Supersaturation of odd linear cycles

Lirong Deng[a] Jie Han[a] Jiaxi Nie[b] Sam Spiro[c]

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Abstract

An r-uniform linear cycle of length ℓ, denoted by C^r_ℓ, is an r-graph with ℓ edges $e_1, e_2, \ldots, e_\ell$ where $e_i = \{v_{(r-1)(i-1)}, v_{(r-1)(i-1)+1}, \ldots, v_{(r-1)i}\}$ (here $v_0 = v_{(r-1)\ell}$). For $0 < \delta < 1$ and n sufficiently large, we show that every n-vertex r-graph G with $n^{r-\delta}$ edges contains at least $n^{(r-1)(2\ell+1)-\delta(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3})-o(1)}$ copies of $C^r_{2\ell+1}$. Further, conditioning on the existence of dense high-girth hypergraphs, we show that there exists n-vertex r-graphs with $n^{r-\delta}$ edges and at most $n^{(r-1)(2\ell+1)-\delta(2\ell+1+\frac{1}{(r-1)\ell-1})+o(1)}$ copies of $C^r_{2\ell+1}$.

Mathematics Subject Classifications: 05C35, 05C38, 05C65,05D40

1 Introduction

Given an r-uniform hypergraph (r-graph for short) F, the Turán number $\mathrm{ex}(F, n)$ is the minimum interger M such that every n-vertex r-graph with more than M edges contains a subgraph isomorphic to F (referred to as a copy of F). Determining or estimating $\mathrm{ex}(F, n)$ is one of the central topics in extremal combinatorics; this kind of problem is known as the *Turán problem.* A natural extension of the Turán problem ask: if the number of edges in an n-vertex r-graph is more than $\mathrm{ex}(n, F)$, then how many copies of F must it necessarily contain? This kind of problem is called the *supersaturation problem.*

For a given r-graph F, we use $v(F)$ and $e(F)$ to denote its numbers of vertices and edges. Erdős and Simonovits [11] proved a general conclusion, that is, if an r-uniform hypergraph G has $\mathrm{ex}(n, F) + cn^r$ hyperedges, then G contains at least $\varepsilon n^{v(F)}$ copies of F where ε is depending on c, see also [23, 41, 37, 40] and references therein. When considering the case of "weakly supersaturated graphs", specifically when $e(G) = \mathrm{ex}(n, F) + o(n^r)$, the scenario becomes more intricate, see [27, 29, 38] and references therein. In the case of graphs ($r = 2$), Erdős and Simonovits made the following conjecture for the supersaturation problem for bipartite graphs:

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[a] School of Mathematics and Statistics, Beijing Institute of Technology. Email: (`lirong.deng@bit.edu.cn`, `han.jie@bit.edu.cn`).

[b] School of Mathematics, Georgia Institute of Technology (`jnie47@gatech.edu`).

[c] Department of Mathematics, Georgia State University(`sspiro@gsu.edu`).

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Conjecture 1 ([46])**.** Let F be a bipartite graph such that $\mathrm{ex}(n, F) = O(n^{2-\alpha})$ for some $0 < \alpha < 1$. There exist positive constants c and $c' > 0$ such that if G is an n-vertex graph such that $e(G) \geqslant cn^{2-\alpha}$, then G contains at least

$$c' \frac{e(G)^{e(F)}}{n^{2e(F)-v(F)}}$$

copies of F.

Note that the number $c' \frac{e(G)^{e(F)}}{n^{2e(F)-v(F)}}$ in Conjecture 1.1 is achieved by the random graph with $e(G)$ edges: it contains in expectation $\binom{n}{v(F)}\big(\frac{e(G)}{\binom{n}{2}}\big)^{e(F)} \approx c' \frac{e(G)^{e(F)}}{n^{2e(F)-v(F)}}$ copies of F. In [13], Simonovits proved Conjecture 1 for even cycles $C_{2\ell}$. In [10], Erdős and Simonovits proved it for the path graph P_4. Furthermore, in [11], they extended their results to many other cases, including the cube graph and the complete bipartite graph $K_{p,q}$. For additional references, the reader is referred to Lovász and Simonovits [26], Razborov [39], Lovász [25], and Reiher [41].

It turns out that the supersaturation problem is closely related to the *Sidorenko problem.* A *homomorphism* from an r-graph F to an r-graph H is a map $\phi : V(F) \to V(H)$ such that $\phi(e)$ is an edge of H whenever e is an edge of F. Let $\hom(F, H)$ denote the number of homomorphisms from F to H and define the *homomorphism density*

$$t_F(H) = \frac{\hom(F, H)}{v(H)^{v(F)}},$$

which is equivalently the probability that a random map $\phi : V(F) \to V(H)$ is a homomorphism. From this, we can derive an important definition regarding homomorphism density.

Definition 2. We say that an r-graph F is *Sidorenko* if for every r-graph H we have

$$t_F(H) \geqslant t_{K_r^r}(H)^{e(F)},$$

where K_r^r is the r-graph consisting of a single edge.

Observe that if F is not r-partite, then F is not Sidorenko (due to $H = K_r^r$, for example). For bipartite graphs, Sidorenko made the following conjecture.

Conjecture 3 ([44, 45])**.** Every bipartite graph is Sidorenko.

A considerable body of research has been devoted to the investigation of Sidorenko's conjecture [3, 4, 5, 6, 8, 15, 17, 20, 22, 24, 47]. Despite these efforts, the conjecture remains widely open.

It is well-established that Sidorenko's conjecture does not generalize to hypergraphs. Indeed, Sidorenko [44] showed that r-uniform linear triangles are not Sidorenko for $r \geqslant 3$. This result has historically limited interest in identifying which r-graphs satisfy the Sidorenko property. However, this trend has recently shifted with the seminal work of Conlon, Lee, and Sidorenko [7], who unveiled a profound connection between non-Sidorenko hypergraphs and Turán numbers.

Theorem 4 ([7]). *If F is not Sidorenko, then there exists $c = c(F)$ such that*

$$\mathrm{ex}(n, F) = \Omega\left(n^{r-\frac{v(F)-r}{e(F)-1}+c}\right).$$

We note that this improves upon the lower bound $\mathrm{ex}(n, F) = \Omega(n^{r-\frac{v(F)-r}{e(F)-1}})$ which holds for all F by a standard random deletion argument.

Nie and Spiro [35] introduced the following notion to measure how far an r-graph is from being Sidorenko.

Definition 5. Given an r-partite r-graph F, let

$$s(F) := \sup\left\{s : \exists H,\ t_F(H) = t_{K_r^r}(H)^{s+e(F)} > 0\right\}.$$

We call $s(F)$ the *Sidorenko gap* of F.

Clearly, F is Sidorenko if and only if $s(F) = 0$. For non-Sidorenko hypergraphs, Nie and Spiro [35] determined $s(F)$ for F being the r-expansion of k-uniform clique on $k + 1$ vertices (see [35] for precise definition).

One can show that if F is not Sidorenko, then the analogue of Conjecture 1 for F is not true. More precisely, we can prove the following.

Proposition 6. *For every $r \geqslant 3$, let F be an r-graph. For any integer n and reals $\varepsilon, \delta \in (0, 1)$, there exists $N > n$ such that there exists an N-vertex r-graphs H_N with at least $N^{r-\delta}$ edges such that H_N contains at most*

$$N^{v(F)-\delta(s(F)+e(F)-\varepsilon)}$$

copies of F.

Inspired by Proposition 6, we make the following conjecture.

Conjecture 7. For every $r \geqslant 3$, let F be an r-partite r-graph such that $\mathrm{ex}(F, n) = O(n^{r-\alpha})$ for some $0 < \alpha < r - 1$. Then there exists positive constant c such that if G is an n-vertex r-graph such that $e(G) \geqslant cn^{r-\alpha}$, then G contains at least

$$n^{v(F)-o(1)}\left(\frac{e(G)}{n^r}\right)^{e(F)+s(F)}$$

copies of F.

An r-uniform linear cycle of length ℓ, denoted by C_ℓ^r, is an r-graph with ℓ edges $e_1, e_2, \dots, e_\ell$ where $e_i = \{v_{(r-1)(i-1)}, v_{(r-1)(i-1)+1}, \dots, v_{(r-1)i}\}$ (here $v_0 = v_{(r-1)\ell}$). Conjecture 7 have been implicitly confirmed for $C_{2\ell}^r$, $\ell \geqslant 2$ [33, 31], and also for the non-Sidorenko hypergraph the r-expansion of k-uniform clique on $k + 1$ vertice [1] [32].

In this paper, we study the supersaturation problem and the Sidorenko gap problem of odd linear cycles. The problems for C_3^r have been solved in [36, 32, 35]; it is proved that $s(C_3^r) = \frac{1}{r-2}$ and Conjecture 7 has been confirmed for C_3^r. The problems for $C_{2\ell+1}^r$ where $\ell \geqslant 2$ are still wide open. In [35], Nie and Spiro showed that $s(C_{2\ell+1}^r) \leqslant \frac{2\ell-1}{r-2}$. Further, they mentioned (without a proof) a conditional lower bound $s(C_{2\ell+1}^r) \geqslant \frac{1}{(r-1)\ell-1}$. Here we prove this lower bound formally and as a consequence obtain a conditional upper bound for the supersaturation problem. In this paper $o(1)$ means a function that goes to 0 when n tends to infinity. A Berge-Cycle of length k in a hypergraph is a set of k distinct vertices $\{v_1, v_2, \dots, v_k\}$ and k distinct edges $\{e_1, e_2, \dots, e_k\}$ such that $\{v_i, v_{i+1}\} \subset e_i$ with indices taken modulo k. We call the length of the shortest Berge-Cycle contained in a hypergraph the girth of this hypergraph.

Theorem 8. *For every $r \geqslant 3$ and $\ell \geqslant 2$, if for all positive integers n there exist n-vertex r-graphs of girth $2\ell + 2$ with $n^{1+1/\ell-o(1)}$ edges, then*

$$s(C_{2\ell+1}^r) \geqslant \frac{1}{(r-1)\ell-1},$$

and thus, for every constant $0 < \delta < 1$, there exists n-vertex r-graphs with at least $n^{r-\delta}$ edges containing at most

$$n^{(r-1)(2\ell+1)-\delta(2\ell+1+\frac{1}{(r-1)\ell-1})+o(1)}$$

copies of $C_{2\ell+1}^r$.

The condition of the existence of dense high-girth hypergraphs here if true would be best possible [2]. It can be viewed as a hypergraph analogue of the Erdős-Simonovits conjecture for the existence of dense high-girth graphs, which is widely believed to be true among the research community. In the case when $\ell = 1$, such condition is true due to the seminal Ruzsa-Szemerédi construction [42] and its generalization [16]. See also [21, 43, 18] for more on this topic.

On the other hand, concerning lower bounds for the supersaturation problem, as a consequence of [32, Theorem 2.5], we know that an r-graph with $e(G) \geqslant n^{r-1+o(1)}$ contains at least

$$n^{(r-1)(2\ell+1)-o(1)} \left(\frac{e(G)}{n^r}\right)^{2\ell+1+\frac{2\ell-1}{r-2}}$$

copies of $C_{2\ell+1}^r$. We prove the following better lower bounds for $\ell \geqslant 2$.

Theorem 9. *For every $r \geqslant 3$, $\ell \geqslant 2$, there exists a constant C such that if G is an n-vertex r-graph such that $e(G) \geqslant n^{r-1}(\log n)^C$, then, as $n \to \infty$, G contains at least*

$$n^{(r-1)(2\ell+1)-o(1)} \left(\frac{e(G)}{n^r}\right)^{\left(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3}\right)}$$

copies of $C_{2\ell+1}^r$. As a consequence, we have

$$s(C_{2\ell+1}^r) \leqslant \frac{4\ell-1}{(r-1)(2\ell+1)-3}.$$

Theorem 9 is our main result. In fact, it is obtained by enhancing a weaker result using the method of codegree dichotomy(See Lemma 17). We present this weaker result here.

Proposition 10. *For every $r \geqslant 3$, $\ell \geqslant 2$, there exists a constant C such that if G is an n-vertex r-graph such that $e(G) \geqslant n^{r-1}(\log n)^C$, then, as $n \to \infty$, G contains at least*

$$n^{(r-1)(2\ell+1)-o(1)} \left(\frac{e(G)}{n^r}\right)^{(2\ell+1+\frac{2}{r-1})}$$

copies of $C^r_{2\ell+1}$. As a consequence, we have

$$s(C^r_{2\ell+1}) \leqslant \frac{2}{r-1}.$$

It is worth noting that Theorem 9 and Proposition 10 give the same bound when $r = 3$.

1.1 Notation

A k-shadow of H is a set X of k vertices such that $X \subset e$ for some edge e of H. The codegree of X is the number of edges of H containing X, which is written as $d_H(X)$; when $X = \{u, v\}$, we simplify the notation to $d_H(u, v)$. We denote by $\Delta_j(H)$ the maximum $d_H(\delta)$ among all j-shadows δ. Whenever H is clear from the context we will drop it from the notation. For an r-partite r-graph G with vertex partition $(V_1, V_2, \cdots, V_r)$, for $1 \leqslant i < j \leqslant r$, we write $\partial_{V_i,V_j}(H)$ for the set of 2-shadows (v_i, v_j) with $v_i \in V_i$ and $v_j \in V_j$. Usually we simplify the notation to $\partial_{i,j}(H) = \partial_{V_i,V_j}(H)$. Throughout this paper, We will use a notation $\mathrm{plog}_{\ell,r}(n)$ to denote any function which is of order $\Theta((\log n)^{f(\ell,r)})$.

For an r-graph H and a positive integer t, the *t-blow-up* $H[t]$ is the r-graph on vertex set $V(H) \times [t]$ with hyperedges $\{\{(v_1, t_1), \ldots, (v_r, t_r)\} : \{v_1, \ldots, v_r\} \in H, 1 \leqslant t_1, \ldots, t_r \leqslant t\}$.

1.2 Related work

Fox, Sah, Sawhney, Stoner, and Zhao [14] studied the supersaturation problem of C^3_3, but in a slightly different flavor: they showed that for any $t > 0$, a sufficiently large 3-graph G with edge density t must contain at least $|V(G)|^4 t^{4-o(1)}$ copies of C^3_3.

Jiang and Yepremyan [19] studied the supersaturation of even linear cycles in linear hypergraphs. Recently, Mubayi and Solymosi studied the supersaturation of linear 5-cycles in linear hypergraphs [30].

2 Supersaturation

The proof idea for Proposition 6 is to use the blow-up technique to link the homomorphic copies of F in H with the copies of F in $H[t]$, and then employ the Tensor product of hypergraphs to ensure that the hypergraph can be sufficiently large.

2.1 Proof of Proposition 6

Proof of Proposition 6. By the definition of $s(F)$, for any $\varepsilon > 0$, there exists some $\delta' > 0$ and an m-vertex r-graph H with $m^{r-\delta'}$ edges such that

$$\hom(F, H) \leqslant m^{v(F)-\delta'(e(F)+s(F)-\frac{\varepsilon}{2})}.$$

Moreover, we define the tensor product $H \otimes H = H^2$ to be r-graph on $V(H) \times V(H)$, where $((x_1, y_1), \dots, (x_r, y_r)) \in E(H^2)$ if and only if $(x_1, \dots, x_r)$, $(y_1, \dots, y_r) \in E(H)$. Clearly we have $v(H^2) = v(H)^2 = m^2$ and $e(H^2) = e(H)^2 = m^{2(r-\delta')}$. Furthermore, each copy of F in H^2 induces two (possibly the same) copies of F in H. Hence, we have $\hom(F, H^2) \leqslant \hom(F, H)^2 = m^{2(v(F)-\delta'(s(F)+e(F)-\frac{\varepsilon}{2}))}$, which means H^2 still satisfies the upper bound of the number of F-copies. Therefore, by repeatedly applying the tensor product operation, we may assume that m is sufficiently large.

Now we construct the desired r-graph promised by the proposition based on H. If $\delta \leqslant \delta'$, then let $H[t]$ denote the *t-blow-up* of H, where the vertex set is given by

$$\{v_{ij}, i \in [m], j \in [t]\}$$

and let ϕ be an arbitrary homomorphism from F to H. We define an injection ϕ' from F to $H[t]$, if there exists $v_a, v_b \in V(F)$, such that $\phi(v_a) = \phi(v_b) = v_i \in V(H)$, then we assign $\phi'(v_1)$ and $\phi'(v_2)$ to different vertices within the same t-set. i.e. $\phi'(v_1) = v_{ij} \in V(H[t])$, $\phi'(v_2) = v_{ik} \in V(H[t])$, where $1 \leqslant j \neq k \leqslant t$. Clearly, such a ϕ' corresponds to a copy of F in $H[t]$. Conversely, each labeled copy of F in $H[t]$ also corresponds to such a homomorphism. When t is sufficiently large, there are at most $\hom(F, H) \cdot t^{v(F)}$ such injections. Moreover, let $n = mt$ and $\delta'' = \frac{\log m}{\log n}\delta'$. Note that $m^{\delta'} = n^{\delta''}$. Clearly we have that $H[t]$ has n vertices and $m^{r-\delta'} \cdot t^r = n^{r-\delta''}$ edges. We find that $H[t]$ contains at most $m^{v(F)-\delta'(e(F)+s(F)-\frac{\varepsilon}{2})} \cdot t^{v(F)} \leqslant n^{v(F)-\delta''(e(F)+s(F)-\varepsilon)}$ labeled copies of F. Setting $t = m^{\frac{\delta'}{\delta}-1}$ yields $\delta'' = \delta$. Thus, we obtain the desired result.

If $\delta > \delta'$, then we consider $H[p]$, a random induced subgraph of H, where p will be specified later. That is, $H[p]$ is obtained from H by keeping each vertex in H independently and randomly with probability p. Set $n = mp$ and let Y be the number of F-copies in $H[p]$. Let $\delta'' = \frac{\delta' \log m + 2r \log \frac{3}{2}}{\log n}$ and note that $m^{-\delta'} = (\frac{3}{2})^{2r} n^{-\delta''}$. Clearly we have $\mathbb{E}(v(H[p])) = mp = n$, $\mathbb{E}(e(H[p])) = m^{r-\delta'} p^r = (\frac{3}{2})^{2r} n^{r-\delta''}$ and

$$\begin{aligned}
\mathbb{E}(Y) &\leqslant \hom(F, H) \cdot p^{v(F)} \\
&\leqslant m^{v(F)-\delta'(e(F)+s(F)-\frac{\varepsilon}{2})} \cdot p^{v(F)} \\
&= (3/2)^{2r(e(F)+s(F)-\frac{\varepsilon}{2})} n^{v(F)-\delta''(e(F)+s(F)-\frac{\varepsilon}{2})}.
\end{aligned}$$

Setting $p = m^{\frac{\delta'}{\delta}-1} \cdot (\frac{3}{2})^{\frac{2r}{\delta}}$ yields $\delta'' = \delta$. Since the number of vertices N in $H[p]$ follows a binomial distribution $N \sim \mathrm{Bin}(m, p)$, by Chernoff's Bound, we have

$$\Pr\left(|N - n| \geqslant \frac{n}{2}\right) \leqslant 2\exp\left(-\frac{n}{12}\right).$$

We bound the variance $\mathrm{Var}(e(H[p]))$ by summing the covariances of the edge inclusion indicator variables:

$$\mathrm{Var}(e(H[p])) \leqslant \mathbb{E}[e(H[p])] + \sum_{i=1}^{r-1} \sum_{|e\cap f|=i} p^{2r-i} \leqslant \mathbb{E}[e(H[p])] + \sum_{i=1}^{r-1} e(H)\binom{r}{i} m^{r-i} p^{2r-i}.$$

The last inequality is due to the number of pairs of edges in H intersecting in exactly i vertices is at most $e(H)\binom{r}{i}m^{r-i}$.

We examine the ratio of each term in this sum to $\mathbb{E}[e(H[p])]^2 = e(H)^2 p^{2r}$. For each $1 \leqslant i \leqslant r-1$, we have

$$\frac{e(H)m^{r-i}p^{2r-i}}{e(H)^2 p^{2r}} = \frac{m^{r-i}}{m^{r-\delta'}p^i} = \frac{m^{\delta'}}{(mp)^i}$$

Noting that $mp = \Theta(m^{\delta'/\delta})$, this ratio becomes $\Theta(m^{\delta'(1-i/\delta)})$. Since $0 < \delta < 1$ and $i \geqslant 1$, we have $1 - i/\delta < 0$, which implies that the ratio tends to 0 as $m \to \infty$. Consequently, $\mathrm{Var}(e(H[p])) = o(\mathbb{E}[e(H[p])]^2)$.

By Chebyshev's inequality, as m is sufficiently large, we have

$$\begin{aligned}\Pr\left(e(H[p]) < \frac{1}{2}\mathbb{E}[e(H[p])]\right) &\leqslant \Pr\left(|e(H[p]) - \mathbb{E}[e(H[p])]| \geqslant \frac{1}{2}\mathbb{E}[e(H[p])]\right)\\ &\leqslant \frac{4\mathrm{Var}(e(H[p]))}{\mathbb{E}[e(H[p])]^2} = o(1).\end{aligned}$$

Moreover, for Y, by Markov's inequality, we have

$$\Pr(Y \geqslant 2\mathbb{E}[Y]) \leqslant \frac{1}{2}.$$

Let A, B, C be the events that $N \in [\frac{n}{2}, \frac{3n}{2}]$, $e(H[p]) \geqslant \frac{1}{2}\mathbb{E}[e(H[p])]$, and $Y < 2\mathbb{E}[Y]$. We find that as m is sufficiently large, since $n = mp = m^{\frac{\delta'}{\delta}} \cdot (\frac{3}{2})^{\frac{2r}{\delta}}$, we have $\mathbb{E}(v(H[p])) \geqslant 24\ln 8$. Thus, we infer that each of A and B occurs with probability at least 7/8. Hence, with probability at least 1/4, events A, B and C hold for $H[p]$. This yields the existence of the desired subgraph $H[p]$ with $e(H[p]) \geqslant \frac{1}{2}(\frac{3}{2})^{2r} n^{r-\delta''} \geqslant (\frac{3}{2}n)^{r-\delta}$ which contains at most

$$2\left(\frac{3}{2}\right)^{2r(e(F)+s(F)-\frac{\varepsilon}{2})} n^{v(F)-\delta''(e(F)+s(F)-\frac{\varepsilon}{2})} \leqslant \left(\frac{n}{2}\right)^{v(F)-\delta(e(F)+s(F)-\varepsilon)}$$

copies of F. □

2.2 Proof of Theorem 8

Before presenting the proof, we first provide some necessary definitions. A path of a linear hypergraph is a sequence $v_1e_1v_2e_2 \cdots v_{k-1}e_{k-1}v_k$, where for any $1 \leqslant i \leqslant k-1$, $\{v_i, v_{i+1}\} \subset e_i$, and each v_i appears only once in this sequence. The length of a path is the number of edges of the path. We say that two vertices v_1 and v_2 in a linear hypergraph is connected if there exists a path containing v_1 and v_2, and we say that a linear hypergraph H is connected if any two vertices in H are connected. We say that v_1 is at distance i from v_2 if the length of the shortest path containing v_1 and v_2 is i.

Firstly, we present two simple propositions.

Proposition 11. *Let H be an n-vertex r-graph of girth $2\ell+2$ with $n^{1+1/\ell-o(1)}$ edges. If ϕ is a homomorphism from $C^r_{2\ell+1}$ to H, then $\phi(V(C^r_{2\ell+1}))$ induces a linear tree in H with at most ℓ edges.*

Proof of Proposition 11. Note that H is a linear hypergraph since we regard two edges with an intersection of at least 2 vertices as forming a Berge-Cycle of length 2. Moreover, $\phi(V(C^r_{2\ell+1}))$ induces a linear tree in H since the girth of H is $2\ell+2$, and if $\phi(V(C^r_{2\ell+1}))$ forms a cycle, its length must be at most $2\ell+1$.

Let us prove that the number of edges of $\phi(V(C^r_{2\ell+1}))$ is at most ℓ. Let the linear tree induced by $\phi(V(C^r_{2\ell+1}))$ be denoted as T. Suppose $e(T) \geqslant \ell+1$. Since ϕ is a homomorphism and $e(C^r_{2\ell+1}) = 2\ell+1$, by the pigeonhole principle, there must be at least one edge in $E(T)$ whose preimage consists of exactly one edge, say e_1. We remove e_1 from $C^r_{2\ell+1}$ and denote the resulting hypergraph as C^-. Let the two adjacent edges of e_1 be denoted as e_2 and e_3, with $e_1 \cap e_2 = v_1$ and $e_1 \cap e_3 = v_2$. Since $\{v_1, v_2\} \subset e_1$, $\phi(v_1)$ and $\phi(v_2)$ are distinct vertices of $\phi(V(e_1))$ in T. Therefore, if we remove $\phi(V(e_1) \setminus \{v_1, v_2\})$ from T, v_1 and v_2 will become disconnected, as T is a linear tree. Furthermore, observe that C^- remains connected. Clearly, the homomorphic image of a connected hypergraph is still connected, which leads to a contradiction. Hence, we conclude that $e(T) \leqslant \ell$. $\square$

Proposition 12. *Let H be an n-vertex r-graph of girth $2\ell+2$ with $n^{1+1/\ell-o(1)}$ edges. Then there exists $H' \in H$ with $n^{1-o(1)}$ vertices, $\frac{1}{2}n^{1+1/\ell-o(1)}$ edges, and the number of linear trees in H' with at most ℓ edges is at most $n^{2+o(1)}$.*

Proof of Proposition 12. We iteratively remove vertices from H whose degree is less than $\frac{1}{2}n^{1/\ell-o(1)}$, until the degree of all remaining vertices is greater than $\frac{1}{2}n^{1/\ell-o(1)}$. During the process, at most $n \cdot \frac{1}{2}n^{1/\ell-o(1)} < \frac{1}{2}n^{1+1/\ell-o(1)}$ edges are removed. Thus, we can obtain a subhypergraph $H' \subset H$ with at least $\frac{1}{2}n^{1+1/\ell-o(1)}$ edges, and the girth of H' remains at least $2\ell+2$. Additionally, the minimum degree of H' is $\delta(H') \geqslant \frac{1}{2}n^{1/\ell-o(1)}$.

Suppose $x \in V(H')$ and $d(x) = \Delta(H')$. Let U_i be the set of vertices that are at distance i from x. Let m_i be the number of vertices in U_i. Clearly, $m_1 = \Delta(H')$.

Let $d_{[U,V]}(v)$ denote the number of edges in H' that are restricted to U and V and contain the vertex v. Note that for any $v_{i+1} \in U_{i+1}, 1 \leqslant i \leqslant \ell-1$, $d_{[U_i,U_{i+1}]}(v_{i+1}) = 1$. Otherwise, there would exist vertices $v_a, v_b \in U_i$ and edges $e_1, e_2 \in E(H')$, such that $\{v_a, v_{i+1}\} \subset e_1$ and $\{v_b, v_{i+1}\} \subset e_2$. This would imply the existence of a cycle containing e_1, e_2 with length less than $2\ell+2$, which is a contradiction.

Moreover, $e(H'[U_i]) = 0$, $1 \leqslant i \leqslant \ell$. Otherwise, there exists an edge $e \in E(H'[U_i])$, then there would exist a cycle containing e with length less than $2\ell+2$, which is a contradiction.

Because for any $v_i \in U_i, d(v_i) \geqslant \delta(H') \geqslant \frac{1}{2}n^{1/\ell-o(1)}$ and $d(v_i) = d_{[U_i,U_{i-1}]}(v_i) + d_{H'[U_i]}(v_i) + d_{[U_i,U_{i+1}]}(v_i)$, we have $d_{[U_i,U_{i+1}]}(v_i) \geqslant \frac{1}{2}n^{1/\ell-o(1)} - 1$. Thus, we have

$$m_{i+1} \geqslant \frac{1}{3}n^{1/\ell-o(1)}m_i, \text{ for any } 1 \leqslant i \leqslant \ell-1. \tag{1}$$

Thus, we have $n \geqslant m_\ell \geqslant \Delta(H') \cdot (\frac{1}{3}n^{1/\ell-o(1)})^{\ell-1} = (\frac{1}{3})^{\ell-1}n^{1-1/\ell-o(1)} \cdot \Delta(H')$, which means that

$$\Delta(H') \leqslant n^{1/\ell+o(1)}.$$

Moreover, we have $m_\ell \geqslant (\frac{1}{3})^{\ell-1}n^{1-1/\ell-o(1)} \cdot \Delta(H') \geqslant (\frac{1}{3})^{\ell-1}n^{1-1/\ell-o(1)} \cdot \delta(H') \geqslant (\frac{1}{3})^{\ell-1} \cdot \frac{1}{2}n^{1-o(1)}$, which implies that $|V(H')| = n^{1-o(1)}$.

In conclusion, the number of linear trees with at most ℓ edges is at most

$$n \cdot \Delta(H')^\ell \cdot (r\ell)^\ell \leqslant n^{2+o(1)}.$$

The bound follows from considering all possibilities in the construction: there are at most n choices for the root vertex; then in each extension step, there are at most $r\ell$ choices for the vertex to which a new edge is attached, and at most $\Delta(H')$ choices for the new edge itself. $\square$

Proof of Theorem 8. Let H be an n-vertex r-graph of girth $2\ell+2$ with $n^{1+1/\ell-o(1)}$ edges as assumed in the theorem.

By the definition of the homomorphism density, proposition 11 and proposition 12, we have

$$t_{C^r_{2\ell+1}}(H') = \frac{\hom(C^r_{2\ell+1}, H')}{v(H')^{v(C^r_{2\ell+1})}} = \frac{g(\ell,r)n^{2+o(1)}}{n^{(2\ell+1)(r-1)-o(1)}}.$$

where $g(\ell,r)$ is a function bounded by ℓ and r. And

$$t_{K^r_r}(H') = \frac{r!n^{1+1/\ell-o(1)}}{n^{r-o(1)}}.$$

By definition, $s(C^r_{2\ell+1})$ is at least $\log(t_{C^r_{2\ell+1}}(H'))/\log(t_{K^r_r}(H')) - e(C^r_{2\ell+1})$. Thus, we have

$$s(C^r_{2\ell+1}) \geqslant \frac{\log g(\ell,r) + (2+o(1))\log n - ((2\ell+1)(r-1)-o(1))\log n}{\log r! + (1+1/\ell-o(1))\log n - (r-o(1))\log n} - (2\ell+1),$$

Taking $n \to \infty$, we have

$$s(C^r_{2\ell+1}) \geqslant \frac{2-(2\ell+1)(r-1)}{1+1/\ell-r} - (2\ell+1) = \frac{1}{(r-1)\ell-1}. \qquad \square$$

2.3 Proof of Proposition 10

Our proof for Proposition 10 primarily draws upon the methodology employed by Mubayi and Yepremyan [31] in their treatment of balanced supersaturation for even cycles in r-graphs. We have extended this approach to address the issue of supersaturation for odd cycles in r-graphs.

In our proof, we need a supersaturation result for even cycles in simple graphs by Simonovits [12] (unpublished, see [28, Theorem 1.5] for a proof).

Theorem 13 ([12, 28])**.** *For every $\ell \geqslant 2$, there exist $\delta > 0$ and $k_0 \in \mathbb{N}$ such that the following holds for every $k \geqslant k_0$ and every $n \in \mathbb{N}$. Given a graph G with n vertices and $kn^{1+1/\ell}$ edges there exists a collection $\mathcal{F}$ of copies of $C_{2\ell}$ in G such that $|\mathcal{F}| \geqslant \delta k^{2\ell} n^2$.*

For convenience, we restate Proposition 10.

Proposition 14. *For every $r \geqslant 3$, $\ell \geqslant 2$, there exists a constant C such that if G is an n-vertex r-graph such that $e(G) \geqslant n^{r-1}(\log n)^C$, then, as $n \to \infty$, G contains at least*

$$n^{(r-1)(2\ell+1)-o(1)} \left(\frac{e(G)}{n^r}\right)^{(2\ell+1+\frac{2}{r-1})}$$

copies of $C^r_{2\ell+1}$. As a consequence, we have

$$s(C^r_{2\ell+1}) \leqslant \frac{2}{r-1}.$$

Proof of Proposition 10. Let Q, δ_0, k_0 be the quantities derived from Theorem 13 when applied with ℓ. Let G be a given r-graph with n vertices such that $e(G) \geqslant n^{r-1}(\log n)^C$, where C is a constant. According to a classical result by Erdős and Kleitman [9], G contains an r-partite subgraph H with r-partition $(V_1, V_2, \ldots, V_r)$ such that the number of edges in H satisfies $e(H) \geqslant r!e(G)/r^r$.

Let $\mathcal{Y}$ be the collection of all functions from $\binom{[r]}{2}$ to $[\log \binom{n}{r-2}]$, and for each $i, j \in [r]$ and $y \in \mathcal{Y}$ we simply write y_{ij} for $y(\{i,j\})$. We say that an edge $e \in H$ is of *type* y if for every set $\{i,j\} \subseteq \binom{[r]}{2}$, we have $2^{y_{ij}-1} \leqslant d_H(e \cap (V_i \cup V_j)) < 2^{y_{ij}}$. By the pigeonhole principle, there exists some $y \in \mathcal{Y}$ such that at least $\log(n^r)^{-r^2} e(H)$ edges of H are of *type* y. Let H_0 be the collection of edges of *type* y, and let $\Delta_{ij} := 2^{y_{ij}}$.

Set $H^* = H_0$ and let $U_i := V_i \cap V(H^*)$. If there exists some pairs $v_i \in U_i$, $v_j \in U_j$, $1 \leqslant i < j \leqslant r$ such that $0 < d_{H_0}(v_i, v_j) < 2^{-r^2} \log(n^r)^{-r^2} d_H(v_i, v_j)$, then delete every edge from H^* containing v_i and v_j. We excute this operation iteratively on H^*. After this, we also remove all the vertices which became isolated in the resulting hypergrah. Let H' be the resulting hypergraph, we continue to use the notation U_i for the parts of H'. Noting that it satisfies that if $v_i \in V_i, v_j \in V_j$ such that there exists $e \in E(H')$ with $\{v_i v_j\} \subseteq e$ then

$$\frac{\Delta_{ij}}{2(2\log n^r)^{r^2}} \leqslant \frac{d_H(v_i, v_j)}{(2\log n^r)^{r^2}} \leqslant d_{H'}(v_i, v_j) \leqslant \Delta_{ij}. \tag{2}$$

Furthermore, note that the total number of edges deleted from H_0 to obtain H' is at most

$$\sum_{1 \leqslant i < j \leqslant r} \sum_{v_i \in V_i, v_j \in V_j} \frac{d_H(v_i, v_j)}{(2\log n^r)^{r^2}} \leqslant \frac{r^2 e(H)}{(2\log n^r)^{r^2}} \leqslant \frac{e(H_0)}{2}.$$

Combining the above, we have:

$$e(H') \geqslant \frac{e(H_0)}{2} \geqslant \frac{e(H)}{2\log(n^r)^{r^2}} \geqslant \frac{e(G)}{\operatorname{plog}_{\ell,r}(n)}. \tag{3}$$

Let $\partial_{ij} = \partial_{U_i, U_j}(H')$, without loss of generality, we may assume that $|U_1|$ is the maximum among $|U_1|, \ldots, |U_r|$.

Claim 15. *For arbitrary $u_1 \in U_1$, $u_2 \in U_2$, there exists $3 \leqslant i \leqslant r$ such that there are at least $d_{H'}(u_1, u_2)^{1/(r-2)}$ distinct $x \in U_i$, the set $\{u_1, u_2, x\}$ is part of some edges in H'.*

Proof. Assume that for all $3 \leqslant i \leqslant r$, we can find fewer than $d_{H'}(u_1, u_2)^{1/(r-2)}$ distinct $x \in U_i$ that satisfy the given conditions. Thus, the number of edges containing u_1 and u_2 is less than

$$(d_{H'}(u_1, u_2)^{1/(r-2)})^{r-2} = d_{H'}(u_1, u_2),$$

which is a contradiction. ☐

Claim 16. *There exists some $j \in \{2, \ldots, r\}$ such that*

$$|\partial_{1j}| \geqslant \frac{e(G)^{\frac{1}{r-1}} |U_1|^{\frac{r-2}{r-1}}}{\operatorname{plog}_{\ell,r}(n)}.$$

Proof. Assume that $j = 2$ does not meet the desired bound, as otherwise the proof is trivial. We aim to demonstrate the existence of some $j \in \{3, \ldots, r\}$ that can be made large enough.

By (2), (3), and the definition of ∂_{12}, we have:

$$\frac{e(G)}{\operatorname{plog}_{\ell,r}(n)} \leqslant e(H') \leqslant |\partial_{12}| \Delta_{12}. \tag{4}$$

Consequently,

$$|\Delta_{12}| \geqslant \frac{e(G)}{|\partial_{12}| \operatorname{plog}_{\ell,r}(n)}. \tag{5}$$

Given $u_1 \in U_1, u_2 \in U_2$, if $d_{H'}(u_1, u_2) > 0$, then combining (2) and (5), we arrive at:

$$d_{H'}(u_1, u_2) \geqslant \frac{\Delta_{12}}{2(2 \log n^r)^{r^2}} \geqslant \frac{e(G)}{2(2 \log n^r)^{r^2} |\partial_{12}| \operatorname{plog}_{\ell,r}(n)} = \frac{e(G)}{|\partial_{12}| \operatorname{plog}_{\ell,r}(n)}.$$

Note that u_1 and u_2 are arbitrary vertices in U_1 and U_2, respectively, and they are part of at least one edge. Each such edge takes the form $\{u_1, u_2, u_3, \ldots, u_r\}$, with $u_j \in U_j$ for $j = 3, \ldots, r$. Claim 15 guarantees the existence of j such that for at least $d_{H'}(u_1, u_2)^{1/(r-2)}$ distinct $x \in U_j$, the set $\{u_1, u_2, x\}$ is a 3-shadow. Therefore, for the chosen $u_1 \in U_1$, we identify some $j = j(u_1) \in \{3, \ldots, r\}$ and a subset $X_{j,u_1} \subseteq U_j$ such that u_1, x_j are in an edge for each $x_j \in X_{j,u_1}$, and

$$|X_{j,u_1}| \geqslant d_{H'}(u_1, u_2)^{1/(r-2)} \geqslant \left(\frac{e(G)}{\operatorname{plog}_{\ell,r}(n) |\partial_{12}|} \right)^{1/(r-2)}.$$

Applying the pigeonhole principle, there exists $j \in \{3, \ldots, r\}$ such that for at least $|U_1|/r$ vertices $u_1 \in U_1$, $j(u_1) = j$. Observe that for each $u_1 \in U_1$ and $x \in X_{j,u_1}$, we obtain distinct edges $u_1 x \in \partial_{1j}$. By assumption:

$$|\partial_{12}| \leqslant \frac{e(G)^{\frac{1}{r-1}} |U_1|^{\frac{r-2}{r-1}}}{\operatorname{plog}_{\ell,r}(n)},$$

we have

$$\begin{aligned}|\partial_{1j}| &\geqslant \frac{|U_1|}{r}\left(\frac{e(G)}{\mathrm{plog}_{\ell,r}(n)|\partial_{12}|}\right)^{1/(r-2)} \\ &\geqslant \frac{|U_1|}{r}\left(\frac{e(G)}{\mathrm{plog}_{\ell,r}(n)|U_1|^{\frac{r-2}{r-1}}e(G)^{\frac{1}{r-1}}}\right)^{1/(r-2)} = \frac{e(G)^{\frac{1}{r-1}}|U_1|^{\frac{r-2}{r-1}}}{\mathrm{plog}_{\ell,r}(n)},\end{aligned}$$

as desired. □

By Claim 16, we may assume without loss of generality that

$$|\partial_{12}| \geqslant \frac{e(G)^{\frac{1}{r-1}}|U_1|^{\frac{r-2}{r-1}}}{\mathrm{plog}_{\ell,r}(n)} \geqslant \frac{e(G)^{\frac{1}{r-1}}m^{\frac{r-2}{r-1}}}{\mathrm{plog}_{\ell,r}(n)}, \tag{6}$$

where $m = |U_1| + |U_2|$.

Set $k = \frac{|\partial_{12}|}{m^{1+1/\ell}}$. By (6) and $\ell \geqslant 2, r \geqslant 3$, we can ensure that $k \geqslant k_0$, thus by Theorem 13 applied with k and m, the shadow graph ∂_{12} contains a collection $\mathcal{F}$ of copies of $C_{2\ell}$ satisfying $|\mathcal{F}| \geqslant \delta_0 k^{2\ell} m^2$.

Let C be any 2ℓ-cycle in $\mathcal{F}$ with consecutive edges $x_1x_2\dots x_{2\ell}$ in the natural cyclic order. i.e., it contains edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{2\ell}, x_1\}$. By (2), which states

$$\frac{\Delta_{12}}{2(2\log n^r)^{r^2}} \leqslant d_{H'}(v_1, v_2) \leqslant \Delta_{12},$$

there exists at least $\frac{\Delta_{12}}{2(2\log n^r)^{r^2}}$ different hyperedges containing both x_1 and $x_{2\ell}$. By Claim 15, there exists $3 \leqslant a \leqslant r$ such that there are at least $(\frac{\Delta_{12}}{2(2\log n^r)^{r^2}})^{1/(r-2)}$ different $x_{2\ell+1} \in U_a$ for which there exists an edge $e \in E(H')$ containing both $\{x_{2\ell}, x_{2\ell+1}\}$ and $\{x_{2\ell+1}, x_1\}$. Therefore, every cycle C can be extended to $(\frac{\Delta_{12}}{2(2\log n^r)^{r^2}})^{1/(r-2)}$ different sets of the form $\{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{2\ell}, x_{2\ell+1}\}, \{x_{2\ell+1}, x_1\}\}$. Let the collection of such sets be denoted by $\mathcal{C}$.

For every $C' \in \mathcal{C}$, since $d_{H'}(x_i, x_{i+1}) \geqslant \frac{\Delta_{12}}{2(2\log n^r)^{r^2}}$ for every $1 \leqslant i \leqslant 2\ell - 1$,

$$d_{H'}(x_1, x_{2\ell+1}) \geqslant \frac{\Delta_{1a}}{2(2\log n^r)^{r^2}}$$

and $d_{H'}(x_{2\ell}, x_{2\ell+1}) \geqslant \frac{\Delta_{2a}}{2(2\log n^r)^{r^2}}$, the number of ways to extend C' to some linear $(2\ell+1)$-cycles in H' is at least

$$\left(\frac{\Delta_{12}}{2(2\log n^r)^{r^2}} - 2\ell r n^{r-3}\right)^{2\ell-1}\left(\frac{\Delta_{1a}}{2(2\log n^r)^{r^2}} - 2\ell r n^{r-3}\right)\left(\frac{\Delta_{2a}}{2(2\log n^r)^{r^2}} - 2\ell r n^{r-3}\right). \tag{7}$$

Indeed, as it suffices to choose all these new vertices to be distinct, each time when we pick a new edge of the cycle we must avoid at most $2\ell(r-1)+1 \leqslant 2\ell r$ vertices, which makes at most $2\ell r n^{r-3}$ $(r-2)$-sets unavailable. Furthermore, by (2) and (3), for $1 \leqslant i < j \leqslant r$, we have

$$\frac{e(G)}{\mathrm{plog}_{\ell,r}(n)} \leqslant e(H') \leqslant |\partial_{ij}|\Delta_{ij} \leqslant n^2\Delta_{ij}.$$

Thus, we have

$$\Delta_{ij} \geqslant \frac{n^{-2}e(G)}{\mathrm{plog}_{\ell,r}(n)}. \tag{8}$$

By (8), (7) is at least

$$\left(\frac{\Delta_{12}}{4(2\log n^r)^{r^2}}\right)^{2\ell-1}\left(\frac{\Delta_{1a}}{4(2\log n^r)^{r^2}}\right)\left(\frac{\Delta_{2a}}{4(2\log n^r)^{r^2}}\right) = \Delta_{12}^{2\ell-1}\Delta_{1a}\Delta_{2a}\mathrm{plog}_{\ell,r}(n).$$

Let $\mathrm{Ext}(C')$ be the collection of all cycles $C_{2\ell+1}^{(r)}$ obtained from C' in this manner described above. Define $\mathcal{F}' = \{\mathrm{Ext}(C')|C \in \mathcal{F}\}$, so $\mathcal{F}'$ is a collection of copies of linear $C_{2\ell+1}^{(r)}$ in H'. By the previous discussion and $|\mathcal{F}| \geqslant \delta_0 k^{2\ell}m^2$, we get $|\mathcal{F}'| \geqslant \delta_0 k^{2l}m^2\Delta_{12}^{2\ell-1+\frac{1}{r-2}}\Delta_{1a}\Delta_{2a}\mathrm{plog}_{\ell,r}(n)$. Now putting all estimates together, we obtain

$$\begin{aligned}
|\mathcal{F}'| &\geqslant \delta_0 k^{2l}m^2\Delta_{12}^{2\ell-1+\frac{1}{r-2}}\Delta_{1a}\Delta_{2a}\mathrm{plog}_{\ell,r}(n)\\
&\geqslant \delta_0|\partial_{12}|^{2\ell}m^{-2\ell}\Delta_{12}^{2\ell-1+\frac{1}{r-2}}n^{-4}e(G)^2\mathrm{plog}_{\ell,r}(n) && \text{by } k=\tfrac{|\partial_{12}|}{m^{1+1/\ell}} \text{ and (8)}\\
&\geqslant \mathrm{plog}_{\ell,r}(n)(|\partial_{12}|\Delta_{12})^{2\ell-1+\frac{1}{r-2}}|\partial_{12}|^{1-\frac{1}{r-2}}n^{-4}e(G)^2m^{-2\ell}\\
&\geqslant \mathrm{plog}_{\ell,r}(n)e(G)^{2\ell-1+\frac{1}{r-2}}(e(G)^{\frac{1}{r-1}}m^{\frac{r-2}{r-1}})^{1-\frac{1}{r-2}}n^{-4}e(G)^2m^{-2\ell} && \text{by (6)}\\
&\geqslant \mathrm{plog}_{\ell,r}(n)e(G)^{2\ell+1+\frac{2}{r-1}}n^{-2\ell-4-\frac{r-3}{r-1}} && \text{by } m \leqslant n\\
&\geqslant \mathrm{plog}_{\ell,r}(n)n^{(r-1)(2\ell+1)}\left(\frac{e(G)}{n^r}\right)^{(2\ell+1+\frac{2}{r-1})}\\
&\geqslant n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{(2\ell+1+\frac{2}{r-1})}.
\end{aligned}$$

For the "as a consequence" part of the theorem, by Proposition 6, setting $e(G) = n^{r-\delta}$, we have

$$n^{(r-1)(2\ell+1)-o(1)-\delta(2\ell+1+\frac{2}{r-1})} \leqslant n^{(r-1)(2\ell+1)-\delta(2\ell+1+s(C_{2\ell+1}^r)-\varepsilon)}.$$

Thus,

$$s(C_{2\ell+1}^r) \leqslant \frac{2}{r-1}.$$

This completes the proofs. □

2.4 Proof of Theorem 9

We will use induction on r to prove the supersaturation results for $r \geqslant 4$. The proof splits into two cases according to how large the typical codegrees are: when the typical codegree is small, we find a large collection of $(r-1)$ shadows, use inductive hypothesis on them to obtain a large collection of $C^{r-1}_{2\ell+1}$, and then expand them into a large collection of $C^{r}_{2\ell+1}$; on the other hand, if the typical codegree is large, then we use the method of "Greedy Expansion". We need the following codegree dichotomy lemma.

Lemma 17 ([34], Lemma 3.4)**.** *For $r \geqslant 2$, let H be an r-partite r-graph on n vertices, then for any $e(H)/(4n^{r-1}) < A \leqslant n$, one of the following two statements is true:*

(i) There exists a subgraph $\hat{H} \subseteq H$ with $e(\hat{H}) \geqslant e(H)/2$ such that every $(r-1)$-shadow of $\hat{H}$ has codegree at least A.

(ii) There exist a subgraph $\hat{H} \subseteq H$ with $e(\hat{H}) \geqslant e(H)/(4r\log n)$, an r-partition $V(\hat{H}) = V_1 \cup \cdots \cup V_r$, and

$$\frac{e(H)}{4n^{r-1}} < D \leqslant A$$

such that any $(r-1)$-shadow σ of $\hat{H}$ in $V_1 \times \cdots \times V_{r-1}$ satisfies

$$D/2 \leqslant d_{\hat{H}}(\sigma) < D.$$

For convenience, we restate Theorem 9.

Theorem 18. *For every $r \geqslant 3$, $\ell \geqslant 2$, there exists a constant C such that if G is an n-vertex r-graph such that $e(G) \geqslant n^{r-1}(\log n)^C$, then, as $n \to \infty$, G contains at least*

$$n^{(r-1)(2\ell+1)-o(1)} \left(\frac{e(G)}{n^r}\right)^{\left(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3}\right)}$$

copies of $C^r_{2\ell+1}$. As a consequence, we have

$$s(C^r_{2\ell+1}) \leqslant \frac{4\ell-1}{(r-1)(2\ell+1)-3}.$$

Proof of Theorem 9. We argue by induction on r. The base case $r = 3$ follows from Proposition 10, set $f(r) = \frac{4\ell-1}{(r-1)(2\ell+1)-3}$. Assume that for $r-1$, we have already obtained the desired result.

Let G be a given r-graph with n vertices such that $e(G) \geqslant n^{r-1}(\log n)^C$, where C is a sufficiently large constant. According to a classical result by Erdős and Kleitman [9], G contains an r-partite subgraph H with r-partition $(V_1, V_2, \ldots, V_r)$ such that the number of edges in H satisfies $e(H) \geqslant r!e(G)/r^r$. Apply Lemma 17 on H with

$$A = e(G)^{\frac{2\ell+f(r-1)}{2\ell(r-1)-1+f(r-1)}} n^{-\frac{2\ell+1+(r-1)f(r-1)}{2\ell(r-1)-1+f(r-1)}-o(1)} < n^{\frac{2\ell(r-2)-1}{2\ell(r-1)-1+f(r-1)}-o(1)} < n.$$

Let $\hat{H}$ be the subgraph of H guaranteed by Lemma 17.

If (i) is true, then there exists a subgraph $\hat{H} \subseteq H$ with $e(\hat{H}) \geqslant e(H)/2 \geqslant r!e(G)/2r^r = \Omega(e(G))$, such that every $(r-1)$-shadow of $\hat{H}$ has codegree at least $A > 2(2\ell+1)(r-1)$. Now we describe an algorithm to construct copies of $C^r_{2\ell+1}$ in $\hat{H}$; it involves specifying $2\ell+1$ edges $e_1, e_2, \ldots, e_{2\ell+1}$ which form a copy of $C^r_{2\ell+1}$ in $\hat{H}$, where $e_i = \{v_{i,1}, \ldots, v_{i,r-2}, w_{i-1}, w_i\}$, $1 \leqslant i \leqslant 2\ell+1$, $w_0 = w_{2\ell+1}$. In this algorithm, we can view an edge as an ordered set of vertices. Given an edge with an order on its vertices $e = (a_1, a_2, \ldots, a_r)$ and a vertex v, let $e[2, r] = (a_2, \ldots, a_r)$ and let $e[2, r] \oplus \{v\} = (a_2, \ldots, a_r, v)$.

(I) Start by choosing an edge with an order $e_1 = \{v_{1,1}, \ldots, v_{1,r-2}, w_{2\ell+1}, w_1\}$ in $E(\hat{H})$;

(II) For $1 \leqslant i \leqslant \ell - 1$, we specify $w_{2\ell+1-i}$ and w_{i+1} inductively as following. Let $f_1 = e_1$. Next we greedily pick $f_{2\ell+1} = f_1[2, r] \oplus \{w_{2\ell}\}$ such that $w_{2\ell}$ is different from any specified vertices. Then we pick $f_2 = f_{2\ell+1}[2, r] \oplus \{w_2\}$ such that w_2 is different from any other specified vertices. In general, given f_i, we choose $f_{2\ell+2-i} = f_i[2, r] \oplus \{w_{2\ell+1-i}\}$ such that the only vertex $w_{2\ell+1-i}$ in $f_{2\ell+2-i} \setminus f_i$ is distinct from all specified vertices. Then we choose $f_{i+1} = f_{2\ell+2-i}[2, r] \oplus \{w_{i+1}\}$ such that the only vertex w_{i+1} in $f_{i+1} \setminus f_{2\ell+2-i}$ is distinct from all specified vertices. Further, given f_ℓ, we choose $f_{\ell+1} = f_\ell[2, r] \oplus \{w_{\ell+1}\}$ such that the only vertex $w_{\ell+1}$ in $f_{\ell+1} \setminus f_\ell$ is distinct from all specified vertices. Then we let $f_{\ell+2} = f_{\ell+1}$. So far we have $\{w_{i-1},\ w_i\} \subset f_i$ for every $2 \leqslant i \leqslant 2\ell+1$. (See Figure 1 for an illustration)

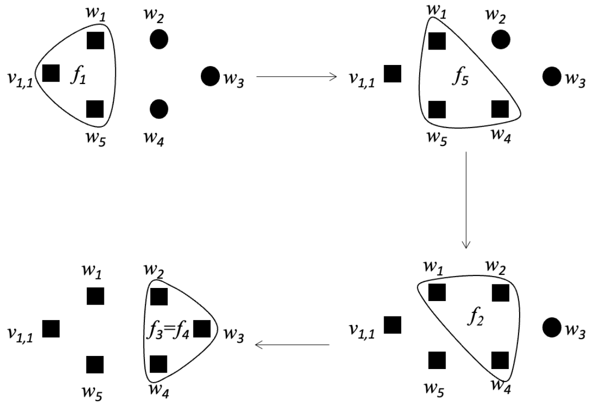

Figure 1: Step (II) of the algorithm for C^3_5. The ■s are specified vertices.

(III) For $2 \leqslant i \leqslant 2\ell+1$, we give f_i a new order such that w_{i-1} and w_i are the last two vertices. Then we specify $v_{i,j}$, $1 \leqslant j \leqslant r-2$, inductively as following. Let $e_{i,0} = f_i$. We choose $e_{i,j}$ such that the only vertex $v_{i,j}$ in $e_{i,j} \setminus e_{i,j-1}$ is distinct from all specified vertices; give vertices of $e_{i,j}$ an order such that $e_{i,j} = e_{i,j-1}[2, r] \oplus \{v_{i,j}\}$. Let $e_i = e_{i,r-2}$.

Note that $e_1, e_2, \ldots, e_{2\ell+1}$ form a copy of $C^r_{2\ell+1}$.In the above algorithm, in the first step, we have $e(\hat{H})$ choices for selecting e_1. In the subsequent steps, each time we have at least $A-(2\ell+1)(r-1)$ choices for selecting a new vertex, since every $(r-1)$-shadow of $\hat{H}$ has codegree at least A. Recall that $A > 2(2\ell+1)(r-1)$. Thus, we can find at least $\Omega(e(G)A^{2\ell(r-1)-1})$ labeled copies of $C^r_{2\ell+1}$ in $\hat{H}$. Note that

$$\begin{aligned}
e(G)A^{2\ell(r-1)-1} &\geqslant e(G)\cdot e(G)^{\frac{(2\ell+f(r-1))(2\ell(r-1)-1)}{2\ell(r-1)-1+f(r-1)}} n^{-\frac{(2\ell+1+(r-1)f(r-1))(2\ell(r-1)-1)}{2\ell(r-1)-1+f(r-1)}-o(1)} \\
&= e(G)\cdot e(G)^{\frac{(2\ell[(r-2)(2\ell+1)3]+4\ell-1)(2\ell(r1)-1)}{(2\ell(r-1)-1)[(r-2)(2\ell+1)-3]+4\ell-1}} n^{-(2\ell+1)-\frac{(r-1)f(r-1)(2\ell(r-1)-1)-(2\ell+1)f(r-1)}{2\ell(r-1)-1+f(r-1)}-o(1)} \\
&= e(G)^{2\ell+1+\frac{(4\ell-1)(2\ell(r-1)-1)-2\ell(4\ell-1)}{(2\ell(r-1)-1)[(r-2)(2\ell+1)-3]+4\ell-1}} n^{-(2\ell+1)-\frac{(r-1)(4\ell-1)(2\ell(r-1)-1)-(2\ell+1)(4\ell-1)}{(2\ell(r-1)-1)[(r-2)(2\ell+1)-3]+4\ell-1}-o(1)} \\
&= e(G)^{2\ell+1+\frac{(4\ell-1)(2\ell(r-2)-1)}{(2\ell(r-2)-1)[(r-1)(2\ell+1)-3]}} n^{-(2\ell+1)-\frac{r(4\ell-1)(2\ell(r-2)-1)}{(2\ell(r-2)-1)[(r-1)(2\ell+1)-3]}-o(1)} \\
&= e(G)^{2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3}} n^{-(2\ell+1)-\frac{r(4\ell-1)}{(r-1)(2\ell+1)-3}-o(1)} \\
&= n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{\left(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3}\right)}.
\end{aligned}$$

Thus we are done in this case.

If (ii) is true, then there exist a subgraph $\hat{H}\subseteq H$ with $e(\hat{H}) \geqslant e(H)/(4r\log n) \geqslant C_2 e(G)/\log n$, where C_2 is a constant, an r-partition $V(\hat{H}) = V_1\cup\cdots\cup V_r$, and $\frac{e(H)}{4n^{r-1}} < D \leqslant A$ such that any $(r-1)$-shadow σ of $\hat{H}$ in $V_1\times\cdots\times V_{r-1}$ satisfies $D/2 \leqslant d_{\hat{H}}(\sigma) < D$. Consider the $(r-1)$-graph H' on $V_1\times\cdots\times V_{r-1}$, whose edge set contains all the $(r-1)$-shadows of $\hat{H}$ in $V_1\times\cdots\times V_{r-1}$. Note that we have $e(H')\cdot D \geqslant e(\hat{H})$. Thus, $e(H') \geqslant \frac{e(\hat{H})}{D} = C_2 e(G)(\frac{(\log n)^{-1}}{D}) \geqslant C_2 e(G)(\frac{(\log n)^{-1}}{n}) \geqslant C_2 n^{r-2}(\log n)^{C-1}$. Then by inductive hypothesis, we have the number of $C^{r-1}_{2\ell+1}$ copies in H' is at least $n^{(r-2)(2\ell+1)-o(1)}\left(\frac{e(G)n^{-o(1)}}{n^{r-1}D}\right)^{(2\ell+1)+f(r-1)}$. Moreover, the number of $C^r_{2\ell+1}$ copies in $\hat{H}$ is at least

$$n^{(r-2)(2\ell+1)-o(1)}\left(\frac{e(G)n^{-o(1)}}{n^{r-1}D}\right)^{(2\ell+1)+f(r-1)}\cdot D^{2\ell+1}, \tag{9}$$

since each $(r-1)$-shadow of $\hat{H}$ can be extended into an r-edge of $\hat{H}$ in at least $D/2-2\ell$ ways. In addition, (9) is equal to

$$\begin{aligned}
&n^{-(2\ell+1)-(r-1)f(r-1)-o(1)} e(G)^{2\ell+1+f(r-1)} D^{-f(r-1)} \\
&\geqslant n^{-(2\ell+1)-(r-1)f(r-1)-o(1)} e(G)^{2\ell+1+f(r-1)} A^{-f(r-1)}.
\end{aligned}$$

Note that

$$\begin{aligned}
RHS &= n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{2\ell+1} n^{\frac{f(r-1)[(2\ell+1)-2\ell(r-1)^2+(r-1)]}{2\ell(r-1)-1+f(r-1)}} e(G)^{\frac{f(r-1)[2\ell(r-1)-1-2\ell]}{2\ell(r-1)-1+f(r-1)}} \\
&= n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{2\ell+1} n^{\frac{-r(4\ell-1)(2\ell(r-2)-1)}{(2\ell(r-2)-1)[(r-1)(2\ell+1)-3]}} e(G)^{\frac{(4\ell-1)(2\ell(r-2)-1)}{(2\ell(r-2)-1)[(r-1)(2\ell+1)-3]}} \\
&= n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{2\ell+1} n^{\frac{-r(4\ell-1)}{(r-1)(2\ell+1)-3}} e(G)^{\frac{4\ell-1}{(r-1)(2\ell+1)-3}} \\
&\geqslant n^{(r-1)(2\ell+1)-o(1)}\left(\frac{e(G)}{n^r}\right)^{(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3})}.
\end{aligned}$$

Thus we are also done in this case.

For the “as a consequence” part of the theorem, by Proposition 6, setting $e(G) = n^{r-\delta}$, we have

$$n^{(r-1)(2\ell+1)-o(1)-\delta(2\ell+1+\frac{4\ell-1}{(r-1)(2\ell+1)-3})} \leqslant n^{(r-1)(2\ell+1)-\delta(2\ell+1+s(C^r_{2\ell+1})-\varepsilon)}.$$

Thus,

$$s(C^r_{2\ell+1}) \leqslant \frac{4\ell-1}{(r-1)(2\ell+1)-3}.$$

This completes the proofs. □

3 Conclusion

- We believe the conditional lower bound in Theorem 8 gives the correct value of $s(C^r_{2\ell+1})$. The current approach in this paper falls short in proving a matching upper bound. New ideas will be necessary to improve the upper bound in Theorem 9.

  Conjecture 19. For $r \geqslant 3$ and $\ell \geqslant 2$, $s(C^r_{2\ell+1}) = \frac{1}{(r-1)\ell-1}$.

- Note that in the proof of the Proposition 10, we used the supersaturation result for even cycles from [28], which is actually a balanced supersaturation result. Indeed, using the full power of this result, we can also derive a hypergraph version of the balanced supersaturation result for linear odd cycles. By applying the container method, we can obtain the random Turán result for 3-uniform linear odd cycles as follows:

  Theorem 20. *For every $\ell \geqslant 2$ a.a.s. the following holds:*

  $$\operatorname{ex}\left(G^3_{n,p}, C^3_{2\ell+1}\right) \leqslant \begin{cases} (1-o(1))pn^3, & \text{if } n^{-3} \ll p \ll n^{-2+\frac{1}{2\ell}} \\ p^{\frac{1}{2\ell+1}} n^{1+\frac{3}{2\ell+1}+o(1)}, & \text{if } n^{-2+\frac{1}{2\ell}} \leqslant p \ll n^{-2+\frac{2}{2\ell-1}} \\ p^{\frac{1}{2}} n^{2+o(1)}, & \text{if } p \gg n^{-2+\frac{2}{2\ell-1}}. \end{cases}$$

  However, the proof of this result (using the container method) is involved and by now standard. Therefore, we choose to omit the complex proof of a suboptimal result.

- One can also consider a stronger version of Conjecture 7 by replacing the $n^{-o(1)}$ factor by a constant. Such a stronger version is (implicitly) known to be true [1] for C^3_3. But the current results for $C^r_{2\ell}$ and the r-expansion of K^k_{k+1} are insufficient for this stronger version. It would be interesting to prove this stronger version of Conjecture 7 for any hypergraph other than C^3_3.

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