{ex}_q(n, \mathcal{F})$ denotes the maximum signless Laplacian spectral radius of $\mathcal{F}$-free graphs on n vertices.

The study of α-spectral Turán problem was initiated by Nikiforov [18], who determined the α-spectral extremal graphs corresponding to complete graphs. For matching M_{k+1}, Yuan and Shao [22] showed that $\mathrm{Ex}_\alpha(n, M_{k+1}) = S_{n,k}$. Further advancing the field, Chen, Liu, and Zhang [3] proved that $\mathrm{Ex}_\alpha(n, P_{2k+2}) = S_{n,k}$ and $\mathrm{Ex}_\alpha(n, P_{2k+3}) = S^+_{n,k}$, shedding light on the α-spectral extremal structures for paths of specific lengths. For cycles, Li and Yu [12] showed that $\mathrm{Ex}_\alpha(n, C_{2k+2}) = S^+_{n,k}$ and $\mathrm{Ex}_\alpha(n, \{C_{2k+1}, C_{2k+2}\}) = S_{n,k}$. Expanding to disjoint cycles, Li, Yu, and Zhang [13] determined the α-spectral extremal graphs and they obtained $\mathrm{Ex}_\alpha(n, \mathcal{F}) = S_{n,2k-1}$, where $\mathcal{F}$ is the family of all disjoint unions of k

cycles. A nice result was achieved by Chen et al. [4], who proved the α-spectral Erdős-Sós theorem. This theorem asserts that every n-vertex graph G with $\lambda_\alpha(G) \geqslant \lambda_\alpha(S_{n,k})$ contains all trees on $2k+2$ vertices unless $G = S_{n,k}$. Moreover, if $\lambda_\alpha(G) \geqslant \lambda_\alpha(S_{n,k}^{+})$, then G contains all trees on $2k+3$ vertices unless $G = S_{n,k}^{+}$. Most recently, Byrne, Desai, and Tait [2] established a unified framework that characterizes the α-spectral extremal graphs for a broad class of graph families.

1.1 The spectral version of the Erdős-Stone-Simonovits theorem

The famous Erdős-Stone-Simonovits theorem gives an asymptotic value of the Turán number for a family of graphs.

Theorem 2 (Erdős-Stone-Simonovits [5, 6]). *If $\mathcal{F}$ is a family of graphs with $\chi(\mathcal{F}) \geqslant 2$, then*

$$\mathrm{ex}(n, \mathcal{F}) = \left(1 - \frac{1}{\chi(\mathcal{F}) - 1} + o(1)\right) \frac{n^2}{2}.$$

As a direct consequence of Theorem 2, we have $\pi(\mathcal{F}) = 1 - \frac{1}{\chi(\mathcal{F})-1}$. Nikiforov [17] later proposed a spectral analog of the Erdős-Stone-Simonovits theorem, stated as follows.

Theorem 3 (Nikiforov [17]). *If F is a graph with $\chi(F) \geqslant 2$, then*

$$\mathrm{ex}_\lambda(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) n.$$

More recently, Zheng, Li, and Su [24] established the signless Laplacian spectral counterpart of this theorem.

Theorem 4 (Zheng, Li, and Su [24]). *If F is a graph with $\chi(F) \geqslant 3$, then*

$$\mathrm{ex}_q(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) 2n.$$

For a bipartite graph F, Zheng, Li, and Su [24] also derived the following result.

Theorem 5 (Zheng, Li, and Su [24]). *If F is a bipartite graph, i.e., $\chi(F) = 2$, then*

$$\mathrm{ex}_q(n, F) = \begin{cases} \big(1 + o(1)\big)\, n & \text{if } F \text{ is not a star;} \\ 2(k-1) & \text{if } F \text{ is a star } K_{1,k}. \end{cases}$$

1.2 The spectral extremal results for color-critical graphs

In 2007, Nikiforov [15] showed that if G is a K_{r+1}-free graph on n vertices, then $\lambda(G) \leqslant \lambda(T_{n,r})$, with equality if and only if $G = T_{n,r}$. Subsequently, He, Jin, and Zhang [9] extended this line of research to the signless Laplacian spectral radius, establishing the following theorem:

Theorem 6 (He, Jin, and Zhang [9]). *If G is an n-vertex K_{r+1}-free graph, then*

$$q(G) \leqslant q(T_{n,r}).$$

Moreover, the equality holds if and only if G is a complete bipartite graph for $r = 2$ and the r-partite Turán graph $T_{n,r}$ for every $r \geqslant 3$.

Later, Nikiforov [18] generalized the aforementioned results and obtained the following theorem.

Theorem 7 (Nikiforov [18]). *Let $r \geqslant 2$ and G be a K_{r+1}-free graph of order n.*

(1) If $0 \leqslant \alpha < 1 - \frac{1}{r}$, then $\lambda_\alpha(G) < \lambda_\alpha(T_{n,r})$, unless $G = T_{n,r}$.

(2) If $1 > \alpha > 1 - \frac{1}{r}$, then $\lambda_\alpha(G) < \lambda_\alpha(S_{n,r-1})$, unless $G = S_{n,r-1}$.

(3) If $\alpha = 1 - \frac{1}{r}$, then $\lambda_\alpha(G) \leqslant \left(1 - \frac{1}{r}\right) n$, with equality if and only if G is a complete r-partite graph.

Color-critical graphs form a broad and fundamental class of graphs. Examples include cliques, odd cycles, book graphs, and even wheels-all of which are color-critical. Let F be a color-critical graph with $\chi(F) = r + 1$. Simonovits [19] determined the Turán number of F, proving that $\mathrm{Ex}(n, F) = T_{n,r}$ for sufficiently large n. Nikiforov [16] established a spectral counterpart to this result, showing that $\mathrm{ex}_\lambda(n, F) = \lambda(T_{n,r})$. Subsequently, Zheng, Li, and Li [23] settled the signless Laplacian spectral extremal problem for F, determining $\mathrm{ex}_q(n, F)$ explicitly.

Theorem 8 (Zheng, Li, and Li [23]). *Let F be a color-critical graph with $\chi(F) = r+1 \geqslant 4$. Then there exists n_0 such that for every F-free graph G on $n \geqslant n_0$ vertices, we have*

$$q(G) \leqslant q(T_{n,r}),$$

where the equality holds if and only if G is the r-partite Turán graph $T_{n,r}$.

2 Main results

Given a family of graphs with $\chi(\mathcal{F}) = r + 1 \geqslant 3$, we first establish the asymptotic value of $\mathrm{ex}_\alpha(n, \mathcal{F})$ as follows.

Theorem 9. *Let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = r + 1 \geqslant 3$ and $0 \leqslant \alpha \leqslant 1 - \frac{1}{r}$. Then*

$$\mathrm{ex}_\alpha(n, \mathcal{F}) = \left(1 - \frac{1}{r} + o(1)\right) n.$$

Remark. Theorem 9 does not hold when $\alpha > 1 - \frac{1}{r}$. Nikiforov [18] proved that $\mathrm{Ex}_\alpha(n, K_{r+1}) = S_{n,r-1}$ for $r \geqslant 2$ and $1 > \alpha > 1 - \frac{1}{r}$. Clearly, $\lambda_\alpha(S_{n,r-1}) \geqslant \alpha(n-1) > \left(1 - \frac{1}{r} + o(1)\right) n$ for sufficiently large n.

Suppose G is an $\mathcal{F}$-free graph of order n. By Lemma 15 (stated in Section 3, which asserts $e(G) \leqslant \lambda_\alpha(G)\frac{n}{2}$) and Theorem 9, we have $e(G) \leqslant \lambda_\alpha(G)\frac{n}{2} \leqslant \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}$. Thus, Theorem 9 implies Theorem 2 in the case where $\chi(\mathcal{F}) \geqslant 3$. For families $\mathcal{F}$ containing a bipartite graph, we establish the following result.

Theorem 10. *Let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = 2$ and $0 \leqslant \alpha < 1$. Then*

(1) $\mathrm{ex}_\alpha(n, \mathcal{F}) = (\alpha + o(1))\, n$ *if $\mathcal{F}$ does not contain $K_{1,k} \cup sK_1$ for any $k \geqslant 1, s \geqslant 0$.*

(2) $\mathrm{ex}_\alpha(n, \mathcal{F}) = O(1)$ *if $\mathcal{F}$ contains $K_{1,k} \cup sK_1$ for some $k \geqslant 1, s \geqslant 0$. Particularly, if $\mathcal{F} = \{K_{1,k}\}$, then* $\mathrm{ex}_\alpha(n, \mathcal{F}) = k - 1$.

When $\alpha = 0$, Theorem 9 combined with Theorem 10 implies Theorem 3. Furthermore, setting $\alpha = \frac{1}{2}$, Theorems 9 and 10 readily yield Theorems 4 and 5.

For color-critical graphs F, we further determine the maximum α-spectral radius among all F-free graphs of order n.

Theorem 11. *Let F be a color-critical graph with $\chi(F) = r + 1 \geqslant 3$, and let G be an F-free graph of order n. For sufficiently large n and $0 \leqslant \alpha < 1 - \frac{1}{r}$, we have*

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{n,r}),$$

where equality holds if and only if $G = T_{n,r}$.

Remark. Theorem 11 does not hold when $\alpha = 1 - \frac{1}{r}$. Indeed, for this specific value of α, Theorem 7 asserts that $\mathrm{Ex}_\alpha(n, K_{r+1})$ consists of all complete r-partite graphs.

Clearly, Theorem 11 extends Theorem 7 for $0 \leqslant \alpha < 1 - \frac{1}{r}$. When $\alpha = 0$, it recovers Nikiforov's spectral result for color-critical graphs [16]; when $\alpha = \frac{1}{2}$, it implies Theorem 8.

The *even wheel* of order $2k + 2$ is defined as $W_{2k+2} := K_1 \vee C_{2k+1}$. Notably, W_{2k+2} is color-critical with $\chi(W_{2k+2}) = 4$. By Theorem 11, we immediately obtain the following result.

Corollary 12. *For $k \geqslant 1$, $0 \leqslant \alpha < \frac{2}{3}$ and sufficiently large n, if G is a W_{2k+2}-free graph of order n, then*

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{n,3}),$$

where equality holds if and only if $G = T_{n,3}$.

The *generalized book graph $B_{r,k}$* is defined as $B_{r,k} := K_r \vee I_k$. One can see that $B_{r,k}$ is a color-critical graph and $\chi(B_{r,k}) = r + 1$. Thus, Theorem 11 implies the following result.

Corollary 13. *Suppose $k \geqslant 1$ and $r \geqslant 2$. For sufficiently large n and $0 \leqslant \alpha < 1 - \frac{1}{r}$, if G is a $B_{r,k}$-free graph of order n, then*

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(T_{n,r}),$$

with equality if and only if $G = T_{n,r}$.

3 Preliminaries

For a graph G and integer $p \geqslant 1$, the *blow-up* of G, denoted by G^p, is the graph obtained from G by replacing each vertex of G by a set of p independent vertices and each edge of G by a complete bipartite graph $K_{p,p}$. A family of graphs $\mathcal{G}$ is called *multiplicative* if $G \in \mathcal{G}$ implies $G^p \in \mathcal{G}$ for all integers $p \geqslant 1$. A family $\mathcal{G}$ is said to be *hereditary* if it is closed under taking induced subgraphs.

Let $\mathcal{F}$ be a family of graphs and $\mathcal{G}$ be a class of $\mathcal{F}$-free graphs. If there exist $\varepsilon > 0$ and $N > 0$ such that every $\mathcal{F}$-free graph G on $n \geqslant N$ vertices with $\delta(G) \geqslant (\pi(\mathcal{F}) - \varepsilon)n$ is a subgraph of some member of $\mathcal{G}$, then we say that $\mathcal{F}$ is *degree-stable* with respect to $\mathcal{G}$.

Given a family of graphs $\mathcal{G}$, let $\mathcal{G}_n$ denote the collection of all n-vertex graphs contained in $\mathcal{G}$. For $0 \leqslant \alpha < 1$, we define

$$\lambda_\alpha(\mathcal{G}_n) = \max_{G \in \mathcal{G}_n} \lambda_\alpha(G).$$

Nikiforov [18] established the following results, which will be invoked in our proofs.

Lemma 14 (Nikiforov [18])**.** *Let G be a graph and $0 \leqslant \alpha \leqslant 1$. Then*

$$\alpha\Delta(G) \leqslant \lambda_\alpha(G) \leqslant \alpha\Delta(G) + (1-\alpha)\lambda(G).$$

Lemma 15 (Nikiforov [18])**.** *Let $0 \leqslant \alpha \leqslant 1$. If G is of order n and m edges, then*

$$\lambda_\alpha(G) \geqslant \sqrt{\frac{1}{n}\sum_{u \in V(G)} d^2(u)} \quad \textit{and} \quad \lambda_\alpha(G) \geqslant \frac{2m}{n}.$$

Equality holds in the second inequality if and only if G is regular. If $\alpha > 0$, equality holds in the first inequality if and only if G is regular.

Lemma 16. *For $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1 - \frac{1}{r}$, let G be a graph of order n and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a non-negative unit eigenvector of $A_\alpha(G)$ corresponding to $\lambda_\alpha(G)$. Suppose $w \in V(G)$ satisfies $x_w = \min\{x_1, \ldots, x_n\}$. Then*

$$\lambda_\alpha(G-w) \geqslant \lambda_\alpha(G)\frac{1-2x_w^2}{1-x_w^2} - \alpha\frac{1-nx_w^2}{1-x_w^2}.$$

Proof. For simplicity, we write $x = x_w$. By the Rayleigh principle, it follows that

$$\begin{aligned}
\lambda_\alpha(G) &= \mathbf{x}^T A_\alpha(G)\mathbf{x} \\
&= \sum_{uv \in E(G)} \left(\alpha x_u^2 + 2(1-\alpha)x_u x_v + \alpha x_v^2\right) \\
&= \sum_{uv \in E(G-w)} \left(\alpha x_u^2 + 2(1-\alpha)x_u x_v + \alpha x_v^2\right) + \sum_{u \in N(w)} \left(\alpha x^2 + 2(1-\alpha)x x_u + \alpha x_u^2\right) \\
&\leqslant \left(1-x^2\right)\lambda_\alpha(G-w) + \alpha d(w)x^2 + 2(1-\alpha)x\sum_{u \in N(w)} x_u + \alpha\sum_{u \in N(w)} x_u^2. \qquad (1)
\end{aligned}$$

Note that $(1-\alpha)\sum_{u\in N(w)} x_u = (\lambda_\alpha(G)-\alpha d(w))x$. Consequently, we obtain the following inequality:

$$\begin{aligned}
& \alpha d(w)x^2 + 2(1-\alpha)x\sum_{u\in N(w)} x_u + \alpha\sum_{u\in N(w)} x_u^2 \\
\leqslant\ & \alpha d(w)x^2 + 2(\lambda_\alpha(G)-\alpha d(w))x^2 + \alpha(1-(n-d(w))x^2) \\
=\ & 2\lambda_\alpha(G)x^2 - \alpha n x^2 + \alpha.
\end{aligned}$$

Combining this with (1), we obtain

$$\lambda_\alpha(G-w) \geqslant \lambda_\alpha(G)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{1-x^2}.$$

□

Lemma 17. *Let $\mathcal{G}$ be a hereditary graph family. For $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1-\frac{1}{r}$, if $\lambda_\alpha(\mathcal{G}_n) > \left(1-\frac{1}{r}\right)n - \left(1-\frac{1}{r}\right)$, then the limit*

$$\pi_\alpha(\mathcal{G}) := \lim_{n\to\infty}\frac{\lambda_\alpha(\mathcal{G}_n)}{n}$$

exists, and it satisfies

$$\pi_\alpha(\mathcal{G}) \leqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}.$$

Proof. Let $G \in \mathcal{G}_n$ satisfy $\lambda_\alpha(G) = \lambda_\alpha(\mathcal{G}_n)$, and let $\mathbf{x} = (x_1,\ldots,x_n)$ be a non-negative unit eigenvector of $A_\alpha(G)$ corresponding to $\lambda_\alpha(G)$. Suppose $w \in V(G)$ such that $x_w = \min\{x_1,\ldots,x_n\}$. Since $\mathcal{G}$ is hereditary, $G-w \in \mathcal{G}_{n-1}$, which implies that $\lambda_\alpha(\mathcal{G}_{n-1}) \geqslant \lambda_\alpha(G-w)$. For simplicity, let $x = x_w$. Note that $x^2 \leqslant \frac{1}{n}$. By Lemma 16, we obtain

$$\lambda_\alpha(\mathcal{G}_{n-1}) \geqslant \lambda_\alpha(\mathcal{G}_n)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{1-x^2}.$$

Thus,

$$\begin{aligned}
\frac{\lambda_\alpha(\mathcal{G}_{n-1})}{n-2} &\geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1+\frac{1}{n-2}\right)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{(n-2)(1-x^2)} \\
&= \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1+\frac{1-nx^2}{(n-2)(1-x^2)}\right) - \alpha\frac{1-nx^2}{(n-2)(1-x^2)} \\
&\geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1},
\end{aligned}$$

where the last inequality holds as $\lambda_\alpha(\mathcal{G}_n) > \left(1-\frac{1}{r}\right)n - \left(1-\frac{1}{r}\right)$ and $0 \leqslant \alpha \leqslant 1-\frac{1}{r}$.

Therefore, $\frac{\lambda_\alpha(\mathcal{G}_{n-1})}{n-2} \geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1} > 0$. This implies that the limit $\pi_\alpha(\mathcal{G}) := \lim_{n\to\infty}\frac{\lambda_\alpha(\mathcal{G}_n)}{n}$ exists. Moreover, $\pi_\alpha(\mathcal{G}) \leqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}$. □

4 Proofs of Theorems 9 and 10

Lemma 18. *For $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1 - \frac{1}{r}$, let G be a graph on n vertices, and $\mathbf{x} = (x_1, \ldots, x_n)$ be a non-negative unit eigenvector of $A_\alpha(G)$ corresponding to $\lambda_\alpha(G)$. Denote $x = \min\{x_1, \ldots, x_n\}$ and $\delta = \delta(G)$. If $x > 0$, then*

$$\lambda_\alpha(G) \leqslant \alpha\delta + (1-\alpha)\sqrt{\delta^2 + \left(\frac{1}{nx^2} - 1\right) n\delta}.$$

Proof. Let $u \in V(G)$ such that $d(u) = \delta$. According to the eigenvector equation with respect to u, we obtain

$$\frac{1}{1-\alpha}(\lambda_\alpha(G) - \alpha\delta)x_u = \sum_{v \in N(u)} x_v.$$

Hence,

$$\begin{aligned} \left(\frac{1}{1-\alpha}\right)^2 (\lambda_\alpha(G) - \alpha\delta)^2 x^2 &\leqslant \left(\frac{1}{1-\alpha}\right)^2 (\lambda_\alpha(G) - \alpha\delta)^2 x_u^2 \\ &\leqslant \delta \sum_{v \in N(u)} x_v^2 \\ &\leqslant \delta(1 - (n-\delta)x^2), \end{aligned}$$

which implies that

$$\lambda_\alpha(G) \leqslant \alpha\delta + (1-\alpha)\sqrt{\delta^2 + \left(\frac{1}{nx^2} - 1\right) n\delta}.$$

□

Let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = r + 1 \geqslant 3$, and let $\mathcal{G}$ denote the collection of all $\mathcal{F}$-free graphs. Clearly, $\mathcal{G}$ is a hereditary graph family, and $\lambda_\alpha(\mathcal{G}_n) = \mathrm{ex}_\alpha(n, \mathcal{F})$. Note that the r-partite Turán graph $T_{n,r} \in \mathcal{G}_n$. By Lemma 15, we thus have

$$\lambda_\alpha(\mathcal{G}_n) \geqslant \lambda_\alpha(T_{n,r}) \geqslant \frac{2e(T_{n,r})}{n} \geqslant \left(1 - \frac{1}{r}\right) n - \frac{r}{4n}.$$

Furthermore, Lemma 17 guarantees the existence of the limit $\lim_{n\to\infty} \frac{\mathrm{ex}_\alpha(n,\mathcal{F})}{n}$. We denote this limit as $\pi'_\alpha(\mathcal{F})$, so that $\pi'_\alpha(\mathcal{F}) = \pi_\alpha(\mathcal{G})$.

Lemma 19. *For $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1 - \frac{1}{r}$, let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = r + 1 \geqslant 3$. Then $\pi'_\alpha(\mathcal{F}) \geqslant \pi(\mathcal{F})$. Furthermore, if $\pi'_\alpha(\mathcal{F}) > 1 - \frac{1}{r}$, then $\pi'_\alpha(\mathcal{F}) = \pi(\mathcal{F})$.*

Proof. We first prove that $\pi'_\alpha(\mathcal{F}) \geqslant \pi(\mathcal{F})$. Let $G \in \mathrm{Ex}(n, \mathcal{F})$. By Lemma 15,

$$\frac{\mathrm{ex}_\alpha(n,\mathcal{F})}{n} \geqslant \frac{\lambda_\alpha(G)}{n} \geqslant \frac{2\mathrm{ex}(n,\mathcal{F})}{n^2},$$

which implies that $\pi'_\alpha(\mathcal{F}) \geqslant \pi(\mathcal{F})$.

Now, suppose $\pi'_\alpha(\mathcal{F}) > 1 - \frac{1}{r}$. We will show $\pi'_\alpha(\mathcal{F}) \leqslant \pi(\mathcal{F})$, thus finishing the proof. Clearly, $\pi'_\alpha(\mathcal{F}) \leqslant 1$. So we assume $\pi'_\alpha(\mathcal{F}) = 1 - \frac{1}{r} + \varepsilon$, where $0 < \varepsilon \leqslant \frac{1}{r}$. Let $\mathcal{G}$ be the collection of all $\mathcal{F}$-free graphs. Let $G_n \in \mathcal{G}_n$ such that $\lambda_\alpha(G_n) = \lambda_\alpha(\mathcal{G}_n)$. Suppose $\mathbf{x}^{(n)} = \left(x_1^{(n)}, \ldots, x_n^{(n)}\right)$ is a non-negative unit eigenvector of $A_\alpha(G_n)$ corresponding to $\lambda_\alpha(G_n)$. Denote $x^{(n)} = \min\left\{x_1^{(n)}, \ldots, x_n^{(n)}\right\}$.

Claim 20. *There exist infinitely many $n \in \mathbb{N}$ such that*

$$\left(x^{(n)}\right)^2 > \frac{1}{n}\left(1 - \frac{1}{\ln n}\right) > 0.$$

Suppose, for contradiction, that there exists some n_0 such that for all $n \geqslant n_0$,

$$0 \leqslant \left(x^{(n)}\right)^2 \leqslant \frac{1}{n}\left(1 - \frac{1}{\ln n}\right).$$

Note that $\mathcal{G}$ is a hereditary graph family. By Lemma 16, we thus have

$$\lambda_\alpha(\mathcal{G}_{n-1}) \geqslant \lambda_\alpha(\mathcal{G}_n)\frac{1 - 2\left(x^{(n)}\right)^2}{1 - \left(x^{(n)}\right)^2} - \alpha\frac{1 - n\left(x^{(n)}\right)^2}{1 - \left(x^{(n)}\right)^2}.$$

Moreover, since $\lambda_\alpha(\mathcal{G}_n) \geqslant \lambda_\alpha(T_{n,r}) \geqslant \left(1 - \frac{1}{r}\right)n - \frac{r}{4n}$, it follows from Lemma 17 that $\frac{\lambda_\alpha(\mathcal{G}_n)}{n-1} \geqslant \pi_\alpha(\mathcal{G}) = \pi'_\alpha(\mathcal{F}) = 1 - \frac{1}{r} + \varepsilon$. Now, let $n \geqslant n_0$ and let $x = x^{(n)}$. Then we get

$$\begin{aligned}
\frac{\lambda_\alpha(\mathcal{G}_{n-1})}{n-2} &\geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1 + \frac{1}{n-2}\right)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{(n-2)(1-x^2)} \\
&= \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1 + \left(1 - \alpha\frac{n-1}{\lambda_\alpha(\mathcal{G}_n)}\right)\frac{1-nx^2}{(n-2)(1-x^2)}\right) \\
&\geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1 + \frac{\varepsilon(1-nx^2)}{(n-2)(1-x^2)}\right) \\
&\geqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n-1}\left(1 + \frac{\varepsilon}{n\ln n}\right),
\end{aligned}$$

where the second inequality holds as $0 \leqslant \alpha \leqslant 1 - \frac{1}{r}$ and $\frac{n-1}{\lambda_\alpha(\mathcal{G}_n)} \leqslant \frac{1}{1-\frac{1}{r}+\varepsilon}$, and the last inequality holds as $x^2 \leqslant \frac{1}{n}\left(1 - \frac{1}{\ln n}\right)$. Thus, for any $i \geqslant n_0$, the following inequality holds:

$$\frac{\lambda_\alpha(\mathcal{G}_{i-1})}{i-2} - \frac{\lambda_\alpha(\mathcal{G}_i)}{i-1} \geqslant \frac{\varepsilon}{i\ln i}\cdot\frac{\lambda_\alpha(\mathcal{G}_i)}{i-1}.$$

Summing both sides over i from n_0 to $2n$, we further obtain

$$\begin{aligned}\frac{\lambda_\alpha(\mathcal{G}_{n_0-1})}{n_0-2}-\frac{\lambda_\alpha(\mathcal{G}_{2n})}{2n-1} &= \sum_{i=n_0}^{2n}\left(\frac{\lambda_\alpha(\mathcal{G}_{i-1})}{i-2}-\frac{\lambda_\alpha(\mathcal{G}_{i})}{i-1}\right)\\ &\geqslant \sum_{i=n_0}^{2n}\frac{\varepsilon}{i\ln i}\cdot\frac{\lambda_\alpha(\mathcal{G}_{i})}{i-1}\\ &\geqslant \varepsilon\pi'_\alpha(\mathcal{F})\sum_{i=n_0}^{2n}\frac{1}{i\ln i},\end{aligned}$$

where the last inequality holds as $\frac{\lambda_\alpha(\mathcal{G}_i)}{i-1}\geqslant\pi_\alpha(\mathcal{G})=\pi'_\alpha(\mathcal{F})$. Taking n sufficiently large, we get a contradiction as $\sum_{i=n_0}^{2n}\frac{1}{i\ln i}$ diverges. This completes the proof of Claim 1. □

By Claim 1, there exists an infinite sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that for each $n\in\{n_k\}_{k=1}^{\infty}$,

$$\left(x^{(n)}\right)^2>\frac{1}{n}\left(1-\frac{1}{\ln n}\right)>0.$$

In light of Lemma 18, we deduce that

$$\begin{aligned}\frac{\lambda_\alpha(G_n)}{n} &\leqslant \frac{\alpha\delta(G_n)}{n}+(1-\alpha)\sqrt{\left(\frac{\delta(G_n)}{n}\right)^2+\left(\frac{1}{n\left(x^{(n)}\right)^2}-1\right)\frac{\delta(G_n)}{n}}\\ &\leqslant \frac{\alpha\delta(G_n)}{n}+(1-\alpha)\left(\frac{\delta(G_n)}{n}+\frac{1}{1-\frac{1}{\ln n}}-1\right)\\ &\leqslant \frac{2\mathrm{ex}(n,\mathcal{F})}{n^2}+(1-\alpha)\left(\frac{1}{1-\frac{1}{\ln n}}-1\right).\end{aligned}$$

Therefore, we obtain

$$\pi'_\alpha(\mathcal{F})=\lim_{n\to\infty}\frac{\lambda_\alpha(G_n)}{n}\leqslant\lim_{n\to\infty}\left(\frac{2\mathrm{ex}(n,\mathcal{F})}{n^2}+(1-\alpha)\left(\frac{1}{1-\frac{1}{\ln n}}-1\right)\right)=\pi(\mathcal{F}).$$

Thus, $\pi'_\alpha(\mathcal{F})=\pi(\mathcal{F})$. The proof is complete. □

We are now ready to prove Theorem 9.

Proof of Theorem 9. It suffices to show that $\pi'_\alpha(\mathcal{F})=1-\frac{1}{r}$. We prove the assertion by contradiction. Recall that $\pi'_\alpha(\mathcal{F})\geqslant\pi(\mathcal{F})=1-\frac{1}{r}$. If $\pi'_\alpha(\mathcal{F})>1-\frac{1}{r}$, then by Lemma 19, we deduce that $\pi'_\alpha(\mathcal{F})=\pi(\mathcal{F})=1-\frac{1}{r}$, which is a contradiction. Thus $\pi'_\alpha(\mathcal{F})=\pi(\mathcal{F})=1-\frac{1}{r}$. This completes the proof. □

Next, we give the proof of Theorem 10.

Proof of Theorem 10. We first consider the case where $\mathcal{F}$ does not contain $K_{1,k} \cup sK_1$ for any $k \geqslant 1, s \geqslant 0$. In this case, $K_{1,n-1}$ is $\mathcal{F}$-free. Let $G \in \mathrm{Ex}_\alpha(n, \mathcal{F})$. By Lemma 14, we have $\lambda_\alpha(G) \geqslant \lambda_\alpha(K_{1,n-1}) \geqslant \alpha(n-1)$. Since $\chi(\mathcal{F}) = 2$, there exists a number $t > 0$ and $F \in \mathcal{F}$ such that $F \subset K_{t,t}$. By the well-known Kővári-Sós-Turán theorem, $\mathrm{ex}(n, K_{t,t}) = O(n^{2-\frac{1}{t}})$. Therefore, by Lemma 14, we obtain

$$\alpha(n-1) \leqslant \lambda_\alpha(G) \leqslant \alpha\Delta(G) + (1-\alpha)\lambda(G) \leqslant \alpha(n-1) + (1-\alpha)\sqrt{2e(G)} = \alpha n + o(n).$$

Then $\mathrm{ex}_\alpha(n, \mathcal{F}) = (\alpha + o(1))n$. Next, we assume $\mathcal{F}$ contains $K_{1,k} \cup sK_1$ for some $k \geqslant 1, s \geqslant 0$. Then $\Delta(G) \leqslant k-1$. It follows from Lemma 14 that

$$\lambda_\alpha(G) \leqslant \alpha\Delta(G) + (1-\alpha)\lambda(G) \leqslant \Delta(G) \leqslant k-1.$$

Thus, $\mathrm{ex}_\alpha(n, \mathcal{F}) = O(1)$. If $\mathcal{F} = \{K_{1,k}\}$, then $K_k \cup I_{n-k}$ is $\mathcal{F}$-free. Thus $\lambda_\alpha(G) \geqslant \lambda_\alpha(K_k \cup I_{n-k}) = k-1$. Hence, $\mathrm{ex}_\alpha(n, \mathcal{F}) = k-1$. The proof is complete. □

5 Proof of Theorem 11

5.1 Some auxiliary lemmas

Keevash, Lenz, and Mubayi [10] established a criterion that allows spectral extremal problems involving the p-spectral radius of hypergraphs to be deduced from their corresponding hypergraph Turán problems. Zheng, Li, and Li [23] later proposed a similar criterion for the signless Laplacian spectral radius. Motivated by these works, we develop a criterion for the α-spectral radius, which reduces α-spectral Turán-type problems to classical extremal problems satisfying the degree-stable property.

Theorem 21. *Let $\mathcal{F}$ be a family of graphs with $\chi(\mathcal{F}) = r+1 \geqslant 3$. Suppose $0 < \varepsilon < \frac{1}{2}$, $\sigma < \frac{\varepsilon^3}{7}$ and $0 \leqslant \alpha \leqslant 1 - \frac{1}{r} - \varepsilon$. Let $\mathcal{G}'_n$ be the set of all $\mathcal{F}$-free graphs on n vertices with minimum degree larger than $(\pi(\mathcal{F}) - \varepsilon)n$. Suppose that there exists $N > 0$ such that for every $n > N$, we have*

$$|\mathrm{ex}(n, \mathcal{F}) - \mathrm{ex}(n-1, \mathcal{F}) - \pi(\mathcal{F})n| \leqslant \sigma n \tag{2}$$

and

$$\left| \lambda_\alpha(\mathcal{G}'_n) - \frac{2\mathrm{ex}(n, \mathcal{F})}{n} \right| \leqslant \sigma. \tag{3}$$

Then there exists $n_0 > 0$ such that for any $\mathcal{F}$-free graph G on $n \geqslant n_0$ vertices, we have

$$\lambda_\alpha(G) \leqslant \lambda_\alpha(\mathcal{G}'_n).$$

In addition, if the equality holds, then $G \in \mathcal{G}'_n$.

Now we prove Theorem 21. Let G be an $\mathcal{F}$-free graph on n vertices. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a non-negative unit eigenvector of $A_\alpha(G)$ corresponding to $\lambda_\alpha(G)$. Suppose $w \in V(G)$ such that $x = x_w = \min\{x_1, \dots, x_n\}$. Let $\delta = \delta(G)$.

From (3), it is readily seen that $\frac{\lambda_\alpha(\mathcal{G}'_n)}{n} - \frac{2\mathrm{ex}(n,\mathcal{F})}{n^2} = o(1)$. Recalling that $\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{2\mathrm{ex}(n,\mathcal{F})}{n^2}$, this immediately implies

$$\lambda_\alpha(\mathcal{G}'_n) = (\pi(\mathcal{F}) + o(1))n. \tag{4}$$

Clearly, $\delta(T_{n,r}) \geqslant (1 - 1/r)\,n - 1 > (1 - 1/r - \varepsilon)\,n$, and $T_{n,r}$ is $\mathcal{F}$-free. Thus, $T_{n,r} \in \mathcal{G}'_n$. Combining Lemma 15 with the known lower bound $e(T_{n,r}) \geqslant \frac{r-1}{2r}n^2 - \frac{r}{8}$, we deduce

$$\lambda_\alpha(\mathcal{G}'_n) \geqslant \left(1 - \frac{1}{r}\right) n - \frac{r}{4n}. \tag{5}$$

Lemma 22. $\lambda_\alpha(\mathcal{G}'_n) \geqslant \lambda_\alpha(\mathcal{G}'_{n-1}) + \pi(\mathcal{F}) - 5\sigma$.

Proof. By the triangle inequality, combining (2) and (3), we deduce that

$$\begin{aligned}
&\left|\lambda_\alpha(\mathcal{G}'_n) - \lambda_\alpha(\mathcal{G}'_{n-1}) - \pi(\mathcal{F})\right| \\
\leqslant\ & \left|\frac{2\mathrm{ex}(n,\mathcal{F})}{n} - \frac{2\mathrm{ex}(n-1,\mathcal{F})}{n-1} - \pi(\mathcal{F})\right| + 2\sigma \\
=\ & \left|\frac{2}{n}(\mathrm{ex}(n,\mathcal{F}) - \mathrm{ex}(n-1,\mathcal{F}) - \pi(\mathcal{F})n) + \pi(\mathcal{F}) - \frac{2\mathrm{ex}(n-1,\mathcal{F})}{n(n-1)}\right| + 2\sigma \\
\leqslant\ & \left|\pi(\mathcal{F}) - \frac{2\mathrm{ex}(n-1,\mathcal{F})}{n(n-1)}\right| + 4\sigma.
\end{aligned}$$

Note that $\lim_{n\to\infty} \frac{2\mathrm{ex}(n-1,\mathcal{F})}{n(n-1)} = \pi(\mathcal{F})$. Thus, for sufficiently large n, we have

$$\lambda_\alpha(\mathcal{G}'_n) \geqslant \lambda_\alpha(\mathcal{G}'_{n-1}) + \pi(\mathcal{F}) - 5\sigma.$$

□

Lemma 23. $x^2 \leqslant \frac{\delta(1-\alpha)^2}{(\lambda_\alpha(G)-\alpha\delta)^2 + \delta(n-\delta)(1-\alpha)^2}$.

Proof. Let $u \in V(G)$ such that $d(u) = \delta$. According to the eigenvector equation with respect to the vertex u, we get

$$(\lambda_\alpha(G) - \alpha\delta)x_u = (1-\alpha)\sum_{v\in N(u)} x_v.$$

Combining this with Power Mean inequality, we obtain

$$\begin{aligned}
\left(\frac{1}{1-\alpha}(\lambda_\alpha(G) - \alpha\delta)x\right)^2 &\leqslant \left(\sum_{v\in N(u)} x_v\right)^2 \leqslant \delta \sum_{v\in N(u)} x_v^2 \\
&\leqslant \delta\left(1 - (n-\delta)x^2\right),
\end{aligned}$$

which implies that

$$x^2 \leqslant \frac{\delta(1-\alpha)^2}{(\lambda_\alpha(G)-\alpha\delta)^2+\delta(n-\delta)(1-\alpha)^2}.$$

□

Lemma 24. *If* $\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n)$ *and* $\delta \leqslant (\pi(\mathcal{F})-\varepsilon)n$*, then for sufficiently large* n*, we have*

$$x^2 < \frac{1-\varepsilon^2}{n}.$$

Proof. If $\delta = 0$, then $x = 0$, the result follows. In the following, we assume $\delta \geqslant 1$. Set $\theta = \frac{\varepsilon}{2}$. By the assumption that $\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n)$ and (5), we get

$$\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n) \geqslant \left(1-\frac{1}{r}\right)n - \frac{r}{4n} \geqslant (\pi(\mathcal{F})-\theta)n. \tag{6}$$

Let

$$f(y,z) = \frac{1}{y}(z-\alpha y)^2 + (n-y)(1-\alpha)^2,$$

where $1 \leqslant y \leqslant (\pi(\mathcal{F})-\varepsilon)n$ and $z \geqslant (\pi(\mathcal{F})-\theta)n$.

A straightforward calculation shows

$$\frac{\partial f(y,z)}{\partial y} = -\left(\frac{z^2}{y^2}+1-2\alpha\right) \text{ and } \frac{\partial f(y,z)}{\partial z} = 2\left(\frac{z}{y}-\alpha\right),$$

which implies that $\frac{\partial f(y,z)}{\partial y} \leqslant 0$ and $\frac{\partial f(y,z)}{\partial z} > 0$ for $1 \leqslant y \leqslant (\pi(\mathcal{F})-\varepsilon)n$ and $z \geqslant (\pi(\mathcal{F})-\theta)n$.

Combining this with (6) and assumption $\delta \leqslant (\pi(\mathcal{F})-\varepsilon)n$, we get

$$f(\delta, \lambda_\alpha(G)) \geqslant f((\pi(\mathcal{F})-\varepsilon)n, (\pi(\mathcal{F})-\theta)n).$$

By Lemma 23, we have

$$\begin{aligned} x^2 n &\leqslant \frac{(1-\alpha)^2}{f(\delta,\lambda_\alpha(G))}n \leqslant \frac{(1-\alpha)^2}{f((\pi(\mathcal{F})-\varepsilon)n,(\pi(\mathcal{F})-\theta)n)}n \\ &= \frac{(1-\alpha)^2(\pi(\mathcal{F})-\varepsilon)}{((1-\alpha)(\pi(\mathcal{F})-\varepsilon)+\varepsilon-\theta)^2+(1-\alpha)^2(\pi(\mathcal{F})-\varepsilon)(1-\pi(\mathcal{F})+\varepsilon)} \\ &= \frac{(1-\alpha)^2(\pi(\mathcal{F})-\varepsilon)}{(1-\alpha)^2(\pi(\mathcal{F})-\varepsilon)+2(1-\alpha)(\pi(\mathcal{F})-\varepsilon)(\varepsilon-\theta)+(\varepsilon-\theta)^2} \\ &\leqslant \frac{1}{1+\frac{2(\varepsilon-\theta)}{1-\alpha}} < 1-\varepsilon^2, \end{aligned}$$

completing the proof. □

Lemma 25. *Suppose G satisfies $\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n)$ and $x^2 < \frac{1-\varepsilon^2}{n}$. Then, for sufficiently large n, we have*

$$\lambda_\alpha(G-w) \geqslant \lambda_\alpha(G)\left(1-\frac{1-\varepsilon^3}{n-1}\right) \tag{7}$$

and

$$\lambda_\alpha(G-w) > \lambda_\alpha(\mathcal{G}'_{n-1}). \tag{8}$$

Proof. By Lemma 16, we obtain

$$\lambda_\alpha(G-w) \geqslant \lambda_\alpha(G)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{1-x^2}.$$

Combining this with the assumption $\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n)$, (5), and $\alpha \leqslant 1-\frac{1}{r}-\varepsilon$, we get

$$\begin{aligned}
\frac{\lambda_\alpha(G-w)}{n-2} &\geqslant \frac{\lambda_\alpha(G)}{n-1}\left(1+\frac{1}{n-2}\right)\frac{1-2x^2}{1-x^2} - \alpha\frac{1-nx^2}{(n-2)(1-x^2)} \qquad (9)\\
&= \frac{\lambda_\alpha(G)}{n-1}\left(1+\left(1-\alpha\frac{n-1}{\lambda_\alpha(G)}\right)\frac{1-nx^2}{(n-2)(1-x^2)}\right)\\
&\geqslant \frac{\lambda_\alpha(G)}{n-1}\left(1+\left(1-\left(1-\frac{1}{r}-\varepsilon\right)\frac{n-1}{\left(1-\frac{1}{r}\right)n-\frac{r}{4n}}\right)\cdot\frac{1-nx^2}{(n-2)(1-x^2)}\right)\\
&\geqslant \frac{\lambda_\alpha(G)}{n-1}\left(1+\frac{\varepsilon}{1-\frac{1}{r}}\cdot\frac{1-nx^2}{(n-2)(1-x^2)}\right)\\
&\geqslant \frac{\lambda_\alpha(G)}{n-1}\left(1+\frac{r\varepsilon^3}{(r-1)(n-1)}\right),
\end{aligned}$$

where the last inequality holds as $x^2 < \frac{1-\varepsilon^2}{n}$.

Then, it follows that

$$\begin{aligned}
\lambda_\alpha(G-w) &\geqslant \lambda_\alpha(G)\left(1-\frac{1}{n-1}\right)\left(1+\frac{r\varepsilon^3}{(r-1)(n-1)}\right)\\
&\geqslant \lambda_\alpha(G)\left(1-\frac{1-\varepsilon^3}{n-1}\right).
\end{aligned}$$

Next we prove (8). By (9), we obtain

$$\begin{aligned}
\frac{\lambda_\alpha(G-w)}{n-2} &\geqslant \frac{\lambda_\alpha(G)}{n-1}\left(1+\frac{1-nx^2}{(n-2)(1-x^2)}\right)-\alpha\frac{1-nx^2}{(n-2)(1-x^2)}\\
&= \frac{\lambda_\alpha(G)}{n-1}+\left(\frac{\lambda_\alpha(G)}{n-1}-\alpha\right)\frac{1-nx^2}{(n-2)(1-x^2)}\\
&\geqslant \frac{\lambda_\alpha(G)}{n-1}+\left(\frac{\left(1-\frac{1}{r}\right)(n-1)}{n-1}-\left(1-\frac{1}{r}-\varepsilon\right)\right)\frac{1-nx^2}{(n-2)(1-x^2)}\\
&= \frac{\lambda_\alpha(G)}{n-1}+\frac{\varepsilon(1-nx^2)}{(n-2)(1-x^2)}\\
&> \frac{\lambda_\alpha(G)}{n-1}+\frac{\varepsilon^3}{n-2},
\end{aligned}\tag{10}$$

where the second inequality holds as $\lambda_\alpha(G)\geqslant\left(1-\frac{1}{r}\right)(n-1)$ and $\alpha\leqslant 1-\frac{1}{r}-\varepsilon$, and the last inequality holds as $x^2<\frac{1-\varepsilon^2}{n}$.

For sufficiently large n, by (4), we have $\lambda_\alpha(\mathcal{G}'_{n-1})\leqslant(\pi(\mathcal{F})+\sigma)(n-1)$. Combining this with (10), Lemma 22 and the assumption $\lambda_\alpha(G)\geqslant\lambda_\alpha(\mathcal{G}'_n)$, we get

$$\begin{aligned}
\lambda_\alpha(G-w) &\geqslant \lambda_\alpha(G)\left(1-\frac{1}{n-1}\right)+\varepsilon^3\\
&\geqslant (\lambda_\alpha(\mathcal{G}'_{n-1})+\pi(\mathcal{F})-5\sigma)\left(1-\frac{1}{n-1}\right)+\varepsilon^3\\
&\geqslant \lambda_\alpha(\mathcal{G}'_{n-1})-7\sigma+\varepsilon^3\\
&> \lambda_\alpha(\mathcal{G}'_{n-1}),
\end{aligned}$$

where the last inequality holds as $\sigma<\frac{\varepsilon^3}{7}$. The proof is complete. $\square$

The following facts, proved by Zheng, Li, and Li [23], will be employed in our proof

Fact 1. If $0<x<\frac{1}{2}$ and $0<a<1$, then $\ln(1-ax)+ax+x^2>0$.

Fact 2. If $x>1$, then $\frac{1}{x}<\ln x-\ln(x-1)$ and $\frac{1}{x^2}<\frac{1}{x-1}-\frac{1}{x}$.

We are now ready to prove Theorem 21.

Proof of Theorem 21. We prove by contradiction. Suppose to the contrary that there exists an $\mathcal{F}$-free graph G of order n such that $\lambda_\alpha(G)\geqslant\lambda_\alpha(\mathcal{G}'_n)$ and $G\notin\mathcal{G}'_n$. Let N be a sufficiently large integer such that Lemma 24 and Lemma 25 hold for $n>N$. Moreover, according to (5), we assume $\lambda_\alpha(\mathcal{G}'_n)\geqslant(1-\varepsilon)\pi(\mathcal{F})n$ for $n>N$. Let $n_0=\left(\frac{Ne}{(1-\varepsilon)\pi(\mathcal{F})}\right)^{\frac{1}{\varepsilon^3}}$. Note that $n_0>N$. Let $\mathbf{x}^{(n)}=\left(x_1^{(n)},\ldots,x_n^{(n)}\right)$ be a non-negative unit eigenvector of $A_\alpha(G)$

corresponding to $\lambda_\alpha(G)$. Suppose $w^{(n)} \in V(G)$ such that $x_{w^{(n)}} = \min\left\{x_1^{(n)}, \ldots, x_n^{(n)}\right\}$. Recall that $\lambda_\alpha(G) \geqslant \lambda_\alpha(\mathcal{G}'_n)$ and $\delta(G) \leqslant (\pi(\mathcal{F}) - \varepsilon)n$. Then, by Lemma 24, $x^2_{w^{(n)}} < \frac{1-\varepsilon^2}{n}$. For $n \geqslant n_0$, denote $G_n = G$ and $G_{n-1} = G_n - w^{(n)}$. Then, it follows from Lemma 25 that

$$\lambda_\alpha(G_{n-1}) \geqslant \lambda_\alpha(G_n)\left(1 - \frac{1-\varepsilon^3}{n-1}\right)$$

and

$$\lambda_\alpha(G_{n-1}) > \lambda_\alpha(\mathcal{G}'_{n-1}).$$

Since $\lambda_\alpha(G_{n-1}) > \lambda_\alpha(\mathcal{G}'_{n-1})$, $G_{n-1} \notin \mathcal{G}'_{n-1}$. So $\delta(G_{n-1}) \leqslant (\pi(\mathcal{F}) - \varepsilon)(n-1)$. Let $\mathbf{x}^{(n-1)} = \left(x_1^{(n-1)}, \ldots, x_{n-1}^{(n-1)}\right)$ be a non-negative unit eigenvector of $A_\alpha(G_{n-1})$ corresponding to $\lambda_\alpha(G_{n-1})$. Suppose $w^{(n-1)} \in V(G_{n-1})$ such that $x_{w^{(n-1)}} = \min\left\{x_1^{(n-1)}, \ldots, x_{n-1}^{(n-1)}\right\}$. Denote $G_{n-2} = G_{n-1} - w^{(n-1)}$. Continue this operation, we can obtain a sequence of graphs G_n, G_{n-1}, $\ldots$, which satisfies

$$\lambda_\alpha(G_{i-1}) \geqslant \lambda_\alpha(G_i)\left(1 - \frac{1-\varepsilon^3}{i-1}\right)$$

and

$$\lambda_\alpha(G_{i-1}) > \lambda_\alpha(\mathcal{G}'_{i-1})$$

for each $n \geqslant i > N$.

Then, for $i = N+1$, we have

$$\begin{aligned}
\lambda_\alpha(G_N) &\geqslant \lambda_\alpha(G_{N+1})\left(1 - \frac{1-\varepsilon^3}{N}\right) \\
&\geqslant \lambda_\alpha(G_n)\prod_{i=N+1}^{n}\left(1 - \frac{1-\varepsilon^3}{i-1}\right) \\
&\geqslant \lambda_\alpha(G_n)\exp\left(-\sum_{i=N+1}^{n}\left(\frac{1-\varepsilon^3}{i-1} + \left(\frac{1}{i-1}\right)^2\right)\right) \\
&\geqslant \lambda_\alpha(G_n)\exp\left(-(1-\varepsilon^3)\ln\frac{n}{N-1} - 1\right) \\
&\geqslant (1-\varepsilon)\pi(\mathcal{F})n\left(\frac{n}{N-1}\right)^{-(1-\varepsilon^3)} e^{-1} \\
&\geqslant (1-\varepsilon)\pi(\mathcal{F})n^{\varepsilon^3}e^{-1} \\
&\geqslant (1-\varepsilon)\pi(\mathcal{F})n_0^{\varepsilon^3}e^{-1} = N,
\end{aligned}$$

where the third inequality holds as Fact 1 and the fourth inequality holds as Fact 2. This contradicts the fact that $\lambda_\alpha(G_N) \leqslant N - 1$. The proof is complete. □

5.2 Proof of Theorem 11

Andrásfai, Erdős, and Sós [1] showed that K_{r+1} is degree-stable with respect to the family of r-partite graphs. Erdős and Simonovits [7] extended the result to all color-critical graphs.

Lemma 26 (Erdős and Simonovits [7])**.** *Let F be a color-critical graph with $\chi(F) = r+1 \geqslant 3$. There is n_0 such that if G is an F-free graph on $n \geqslant n_0$ vertices with $\delta(G) > \frac{3r-4}{3r-1}n$, then G is r-partite.*

Next, we derive an upper bound for $\lambda_\alpha(T_{n,r})$. To this end, we first state the following lemma.

Lemma 27. *Let $\mathcal{G}$ be a hereditary and multiplicative graph family, and let $r \geqslant 2$. If for any $0 \leqslant \alpha \leqslant 1-\frac{1}{r}$, the inequality $\lambda_\alpha(\mathcal{G}_n) > \left(1-\frac{1}{r}\right)n - \left(1-\frac{1}{r}\right)$ holds, then*

$$\lambda_\alpha(G) \leqslant \pi_\alpha(\mathcal{G})n$$

for every $G \in \mathcal{G}_n$.

Proof. Let $H \in \mathcal{G}_n$ satisfy $\lambda_\alpha(H) = \lambda_\alpha(\mathcal{G}_n)$. Since $\mathcal{G}$ is multiplicative, $H^p \in \mathcal{G}_{pn}$ for all $p \geqslant 1$. Using the quotient matrix method, it can be verified that $\lambda_\alpha(H^p) = p\lambda_\alpha(H)$. Consequently,

$$\frac{\lambda_\alpha(\mathcal{G}_n)}{n} = \frac{\lambda_\alpha(H)}{n} = \frac{\lambda_\alpha(H^p)}{pn} \leqslant \frac{\lambda_\alpha(\mathcal{G}_{pn})}{pn}.$$

By Lemma 17 and the assumption $\lambda_\alpha(\mathcal{G}_n) > \left(1-\frac{1}{r}\right)n - \left(1-\frac{1}{r}\right)$, we have $\lim_{p\to\infty}\frac{\lambda_\alpha(\mathcal{G}_{pn})}{pn} = \pi_\alpha(\mathcal{G})$. For any $G \in \mathcal{G}_n$, we thus have

$$\frac{\lambda_\alpha(G)}{n} \leqslant \frac{\lambda_\alpha(\mathcal{G}_n)}{n} \leqslant \pi_\alpha(\mathcal{G}),$$

which implies that $\lambda_\alpha(G) \leqslant \pi_\alpha(\mathcal{G})n$. ☐

Lemma 28. *For $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1-\frac{1}{r}$, we have $\lambda_\alpha(T_{n,r}) \leqslant \left(1-\frac{1}{r}\right)n$.*

Proof. Let $\mathcal{G}$ denote the collection of all K_{r+1}-free graphs. It is straightforward that $\mathcal{G}$ is both hereditary and multiplicative. Since $T_{n,r} \in \mathcal{G}_n$, $\lambda_\alpha(\mathcal{G}_n) \geqslant \left(1-\frac{1}{r}\right)n - \frac{r}{4n}$. By Theorem 9, $\pi_\alpha(\mathcal{G}) = 1-\frac{1}{r}$, and thus Lemma 27 yields $\lambda_\alpha(T_{n,r}) \leqslant \left(1-\frac{1}{r}\right)n$. ☐

We are now ready to give the proof of Theorem 11.

Proof of Theorem 11. Let F be a color-critical graph with $\chi(F) = r+1 \geqslant 3$, and let G be an arbitrary F-free n-vertex graph. Recall that $\mathrm{ex}(n,F) = e(T_{n,r}) = \left(1-\frac{1}{r}\right)\frac{n^2}{2} + O(1)$ and $\pi(F) = 1-\frac{1}{r}$. Let $\varepsilon > 0$ and $\sigma < \frac{\varepsilon^3}{7}$ be two sufficiently small constants. For sufficiently large n, we have

$$|\mathrm{ex}(n,F) - \mathrm{ex}(n-1,F) - \pi(F)n| = O(1) \leqslant \sigma n.$$

Let $\mathcal{G}_n'$ be the set of all F-free graphs on n vertices with minimum degree more than $(\pi(F)-\varepsilon)n$. Note that $1-\frac{1}{r}-\varepsilon > \frac{3r-4}{3r-1}$ as ε is sufficiently small. Then, for sufficiently large n, every graph in $\mathcal{G}_n'$ is an r-partite graph by Lemma 26. Clearly, $T_{n,r} \in \mathcal{G}_n'$, which implies that $\lambda_\alpha(\mathcal{G}_n') = \lambda_\alpha(T_{n,r})$ by Theorem 7. Combining with (5) and Lemma 28, we deduce that $\lambda_\alpha(\mathcal{G}_n') = \lambda_\alpha(T_{n,r}) = \left(1-\frac{1}{r}\right)n + o(1)$. Hence,

$$\left|\lambda_\alpha(\mathcal{G}_n') - \frac{2\mathrm{ex}(n,F)}{n}\right| = o(1) \leqslant \sigma.$$

Therefore, by Theorem 21, $\lambda_\alpha(G) \leqslant \lambda_\alpha(\mathcal{G}_n')$ with equality if and only if $G = T_{n,r}$. This completes the proof. □

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