

On Minimal Polynomials of Elements in Symmetric and Alternating Groups

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Abstract

Let (ρ, V) be an irreducible representation of the symmetric group S_n (or the alternating group A_n), and let g be a permutation on n letters such that each of its cycle lengths divides the length of the longest cycle. We describe completely the minimal polynomial of $\rho(g)$, showing that, in most cases, it equals $x^{o(g)} - 1$, with a few explicit exceptions. As a by-product, we obtain a new proof (using only combinatorics and representation theory) of a theorem of Swanson that gives a necessary and sufficient condition for the existence of a standard Young tableau of a given shape and major index $r \bmod n$, for all r . Thereby, we give a new proof of a celebrated result of Klyachko on Lie elements in a tensor algebra, and of a conjecture of Sundaram on the existence of an invariant vector for n -cycles. We also show that for elements g in S_n or A_n of even order, in most cases, $\rho(g)$ has eigenvalue -1 , with a few explicit exceptions.

Mathematics Subject Classifications: 20C30, 20C15, 05E10, 05E05

1 Introduction

Let G be the symmetric group S_n or the alternating group A_n . For $g \in G$, let $o(g)$ denote the order of g . In this paper, we will compute the minimal polynomial of the operator $\rho(g)$, where ρ is an irreducible representation of G and $g \in G$ is such that all its cycle lengths divide the length of its longest cycle. We prove the following two theorems.

Theorem 1. *For partitions λ and μ of n , let $(\rho_\lambda, V_\lambda)$ denote the irreducible representation of S_n corresponding to λ and $w_\mu \in S_n$ denote a permutation with cycle type μ . Assume that each part of μ divides its largest part. Then the minimal polynomial $p(x)$ of $\rho_\lambda(w_\mu)$ is $x^{o(w_\mu)} - 1$ except in the following cases:*

1. $\lambda = (n)$, $\mu \neq (1^n)$, where $p(x) = (x - 1)$,

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2. $\lambda = (n - 1, 1)$, $\mu = (n)$, where $p(x) = \frac{x^n - 1}{x - 1}$,
3. $\lambda = (2, 1^{n-2})$, $\mu = (n)$, where $p(x) = \frac{x^n - 1}{x - (-1)^{n-1}}$,
4. $\lambda = (1^n)$, $\mu \neq (1^n)$, where $p(x) = (x - (-1)^{n-\text{len}(\mu)})$,
5. $\lambda = (3, 3)$, $\mu = (6)$, where $p(x) = \frac{x^6 - 1}{(x - \zeta)(x - \zeta^{-1})}$, and ζ is a primitive 3rd root of unity,
6. $\lambda = (2, 2, 2)$, $\mu = (6)$, where $p(x) = \frac{x^6 - 1}{(x + \zeta)(x + \zeta^{-1})}$ and ζ is a primitive 3rd root of unity,
7. $\lambda = (2, 2)$, $\mu = (4)$, where $p(x) = \frac{x^4 - 1}{(x - i)(x + i)}$ and i is a primitive 4th root of unity,
8. $\lambda = (2, 2)$, $\mu = (3, 1)$, where $p(x) = \frac{x^3 - 1}{x - 1}$,
9. $\lambda = (2, 2)$, $\mu = (2, 2)$, where $p(x) = \frac{x^2 - 1}{x + 1}$.

The corresponding theorem for the alternating group A_n is as follows:

Theorem 2. *Let (ρ, V) be an irreducible representation of the alternating group A_n and let $g \in A_n$ with cycle type μ . Suppose that every part of μ divides its largest part. Then the minimal polynomial of $\rho(g)$ is $x^{o(g)} - 1$ except in the following cases:*

1. $V = V_{(n)}$, $\mu \neq (1^n)$ where $p(x) = x - 1$,
2. $V = V_{(n-1,1)}$, $\mu = (n)$ and $n \geq 5$ odd, where $p(x) = \frac{x^n - 1}{x - 1}$,
3. $V = V_{(3,1,1)}^+$, $\mu = (5)^+$ or $V = V_{(3,1,1)}^-$, $\mu = (5)^-$, where $p(x) = \frac{x^5 - 1}{(x - \zeta^2)(x - \zeta^3)}$ with $\zeta = \exp(2\pi i/5)$,
4. $V = V_{(3,1,1)}^+$, $\mu = (5)^-$ or $V = V_{(3,1,1)}^-$, $\mu = (5)^+$, where $p(x) = \frac{x^5 - 1}{(x - \zeta)(x - \zeta^4)}$ with ζ as above,
5. $V = V_{(2,2)}^+$, $\mu = (3, 1)^+$ or $V = V_{(2,2)}^-$, $\mu = (3, 1)^-$, where $p(x) = x - \exp(2\pi i/3)$,
6. $V = V_{(2,2)}^+$, $\mu = (3, 1)^-$ or $V = V_{(2,2)}^-$, $\mu = (3, 1)^+$, where $p(x) = x - \exp(4\pi i/3)$,
7. $V = V_{(2,2)}^\pm$, $\mu = (2, 2)$, where $p(x) = x - 1$,
8. $V = V_{(2,1)}^+$, $\mu = (3)^+$ or $V = V_{(2,1)}^-$, $\mu = (3)^-$, where $p(x) = x - \exp(2\pi i/3)$,
9. $V = V_{(2,1)}^+$, $\mu = (3)^-$ or $V = V_{(2,1)}^-$, $\mu = (3)^+$, where $p(x) = x - \exp(4\pi i/3)$.

A. E. Zalesskii posed the following problem, hoping that it would have a positive answer. We refer the reader to [24] for more related problems.

For $G = GL_n(\mathbb{F}_{p^k})$, let $g \in G$ and ϕ be an irreducible representation over \mathbb{C} , the field of complex numbers or over an algebraically closed field of characteristic $q \neq p$.

Problem 3 (Zalesskii [24]). Describe the triples (G, ϕ, g) such that $1 < \deg(\phi(g)) < m(g)$, where $m(g)$ is the order of the element $gZ(G)$ in $G/(Z(G))$, and $\deg(\phi(g))$ denotes the degree of the minimal polynomial of $\phi(g)$.

Note that, in general, $\deg(\phi(g)) \leq m(g)$.

Zalesskii solved Problem 3 when g has order p in [23]. Zalesskii [28] determined the irreducible representations of quasi-simple groups in which some element of prime order p has minimal polynomial whose degree is less than p . For more results in this direction, see [27],[25],[26].

Our Theorems 1, 2 solve the following problem of Tiep and Zalesskii for the symmetric and alternating groups over the field of complex numbers.

Problem 4. [21, Problem 1.1] Determine all possible values for $\deg(\theta(g))$, and if possible, all triples (G, θ, g) with $\deg(\theta(g)) < m(g)$, under the condition that $o(g)$ is a p -power for some prime p .

This work is motivated by one more interesting problem. Before defining the problem, let us introduce some notation.

For a partition λ of n , let $\text{SYT}(\lambda)$ denote the set of all standard Young tableaux with shape λ and the entries belong to $\{1, 2, \dots, n\}$. The *major index* of a standard tableau T , denoted $\text{maj}(T)$, is the sum of all $i \in \{1, 2, \dots, n-1\}$ such that the row of T containing $i+1$ is below the row of T containing i .

Then let

$$a_\lambda^r = |\{T \in \text{SYT}(\lambda) \mid \text{maj}(T) \equiv r \pmod{n}\}|,$$

where r is taken modulo n .

Kraśkiewicz and Weyman [11] related the above number a_λ^r to the representation theory of symmetric groups as follows:

Theorem 5. [11, Theorem 1] *Let C_n be the cyclic subgroup of S_n generated by $w_n = (12 \dots n)$. Let δ^r be the irreducible character of the C_n obtained by sending w_n to $e^{2\pi ir/n}$. Then*

$$a_\lambda^r = \langle \text{Ind}_{C_n}^{S_n} \delta^r, \chi_\lambda \rangle_{S_n} = \langle \delta^r, \text{Res}_{C_n}^{S_n} \chi_\lambda \rangle_{C_n}.$$

Note that $a_\lambda^r = a_\lambda^k$ whenever we have an equality $(r, n) = (k, n)$ of greatest common divisors, since characters of the symmetric group are real-valued. Kraśkiewicz and Weyman [11] also gave a combinatorial interpretation of the multiplicity of eigenvalues of Coxeter elements in the irreducible representations of Weyl groups of type B_n and D_n . In [18], Stembridge extended these results to unitary reflection groups. Through a new approach, Jollenbeck and Schocker [8] gave a combinatorial description using multi major index for all permutations in the symmetric group, which extends the results of Kraśkiewicz and Weyman [11].

For a vector space V , consider the tensor algebra TV of V . Then $TV = \bigoplus_{n \geq 0} T^n V$ and each $T^n V$ is a $GL(V)$ -module. We recall that the irreducible representations of $GL(V)$ are indexed by partitions with at most $\dim(V)$ parts, and the representation corresponding to a partition λ is denoted by W_λ . The irreducible constituents of $T^n V$, as a $GL(V)$ -module,

are those W_λ for which λ is a partition of n with at most $\dim(V)$ parts. Since TV is an associative algebra, it is, in particular, a Lie algebra. Consider the Lie subalgebra LV of TV generated by V . Then it turns out that LV is the free Lie algebra on V (see [14], Theorem 0.4). We have $LV = \bigoplus_{n \geq 1} L_n V$, where $L_n V = LV \cap T^n V$. It is easy to see that each $L_n V$ is also a $GL(V)$ -module. It is natural to ask what the irreducible constituents of $L_n V$ are. The following two theorems of Klyachko answer this question. Firstly, he beautifully related this question to a question about the symmetric group as follows:

Theorem 6. [9, Corollary 1] *Let λ be a partition of n with at most $\dim(V)$ parts. Then we have*

$$m(W_\lambda, L_n V) = m(V_\lambda, \text{Ind}_{C_n}^{S_n} \omega),$$

where ω is any faithful one dimensional character of the cyclic subgroup C_n , $m(W_\lambda, L_n V)$ is the multiplicity of the $GL(V)$ -module W_λ in $L_n V$ and $m(V_\lambda, \text{Ind}_{C_n}^{S_n} \omega)$ is the multiplicity of the S_n -module V_λ in $\text{Ind}_{C_n}^{S_n} \omega$.

He solved the problem for the symmetric group as follows:

Theorem 7. [9, Proposition 2] *Let λ be a partition of $n \geq 3$. Then the restriction to C_n of every irreducible representation V_λ of S_n contains a faithful representation of C_n except when $\lambda = (n), (1^n), (2, 2)$ or $(2, 2, 2)$.*

Combining Theorems 6 and 7, Klyachko obtained the following corollary (the unnumbered corollary following Proposition 2 in [9]).

Theorem 8. [9, Corollary] *Let λ be a partition of $n \geq 7$ having at most $\dim(V)$ parts. Then every irreducible representation W_λ of $GL(V)$ which appears as an irreducible constituent of $T^n V$, also appears as an irreducible constituent of $L_n V$ except when $\lambda = (n), (1^n)$. Equivalently, W_λ is an irreducible constituent of the $GL(V)$ -module $L_n V$ except when $\lambda = (n), (1^n)$.*

For a partition $\lambda \vdash n$ with at most $\dim(V)$ parts, we have, using Theorems 5 and 6,

$$m(W_\lambda, L_n V) = m(V_\lambda, \text{Ind}_{C_n}^{S_n} \omega) = a_\lambda^1. \tag{1}$$

Klyachko's proof of Theorem 7 simply uses representation theory. The goal is to prove the corresponding result for all $r \geq 1$ using representation theory and combinatorics, which we accomplish in this article. We provide a new proof of Theorem 7 using Theorem 1. Using our solution with Theorem 6, we get a new proof of a celebrated Klyachko's theorem 8 on Lie representations of general linear group $GL(V)$. Using Theorems 5 and 7, we have that $a_\lambda^1 > 0$ except when $\lambda = (n), (1^n), (2, 2), (2, 2, 2)$. For a combinatorial proof of this result, see M. Johnson [7]. For more proofs of Theorem 8, see [10], [15].

Our Theorem 1 yields the following for the permutations with cycle type equal to (n) :

Theorem 9. *Let λ be a partition of n . The minimal polynomial $p(x)$ of $\rho_\lambda(w_n)$ is $x^n - 1$ except in the following cases:*

1. $\lambda = (n)$ with $n > 1$ where $p(x) = (x - 1)$,
2. $\lambda = (n - 1, 1)$ where $p(x) = \frac{x^n - 1}{x - 1}$,
3. $\lambda = (2, 1^{n-2})$ where $p(x) = \frac{x^n - 1}{x - (-1)^{n-1}}$,
4. $\lambda = (1^n)$ where $p(x) = (x - (-1)^{n-1})$,
5. $\lambda = (3, 3)$ where $p(x) = \frac{x^6 - 1}{(x - \zeta)(x - \zeta^{-1})}$ and ζ is a primitive 6th root of unity,
6. $\lambda = (2, 2, 2)$ where $p(x) = \frac{x^6 - 1}{(x + \zeta)(x + \zeta^{-1})}$ with ζ as above,
7. $\lambda = (2, 2)$ where $p(x) = \frac{x^4 - 1}{x^2 + 1}$.

This result was proved by Swanson [20] and independently by Yang and Staroletov [22] (who proved a more general result). Their proofs use asymptotics of normalized characters. In this article, we provide a new proof (4) of this result (in the spirit of Klyachko's proof of Theorem 7) using only representation theory and combinatorics, which, we believe, is more accessible. We remark that a purely combinatorial proof of the above result is still an open problem.

We would also like to note that our main theorem 1 reconfirms the following conjecture of Sundaram, which was proved by Swanson, as a special case of Theorem 9.

Theorem 10. [19, Remark 4.8] *Let λ be a partition of n . Then 1 is an eigenvalue of the operator $\rho_\lambda(w_n)$ (equivalently, $\rho_\lambda(w_n)$ has a nonzero invariant vector in V_λ) except in the following cases.*

1. $\lambda = (n - 1, 1)$ with $n \geq 2$,
2. $\lambda = (2, 1^{n-2})$ with $n \geq 3$ odd,
3. $\lambda = (1^n)$ with n even.

For an element g in a group G , we say that g has a nonzero invariant vector in an irreducible representation (ρ, V) of G if there exists a nonzero vector $v \in V$ such that $\rho(g)v = v$.

Together with Amrutha P and A. Prasad [2, 3], we have shown the existence of a nonzero invariant vector when G is the symmetric group, g is any permutation and V is any irreducible representation of S_n .

Theorem 11. [2, Main theorem] *Let λ, μ be partitions of n . Then w_μ has a nonzero invariant vector in V_λ (equivalent to 1 being an eigenvalue of $\rho_\lambda(w_\mu)$) except in the following cases:*

1. $\lambda = (1^n)$, μ is any partition of n for which $w_\mu \notin A_n$,
2. $\lambda = (n - 1, 1)$, $\mu = (n)$, $n \geq 2$,

3. $\lambda = (2, 1^{n-2}), \mu = (n), n \geq 3$ is odd,
4. $\lambda = (2^2, 1^{n-4}), \mu = (n-2, 2), n \geq 5$ is odd,
5. $\lambda = (2, 2), \mu = (3, 1),$
6. $\lambda = (2^3), \mu = (3, 2, 1),$
7. $\lambda = (2^4), \mu = (5, 3),$
8. $\lambda = (4, 4), \mu = (5, 3),$
9. $\lambda = (2^5), \mu = (5, 3, 2).$

The proof is combinatorial, except for the base case, which is Theorem 9. By proving Theorem 9 using representation theory and combinatorics, we complete the proof of Theorem 11 within the same framework.

Similarly, for alternating groups, we proved, with Amrutha P and A. Prasad, the following result:

Theorem 12. [1, Theorem C] *Let μ be a partition of n . Then 1 is an eigenvalue of w_μ in an irreducible representation V of the alternating group A_n except in the following cases.*

1. $V = V_{(n-1,1)}$ and $\mu = (n)$ with $n > 3$ is odd.
2. $V = V_{(2,1)}^\pm$ and $\mu = (3),$
3. $V = V_{(2,2)}^\pm$ and $\mu = (3, 1),$
4. $V = V_{(4,4)}$ and $\mu = (5, 3).$

The proof is combinatorial except for the base case which is the following Theorem 13, itself a special case of Theorem 2. By proving Theorem 13 using representation theory and combinatorics, we complete the proof of Theorem 12 within the same framework.

Theorem 13. *Let n be a positive odd integer. Then the minimal polynomial $p(x)$ of w_n in an irreducible representation V of the alternating group A_n has degree less than n in precisely the following cases.*

1. $V = V_{(n)},$ where $p(x) = x - 1,$
2. $V = V_{(n-1,1)},$ where $p(x) = \frac{x^n - 1}{x - 1},$
3. $V = V_{(3,1,1)}^\pm,$ where $p(x) = \frac{x^5 - 1}{(x - \zeta)(x - \zeta^{-1})},$ where ζ is a primitive 5th root of unity.

Finally, we prove the counterpart of Theorem 11, which is of independent interest.

Theorem 14. *Suppose that w_μ has even order. Then there is a nonzero vector v in V_λ such that $\rho_\lambda(w_\mu)v = -v$ (equivalently, -1 is an eigenvalue of $\rho_\lambda(w_\mu)$) except in the following cases.*

1. $\lambda = (n)$,
2. $\lambda = (n - 2, 2)$ and $\mu = (n - 2, 2)$ with $n \geq 5$ odd,
3. $\lambda = (2, 1^{n-2})$ and $\mu = (n)$ with $n \geq 4$ even,
4. $\lambda = (1^n)$ and μ is such that w_μ is an even permutation,
5. $\lambda = (2, 2)$ and $\mu = (2, 2)$.
6. $\lambda = (3, 3)$ and $\mu = (3, 2, 1)$,
7. $\lambda = (5, 5)$ and $\mu = (5, 3, 2)$.

The analogous theorem for the alternating group is as follows:

Theorem 15. *Let w_μ be an element in the alternating group A_n with cycle type μ and suppose that w_μ has an even order. Then for every non-linear irreducible representation (ρ, V) , there exists a nonzero vector v in V such that $\rho(w_\mu)v = -v$ except when $V = V_{(2,2)}^\pm$.*

Remark 16. After this article was posted on arXiv, Staroletov[17] used similar techniques to improve Theorem 1 of this paper to give a complete description of the eigenvalues of the operator $\rho(g)$ for all irreducible representations of the symmetric group S_n and for all elements g in S_n . Our proof of Theorem 9 is the first combinatorial proof, which is also in the spirit of Klyachko's proof of Theorem 7. Theorem 1.1 of this paper solves Problem 4 of Tiep and Zalesskii for the first time.

The article is organized as follows: In Sect. 2, we introduce the notation and give basic definitions. In Sect. 3, we provide some important lemmas and a proposition to prove Theorem 9. In Sect. 4, we prove Theorem 9. In Sect. 5, we prove Theorems 1, 14, and 15.

2 Notation and Preliminaries

Let G be a finite group and H be its subgroup. Let $\text{Irr}(G)$ denote the set of all irreducible characters of G . For an irreducible character Ψ of a subgroup H , let $\text{Irr}(G|\Psi) = \{\chi \in \text{Irr}(G) \mid \langle \text{Res}_H^G \chi, \Psi \rangle_H \neq 0\}$. Let (χ, V) and (Ψ, W) be representations of G . Write $\mathbb{1}_H$ for the trivial character of H , and we write $\mathbb{1}$ when H is clear from the context. If (χ, V) is a subrepresentation of (Ψ, W) , then we denote this by $\chi \leq \Psi$ or $V \leq W$.

For a composition α of n , we define a natural permutation w_α of $[n]$ as follows:

$$w_\alpha = (1 \ 2 \ \dots \ \alpha_1)(\alpha_1 + 1 \ \alpha_1 + 2 \ \dots \ \alpha_1 + \alpha_2) \dots \left(\sum_{i=1}^{k-1} \alpha_i + 1 \ \sum_{i=1}^{k-1} \alpha_i + 2 \ \dots \ \sum_{i=1}^k \alpha_i = n \right),$$

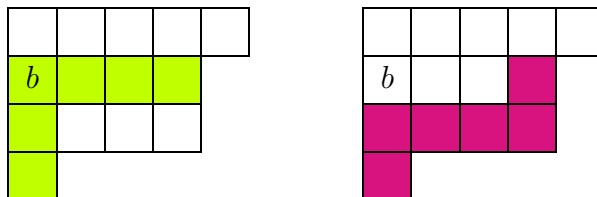
where k is the length of α . Let C_α denote the cyclic subgroup generated by w_α . When $\alpha = (n)$, we write C_n instead of $C_{(n)}$. Note that the cycle type μ of w_α is the partition obtained from α by reordering the parts of α in decreasing order. Hence, C_α and C_μ are conjugate subgroups in the symmetric group S_n .

We make frequent use of the following observation. Let δ be an irreducible character of C_α and let $\tilde{\delta}$ be an irreducible character of $C_{\alpha_1} \times \cdots \times C_{\alpha_k}$ such that $\text{Res}_{C_\alpha}^{C_{\alpha_1} \times \cdots \times C_{\alpha_k}} \tilde{\delta} = \delta$. Then

$$\text{Ind}_{C_\alpha}^{S_n} \delta \geq \text{Ind}_{C_{\alpha_1} \times C_{\alpha_2} \times \cdots \times C_{\alpha_k}}^{S_n} \tilde{\delta} = \text{Ind}_{S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_k}}^{S_n} \text{Ind}_{C_{\alpha_1} \times C_{\alpha_2} \times \cdots \times C_{\alpha_k}}^{S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_k}} \tilde{\delta},$$

The representations of the symmetric group S_n are indexed by partitions of n . We denote the irreducible representation of S_n indexed by the partition λ by $(\rho_\lambda, V_\lambda)$ and its character by χ_λ . For specific character values we use the recursive Murnaghan-Nakayama rule. We shall introduce some notation before defining the rule. We always use the English notation for the Young diagrams of partitions. For a cell b in a Young diagram of a partition λ , the hook length h_b of b is one plus the number of cells in the Young diagram which lie directly below it or directly to its right. For a cell b of the Young diagram of the partition λ , the rim hook rim_b (border strip in [12]) of the cell $b = (i, j)$ is the set of cells (u, v) such that $u \geq i$, $v \geq j$ and the Young diagram of λ does not contain a cell in the position $(i + 1, j + 1)$. The height $\text{ht}(\text{rim}_b)$ of rim_b is one minus the number of rows of λ which it intersects. The following example illustrates the above definitions.

Example 17. Let $\lambda = (5, 4, 4, 1)$, $b = (2, 1)$. Then the hook length h_b of b is 6, the rim hook rim_b of b is the set of cells $\{(2, 4), (3, 4), (3, 3), (3, 2), (3, 1), (4, 1)\}$ and the height $\text{ht}(\text{rim}_b)$ of rim_b is $3 - 1 = 2$. This is illustrated below where the first diagram considers the hook length and the second diagram considers the rim hook.



Fix any part of μ , say μ_t . The fastest way of computing these character values is using the recursive Murnaghan-Nakayama rule, which is as follows:

$$\chi_\lambda(w_\mu) = \sum_{b \in \lambda \text{ with } h_b = \mu_t} (-1)^{\text{ht}(\text{rim}_b)} \chi_{\lambda - \text{rim}_b}(w_{\tilde{\mu}}),$$

where b varies over the cells b of the Young diagram of λ such that the hook length h_b of the cell b is equal to μ_t and $\lambda - \text{rim}_b$ is the partition of shape obtained from the shape of λ by removing the cells of rim_b .

Let us discuss the relation between the characters of symmetric groups and symmetric functions. For more details and proofs see [12],[16]. Let Λ denote the ring of symmetric functions in the variables x_1, x_2, \dots . For a partition of λ , let p_λ , s_λ denote the power sum and Schur symmetric functions respectively.

Let h be a class function of the symmetric group S_n , the Frobenius characteristic ch from the direct sum $\bigoplus_{n=1}^{\infty} R(S_n)$ of the class functions $R(S_n)$ on S_n is defined as follows:

$$\begin{aligned} \text{ch} : \bigoplus_{n \geq 1} R(S_n) &\rightarrow \Lambda \\ g &\mapsto \sum_{\mu \vdash n} \frac{h(w_\mu) p_\mu}{z_\mu}, \end{aligned}$$

where $z_\mu = \prod_i i^{m_i} m_i!$ with m_i being the number of parts of μ equal to i . In particular, it is known that the Frobenius characteristic of χ_λ is the Schur function s_λ . Then ch is a ring homomorphism in the following sense:

$$\text{ch} \text{Ind}_{S_m \times S_t}^{S_{m+t}} \chi_\mu \otimes \chi_\nu = s_\mu s_\nu,$$

for all partitions $\mu \vdash m$ and $\nu \vdash t$.

The product $s_\mu s_\nu$ can be computed combinatorially by the Littlewood-Richardson rule.

Definition 18 (Littlewood-Richardson rule). Let μ and ν be partitions of m and t respectively. Then

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda,$$

where $c_{\mu\nu}^\lambda$ is the number of semistandard tableaux of shape λ/μ and type ν such that the reverse row reading word is a lattice permutation, see [12, Section I.9]. Also, a semistandard tableau of shape λ/μ and type ν whose reverse row reading word is a lattice permutation, is called an LR-tableau of shape λ/μ and type ν .

Example 19. Let $\lambda = (4, 3, 1)$, $\mu = (2, 1)$ and $\nu = (3, 2)$. Then there are three semistandard tableaux of shape λ/μ and type ν , of which the first two are LR-tableaux:

		1	1				1	1				1	2
	1	2				2	2				1	2	
2					1					1			

For symmetric functions f and g , we say that $f - g$ is Schur positive if $f - g$ is a nonnegative integer linear combination of Schur functions, equivalently $f - g$ is equal to $\text{ch}(h)$ for some character h . If w_μ has even order, define $g_\mu = \text{Ind}_{C_\mu}^{S_n}(-\mathbf{1})$, where $-\mathbf{1}$ denotes the linear character of C_μ obtained by sending w_μ to -1 . Since $C_\mu \leq C_{\mu_1} \times \cdots \times C_{\mu_k}$, we have

$$\begin{aligned} \text{Ind}_{C_\mu}^{S_n} \mathbf{1} &= \text{Ind}_{C_{\mu_1} \times \cdots \times C_{\mu_k}}^{S_n} \text{Ind}_{C_\mu}^{C_{\mu_1} \times \cdots \times C_{\mu_k}} \mathbf{1} \geq \text{Ind}_{C_{\mu_1} \times \cdots \times C_{\mu_k}}^{S_n} \mathbf{1} \\ \text{Ind}_{C_\mu}^{S_n} \mathbf{1} &\geq \text{Ind}_{C_{\mu_1} \times \cdots \times C_{\mu_k}}^{S_n} \mathbf{1}. \end{aligned}$$

Define $f_\mu = \text{Ind}_{C_\mu}^{S_n} \mathbf{1}$. Therefore,

$$f_\mu \geq \prod_{i \geq 1} f_{\mu_i} \tag{2}$$

Let $\mu \vdash n$, such that $\mu_i | \mu_1$ for all $i = 1, \dots, k$. Let η be any irreducible character of C_μ . We have an irreducible character $\tilde{\eta}$ for C_{μ_1} , obtained by mapping w_{μ_1} to $\eta(w_\mu)$. Then

$$\text{Ind}_{C_\mu}^{S_n} \eta \geq \text{Ind}_{C_{\mu_1} \times \cdots \times C_{\mu_k}}^{S_n} \tilde{\eta} \times \mathbf{1} \times \cdots \times \mathbf{1} \tag{3}$$

Let λ' be the partition conjugate to the partition λ . Then we have an involution ω defined by mapping each Schur function s_λ to $s_{\lambda'}$. It turns out that ω is a ring isomorphism.

We now describe the conjugacy classes and irreducible representations of the alternating group A_n . Let DOP_n denote the set of all partitions of n consisting of distinct odd parts. Let $\mu \vdash n$ with w_μ being an even permutation. Then the conjugacy class of S_n consisting of permutations with cycle type μ remains a single conjugacy class in A_n if $\mu \notin \text{DOP}_n$ and splits into two distinct conjugacy classes of equal size in A_n if $\mu \in \text{DOP}_n$. For $\mu \in \text{DOP}_n$, set $w_\mu^+ = w_\mu$ and $w_\mu^- = w w_\mu w^{-1}$ for some $w \in S_n \setminus A_n$. Then w_μ^+ and w_μ^- are representatives of the two distinct conjugacy classes of A_n obtained from the splitting of the conjugacy class of S_n consisting of permutations with cycle type μ . In this case, we can write $\mu = (2m_1 + 1, 2m_2 + 1, \dots, 2m_k + 1)$ with $m_1 > m_2 > \dots > m_k$. We define $\phi(\mu)$ to be the self-conjugate partition which is equal to $(m_1, m_2, \dots, m_k \mid m_1, m_2, \dots, m_k)$ in Frobenius notation.

For $\lambda \neq \lambda'$, we have $\text{Res}_{A_n}^{S_n} V_\lambda \cong \text{Res}_{A_n}^{S_n} V_{\lambda'}$ and $\text{Res}_{A_n}^{S_n} V_\lambda$ is an irreducible representation of A_n . We denote this irreducible representation (resp. its character) again by V_λ (resp. χ_λ).

For $\lambda = \lambda'$, we have $\text{Res}_{A_n}^{S_n} V_\lambda = V_\lambda^+ \oplus V_\lambda^-$, where V_λ^+ and V_λ^- are two non-isomorphic irreducible representations of A_n determined by the characters χ_λ^+ and χ_λ^- , respectively, defined as follows. Let $\sigma \in A_n$ with cycle type μ . Then

$$\chi_\lambda^+(\sigma) = \begin{cases} \frac{1}{2}(\epsilon_\mu + \sqrt{\epsilon_\mu \prod_i \mu_i}) & \text{if } \mu \in \text{DOP}_n, \lambda = \phi(\mu), \text{ and } \sigma \text{ is conjugate to } w_\mu^+, \\ \frac{1}{2}(\epsilon_\mu - \sqrt{\epsilon_\mu \prod_i \mu_i}) & \text{if } \mu \in \text{DOP}_n, \lambda = \phi(\mu), \text{ and } \sigma \text{ is conjugate to } w_\mu^-, \\ \frac{1}{2}\chi_\lambda(w_\mu) & \text{otherwise,} \end{cases} \quad (4)$$

$$\chi_\lambda^-(\sigma) = \begin{cases} \frac{1}{2}(\epsilon_\mu - \sqrt{\epsilon_\mu \prod_i \mu_i}) & \text{if } \mu \in \text{DOP}_n, \lambda = \phi(\mu), \text{ and } \sigma \text{ is conjugate to } w_\mu^+, \\ \frac{1}{2}(\epsilon_\mu + \sqrt{\epsilon_\mu \prod_i \mu_i}) & \text{if } \mu \in \text{DOP}_n, \lambda = \phi(\mu), \text{ and } \sigma \text{ is conjugate to } w_\mu^-, \\ \frac{1}{2}\chi_\lambda(w_\mu) & \text{otherwise,} \end{cases} \quad (5)$$

where $\epsilon_\mu = (-1)^{\sum_i (\mu_i - 1)/2}$.

For more details on the representations of the alternating groups A_n , see [6],[13].

We mention a few important things regarding wreath products of groups which we use in this paper. For a subgroup $H \leq S_n$ and $K \leq S_m$, we have a natural isomorphism (see [6, 4.1.18]) from $H \wr K$ into S_{mn} as follows:

$$\Gamma : H \wr K \rightarrow S_{mn} : (f, \pi) \mapsto \left(\begin{array}{c} (j-1)m + i \\ (\pi(j) - 1)m + f(\pi(j))(i) \end{array} \right)_{1 \leq i \leq m, 1 \leq j \leq n}.$$

We now have the following very important observation.

Lemma 20. *Let $C_n \leq S_n$ and $C_m \leq S_m$. Then $\Gamma(C_m \wr C_n) \leq S_{mn}$ contains an mn -cycle.*

Proof. Let $\sigma_{mn} = (((123 \dots m), e, e, \dots, e); (12 \dots n))$. Then we have

$$\Gamma(\sigma_{mn}) = (1 \ m+1 \ 2m+1 \ \dots \ (n-1)m+1 \ 2 \ m+2 \ 2m+2 \ \dots \ (n-1)m+2 \ \dots \ m \ 2m \ \dots \ nm).$$

□

For the representations of wreath products, see [6].

Let us make the following useful definition.

Definition 21 (Character reduced to a cycle). Let δ be an irreducible character of the cyclic subgroup C_n and $n = mp$. Then we define $\delta^{(p)}$ to be the irreducible character of C_m which sends w_m to $\delta(w_n)^p$.

The following corollary of Theorem 11 will be helpful in the proof of the main theorems.

Corollary 22. *The only pairs of partitions (λ, μ) of a given integer n for which w_μ does not admit a nonzero vector v in V_λ such that $\rho_\lambda(w_\mu)v = \text{sgn}(w_\mu)v$ are the following:*

1. $\lambda = (n)$, μ is any partition of n for which $w_\mu \notin A_n$,
2. $\lambda = (2, 1^{n-2})$, $\mu = (n)$, $n \geq 2$,
3. $\lambda = (n-1, 1)$, $\mu = (n)$, $n \geq 3$ is odd,
4. $\lambda = (n-2, 2)$, $\mu = (n-2, 2)$, $n \geq 5$ is odd,
5. $\lambda = (2, 2)$, $\mu = (3, 1)$,
6. $\lambda = (3, 3)$, $\mu = (3, 2, 1)$,
7. $\lambda = (4, 4)$, $\mu = (5, 3)$,
8. $\lambda = (2^4)$, $\mu = (5, 3)$,
9. $\lambda = (5, 5)$, $\mu = (5, 3, 2)$.

Proof. The proof follows from Theorem 11 and the fact that $V_\lambda \otimes V_{(1^n)} = V_{\lambda'}$. □

3 Ingredients for the Proof of the Main Theorems

The following proposition plays a key role in the proof of Theorem 9 and is motivated by [4, Proposition 3.5].

Proposition 23. *Let δ be an irreducible character of the cyclic subgroup C_n of S_n . Suppose p is a prime and $n = mp$ with $m > 1$.*

Assume the following:

- μ^1, \dots, μ^p are partitions of m with at least two of them distinct,
- $\langle \text{Res}_{C_m}^{S_m} \chi_{\mu^i}, \mathbf{1}_{C_m} \rangle \neq 0$ for all $i \geq 2$ and $\langle \text{Res}_{C_m}^{S_m} \chi_{\mu^1}, \delta^{(p)} \rangle_{C_m} \neq 0$,
- $\lambda \vdash n$ such that $\chi_{\mu^1} \times \dots \times \chi_{\mu^p}$ is an irreducible constituent of $\text{Res}_{S_m^{x_p}}^{S_n} \chi_\lambda$.

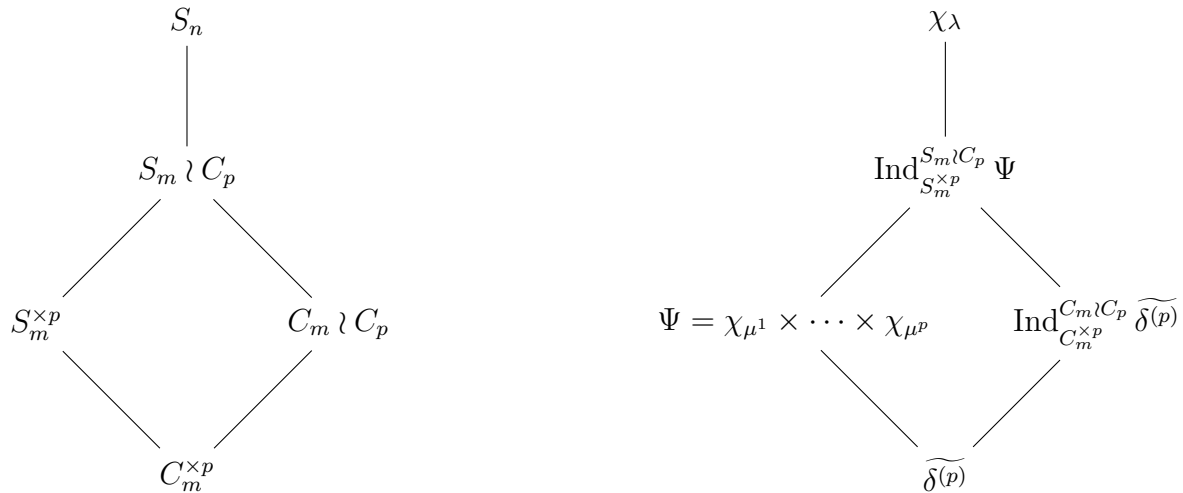
Then:

- $\text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \widetilde{\delta^{(p)}}$ is a subrepresentation of $\text{Res}_{C_m \wr C_p}^{S_n} \chi_\lambda$, where $\widetilde{\delta^{(p)}} := \delta^{(p)} \times \mathbb{1}_{C_m} \times \cdots \times \mathbb{1}_{C_m}$,

- As a consequence, we have

$$\langle \text{Res}_{C_n}^{S_n} \chi_\lambda, \delta \rangle > 0.$$

Proof. Consider the following diagrams, which outline the structure of the proof. The nodes of the first diagram consist of subgroups of S_n and those of the second diagram consist of representations of the corresponding subgroups in the first diagram.



Let us discuss the nodes of the above diagrams more carefully now. Define $\Psi = \chi_{\mu^1} \times \cdots \times \chi_{\mu^p}$. The group $S_m \wr C_p$ is a subgroup of S_n via the isomorphism Γ (2). Note that $I_{S_m \wr C_p}(\Psi) = S_m^{\times p}$, the inertia group of Ψ in $S_m \wr C_p$, since $\mu^i \neq \mu^j$ for some $i \neq j$. Therefore, by Clifford's theorem [5, Theorem 6.11], $\text{Ind}_{S_m^{\times p}}^{S_m \wr C_p} \Psi$ is an irreducible representation of $S_m \wr C_p$. We now establish the chain of subrepresentations. Firstly,

$$\text{Res}_{S_m \wr C_p}^{S_n} \chi_\lambda \geq \text{Ind}_{S_m^{\times p}}^{S_m \wr C_p} \Psi. \quad (6)$$

Using Mackey's restriction formula, we have

$$\text{Res}_{C_m \wr C_p}^{S_m \wr C_p} \text{Ind}_{S_m^{\times p}}^{S_m \wr C_p} \Psi = \sum_{g \in S_m^{\times p} \backslash S_m \wr C_p / C_m \wr C_p} \text{Ind}_{(S_m^{\times p})^g \cap C_m \wr C_p}^{C_m \wr C_p} \text{Res}_{(S_m^{\times p})^g \cap C_m \wr C_p}^{(S_m^{\times p})^g} (\Psi)^g,$$

where for a subgroup H , $H^g = gHg^{-1}$ and for a character η of H , η^g is a character of gHg^{-1} defined by $\eta^g(x) = \eta(g^{-1}xg)$. Considering the summand corresponding to the double coset representative $g = e$, we obtain

$$\begin{aligned} \text{Res}_{C_m \wr C_p}^{S_m \wr C_p} \text{Ind}_{S_m^{\times p}}^{S_m \wr C_p} \Psi &\geq \text{Ind}_{(S_m^{\times p}) \cap C_m \wr C_p}^{C_m \wr C_p} \text{Res}_{(S_m^{\times p}) \cap C_m \wr C_p}^{S_m^{\times p}} \Psi \\ &= \text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \text{Res}_{C_m^{\times p}}^{S_m^{\times p}} \Psi. \end{aligned}$$

Using the assumptions on μ^i and the definition of $\widetilde{\delta(p)}$, the above equation becomes

$$\text{Res}_{C_m \wr C_p}^{S_m \wr C_p} \text{Ind}_{S_m^{\times p}}^{S_m \wr C_p} \Psi \geq \text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \widetilde{\delta(p)}. \quad (7)$$

Using Equations (6) and (7), we obtain

$$\text{Res}_{C_m \wr C_p}^{S_n} \chi_\lambda \geq \text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \widetilde{\delta(p)},$$

which proves the former statement of the theorem.

From Lemma 20, we know that $d_n := \Gamma(\sigma_{mp})$ is an n -cycle in S_n . Let D_n be the cyclic subgroup of S_n generated by d_n . Once again, applying Mackey's restriction formula, we have

$$\begin{aligned} \text{Res}_{D_n}^{C_m \wr C_p} \text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \widetilde{\delta(p)} &= \sum_{g \in C_m^{\times p} \backslash C_m \wr C_p / D_n} \text{Ind}_{(C_m^{\times p})^g \cap D_n}^{D_n} \text{Res}_{(C_m^{\times p})^g \cap D_n}^{(C_m^{\times p})^g} \widetilde{\delta(p)}^g \\ &\geq \text{Ind}_{C_m^{\times p} \cap D_n}^{D_n} \text{Res}_{C_m^{\times p} \cap D_n}^{C_m^{\times p}} \widetilde{\delta(p)} \\ &= \text{Ind}_{\langle d_n^p \rangle}^{D_n} \text{Res}_{\langle d_n^p \rangle}^{C_m^{\times p}} \widetilde{\delta(p)} \end{aligned}$$

$$\text{Res}_{D_n}^{C_m \wr C_p} \text{Ind}_{C_m^{\times p}}^{C_m \wr C_p} \widetilde{\delta(p)} \geq \text{Ind}_{\langle d_n^p \rangle}^{D_n} \eta, \quad (8)$$

where η is the one dimensional character $\text{Res}_{\langle d_n^p \rangle}^{C_m^{\times p}} \widetilde{\delta(p)}$ of $\langle d_n^p \rangle$.

From Equations (7) and (8), we have

$$\text{Res}_{D_n}^{S_n} \chi_\lambda \geq \gamma,$$

for all irreducible characters γ of D_n such that $\text{Res}_{\langle d_n^p \rangle}^{D_n} \gamma = \eta$. Since all irreducible characters of S_n are real valued, we have

$$\text{Res}_{D_n}^{S_n} \chi_\lambda \geq \gamma^\tau,$$

for all irreducible characters γ^τ of D_n such that $\text{Res}_{\langle d_n^p \rangle}^{D_n} \gamma = \eta$ and for all Galois automorphisms τ of $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n -th root of unity. Since D_n and C_n are conjugate subgroups of S_n , we have

$$\text{Res}_{C_n}^{S_n} \chi_\lambda \geq \delta.$$

□

Corollary 24. *Let δ be an irreducible character of the cyclic subgroup C_n of S_n and let $n = 4m$ with $m > 1$. Assume the following:*

- $\mu^1, \mu^2, \mu^3, \mu^4$ are partitions of m with at least three of them mutually distinct,

- $\langle \text{Res}_{C_m}^{S_m} \chi_{\mu^i}, \mathbf{1}_{C_m} \rangle \neq 0$ for all $i \geq 2$ and $\langle \text{Res}_{C_m}^{S_m} \chi_{\mu^1}, \delta^{(4)} \rangle_{C_m} \neq 0$,
- $\lambda \vdash n$ such that $\chi_{\mu^1} \times \cdots \times \chi_{\mu^4}$ is an irreducible constituent of $\text{Res}_{S_m^{\times 4}}^{S_n} \chi_\lambda$.

Then:

- $\text{Ind}_{C_m^{\times 4}}^{C_m \wr C_4} \widetilde{\delta^{(4)}}$ is a subrepresentation of $\text{Res}_{C_m \wr C_4}^{S_m} \chi_\lambda$, where $\widetilde{\delta^{(4)}} := \delta^{(4)} \times \mathbf{1}_{C_m} \times \cdots \times \mathbf{1}_{C_m}$,
- As a consequence, we have

$$\langle \text{Res}_{C_n}^{S_n} \chi_\lambda, \delta \rangle > 0.$$

Proof. The corollary follows from the proof of Proposition 23 and the following observation. Since at least three elements of $\{\mu^1, \mu^2, \mu^3, \mu^4\}$ are mutually distinct, the inertia group of $\Psi = \chi_{\mu^1} \times \chi_{\mu^2} \times \chi_{\mu^3} \times \chi_{\mu^4}$ in $S_m \wr C_4$ is $S_m^{\times 4}$ and thus $\text{Ind}_{S_m^{\times 4}}^{S_m \wr C_4} \Psi$ is an irreducible representation of $S_m \wr C_4$. \square

Corollary 25. *Let δ be an irreducible character of the cyclic subgroup C_n of S_n and let $n = 2p$ where $p > 10$ is a prime. Consider the following two cases:*

- If $\delta^{(p)}$ is the sign character $\chi_{(1,1)}$ of S_2 (equivalently, of C_2), set $\mu^1 = (1, 1)$ and $\mu^2 = \mu^3 = \dots = \mu^p = (2)$.
- Otherwise, $\delta^{(p)}$ is the trivial character $\chi_{(2)}$ of S_2 , set $\mu^1 = \mu^2 = (1, 1)$ and $\mu^3 = \mu^4 = \dots = \mu^p = (2)$.

Then:

- $\chi_{\mu^1} \times \cdots \times \chi_{\mu^p}$ is an irreducible constituent of $\text{Res}_{S_2^{\times p}}^{S_n} \chi_\lambda$ where $\lambda = (p, p)$.
- $\text{Ind}_{S_2^{\times p}}^{S_2 \wr C_p} \chi_{\mu^1} \times \cdots \times \chi_{\mu^p}$ is a subrepresentation of $\text{Res}_{S_2 \wr C_p}^{S_n} \chi_\lambda$.
- As a consequence, we have

$$\langle \text{Res}_{C_n}^{S_n} \chi_\lambda, \delta \rangle > 0.$$

Proof. The results follow from a simple calculation of LR-coefficient and from the proof of Proposition 23. \square

We have a couple of interesting lemmas.

Lemma 26. *For every partition λ of $p + q$, and every partition α of p that is contained in λ , there exists a partition β of q such that $s_\alpha s_\beta \geq s_\lambda$.*

Proof. We reproduce the proof of [2, Lemma 2.1], as it will be used extensively in this article. Let $T_{\lambda/\alpha}$ denote the skew-tableau obtained by putting i in the i th cell (from the top) of each column of λ/α . Let β be the weight of $T_{\lambda/\alpha}$. Clearly, $T_{\lambda/\alpha}$ is semistandard. Since every $i + 1$ occurs below an i in the same column, the reverse row reading word is a

lattice permutation. By the Littlewood-Richardson rule, $s_\alpha s_\beta \geq s_\lambda$ holds. For example, if $\lambda = (5, 4, 4, 1)$ and $\alpha = (3, 2, 1)$ then

$$T_{\lambda\alpha} = \begin{array}{cccccc} & & & & 1 & 1 \\ & & & & 1 & 2 \\ & & & 1 & 2 & 3 \\ & & 1 & & & \\ 1 & & & & & \end{array},$$

and β is $(5, 2, 1)$. □

Define

$$a_n = \begin{cases} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n-1,1), (2,1^{n-2})}} s_\lambda & \text{if } n \text{ is odd,} \\ \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n-1,1), (1^n)}} s_\lambda & \text{if } n \text{ is even.} \end{cases} \quad (9)$$

Lemma 27. *Given integers $p \geq 2$, $q \geq 1$, and a partition $\lambda \vdash (p+q)$ different from (1^{p+q}) , there exists a partition $\beta \vdash q$ such that $a_q \geq s_\beta$ and $\beta \subset \lambda$.*

Proof. See [2, Lemma 2.3], with the understanding that a_q and $f_q = \text{Ind}_{C_q}^{S_q} \mathbf{1}$ have the same set of Schur functions appearing with positive coefficients in their Schur basis expansion. □

4 Proof of Theorem 9

Lemma 28. *If $p > 5$ is an odd positive integer, then*

$$\sum_{\substack{\alpha, \beta \vdash p \\ \alpha \neq \beta \\ \alpha \neq (p-1,1), (2,1^{p-2}) \\ \beta \neq (p-1,1), (2,1^{p-2})}} s_\alpha s_\beta \geq s_\lambda \quad (10)$$

for all partitions λ of $2p$ except when $\lambda = (2p), (2p-1, 1), (p, p), (2^p), (2, 1^{2p-2}), (1^{2p})$.

If $p > 5$ is an odd integer, then

$$\sum_{\substack{\alpha, \beta \vdash p \\ \alpha \neq \beta \\ \alpha \neq (p), (1^p) \\ \beta \neq (p-1,1), (2,1^{p-2})}} s_\alpha s_\beta \geq s_\lambda \quad (11)$$

for all partitions λ of $2p$ except when $\lambda = (2p), (p, p), (2^p), (1^{2p})$.

If $p > 5$ is an even positive integer, then

$$\sum_{\substack{\alpha, \beta \vdash p \\ \alpha \neq \beta \\ \alpha \neq (p-1,1), (1^p) \\ \beta \neq (p-1,1), (1^p)}} s_\alpha s_\beta \geq s_\lambda \quad (12)$$

for all partitions λ of $2p$ except when

$$\lambda = (2p), (2p-1, 1), (p, p), (3, 1^{2p-3}), (2^p), (2, 2, 1^{2p-4}), (2, 1^{2p-2}), (1^{2p}).$$

If $p > 5$ is an even positive integer, then

$$\sum_{\substack{\alpha, \beta \vdash p \\ \alpha \neq \beta \\ \alpha \neq (p), (1^p) \\ \beta \neq (p-1, 1), (1^p)}} s_\alpha s_\beta \geq s_\lambda \tag{13}$$

for all partitions λ of $2p$ except when $\lambda = (2p), (p, p), (3, 1^{2p-3}), (2^p), (2, 2, 1^{2p-4}), (2, 1^{2p-2}), (1^{2p})$.

Proof. Let us fix some notations. For a positive integer p , let

$$A_p = \begin{cases} \{(p-1, 1), (2, 1^{p-2})\}, & \text{if } p \text{ is odd,} \\ \{(p-1, 1), (1^p)\}, & \text{if } p \text{ is even.} \end{cases}$$

$$B_p = \{(p), (1^p)\}.$$

We have two equations in the statement of the lemma to consider, depending on whether p is odd or even. If p is odd, for Equation (10) (resp. (11)), we must produce partitions α, β of p such that $\alpha \neq \beta$, $\alpha, \beta \notin A_p$ (resp. $\alpha \notin B_p$ and $\beta \notin A_p$) and similarly for p even.

The proof proceeds by a case-by-case analysis.

Let λ be a partition of $2p \geq 12$. The theorem can be directly verified when $\lambda = (2p)$ or (1^{2p}) . Thus, we may assume that λ is neither $(2p)$ nor (1^{2p}) .

- $\lambda \supset (p-1, 1)$

- Suppose that $\lambda_2 > 1$. Then choose $\alpha = (p-2, 2)$, and choose β as in the proof of Lemma 26. Since $\alpha \notin A_p \cup B_p$, if $\beta \notin A_p$ and $\beta \neq \alpha$, then we are done. Otherwise, one of the following situations occurs.

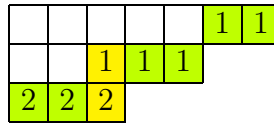
- * $\beta = (p-1, 1)$. Since each column of $T_{\lambda\alpha}$ in the proof of Lemma 26 is filled with integers $1, 2, \dots$ in increasing order, λ/α has $p-2$ columns with one cell and one column with two cells. Either the first column, the third column, or the $(p-1)$ st column can have two cells.

If the first column of λ/α has two cells, then we may change β to $(p-2, 1, 1)$ by constructing a skew-tableau as shown in the example below.

					1	1
		1	1	1		
2						
3						

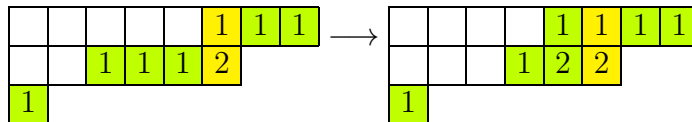
Note that the reverse reading word will be a lattice permutation because $\lambda \supset (p-1, 1)$, so there will be at least one cell in the first row of λ/α .

If the third column of λ/α has two cells, then we may change β to $(p-3, 3)$ as shown in the example below.

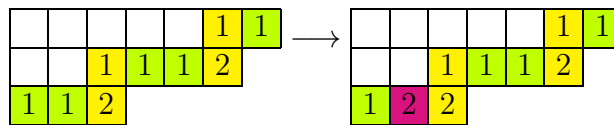


Note that the reverse reading word will be a lattice permutation because at least three 1's occur above the third row (since $p \geq 6$).

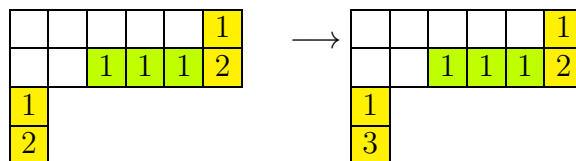
If the $(p-1)$ st column of λ/α has two cells, we may change α to $(p-3, 3)$ (note that $p \geq 6$, so α is a partition). We may choose $\beta = (p-2, 2)$ as shown in the example below.



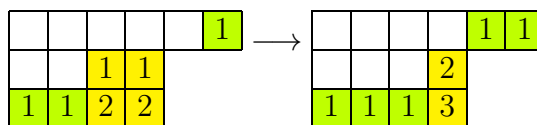
- * $\beta = (2, 1^{p-2})$ with p odd. This would mean that λ/α has two columns, having 1 and $p-1$ cells respectively. But since $\lambda \supset (p-1, 1)$, the $(p-1)$ st cell in the first row lies in a column of length one. Since $p \geq 6$, the other column of λ/α has to be the first one. Therefore, λ/α is a vertical strip, and we can replace β with (1^p) .
- * $\beta = (1^p)$ with p even. In this case, λ/α has only one column. But λ/α contains a cell at position $(1, p-1)$ and hence all the cells of λ/α lie in the $p-1$ st column. It follows that this case cannot occur.
- * $\beta = (p-2, 2)$. In this case, the shape λ/α contains $p-4$ columns with a single cell and exactly two columns with two cells. If those two column numbers are $p-1$ and p , then we get $\lambda = (p, p)$, for which none of the equations in the statement of the lemma hold. If those two column numbers are 3 and $p-1$, then we may change β to $(p-3, 3)$ as shown in the example below.



If those two column numbers are 1 and $p-1$, then $\lambda = (p-1, p-1, 1, 1)$. Therefore, we may change β to $(p-3, 1, 1)$ as shown in the example below.



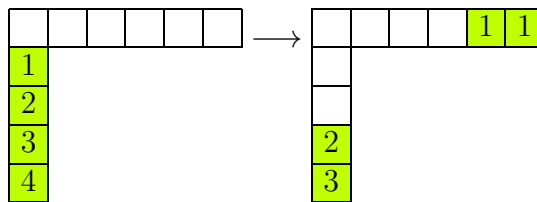
If those two columns numbers are 3 and 4, then we can choose $\alpha = (p-3, 3)$ and $\beta = (p-2, 1, 1)$ as shown in the example below.



The first row of λ/α contains at least one cell (since $p \geq 6$), hence the reverse row reading word is a lattice permutation. If those two columns numbers are 1 and 3, then we may choose $\beta = (p-2, 1, 1)$ by incrementing the entry 2 in the cell $(4, 1)$ by 1. Finally, the remaining case is that the column numbers are 1 and 2. Now we may choose $\beta = (p-4, 2, 2)$ by incrementing all the entries in the first two columns by 1. Again, the reverse row reading word is a lattice permutation since $p \geq 6$.

– Suppose $\lambda_2 = 1$ and $\lambda_1 \geq p$. Let us replace β by (p) and choose α as in the proof of Lemma 26. Since $\beta \notin A_p$, if $\alpha \notin A_p$ (rep. $\alpha \notin B_p$) and $\beta \neq \alpha$, then we are done. Otherwise, one of the following cases occurs.

- * $\alpha = (p-1, 1)$. In this case, λ/β has $p-1$ columns, of which one column has exactly two cells (and the others have only one cell). The column with two cells has to be the first column of λ/β , since λ is a hook. Incrementing its entries by 1 allows us to replace β by $(p-2, 1, 1)$.
- * $\alpha = (2, 1^{p-2})$ with p odd. Then λ/β has two columns, having 1 and $p-1$ cells respectively. Since λ is a hook, the column with one cell must lie in the first row and the column with $p-1$ cells has to be the first column. Incrementing the entries of the first column by 1 allows us to replace α by (1^p) .
- * $\alpha = (1^p)$ with p even. All the cells of λ/β must lie in the first column. We may replace β by $(p-2, 1, 1)$ and α by $(2, 1^{p-2})$ as shown in the following example.



- * $\alpha = (p)$. Then $\lambda = (p+r, p-r)$ with $0 \leq r < p$. For $2 \leq r \leq p-2$, we may choose $\alpha = (p-2, 2)$ as shown in the example below.



Otherwise λ must be one of $(2p-1, 1)$, $(p+1, p-1)$, (p, p) . Suppose that $\lambda = (p, p)$. The fact that $c_{\alpha\beta}^\lambda > 0$ with $\alpha, \beta \vdash p$ implies that $\alpha = \beta = (p-k, k)$ and $c_{\alpha\beta}^\lambda = 1$. Hence, when $\lambda = (p, p)$, no equation in the lemma is satisfied. Now if $\lambda = (2p-1, 1)$, then we may choose $\alpha = (p-1, 1)$ and $\beta = (p)$ for Equations (11) and (13). The other equations are not satisfied when $\lambda = (2p-1, 1)$.

If $\lambda = (p+1, p-1)$, then we may choose $\alpha = (p-2, 2)$ and $\beta = (p-3, 3)$ as shown in the example below.

				1	1	1
		2	2	2		

- Suppose $\lambda_2 = 1$ and $\lambda_1 = p - 1$. In other words, $\lambda = (p - 1, 1^{p+1})$. Replace α by $(p - 2, 1, 1)$. We could replace β by $(2, 1^{p-2})$ or (1^{2p}) depending on parity of p .

Case 2: Suppose that $\lambda \supset (2, 1^{p-2})$.

If p is odd, then Equations (10) and (11) are invariant under the involution ω which takes s_λ to $s_{\lambda'}$. Thus, in this case we can conjugate λ so that λ' contains $(p - 1, 1)$. Now we can use the previous case. Finally, the partitions for which the equations are not satisfied are conjugate to those partitions for which the equations are not satisfied in the previous case. Namely, the partitions $(2, 1^{2p-2}), (2^p), (1^{2p})$ for Equations (10), and $(2^p), (1^{2p})$ for (11).

Now we shall consider the remaining equations (12) and (13); thus, p is even.

- Suppose that $\lambda_2 \geq 2$. Thus, we can replace α with $(2, 2, 1^{p-4})$. Choose β as in the proof of Lemma 26. If $\beta \notin B_p$ and $\beta \neq \alpha$, then we are done. Otherwise, one of the following cases must arise.
 - $\beta = (p - 1, 1)$. Here, the skew shape λ/α has $p - 1$ columns, with exactly one column containing two cells. The column with two cells has to be one of the first three columns.

If the first or second column of λ/α has two cells, increment the entries of those two cells by 1 to replace β by $(p - 2, 1, 1)$. Since $p \geq 6$, λ/α has at least one cell in the first row, so the reverse row reading word is a lattice permutation.

Suppose the third column of λ/α has two cells. Since $\lambda \supset (2, 1^{p-2})$, the first column of λ/α must have exactly one cell. Changing the entry of this cell from 1 to 3 gives us an LR-tableau of weight $(p - 2, 1, 1)$.
 - $\beta = (1^p)$ with p even. In this case, λ must be equal to $(2, 2, 1^{2p-4})$. One can easily check that indeed when $\lambda = (2, 2, 1^{2p-4})$ Equations (12) and (13) are not satisfied.
 - $\beta = (2, 2, 1^{p-4})$. In this case, λ/α must have exactly two columns with one having 2 cells and the other having $p - 2$ cells. Notice that one of these columns must be the first column since λ/α contains at least one cell in the first column. If the second column contains $p - 2$ cells, then $\lambda = (2^p)$. One can easily see that when $\lambda = (2^p)$, none of the equations in the lemma are satisfied, by applying the involution ω and our argument for $\lambda = (p, p)$. Therefore, the first column must contain $p - 2$ cells. Hence, $\lambda = (2^4, 1^{2p-8})$ or $(3, 3, 1^{2p-6})$. In both cases the skew shape λ/α is a vertical strip, and we may replace β by (1^p) .

- Suppose $\lambda_2 = 1$ (so that λ is a hook) and $\lambda_1 \geq 3$. Replace α by $(3, 1^{p-3})$ and choose β as in the proof of Lemma 26. Since $\alpha \notin A_p \cup B_p$, if $\beta \notin A_p$ and $\beta \neq \alpha$, then we are done. Otherwise, one of the following cases must occur.
 - $\beta = (p-1, 1)$. In this case, λ has $p-1$ columns, one having two cells and the other having only one cell. Since λ is a hook, only the first column of λ/α can have two cells. Incrementing the entries in these cells by one will allow us to change β to $(p-2, 1, 1)$.
 - $\beta = (1^p)$. In this case, λ must be $(3, 1^{2p-3})$. One can easily see that when $\lambda = (3, 1^{2p-3})$ Equations (13) and (12) are not satisfied.
 - $\beta = (3, 1^{p-3})$. In this case λ/α has exactly three columns with one having $p-2$ cells and the other two columns having one cell respectively. Since λ is a hook, thus $\lambda = (5, 1^{2p-5})$. We may replace β by (1^p) .
- Suppose $\lambda_2 = 1$ and $\lambda_1 = 2$. In this case λ must be $(2, 1^{2p-2})$, for which both Equations (12) and (13) are unsatisfied.

Finally, let us consider the case when λ does not contain both $(p-1, 1)$ and $(2, 1^{p-2})$. Equivalently, $\lambda_1, \lambda'_1 < p-1$. Note that we have assumed $\lambda \neq (2p), (1^{2p})$. Thus, $l(\lambda) > 2$ and $\lambda_1 > 2$. Let α be a partition of p contained in λ which is maximal in the dominance order. Clearly, $\alpha \notin A_p \cup B_p$. Let us choose β as in the proof of Lemma 26. If $\beta \neq \alpha$, then we are done since $\beta \notin A$.

Otherwise, $\alpha = \beta$. Let us replace α using the following procedure.

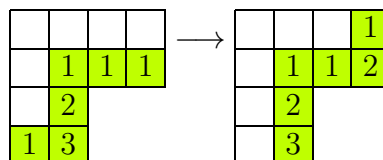
1. Choose all the cells in the first row of λ .
2. Since $\lambda_1 < p-1$, we need to choose more cells to construct a partition α of p . In this step, we choose cells in the first column from the top.
3. Choose the rest of the cells for α in any manner the reader wishes if required.

Now we have constructed our α . Let us choose β as in the proof of Lemma 26.

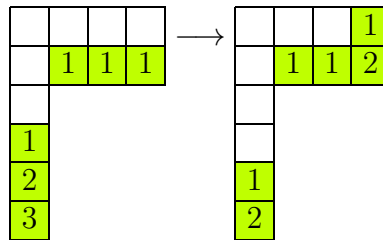
If step 3 of the algorithm was used to construct α , then clearly $\beta_1 < \alpha_1$ which implies $\alpha \neq \beta$. Thus, we are done.

Otherwise, $\alpha = (\lambda_1, 1^{p-\lambda_1})$. If $\lambda_3 > 2$, then clearly $\beta_2 > 1$. Therefore, $\alpha \neq \beta$ and we are done. Hence, we may assume that $\lambda_3 \leq 2$.

Suppose that $\lambda_3 = 2$. We also have $\lambda_1 = \lambda_2$ since $\alpha_1 = \beta_1$. Note that the first column of λ/α contains exactly one cell due to $\alpha_1 = \beta_1$, $\alpha_2 = 1$ and $\lambda_3 = 2$. Since $\alpha = \beta = (\lambda_1, 1^{p-\lambda_1})$, we must have $\lambda = (\lambda_1, \lambda_1, 2^{p-\lambda_1})$. In this case we may choose $\alpha = (\lambda_1 - 1, 1^{p-\lambda_1+1})$ and $\beta = (\lambda_1 - 1, 2^2, 1^{p-\lambda_1+3})$ as shown in the example below. Note that $\alpha, \beta \notin \{(p), (p-1, 1), (2, 1^{p-2}), (1^p)\}$ due to $\lambda_1, \lambda'_1 < p$.



Finally let $\lambda_3 = 1$ and hence $\lambda = (\lambda_1, \lambda_1, 1^{2p-2\lambda_1})$. We may choose $\alpha = (\lambda_1 - 1, 1^{p-\lambda_1+1})$ and $\beta = (\lambda_1, 2, 1^{p-\lambda_1-2})$ as shown in the example below.



This completes the proof. □

Lemma 29. *Let $p \geq 7$ be a positive odd integer. Then*

$$\sum_{\substack{\alpha \vdash 2p \\ \alpha \neq (2p), (2p-1, 1), (p, p) \\ \alpha \neq (2^p), (2, 1^{2p-2}), (1^{2p})}} s_\alpha \sum_{\substack{\beta \vdash p \\ \beta \neq (p-1, 1), (2, 1^{p-2})}} s_\beta \geq s_\lambda \tag{14}$$

for all partitions λ of $3p$ except when $\lambda = (3p), (3p - 1, 1), (2, 1^{3p-2}), (1^{3p})$, and

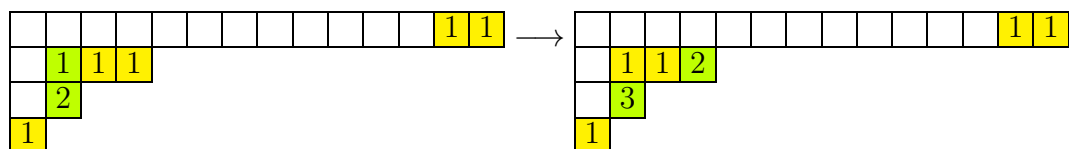
$$\sum_{\substack{\alpha \vdash 2p \\ \alpha \neq (2p), (p, p) \\ \alpha \neq (2^p), (1^{2p})}} s_\alpha \sum_{\substack{\beta \vdash p \\ \beta \neq (p-1, 1), (2, 1^{p-2})}} s_\beta \geq s_\lambda \tag{15}$$

for all partitions λ of $3p$ except when $\lambda = (3p), (1^{3p})$.

Proof. We shall prove the first statement of the lemma. The latter statement follows from the former statement and the fact that (15) holds when $\lambda = (3p - 1, 1), (2, 1^{3p-2})$.

Let $H := \{(2p), (2p - 1, 1), (p, p), (2^p), (2, 1^{2p-2}), (1^{2p})\}$. For $\lambda = (3p)$ or (1^{3p}) , the lemma is obvious. We may assume that $\lambda \vdash 3p$ and $\lambda \neq (3p), (1^{3p})$. We can choose $\alpha \vdash 2p$, $\beta \vdash p$ such that $s_\alpha s_\beta \geq s_\lambda$ with $\beta \neq (p - 1, 1), (2, 1^{p-2})$ by Lemma 27. If $\alpha \notin H$, then we are done. Otherwise, one of the following cases must occur.

- $\alpha = (2p)$ or $(2p - 1, 1)$. Suppose that $\lambda'_1 > 2$. Then we may replace α by $(2p - 2, 1, 1)$ and choose β as in the proof of Lemma 26. If $\beta \neq (p - 1, 1), (2, 1^{p-2})$, then we are done. Otherwise, one of the following cases must occur.
 - $\beta = (p - 1, 1)$. Then λ/α has exactly $p - 1$ columns with one of them having two cells. Either the first column or the second column can have two cells. If the first column has two cells, then we may increment the entries in the first column by 1, and we replace β by $(p - 2, 1, 1)$. If the second column has two cells, then we can increment the last entry 1 in the second row. Thus, we can replace β by $(p - 2, 1, 1)$ as shown in the example below.

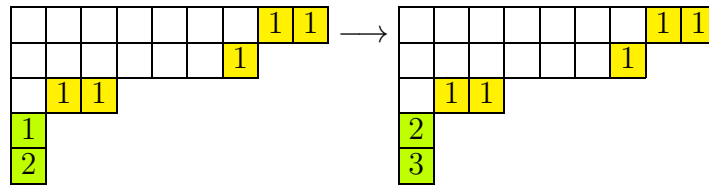


- $\beta = (2, 1^{p-2})$. In this case λ/α has exactly two columns with one having $p-2$ cells and the other having a single cell. Since λ/α contains at least one cell in its first row, we conclude that $\lambda = (2p, 1^p)$. We shall replace β by (1^p) .

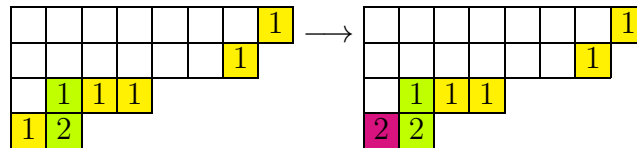
Suppose that $\lambda'_1 = 2$, i.e., λ is a two-row partition $(2p+r, p-r)$ with $1 \leq r < p$. If $r = p-1$, then $\lambda = (3p-1, 1)$. In this case we can easily see that Equation (14) does not hold. For $1 \leq r < p-1$ we may replace α by $(2p-2, 2)$ and β by (p) .

- $\alpha = (p, p)$. Suppose that $\lambda'_1 \geq 3$. We replace α by $(p, p-1, 1)$ and choose β as in the proof of Lemma 26. If $\beta \neq (p-1, 1), (2, 1^{p-2})$, then there is nothing to prove. Otherwise, one of the following cases occurs.

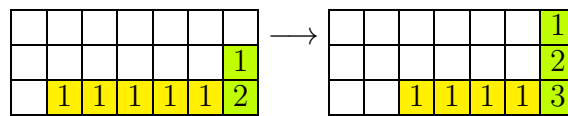
- $\beta = (p-1, 1)$. In this case, λ/α has $p-2$ columns with $p-1$ columns having a single cell and one column having two cells. Then one of the first, second, p th or $p+1$ th column has two cells. If the first column has two cells, then we may replace β by $(p-2, 1, 1)$ as shown in the example below.



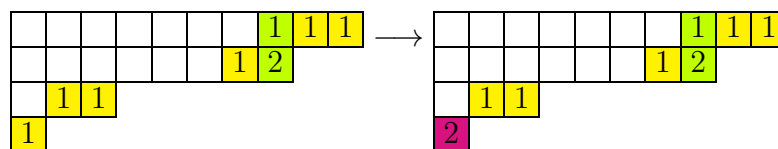
If the second column has two cells, then we may replace β by $(p-2, 2)$ as shown in the example below.



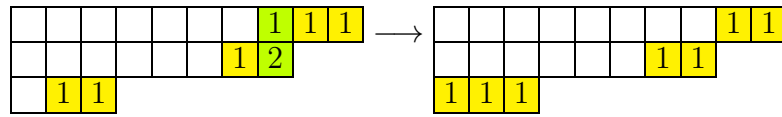
If the p th column has two cells, then we may replace α by $(p-1, p-1, 2)$ and β by $(p-2, 1, 1)$ as shown in the example below.



If the $p+1$ th column has two cells, then we may replace β by $(p-2, 2)$ if $(4, 1)$ th cell lies in λ/α as shown in the example below.



Otherwise, we may replace α by $(p + 1, p - 1)$ and β by (p) as shown in the example below.



- $\beta = (2, 1^{p-2})$. In this case, λ/α has exactly two columns, with one having $p - 1$ cells and the other having a single cell. Since λ/α has a cell in its second row, thus $\lambda = (p, p, 1^p)$. We may replace $\beta = (1^p)$ since λ/α is a vertical strip.

Note that the LHS of Equation (14) is invariant under the involution ω . Thus, for the other cases $\alpha = (2^p), (2, 1^{2p-2}), (1^{2p})$ we use the above cases and the involution ω to prove the lemma. □

Lemma 30. *Let $p \geq 7$ be a positive odd integer. Then*

$$\left(\sum_{\substack{\beta \vdash p \\ \beta \neq (p-1,1), (2,1^{p-2})}} s_\beta \right)^j \sum_{\substack{\alpha \vdash 2p \\ \alpha \neq (2p), (2p-1,1), (p,p) \\ \alpha \neq (2^p), (2,1^{2p-2}), (1^{2p})}} s_\alpha \geq s_\lambda \quad (16)$$

for all $j \geq 1$ and for all partitions λ of $n = (j + 2)p$, except possibly when $\lambda = (n), (n - 1, 1), (2, 1^{n-2}), (1^n)$ and

$$\left(\sum_{\substack{\beta \vdash p \\ \beta \neq (p-1,1), (2,1^{p-2})}} s_\beta \right)^j \sum_{\substack{\alpha \vdash 2p \\ \alpha \neq (2p), (p,p) \\ \alpha \neq (2^p), (1^{2p})}} s_\alpha \geq s_\lambda \quad (17)$$

for all $i \geq 1$ and for all partitions λ of $n = (j + 2)p$, except when $\lambda = (n), (1^n)$.

Proof. The proof follows by applying Lemma 29 iteratively. □

Proof of Theorem 9. We prove the theorem by induction on n . We can easily verify the theorem when $n \leq 26$ by using Sage for example. So assume that $n > 26$. Let us also assume that the theorem holds for $k < n$ by induction. We prove the theorem when $k = n$.

If n is a prime, then a small calculation yields the theorem. One such calculation would be the following. Consider the matrix $\rho_\lambda(w_{(n)})$. When λ is not a hook partition, we have that $\chi_\lambda(w_{(n)}) = \text{trace}(\rho_\lambda(w_{(n)})) = 0$ by the Murnaghan-Nakayama rule. Recall that the characteristic polynomial of $\rho_\lambda(w_{(n)})$ is in $\mathbb{Z}[x]$. Since the sum of all the eigenvalues of $\rho_\lambda(w_{(n)})$ is 0 and the only possible eigenvalues are p th roots of unity, we have that all p th roots of unity occur as eigenvalues of $\rho_\lambda(w_{(n)})$. When λ is a hook partition (\neq

$(n), (n-1, 1), (2, 1^{n-2}), (1^n)$, we notice that $\chi_\lambda(w_{(n)}) = \text{trace}(\rho_\lambda(w_{(n)})) = (-1)^{n-\lambda_1}$. Since $\dim(V_\lambda) > n-1$ and sum of non-trivial p th roots of unity is -1 , we conclude that all p th roots of unity occurs as an eigenvalue of $\rho_\lambda(w_{(n)})$. When λ is equal to one of $(n), (n-1, 1), (2, 1^{n-2}),$ or (1^n) , one can easily verify the theorem.

Now we may assume that n is a composite number.

Suppose that n is odd. Write $n = mp$ where p is the smallest prime divisor of n . Since $n \geq 26$, we have that m is an odd positive integer greater than or equal to 7. Let $\lambda \vdash n$. If λ is equal to one of $(n), (n-1, 1), (2, 1^{n-2}), (1^n)$, then one can easily verify the theorem by direct computation. Otherwise, using Lemma 30 and Proposition 23, the theorem follows.

Suppose that n is even. We write $n = mp$ where $m = n/2$ and $p = 2$. Note that $m \geq 10$. Let $\lambda \vdash n$ and not equal to any of $(n), (n-1, 1), (n/2, n/2), (3, 1^{n-3}), (2^{n/2}), (2, 2, 1^{n-4}), (2, 1^{n-2}), (1^n)$. Then using Lemma 28 and Proposition 23, the theorem follows.

Finally, if λ is equal to any of $(n), (n-1, 1), (n/2, n/2), (3, 1^{n-3}), (2^{n/2}), (2, 2, 1^{n-4}), (2, 1^{n-2}), (1^n)$, one can directly compute (for example, the inner product of characters) to prove the theorem except maybe when $\lambda = (n/2, n/2)$ or $(2^{n/2})$.

Suppose that $\lambda = (n/2, n/2)$. If n is divisible by 4, then $n = 4m$ (note that $m \geq 6$) and we may choose $\mu^1 = (m, m), \mu^2 = (m+1, m-1), \mu^3 = (m+2, m-2), \mu^4 = (m+1, m-1)$ in Corollary 24, to get the theorem.

Otherwise, $n = mp$ where p is the least prime dividing n other than 2. If $m \geq 6$, then we may choose $\mu^1 = (m-2, 2)$ and $\mu^2 = \mu^3 = \dots = \mu^p = (m)$ in Proposition 23, and we are done. Otherwise, $m = 2$ since m is even and not divisible by 4. Now the theorem follows from Corollary 25.

Since $\chi_{(n/2, n/2)} \otimes \chi_{(1^n)} = \chi_{(2^{n/2})}$, and we have shown that $\rho_{(n/2, n/2)}(w_n)$ has minimal polynomial $x^n - 1$, it follows that the theorem is true when $\lambda = (2^{n/2})$.

This completes the proof. □

5 Proof of Theorem 1

Let us recall that $f_n = \text{ch Ind}_{C_n}^{S_n} \mathbf{1}$. From Theorem 9, we have that $f_n \geq a_n$, where a_n is defined in Equation (9). The following are the necessary lemmas to prove Theorem 1.

Lemma 31. *Let $\gamma = (p, q)$ be a partition of n with $p \geq 3$ and $q \geq 1$. Then*

$$\text{ch Ind}_{C_{(p,q)}}^{S_n} \widetilde{\zeta \times \mathbf{1}} \geq s_\lambda \tag{18}$$

for all partitions $\lambda \vdash n$ except when $\lambda = (n), (1^n)$, where ζ is any non-real one dimensional character of C_p , $\mathbf{1}$ denotes the trivial character of C_q and $\widetilde{\zeta \times \mathbf{1}} := \text{Res}_{C_{(p,q)}^{C_p \times C_q}} \zeta \times \mathbf{1}$.

Proof. Clearly,

$$\text{Ind}_{C_{(p,q)}}^{S_n} \widetilde{\zeta \times \mathbf{1}} = \text{Ind}_{C_p \times C_q}^{S_n} \text{Ind}_{C_{(p,q)}^{C_p \times C_q}} \widetilde{\zeta \times \mathbf{1}} \geq \text{Ind}_{C_p \times C_q}^{S_n} \zeta \times \mathbf{1}, \tag{19}$$

where the last inequality \geq means that the RHS is a subrepresentation of LHS. Using Theorem 9 and induction in stages, we obtain

$$\text{ch Ind}_{C_p \times C_q}^{S_n} \zeta \times \mathbf{1} = \text{ch Ind}_{S_p \times S_q}^{S_n} \text{Ind}_{C_p \times C_q}^{S_p \times S_q} \zeta \times \mathbf{1} \geq \sum_{\substack{\alpha \vdash p, \beta \vdash q \\ \alpha \neq (p), (1^p) \\ f_q \geq s_\beta}} s_\alpha s_\beta \geq s_\lambda. \quad (20)$$

Let $\lambda \vdash n$ be different from (n) and (1^n) . For almost all partitions λ , we provide a pair of partitions α, β such that $\alpha \neq (p), (1^p)$, $f_q \geq s_\beta$ and $s_\alpha s_\beta \geq s_\lambda$. Hence, the following equation holds for almost all λ :

$$\sum_{\substack{\alpha \vdash p, \beta \vdash q \\ \alpha \neq (p), (1^p) \\ f_q \geq s_\beta}} s_\alpha s_\beta \geq s_\lambda. \quad (21)$$

Using Equations (19), (20), and (21) we obtain that $\text{ch Ind}_{C_{(p,q)}}^{S_n} \widetilde{\zeta} \times \mathbf{1} \geq s_\lambda$. For the remaining partitions λ , we shall do the direct computation, as we will see.

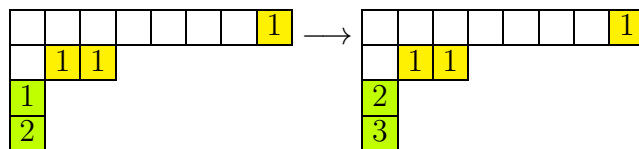
Suppose that $q \geq 6$; then one can easily use the proof of the statements corresponding to Equations (11) and (13) in Lemma 28 to get the results where the partitions which do not satisfy Equations (11) and (13) in Lemma 28 can be easily checked in the statement of the lemma separately. Suppose that $q \leq 5$ and $p \leq 14$. For these finitely many cases, one can directly verify that Equation (21) holds for all $\lambda \vdash n$ except when $\lambda = (2, 1), (3, 3), (2, 2, 2)$ and possibly when $\lambda \in B$, where $B = \{(n), (2, 1^{n-2}), (1^n)\}$. In these cases, we may verify Equation (18) directly.

Suppose that $q \leq 5$ and $p \geq 15$. Let us choose $\alpha \vdash p$ and $\beta \vdash q$ such that $f_q \geq s_\beta$ and $s_\alpha s_\beta \geq s_\lambda$ using Lemma 27. If $\alpha \neq (p), (1^p)$, then we are done.

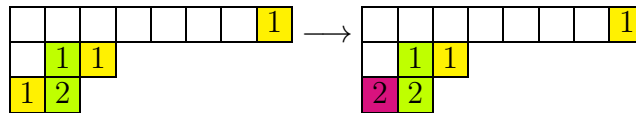
Case 1: $\alpha = (p)$.

Since $\lambda \neq (n)$, we may replace α by $(p - 1, 1)$ and choose β as in the proof of Lemma 26. If $f_q \geq s_\beta$, then we are done. Otherwise, one of the following cases must occur.

- Suppose that $\beta = (q - 1, 1)$. Then λ/α contains $q - 1$ columns, of which precisely one column contains two cells. The column of λ/α which contains two cells can be either the first or the second since $p \geq 15$ and $q \leq 5$. If the first column contains two cells, then incrementing the entries in the first column of λ/α by 1 allows us to replace β by $(q - 2, 1, 1)$ as shown below. Note that $q \geq 3$ in this case is due to the existence of at least one cell in the first row of λ/α .



If the second column contains two cells, then incrementing the entry in the first column of λ/α allows us to replace β by $(q - 2, 2)$ as shown below except when $q = 3$.



If $q = 3$, then we may replace α by $(p - 2, 2)$ and β by (3) .

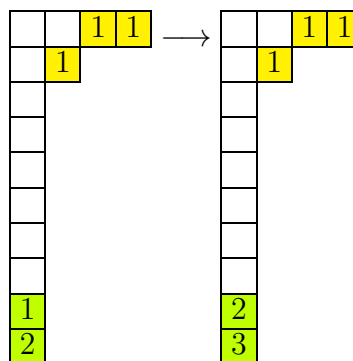
- Suppose that $\beta = (2, 1^{q-2})$ and q odd. Then $\lambda = (p, 1^q)$. We may replace β by (1^q) since λ/α is a vertical strip.
- Suppose that $\beta = (1^q)$ and q even. Then λ/α must be contained in the p th column but $q \geq 2$ since q is even. Hence, this case cannot occur.

Case 2: $\alpha = (1^p)$.

Since $\lambda \neq (1^n)$, we may replace α by $(2, 1^{p-2})$ and choose β as in the proof of Lemma 26. If $f_q \geq s_\beta$, then we are done. Otherwise, one of the cases below must occur.

- Suppose that $\beta = (q - 1, 1)$. Then the Young diagram of λ/α contains $q - 1$ columns, of which precisely one column contains two cells. The column of λ/α which contains two cells can be either the first, second or third column.

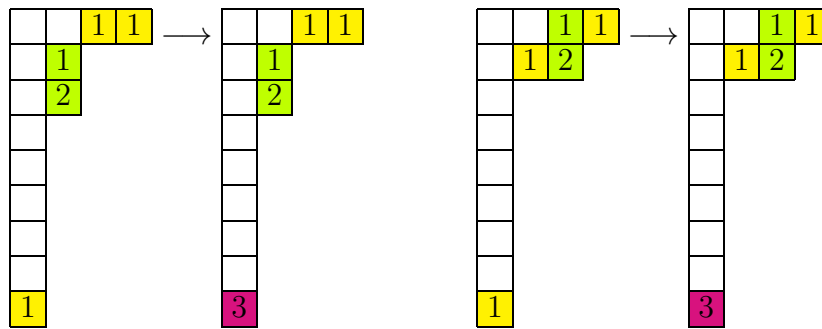
If the first column contains two cells, then incrementing the entries in the first column by 1 allows us to replace β by $(q - 2, 1, 1)$ as shown in the example below except when $q \leq 2$.



If $q \leq 2$, then $\lambda = (2, 1^{n-1})$ and in this case one can compute directly to show that Equation (18) holds.

If the second column contains two cells or the third column contains two cells, then incrementing the entry in the first column by 1 (since the first column of λ/α contains precisely one cell), allows us to replace β by $(q - 2, 1, 1)$ as shown in the

examples below. Note that in this case $q \geq 3$.



- Suppose that $\beta = (2, 1^{q-2})$ and q odd. Then $\lambda = (3, 1^{n-2}), (2, 2, 1^{n-4}), (2^q, 1^{p-q})$. Since λ/α is vertical strip in all these cases (using $p \geq 15$ and $q \leq 5$ in the last case), we may replace β by (1^q) . This completes the proof for this case.
- Suppose that $\beta = (1^q)$ and q even. Then λ must be equal to $(2, 1^{n-2})$. In this case, one can simply check directly, for example using the inner product of characters.

For the remaining, finitely many pairs (p, q) , one can directly compute using Sage for example. \square

We have a nice corollary which follows from the proof of Lemma 31.

Corollary 32. *Let $\mu = (p, q)$ be a partition of n with $p \geq 3$ and $q \geq 1$. Then*

$$\sum_{\substack{\alpha \vdash p, \beta \vdash q \\ \alpha \neq (p), (1^p) \\ f_q \geq s_\beta}} s_\alpha s_\beta \geq s_\lambda \quad (22)$$

holds for all partitions $\lambda \vdash n$ except when $\lambda = (3, 3), (2, 2, 2)$ with $p = q = 3$ and possibly when $\lambda \in \{(n), (2, 1^{n-2}), (1^n)\}$.

Lemma 33. *Let (p, q) be a partition of n with $w_{(p,q)}$ having even order and $p \geq 6, q \geq 1$. Then $\text{ch Ind}_{C_{(p,q)}}^{S_n} \mathbf{1} \geq s_\lambda$ for all partitions λ of n except when λ equal to (n) and possibly (1^n) .*

Proof. Suppose that $w_{(p,q)}$ is an odd permutation. Then the theorem follows from Corollary 22. So we may assume that $w_{(p,q)}$ is an even permutation, and thus both p, q and $p + q = n$ are even.

Since $g_{(p,q)} = \text{ch Ind}_{C_{(p,q)}}^{S_n} \mathbf{1} \geq \text{ch Ind}_{C_p \times C_q}^{S_n} (-\mathbf{1} \times \mathbf{1})$ and $\text{ch Ind}_{C_{(n)}}^{S_n} \mathbf{1} \geq s_\lambda$ for all partitions of λ except when $\lambda = (n), (2, 1^{n-2})$ by Theorem 9, we have

$$g_{(p,q)} \geq \sum_{\alpha \neq (p), (2, 1^{p-2})} s_\alpha \sum_{\alpha \neq (q-1, 1), (1^q)} s_\beta.$$

Let λ be a partition of n and $\lambda \notin \{(n), (n-1, 1), (2, 1^{n-2}), (1^n)\}$. Using Lemmas 26 and 27, we get a pair of partitions α and β such that $s_\alpha s_\beta \geq s_\lambda$ and $f_q \geq s_\beta$. If $g_p \geq s_\alpha$, then there is nothing to prove. Otherwise, one of the following cases must occur.

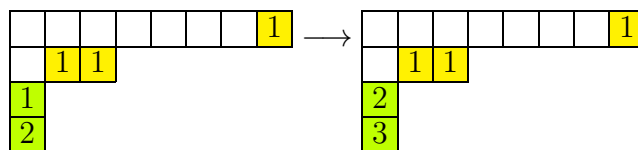
Case 1: $\alpha = (p)$.

Since $\lambda \neq (n)$ we may replace α by $(p-1, 1)$ and choose β as in the proof of 26. If $f_q \geq s_\beta$, we are done. Otherwise, one of the following cases occurs.

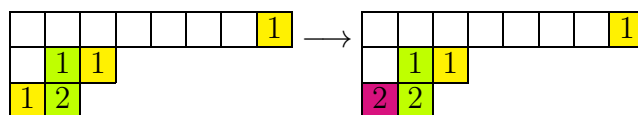
- Suppose that $\beta = (q-1, 1)$.

Then λ/α contains $q-1$ columns, of which precisely one column contains two cells. The column of λ/α which contains two cells can be one of the first, the second, or the p th column.

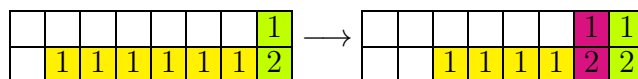
If the first column of λ/α contains two cells, then incrementing the entries in the first column by 1 allows us to replace β by $(q-2, 1, 1)$ as shown in the example below. Note that $q \geq 4$.



If the second column of λ/α contains two cells, then incrementing the entry in the first column by 1 allows us to replace β by $(q-2, 2)$ as shown in the example below.



If the p th column of λ/α contains two cells, then we may replace α by $(p-2, 2)$ and β by $(p-2, 2)$ as shown in the example below.



Note that in this case, p must be equal to q .

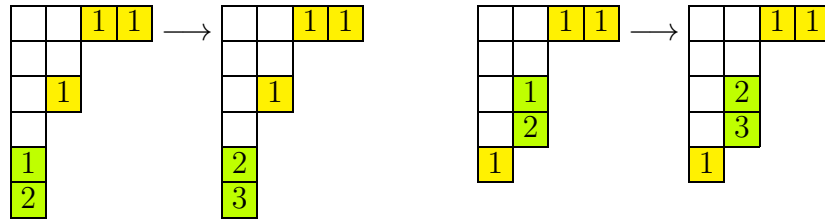
- Suppose that $\beta = (1^q)$ and q even. Then λ/α has exactly one column and there exists a cell in the p th column of λ/α . This implies there is no possible Young diagram except when $q = 1$. Since q is even, this case is also excluded.

Case 2: $\alpha = (2, 1^{p-2})$.

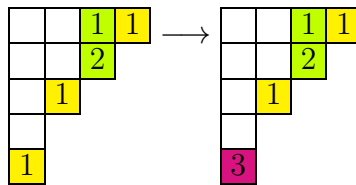
Now we shall consider the following cases.

- Suppose $\lambda_2 \geq 2$. We shall replace α by $(2, 2, 1^{p-4})$. Choose β as in the proof of Lemma 26. If $f_q \geq s_\beta$, then we are done. Otherwise, one of the following cases must occur.

- $\beta = (q - 1, 1)$. In this case, λ/α has $q - 1$ columns, with one column having two cells. The column with two cells has to be one of the first three columns. If the first or second column of λ/α has two cells, increment the entries of those two cells by 1 to replace β by $(q - 2, 1, 1)$. Since $p \geq 4$, λ/α has at least one cell in the first row, the resulting skew-tableau is an LR-tableau.



Suppose the third column of λ/α has two cells. Since $\lambda \supset (2, 1^{p-2})$, the first column of λ/α must have exactly one cell. Changing the entry of this cell from 1 to 3 allows us to replace β by $(q - 2, 1, 1)$.



- Suppose that $\beta = (1^q)$. In this case, λ must be equal to $(2, 2, 1^{n-4})$. Then we may choose $\alpha = (1^p)$ and $\beta = (2, 1^{q-2})$ except when $q = 2$. In this case, $\lambda = (2, 2, 1^{p-4})$. One can verify that when $\lambda = (2, 2, 1^{p-4})$ the theorem is true.
- Suppose $\lambda_2 = 1$ (so that λ is a hook) and $\lambda_1 \geq 3$. Replace α by $(3, 1^{p-3})$ and choose β according to the proof of Lemma 26. If $f_q \geq s_\beta$, then we are done. Otherwise, one of the following cases must occur.
 - Suppose that $\beta = (q - 1, 1)$. In this case, λ/α has $q - 1$ columns, one having two cells and the other having only one cell. Since λ is a hook, only the first column of λ/α can have two cells. Incrementing the entries in the first column by one will allow us to replace β to $(q - 2, 1, 1)$.
 - Suppose that $\beta = (1^q)$. In this case, λ must be $(3, 1^{n-3})$. We may replace α by (1^p) and, β by $(2, 1^{q-2})$.

This completes the proof. □

Proof of Theorem 1. Let $\mu \vdash n \geq 11$ such that all its parts divide μ_1 . We determine the pair (λ, μ) for which the degree of the minimal polynomial of $\rho_\lambda(w_\mu)$ is not equal to μ_1 . It suffices to check whether all μ_1 th roots of unity are eigenvalues of $\rho_\lambda(w_\mu)$.

We know that $\rho_\lambda(w_\mu)$ has eigenvalue 1 except for the pairs (λ, μ) given in Theorem 11.

If the length of μ is 1, then the theorem follows from Theorem 11. So let us assume that the length of μ is greater than or equal to 2, and we may also assume that $\mu \neq (1^n)$.

Let us consider a non-real μ_1 th root of unity t ; thus $\mu_1 \geq 3$. Using Lemma 31, we have that t is an eigenvalue of $\rho_\lambda(w_{(\mu_1, \mu_2)})$ except when $\lambda = (n), (1^n)$.

Now we note that $\mu_1 + \mu_2 \geq 6$ or $\mu_i \leq 1$ for all $i \geq 2$ since $\mu_2 | \mu_1$.

In the former case, we can use Corollary 32 with p replaced by $\mu_1 + \mu_2$ and q replaced by μ_3 , and verify for the partitions listed as exceptions in Corollary 32 directly. We conclude that t is an eigenvalue of $\rho_\lambda(w_{(\mu_1, \mu_2, \mu_3)})$. Continuing this process with p replaced by $\sum_{i=1}^k \mu_i$ and q replaced by μ_{k+1} in Corollary 32 and using induction we complete the proof in this case.

For the latter case, we have $\mu_i \leq 1$ for all $i \geq 2$. Since $\mu_1 | \mu_2$, we have $3 \leq \mu_1 \leq 5$. Hence, $\mu = (\mu_1, 1, 1, 1, \dots, 1)$, and we see that $\rho_\lambda(\mu_1, 1, 1, 1)$ has minimal polynomial $x^{\mu_1+3} - 1$ for all partitions λ of $\mu_1 + 3$ except when $\lambda = (\mu_1 + 3)$ or (1^{μ_1+3}) . Now we could replace p by $\mu_1 + 3$ and q by 1 in Corollary 32 and then use induction to complete the proof as in the previous case.

Finally, let us consider the eigenvalue -1 in the case of μ_1 even. We assumed that $\mu \neq (1^n)$ and the length of μ is greater than or equal to two. If w_μ is an odd permutation, then using Corollary 22, we have that -1 is an eigenvalue of $\rho_\lambda(w_\mu)$. Thus, we may assume that w_μ is an even permutation, hence $w_{\tilde{\mu}}$ is an odd permutation where $\tilde{\mu} = (\mu_2, \mu_3, \dots)$.

Suppose that $\mu_1 = 2$, hence w_μ is an involution. Then we can easily see that -1 must be an eigenvalue for all faithful representations of S_n . The only non-faithful representations are precisely when $\lambda \in \{(n), (1^n), (2, 2)\}$. Indeed, if ρ_λ is not faithful, then the kernel K forms a normal subgroup which is not equal to the trivial subgroup $\{e\}$. If $K = S_n$, then ρ_λ is the trivial representation. Hence, in this case, $\lambda = (n)$. Otherwise, K is a non-trivial normal subgroup of S_n . But the only non-trivial normal subgroups of S_n are A_n and, if $n = 4$ we have one more, namely, the Klein-four subgroup K_4 which consists of all permutations of cycle type equal to $(2, 2)$ and identity $\{e\}$. Therefore, if $K = A_n$, then we have only two possible irreducible representations for S_n/A_n . In this case, λ must be equal either (n) or (1^n) . If $n = 4$ and $K = K_4$, then there are only three possible irreducible representations for S_n/K_4 . In this case, λ must be equal to one of the elements in the set $\{(4), (2, 2), (1^4)\}$. One can directly verify the theorem when $\lambda \in \{(n), (1^n), (2, 2)\}$.

Therefore, we may assume that $\mu_1 \geq 4$. Suppose that the length of $\mu = 2$. Then we may use Lemma 33 and checking the remaining finitely many cases directly will yield the theorem. Now we may assume that the length of μ is at least three. Using Corollary 32 with p replaced by $\mu_1 + \mu_2$ and q replaced by μ_3 , we can conclude that the theorem holds when the length of the partition μ is three.

Continuing this process with p replaced by $\sum_{i=1}^k \mu_i$ and q replaced by μ_{k+1} in Corollary 32 and using induction, we complete the proof in this case.

This completes the proof. □

Finally, we present some applications of the lemmas we have proved.

Proof of Theorem 14. Let $n \geq 11$. For $n \leq 11$ we verify the theorem by direct computation. Suppose that w_μ is an odd permutation. Then the theorem follows from Corollary 22. Otherwise, at least two parts of μ are even, say μ_i, μ_j for some $i > j$.

Suppose that $\mu_1 \geq 4$. Then we may consider the permutation w_ν where $\nu = (\mu_1, \mu_i)$ (or equal to (μ_1, μ_j) if $i = 1$). We shall work with $\nu = (\mu_1, \mu_i)$, since for the other case the proof is similar. Using Lemma 33 and direct computation for the finitely many remaining cases, we get $g_\nu \geq s_\lambda$ for all partitions of $\lambda \vdash \mu_1 + \mu_i$ except possibly when $\lambda = (\mu_1 + \mu_i)$ or $(1^{\mu_1 + \mu_i})$. Now we may use Corollary 32 with p replaced by $\sum_{l=1}^k \mu_l$ and q by μ_{k+1} . The theorem follows by induction.

Suppose that $\mu_1 \leq 3$ and hence $\mu = (3^k, 2^l, 1^t)$ with $l \geq 2$. If $k = 0$, then the theorem follows from Theorem 1. Assume that $k \geq 1$. Since $g_{(3,2,2)} \geq s_\lambda$ for all partitions $\lambda \vdash 7$ except when $\lambda = (7), (1^7)$, we can use Corollary 32 with p replaced by 7 and q replaced by any part of μ other than $\mu_1 = 3, \mu_{k+1} = 2, \mu_{k+2} = 2$. Now the theorem follows by iterating this process using Corollary 32. This completes the proof. \square

6 Proof of theorems on alternating groups

Proof of Theorem 15. Let V be a non-trivial irreducible representation of A_n . If $V = V_\lambda$ with $\lambda \neq \lambda'$, then $\rho_\lambda(w_\mu)$ has eigenvalue -1 from Theorem 14. Otherwise, $V = V_\lambda^\pm$ with $\lambda = \lambda'$. In this case, we have $\chi_\lambda^\pm(w_\mu^i) = \frac{\chi_\lambda(w_\mu^i)}{2}$. Computing the inner product of the characters $\text{Res}_{C_\mu}^{A_n} \chi_\lambda^\pm, \delta$ where $\delta \in \text{Irr}(C_\mu)$ yields that w_μ has eigenvalue -1 in V_λ^+ and V_λ^- if and only if it has eigenvalue -1 in V_λ . Now the theorem follows from Theorem 14. \square

Proof of Theorem 13. One can prove this theorem independently by arguments similar to those of the proof of Theorem 9 for the symmetric group. But for the sake of completeness, we shall give a short proof that uses Theorem 9.

We verify the theorem for $n \leq 25$ by direct computation or using Sage. Now we assume that $n \geq 27$ is an odd positive integer. Let (ρ, V) be an irreducible representation of the alternating group A_n . If V is not equal to both $V_{\left(\frac{n+1}{2}, \frac{n-1}{2}\right)}^+$ and $V_{\left(\frac{n+1}{2}, \frac{n-1}{2}\right)}^-$, then the theorem follows from Theorem 9. Because the character value is $\rho(w_\mu^i) = \chi_\lambda(w_\mu^i)$ for all i or $\rho(w_\mu^i) = \frac{\chi_\lambda(w_\mu^i)}{2}$ for all i . In either case, the inner product of the characters yields that t is an eigenvalue for w_n^\pm in V if and only if t is an eigenvalue for w_n in V_λ . The theorem follows from Theorem 9 in this case.

Finally, let $V = V_{\left(\frac{n+1}{2}, \frac{n-1}{2}\right)}^\pm$. Suppose that p is prime. Then one can easily verify that the minimal polynomial of $\rho(w_n)$ is $x^n - 1$. Otherwise, let us write $n = mp$ where p is the smallest prime dividing n , and hence $m \geq 5$. Note that $C_m \wr C_p \subset A_n$. We may choose $\mu^1 = (m-2, 1, 1)$ and $\mu^2 = \mu^3 = \dots = \mu^p = (m)$. Then we see that $\text{Ind}_{A_m^{\times p}}^{A_n} \chi_{\mu^1} \times \dots \times \chi_{\mu^p} \geq \chi_\lambda^\pm$. Now the theorem follows using induction and Proposition 23. \square

Proof of Theorem 2. Let μ be a partition of $n \geq 26$ with all its parts dividing μ_1 such that w_μ is an even permutation. If the length of μ is one, then the theorem follows from Theorem 13. If $\mu \notin \text{DOP}_n$, then the theorem follows from Theorem 1. Otherwise, $\mu \in \text{DOP}_n$. Recall that for $\mu = (2m_1 + 1, 2m_2 + 1, \dots, 2m_k + 1)$, we defined $\phi(\mu) = (m_1, m_2, \dots, m_k \mid m_1, m_2, \dots, m_k)$. If $\lambda \neq \phi(\mu)$, then the result follows from Theorem 1. Hence, let $\lambda = \phi(\mu)$. We have

$$\text{Ind}_{C_\mu}^{A_n} \delta \geq \text{Ind}_{C_{\mu_1} \times C_{\mu_2} \times \dots \times C_{\mu_k}}^{A_n} \tilde{\delta} = \text{Ind}_{A_{\mu_1} \times A_{\mu_2} \times \dots \times A_{\mu_k}}^{A_n} \text{Ind}_{C_{\mu_1} \times C_{\mu_2} \times \dots \times C_{\mu_k}}^{A_{\mu_1} \times A_{\mu_2} \times \dots \times A_{\mu_k}} \tilde{\delta},$$

where δ is any linear character of C_μ and $\tilde{\delta} = \bar{\delta} \times \mathbf{1} \times \dots \times \mathbf{1}$ with $\bar{\delta}(w_{\mu_1}^i) = \delta(w_\mu^i)$ for all i . Let $\nu^i = \phi(\mu_i)$ for $i = 1, 2, \dots, k$ where k is the number of parts of μ . If 3 or 5 is not a part of μ , then we are done, since

$$\begin{aligned} \text{Ind}_{C_\mu}^{A_n} \delta &\geq \text{Ind}_{A_{\mu_1} \times A_{\mu_2} \times \dots \times A_{\mu_k}}^{A_n} \text{Ind}_{C_{\mu_1} \times C_{\mu_2} \times \dots \times C_{\mu_k}}^{A_{\mu_1} \times A_{\mu_2} \times \dots \times A_{\mu_k}} \tilde{\delta} \\ &\geq \text{Ind}_{A_{\mu_1} \times A_{\mu_2} \times \dots \times A_{\mu_k}}^{A_n} \chi_{\nu^1} \times \dots \times \chi_{\nu^k} \\ &\geq \chi_{\phi(\mu)} \end{aligned}$$

In particular, $\text{Ind}_{C_\mu}^{A_n} \delta \geq \chi_{\phi(\mu)}^+$ and $\text{Ind}_{C_\mu}^{A_n} \delta \geq \chi_{\phi(\mu)}^-$. We are done.

So let us assume that 3 or 5 is a part, say μ_j , of μ . Note that either 3 or 5 is a part of μ . Then we may choose $\nu^{j+1} = (\frac{\mu_{j+1}+1}{2}, 2, 1^{\frac{\mu_{j+1}-5}{2}})$ and if $\mu_j = 3$, then $\nu^j = (3)$, otherwise $\nu^j = (4, 1)$. We let $\nu^t = \phi(\mu_t)$ for all $t \neq j, j+1$. Similarly, we have, $\text{Ind}_{C_\mu}^{A_n} \delta \geq \chi_\lambda^+$ and $\text{Ind}_{C_\mu}^{A_n} \delta \geq \chi_\lambda^-$. This completes the proof. \square

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References

- [1] Amrutha P, A. Prasad, and Velmurugan S. Cyclic characters of alternating groups, 2024. [arXiv:2403.05109](https://arxiv.org/abs/2403.05109). 12

- [2] Amrutha P, A. Prasad, and Velmurugan S. Existence of elementwise invariant vectors in representations of symmetric groups. *Algebr. Comb.*, 7(4):915–929, 2024. 1, 11, 26, 3
- [3] Amrutha P, A. Prasad, and Velmurugan S. Locally invariant vectors in representations of symmetric groups. *Sém. Lothar. Combin.*, 91B.66, 2024. Proceedings of the 36th International Conference on Formal Power Series and Algebraic Combinatorics. 1
- [4] E. Giannelli and S. Law. On permutation characters and Sylow p -subgroups of S_n . *J. Algebra*, 506:409–428, 2018. 3, 6
- [5] I. M. Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423]. 3
- [6] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, MA, 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson. 2, 2
- [7] M. Johnson. Standard tableaux and Klyachko’s theorem on Lie representations. *J. Combin. Theory Ser. A*, 114(1):151–158, 2007. 1
- [8] A. Jöllenbeck and M. Schocker. Cyclic characters of symmetric groups. *J. Algebraic Combin.*, 12(2):155–161, 2000. 1
- [9] A. A. Klyachko. Lie elements in a tensor algebra. *Sibirsk. Mat. Ž.*, 15:1296–1304, 1430, 1974. 6, 7, 1, 8
- [10] L. G. Kovács and R. Stöhr. A combinatorial proof of Klyachko’s theorem on Lie representations. *J. Algebraic Combin.*, 23(3):225–230, 2006. 1
- [11] W. Kraškiewicz and J. Weyman. Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.*, (63):265–284, 2001. 1, 5, 1
- [12] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. Oxford University Press, second edition, 2015. 2, 2, 18
- [13] A. Prasad. *Representation theory*, volume 147 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Delhi, 2015. A combinatorial viewpoint. 2
- [14] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications. 1
- [15] M. Schocker. Embeddings of higher Lie modules. *J. Pure Appl. Algebra*, 185(1-3):279–288, 2003. 1
- [16] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 208 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, [2024] ©2024. With an appendix by Sergey Fomin. 2

- [17] A. Staroletov. On eigenvalues of permutations in irreducible representations of symmetric and alternating groups. *Bull. Lond. Math. Soc.*, 58(1):Paper No. e70235, 19, 2026. 16
- [18] J. R. Stembridge. On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math.*, 140(2):353–396, 1989. 1
- [19] S. Sundaram. The conjugacy action of S_n and modules induced from centralisers. *J. Algebraic Combin.*, 48(2):179–225, 2018. 10
- [20] J. P. Swanson. On the existence of tableaux with given modular major index. *Algebr. Comb.*, 1(1):3–21, 2018. 1
- [21] P. H. Tiep and A. E. Zalesskii. Hall-Higman-type theorems for semisimple elements of finite classical groups. *Proc. Lond. Math. Soc. (3)*, 97(3):623–668, 2008. 4
- [22] N. Yang and A. M. Staroletov. The minimal polynomials of powers of cycles in the ordinary representations of symmetric and alternating groups. *J. Algebra Appl.*, 20(11):Paper No. 2150209, 18, 2021. 1
- [23] A. E. Zalesskii. The spectra of elements of order p in complex representations of finite Chevalley groups of characteristic p . *Vests̄i Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk*, (6):20–25, 124, 1986. 1
- [24] A. E. Zalesskii. Eigenvalues of matrices of complex representations of finite groups of lie type. In L. L. Avramov and K. B. Tchakerian, editors, *Algebra Some Current Trends*, pages 206–218, Berlin, Heidelberg, 1988. Springer Berlin Heidelberg. 1, 3
- [25] A. E. Zalesskii. Eigenvalues of prime-order elements in projective representations of alternating groups. *Vests̄i Akad. Navuk Belarus̄i Ser. Fiz.-Mat. Navuk*, (3):41–43, 131, 1996. 1
- [26] A. E. Zalesskii. Minimal polynomials and eigenvalues of p -elements in representations of quasi-simple groups with a cyclic Sylow p -subgroup. *J. London Math. Soc. (2)*, 59(3):845–866, 1999. 1
- [27] A. E. Zalesskii. The number of distinct eigenvalues of elements in finite linear groups. *J. London Math. Soc. (2)*, 74(2):361–378, 2006. 1
- [28] A. E. Zalesskii. Minimum polynomials of the elements of prime order in representations of quasi-simple groups. *J. Algebra*, 320(6):2496–2525, 2008. 1