Positive combinatorial formulae for involution matrix loci and orbit harmonics

Hai Zhu

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Abstract

Let $\mathcal{M}_{n,a}$ be the set consisting of involutions in the symmetric group $\mathfrak{S}_n$ with exactly a fixed points, and apply the orbit harmonics method to obtain a graded $\mathfrak{S}_n$-module $R(\mathcal{M}_{n,a})$. Liu, Ma, Rhoades, and Zhu figured out a signed combinatorial formula for the graded Frobenius image $\mathrm{grFrob}(R(\mathcal{M}_{n,a});q)$ of $R(\mathcal{M}_{n,a})$. We cancel these minus signs using lattice paths. Finally, we also find two positive combinatorial formulae for $\mathrm{grFrob}(R(\mathcal{M}_{n,a});q)$. As an application, we deduce a series of $\mathfrak{S}_n$-equivariant isomorphisms between graded components $R(\mathcal{M}_{n,a})_d$ and $R(\mathcal{M}_{n,a'})_d$ for some integers $a \neq a'$ and d. Our positive formulae also yield potential attempts to find a linear basis for $R(\mathcal{M}_{n,a})$ and a statistic $\mathrm{stat} : \mathcal{M}_{n,a} \to \mathbb{Z}_{\geqslant 0}$ to interpret the Hilbert series $\mathrm{Hilb}(R(\mathcal{M}_{n,a});q)$ of $R(\mathcal{M}_{n,a})$.

Mathematics Subject Classifications: 05E05, 05E10

1 Introduction

Orbit harmonics is a vital method in combinatorial representation theory. Let $N > 0$ be an integer. Take a finite locus $\mathcal{Z} \subseteq \mathbb{C}^N$ and apply the orbit harmonics method to $\mathcal{Z}$. The output is $R(\mathcal{Z})$, a quotient of the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_N]$ by a homogeneous ideal. If $\mathcal{Z}$ further carries an action of some subgroup $G \leqslant \mathrm{GL}(\mathbb{C}^N)$, then we have an isomorphism of G-modules $R(\mathcal{Z}) \cong \mathbb{C}[\mathcal{Z}]$ where $\mathbb{C}[\mathcal{Z}]$ is the space of all functions $f : \mathcal{Z} \to \mathbb{C}$. That is, $R(\mathcal{Z})$ is a graded refinement of the above-mentioned action $G \curvearrowright \mathcal{Z}$. Furthermore, $R(\mathcal{Z})$ provides algebraic tools to understand combinatorial properties of $\mathcal{Z}$.

Orbit harmonics interacts with many fields in mathematics, such as cohomology theory [3], Macdonald theory [4, 5], cyclic sieving [10], Donaldson-Thomas theory [11], and Ehrhart theory [12].

Let $\mathrm{Mat}_{n\times n}(\mathbb{C})$ be the affine space of n by n complex matrices. Each permutation $w \in \mathfrak{S}_n$ can be identified with the 0-1 matrix M with $M_{i,j} = 1$ if and only if $w(i) = j$, so $\mathfrak{S}_n$ is identified with a finite locus in $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Rhoades [13] initiated the implementation of the orbit harmonics method into matrix loci $\mathcal{Z} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$, studying the permutation

Department of Mathematics, UC San Diego, La Jolla, California, USA (haz138@ucsd.edu).

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matrix locus $\mathcal{Z} = \mathfrak{S}_n$ carrying an action of the subgroup $\mathfrak{S}_n \times \mathfrak{S}_n \leqslant \mathrm{GL}(\mathrm{Mat}_{n\times n}(\mathbb{C}))$. Thanks to this work, Chen's log-concavity conjecture [1] can be studied from an algebraic perspective. Liu [7] extended Rhoades [13] to colored permutations.

Permutation matrix loci were revisited by Liu et al. [6]. They considered $\mathcal{Z} = \mathcal{M}_{n,a}$, the matrix locus corresponding to involutions in $\mathfrak{S}_n$ with a fixed points, carrying the conjugate action of $\mathfrak{S}_n$. They found an explicit combinatorial expression of the graded Frobenius image $\mathrm{grFrob}(R(\mathcal{M}_{n,a}); q)$ (see Subsection 2.4 for its definition) of $R(\mathcal{M}_{n,a})$ (Theorem 4). However, this is a signed formula, while combinatorialists usually prefer sign-free expressions. We combinatorially cancel these minus signs by constructing bijections among index sets, and we assign lattice paths to horizontal strips (which are special pairs of partitions) to construct these bijections.

We find two positive combinatorial formulae for $\mathrm{grFrob}(R(\mathcal{M}_{n,a}); q)$. Our main theorem is the second one which is more compatible with the Schur expansion of $\mathrm{Frob}(\mathbb{C}[\mathcal{M}_{n,a}])$ $= \mathrm{Frob}(R(\mathcal{M}_{n,a})) = h_{(n-a)/2}[h_2] \cdot h_a$ using Pieri's rule. This special plethysm, $h_{(n-a)/2}[h_2]$, is introduced in Subsection 2.3 along with its Schur expansion.

Theorem 1. *For integers n, a such that $n > 0$, $0 \leqslant a \leqslant n$, $a \equiv n \mod 2$, we have that*

$$\mathrm{grFrob}(R(\mathcal{M}_{n,a}); q) = \sum_{\lambda \vdash n} \sum_{\mu \vdash n-a} q^{\frac{n+a-\mathrm{wid}(\lambda/\mu)}{2}} \cdot s_\lambda$$

where the second sum runs over all even partitions μ such that λ/μ is a horizontal strip.

In Theorem 1, the statistic wid() associated with horizontal strips is given by Definition 9, and a partition μ is called an even partition if all of its parts μ_i are even. As an application, we derive lots of isomorphisms with the form $R(\mathcal{M}_{n,a})_d \cong R(\mathcal{M}_{n,a'})_d$ in Corollary 25.

The rest of this paper is structured as follows. In Section 2, we give background material on orbit harmonics, the representation theory of symmetric groups, and some notations used to describe our main results. In Section 3, we give our first sign-free graded character formula for $R(\mathcal{M}_{n,a})$. In Section 4, we show our main theorem (Theorem 1), which is our second sign-free graded character formula for $R(\mathcal{M}_{n,a})$. In Section 5, we apply our first graded character formula to show some $\mathfrak{S}_n$-module isomorphisms. In Section 6, we state some open problems and their potential solution strategies arising from our results.

2 Background

2.1 Orbit harmonics

Let $\mathbf{x}_N = (x_1, x_2, \ldots, x_N)$ be a sequence of variables and let $\mathbb{C}[\mathbf{x}_N]$ be the polynomial ring over these variables. Given a finite locus $\mathcal{Z} \subseteq \mathbb{C}^N$, we define its *vanishing ideal* $\mathbf{I}(\mathcal{Z}) \subseteq \mathbb{C}[\mathbf{x}_N]$ by

$$\mathbf{I}(\mathcal{Z}) := \{f \in \mathbb{C}[\mathbf{x}_N] \,:\, f(z) = 0 \text{ for all } z \in \mathcal{Z}\}.$$

Multivariate Lagrange interpolation implies that $\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z})$ as vector spaces, where $\mathbb{C}[\mathcal{Z}]$ is the space of all functions $f : \mathcal{Z} \to \mathbb{C}$.

Now we construct a graded version of the vector spaces above. For an ideal $I \subseteq \mathbb{C}[\mathbf{x}_N]$, the *associated graded ideal* $\mathrm{gr}I$ of I is defined as the ideal generated by the highest homogeneous components of non-constant polynomials $g \in I$. That is, for each nonzero polynomial $f \in \mathbb{C}[\mathbf{x}_N]$, write $f = \sum_{d=0}^{m} f_d$ where f_d is homogeneous of degree d for $0 \leqslant d \leqslant m$ and $f_m \neq 0$. Define $\tau(f) := f_m$. Then we have

$$\mathrm{gr}I := \langle \tau(f) \,:\, 0 \neq f \in I \rangle.$$

Define the *orbit harmonics ring* $R(\mathcal{Z})$ of $\mathcal{Z}$ by

$$R(\mathcal{Z}) := \mathbb{C}[\mathbf{x}_N]/\mathrm{gr}\mathbf{I}(\mathcal{Z}).$$

The orbit harmonics method possesses an interesting and essential property: a chain of vector space isomorphisms

$$\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z}) \cong R(\mathcal{Z}).$$

If further $\mathcal{Z}$ is stable under the action of a matrix group $G \leqslant \mathrm{GL}(\mathbb{C}^N)$, the isomorphisms in this chain become G-module isomorphisms. The first isomorphism arises from multivariate Lagrange interpolation but the second isomorphism is not explicit (i.e. without an explicit G-equivariant linear map). In fact, the second isomorphism is an abstract isomorphism arising from the following G-equivariant linear maps on graded components for each degree $d \geqslant 0$:

$$\mathbb{C}[\mathbf{x}_N]_d/\mathrm{gr}\mathbf{I}(\mathcal{Z})_d \xrightarrow{\cong} \frac{\mathbb{C}[\mathbf{x}_N]_{\leqslant d}/(\mathbf{I}(\mathcal{Z}) \cap \mathbb{C}[\mathbf{x}_N]_{\leqslant d})}{\mathbb{C}[\mathbf{x}_N]_{\leqslant d-1}/(\mathbf{I}(\mathcal{Z}) \cap \mathbb{C}[\mathbf{x}_N]_{\leqslant d-1})} \tag{1}$$

induced by

$$\begin{aligned} \mathbb{C}[\mathbf{x}_N]_d &\longrightarrow \mathbb{C}[\mathbf{x}_N]_{\leqslant d}/(\mathbf{I}(\mathcal{Z}) \cap \mathbb{C}[\mathbf{x}_N]_{\leqslant d}) \\ f &\longmapsto f \mod (\mathbf{I}(\mathcal{Z}) \cap \mathbb{C}[\mathbf{x}_N]_{\leqslant d}) \end{aligned}$$

which maps $\mathrm{gr}\mathbf{I}(\mathcal{Z})_d$ into $\mathbb{C}[\mathbf{x}_N]_{\leqslant d-1}/(\mathbf{I}(\mathcal{Z}) \cap \mathbb{C}[\mathbf{x}_N]_{\leqslant d-1})$. The bijectivity of (1) is a standard result (see, for example, [6, Lemma 2.2]).

2.2 Partitions and Young tableaux

Given a positive integer $n \in \mathbb{N}$, a *partition* λ of n is a weakly decreasing sequence of nonnegative integers $\{\lambda_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \lambda_i = n$. In this case, we write $\lambda \vdash n$. The integer λ_i is called *the i-th part of* λ, and λ is called an *even partition* if all of its parts are even. We usually identify a partition λ with its *Young diagram* obtained by placing λ_i left-justified boxes in the i-th row ($i = 1, 2, \ldots$). A *standard Young tableau* of shape λ is a bijective filling of $[n] := \{1, 2, \ldots, n\}$ into the boxes of λ, such that the entries are rightwards increasing along each row and downwards increasing along each column. A *semi-standard*

Young tableau of shape λ is a filling of positive integers $\mathbb{N}$ into the boxes of λ, such that the entries are rightwards weakly increasing along each row and downwards strongly increasing along each column. Write $\mathrm{SYT}(\lambda) := \{\text{standard Young tableaux of shape } \lambda\}$ and $\mathrm{SSYT}(\lambda) := \{\text{semi-standard Young tableaux of shape } \lambda\}$. It is clear that $\mathrm{SYT}(\lambda) \subseteq \mathrm{SSYT}(\lambda)$. For a semi-standard Young tableau $P \in \mathrm{SSYT}(\lambda)$, we write $\mathrm{sh}(P) = \lambda$, which means that P is of shape λ. For instance, $\lambda = (7, 4, 4, 2) \vdash 17$ is a partition. Its Young diagram and one element in $\mathrm{SYT}(\lambda)$ is shown below.

1	3	7	10	11	15	17
2	4	8	12			
5	6	9	13			
14	16					

Robinson-Schensted-Knuth correspondence [2, Section 4] is a one-to-one correspondence RSK from $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathbb{Z}_{\geqslant 0}$ (all but finitely many entries are 0) to pairs of semi-standard Young tableaux of the same shape. RSK has several equivalent definitions, among which the most classical one uses the row insertion operation. In particular, the restriction of RSK to permutation matrices yields an elegant bijection

$$\mathfrak{S}_n \xrightarrow{1:1} \bigsqcup_{\lambda \vdash n} (\mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)).$$

We will use the following standard result for RSK in Section 6. See [2, Section 4.2] for details.

Proposition 2. *Let A be an $n \times n$ matrix with nonnegative integer entries, and write $\mathrm{RSK}(A) = (P, Q)$. Then $P = Q$ if and only if A is symmetric. In this case, the number of odd columns of $\mathrm{sh}(P)$ equals the trace of A.*

2.3 Symmetric functions

Write $\Lambda = \bigoplus_{n \geqslant 0} \Lambda_n$ for the graded algebra of symmetric functions in an infinite variable set $\mathbf{x} = \{x_1, x_2, \dots\}$ with coefficient field $\mathbb{C}(q)$. Each graded component Λ_n has several well-known linear basis, two of which are the complete homogeneous functions $\{h_\lambda\}_{\lambda \vdash n}$ and the Schur functions $\{s_\lambda\}_{\lambda \vdash n}$. The definitions of both can be found in [8]. We state both definitions below for completeness. For any integer $d \geqslant 0$, define h_d by

$$h_d := \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_d} \prod_{j=1}^{d} x_{i_j}$$

with the convention that $h_0 = 1$. For $\lambda \vdash n$, define $h_\lambda := \prod_{i=1}^{\infty} h_{\lambda_i}$ which is actually a finite product because $h_0 = 1$. In addition, define s_λ by

$$s_\lambda := \sum_{T \in \mathrm{SSYT}(\lambda)} \prod_{i \geqslant 1} x_i^{m_i(T)}$$

where $m_i(T)$ is the number of occurrences of i in T.

For $f = \sum_\lambda c_\lambda(q) \cdot s_\lambda \in \Lambda$ where $c_\lambda(q) \in \mathbb{C}(q)$ and a condition P on partitions, we introduce the truncation notation

$$\{f\}_P := \sum_{\lambda \text{ satisfies } P} c_\lambda(q) \cdot s_\lambda.$$

The multiplication of Schur functions by complete homogeneous functions satisfies Pieri's rule (see [8] for its proof): For any partition μ and any integer $a \geqslant 0$, we have that

$$s_\mu \cdot h_a = \sum_{\lambda/\mu} s_\lambda$$

summing over horizontal strips λ/μ of size a. Here, a *horizontal strip* λ/μ is a pair of partitions $\mu \subseteq \lambda$ (which means the Young diagram containment $\mu \subseteq \lambda$) such that the complement diagram $\lambda \setminus \mu$ has at most one box in each column. The size $|\lambda/\mu|$ of λ/μ is given by $|\lambda/\mu| := |\lambda| - |\mu|$. Interestingly, we can deduce the above-stated Pieri's rule from its tableau version [9, Lemma 4.11]:

Proposition 3 (A tableau version of Pieri's rule)**.** *Let μ be a partition and $a \geqslant 0$ be an integer. Then we have the bijection*

$$\mathrm{SSYT}(\mu) \times \mathrm{SSYT}((a)) \xrightarrow{1:1} \bigsqcup_{\substack{\textit{horizontal strip } \lambda/\mu \\ |\lambda/\mu| = a}} \mathrm{SSYT}(\lambda)$$

where each pair $(P, Q) \in \mathrm{SSYT}(\mu) \times \mathrm{SSYT}((a))$ is mapped to the tableau R obtained by inserting all entries of Q (one by one and from left to right) into P using the row insertion algorithm.

In addition to multiplication, Λ carries an operation called *plethysm*

$$\begin{aligned} \Lambda \times \Lambda &\longrightarrow \Lambda \\ (f, g) &\longmapsto f[g]. \end{aligned}$$

A standard property of plethysm is that: For any integers $d \geqslant 0$,

$$h_d[h_2] = \sum_{\substack{\lambda \vdash 2d \\ \lambda \text{ is even}}} s_\lambda.$$

See [8] for more details about plethysm.

2.4 Representation theory of symmetric groups

Let $\mathfrak{S}_n$ be the symmetric group of $[n]$. All irreducible representations of $\mathfrak{S}_n$ are classified by *Specht modules* $\{V^\lambda\}_{\lambda \vdash n}$ (see [2, Section 7] for details). V^λ satisfies a standard property

$$\dim V^\lambda = |\mathrm{SYT}(\lambda)|.$$

Consider a finite-dimensional $\mathfrak{S}_n$-module V. Suppose that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda V^\lambda$ for some $c_\lambda \in \mathbb{Z}_{\geqslant 0}$. We can describe the module structure of V using its *Frobenius image* $\mathrm{Frob}(V) \in \Lambda_n$ given by

$$\mathrm{Frob}(V) \coloneqq \sum_{\lambda \vdash n} c_\lambda \cdot s_\lambda.$$

Furthermore, if $V = \bigoplus_{d=0}^{m} V_d$ is a graded $\mathfrak{S}_n$-module, its graded module structure can be described using its *graded Frobenius image* $\mathrm{grFrob}(V;q) \in \Lambda_n$ given by

$$\mathrm{grFrob}(V;q) \coloneqq \sum_{d=0}^{m} q^d \cdot \mathrm{Frob}(V_d).$$

2.5 Basic settings for our main result

Let $\mathcal{M}_{n,a} \coloneqq \{w \in \mathfrak{S}_n \,:\, w^2 = 1$ and w has exactly a fixed points$\}$ and identify $w \in \mathcal{M}_{n,a}$ with its permutation matrix, a symmetric 0-1 matrix with trace a. Therefore, $\mathcal{M}_{n,a} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a finite matrix locus carrying the conjugate action of $\mathfrak{S}_n$ given by $g(w) = gwg^{-1}$.

Let $\mathbf{x}_{n\times n} \coloneqq (x_{i,j})_{1\leqslant i,j\leqslant n}$ be the variable matrix and let $\mathbb{C}[\mathbf{x}_{n\times n}]$ be the polynomial ring over these variables. Applying the orbit harmonics method to $\mathcal{M}_{n,a}$, we obtain a graded $\mathfrak{S}_n$-module

$$R(\mathcal{M}_{n,a}) = \mathbb{C}[\mathbf{x}_{n\times n}]/\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a}).$$

Zhu [14, Proposition A.7] found an explicit generating set of the defining ideal $\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a})$ of $R(\mathcal{M}_{n,a})$:

- all sums $x_{i,1} + \cdots + x_{i,n}$ of variables in a single row,
- all sums $x_{1,j} + \cdots + x_{n,j}$ of variables in a single column,
- all products $x_{i,j} \cdot x_{i,j'}$ for $1 \leqslant i \leqslant n, 1 \leqslant j, j' \leqslant n$ of variables in a single row,
- all products $x_{i,j} \cdot x_{i',j}$ for $1 \leqslant i, i' \leqslant n, 1 \leqslant j \leqslant n$ of variables in a single column,
- all diagonally symmetric differences $x_{i,j} - x_{j,i}$ of variables,
- the diagonal sum $\sum_{i=1}^{n} x_{i,i}$,
- all products $\prod_{i\in S} x_{i,i}$ for $S \subseteq [n]$ with $|S| = a+1$, and
- all monomials of degree greater than $\frac{n-a}{2}$.

Liu, Ma, Rhoades, and Zhu figured out a signed combinatorial formula [6, Theorem 5.21] for $\mathrm{grFrob}(R(\mathcal{M}_{n,a});q)$, which will play an essential role in Section 3:

Theorem 4. *Suppose $a \equiv n \mod 2$. The graded Frobenius image of $R(\mathcal{M}_{n,a})$ is given by*

$$\operatorname{grFrob}(R(\mathcal{M}_{n,a}); q) = \sum_{d=0}^{(n-a)/2} \{h_d[h_2] \cdot h_{n-2d} - h_{d-1}[h_2] \cdot h_{n-2d+2}\}_{\lambda_1 \leqslant n-2d+a} \cdot q^d \qquad (2)$$

where we interpret $h_{-1} := 0$.

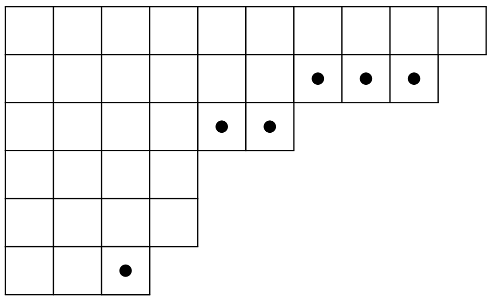
To study the index set of the Schur expansion of the right-hand side of Equation (2) using Pieri's rule, we need the following constructions.

Definition 5. Given a horizontal strip λ/μ, we associate a lattice path $\mathcal{LP}(\lambda/\mu)$ of length ∞ with it sequentially as follows:

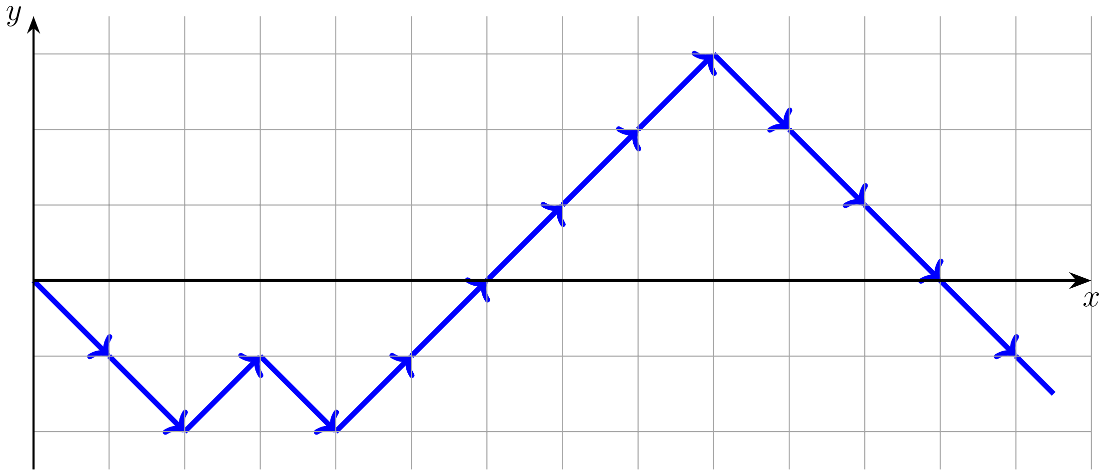
- Let $\mathcal{LP}(\lambda/\mu)$ start at the origin $(0, 0)$.
- Scan all the columns of λ rightwards. If one column intersects λ/μ, append an NE step (i.e. the vector $(1, 1)$) to $\mathcal{LP}(\lambda/\mu)$. Otherwise, append an SE step (i.e. the vector $(1, -1)$) to $\mathcal{LP}(\lambda/\mu)$.
- Append infinitely many SE steps to $\mathcal{LP}(\lambda/\mu)$.

For convenience, we denote *restrictions* of a lattice path $\mathcal{L}$ by $\mathcal{L}\mid_{a \leqslant x \leqslant b}$ and $\mathcal{L}\mid_{x=c}$. In particular, $\mathcal{L}\mid_{x=c}$ is actually a point, and the y-coordinate of this point is called its *height*.

Example 6. Consider $\lambda = (10, 9, 6, 4, 4, 3)$ and $\mu = (10, 6, 4, 4, 4, 2)$. That is,

$\lambda/\mu =$  .

Then the lattice path $\mathcal{LP}(\lambda/\mu)$ is shown as follows.

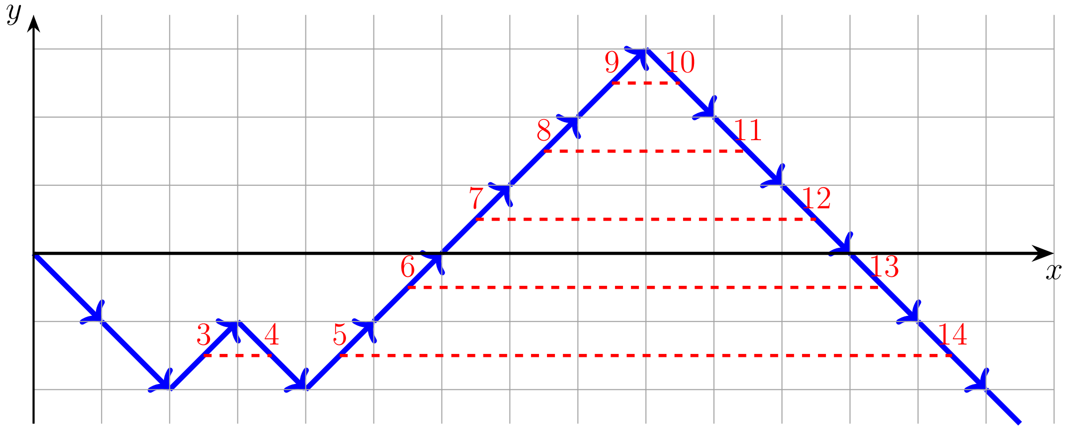


Note that we can also reconstruct a horizontal strip λ/μ if we know λ and $\mathcal{LP}(\lambda/\mu)$.

Definition 7. For a given lattice path $\mathcal{L}$ starting at $(0, 0)$ and consisting of infinitely many NE steps and SE steps, a step of $\mathcal{L}$ is called the *i-th step* if the x-coordinate of its right endpoint equals i. An ordered pair (i, j) of positive integers $0 < i < j$ is called a *reflection pair* of $\mathcal{L}$ if (i, j) satisfies the following three conditions:

- The i-th step of $\mathcal{L}$ (denoted by $\mathcal{L}_i$) is an NE step.
- The j-th step of $\mathcal{L}$ (denoted by $\mathcal{L}_j$) is an SE step.
- The horizontal line segment connecting the midpoints of the i-th and j-th steps of $\mathcal{L}$ intersects $\mathcal{L}$ only at its endpoints.

Example 8. All the reflection pairs of the lattice path $\mathcal{LP}(\lambda/\mu)$ shown in Example 6 are $(3, 4)$, $(5, 14)$, $(6, 13)$, $(7, 12)$, $(8, 11)$, and $(9, 10)$. See the picture below for details.



Definition 9. The *width* $\mathrm{wid}(\lambda/\mu)$ of a horizontal strip λ/μ is given by

$$\mathrm{wid}(\lambda/\mu) := \max\{\lambda_1, \max\{j : (i, j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\mu) \text{ for some } i\}\}$$

by the convention that $\max \varnothing = 0$.

Remark 10. There is another equivalent way to define $\mathrm{wid}(\lambda/\mu)$. Let $\{c_i\}_{i=1}^{\lambda_1}$ be the sequence given by

$$c_i := \begin{cases} -1, & \text{if the } (\lambda_1 - i + 1)\text{-th column of } \lambda \text{ intersects } \lambda/\mu \\ 1, & \text{otherwise} \end{cases}$$

and let M be the maximum prefix summation of $\{c_i\}_{i=1}^{\lambda_1}$. Then

$$\mathrm{wid}(\lambda/\mu) = \lambda_1 + \max\{0, M\}.$$

Example 11. Let λ/μ be the horizontal strip in Example 6. Then Example 8 indicates that

$$\max\{j : (i, j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\mu) \text{ for some } i\} = 14$$

and thus

$$\mathrm{wid}(\lambda/\mu) = \max\{\lambda_1, 14\} = \max\{10, 14\} = 14.$$

3 Preliminary version: a positive combinatorial formula depending on the graded component

In this section, we give a positive combinatorial version of Theorem 4 by cancellation. Our strategy is embedding all the terms of the Schur expansion of $\{h_{d-1}[h_2]\cdot h_{n-2d+2}\}_{\lambda_1\leqslant n-2d+a}$ into the Schur expansion of $\{h_d[h_2]\cdot h_{n-2d}\}_{\lambda_1\leqslant n-2d+a}$. That is, we provide a positive combinatorial formula for the Schur expansion of $\{h_d[h_2]\cdot h_{n-2d}-h_{d-1}[h_2]\cdot h_{n-2d+2}\}_{\lambda_1\leqslant n-2d+a}$.

We start by introducing some notations. For any partition $\lambda\vdash n$ and any integer $d\geqslant 0$, we introduce two sets of horizontal strips given by:

$$\mathrm{HS}(d,\lambda)\coloneqq\{\lambda/\mu\ :\ \mu\vdash 2d \text{ is an even partition such that } \lambda/\mu \text{ is a horizontal strip}\} \tag{3}$$

$$\mathrm{PHS}(d,\lambda)\coloneqq\{\lambda/\mu\in\mathrm{HS}(d,\lambda)\ :\ \text{the first } \lambda_1 \text{ steps of } \mathcal{LP}(\lambda/\mu) \text{ are weakly higher than the } x\text{-axis}\}. \tag{4}$$

Example 12. Recall the horizontal strip λ/μ in Example 6 given by $\lambda=(10,9,6,4,4,3)$ and $\mu=(10,6,4,4,4,2)$. Since $\mu\vdash 30$ is an even partition, we have that $\lambda/\mu\in\mathrm{HS}(15,\lambda)$. However, $\mathcal{LP}(\lambda/\mu)$ goes below the x-axis at the first step, so $\lambda/\mu\notin\mathrm{PHS}(d,\lambda)$.

In the rest of this section, *fix* two integers n and a such that $n>0$, $0\leqslant a\leqslant n$ and $a\equiv n \mod 2$. Let $0\leqslant d\leqslant\frac{n-a}{2}$ be an integer and $\lambda\vdash n$ be a partition such that $\lambda_1\leqslant n-2d+a$.

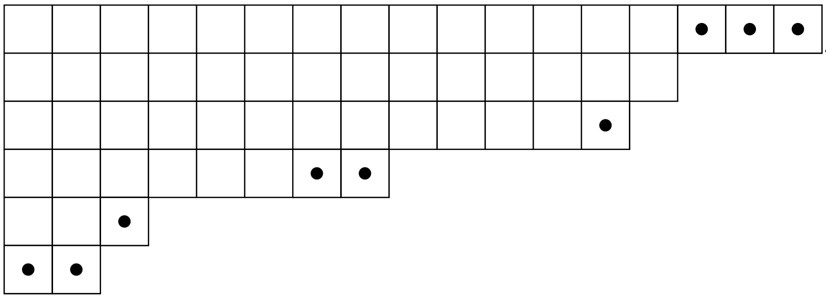
We construct a map (in fact a bijection which will be shown in Lemma 16)

$$\Phi_{d,\lambda}:\mathrm{HS}(d,\lambda)\setminus\mathrm{PHS}(d,\lambda)\longrightarrow\mathrm{HS}(d-1,\lambda) \tag{5}$$

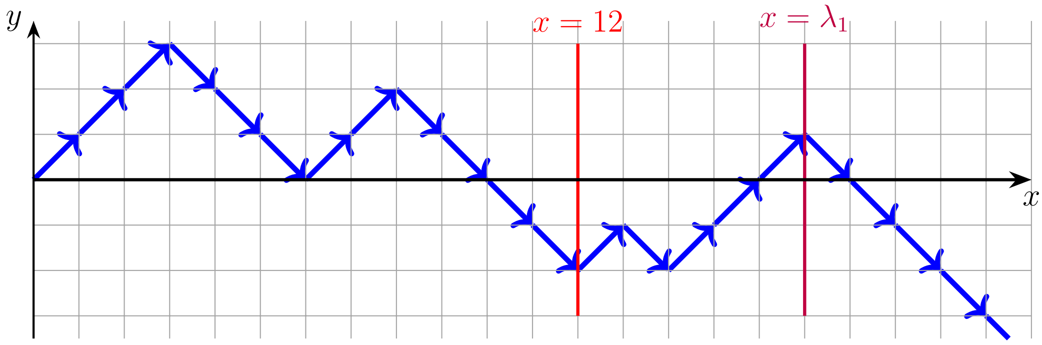
with the convention that $\Phi_{0,\lambda}:\varnothing\to\varnothing$. Let $\lambda/\mu\in\mathrm{HS}(d,\lambda)\setminus\mathrm{PHS}(d,\lambda)$.

- Let $x=m$ be the leftmost minimum of $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant\lambda_1}$.
- The fact that $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant\lambda_1}\notin\mathrm{PHS}(d,\lambda)$ implies that $m>0$. Furthermore, μ is an even partition, indicating that m is even. Therefore, $m\geqslant 2$.
- As a result of $m\geqslant 2$, the $(m-1)$-th step and m-th step of $\mathcal{LP}(\lambda/\mu)$ exist. Moreover, both of these two steps are SE steps since $x=m$ is the leftmost minimum of $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant\lambda_1}$. Therefore, removing the lowest boxes respectively from the $(m-1)$-th and m-th columns of μ yields a new even partition $\nu\vdash 2d-2$ such that λ/ν remains a horizontal strip.
- Let $\Phi_{d,\lambda}(\lambda/\mu)\coloneqq\lambda/\nu$.

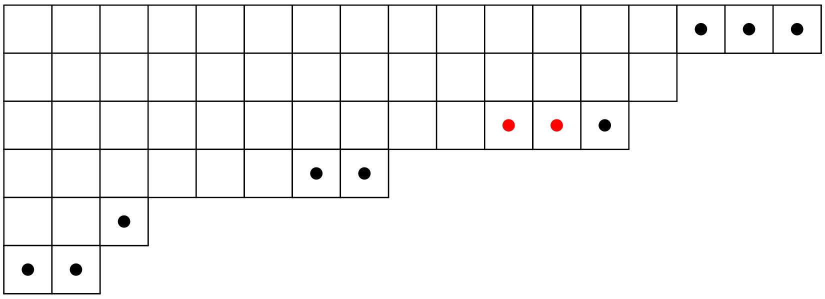
Example 13. We use a concrete example to illustrate how to obtain $\Phi_{d,\lambda}(\lambda/\mu)$. Consider $\lambda = (17, 14, 13, 8, 3, 2)$ and $\mu = (14, 14, 12, 6, 2)$, yielding a horizontal strip

$\lambda/\mu =$  .

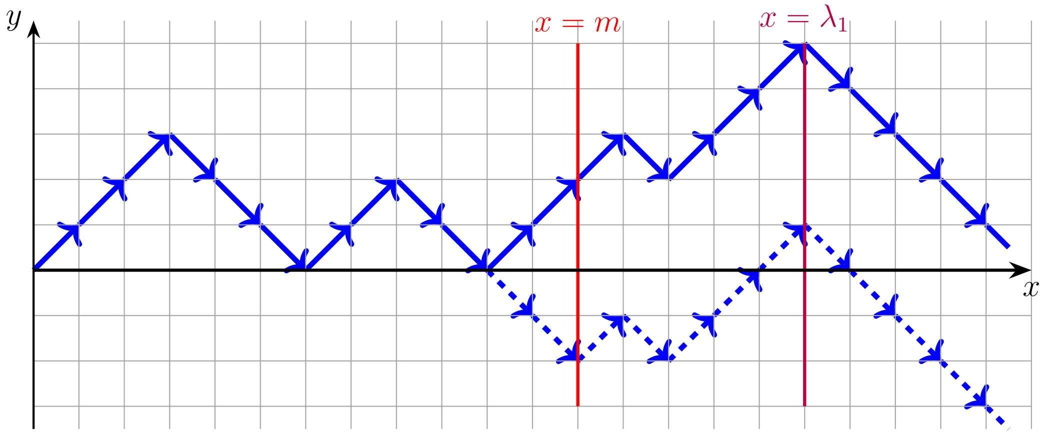
Then the lattice path $\mathcal{LP}(\lambda/\mu)$ is shown as follows.



Note that $x = 12$ is the first lowest point of $\mathcal{LP}(\lambda/\mu)_{0\leqslant x\leqslant\lambda_1}$ and hence $m = 12$. Now we remove the lowest boxes respectively from the 11-th and 12-th columns of μ, or, equivalently, add two boxes $(3, 11)$ and $(3, 12)$ into the horizontal strip λ/μ, obtaining a new horizontal strip

$\Phi_{d,\lambda}(\lambda/\mu) = \lambda/\nu =$  .

Remark 14. It is helpful to see $\Phi_{d,\lambda}$ as a "turning upwards" operation on lattice paths, which will be used in the proof of Lemma 16. Intuitively, the lattice path $\mathcal{LP}(\Phi_{d,\lambda}(\lambda/\mu))$ arises from $\mathcal{LP}(\lambda/\mu)$ by turning the $(m-1)$-th and m-th steps upwards as in the following figure. We will see that $x = m - 2$ is the last lowest point of $\mathcal{LP}(\Phi_{d,\lambda}(\lambda/\mu))\,|_{0\leqslant x\leqslant\lambda_1}$.



We have an immediate observation about m.

Lemma 15. *The value $x = m - 2$ is the rightmost minimum of $\mathcal{LP}(\lambda/\nu) \mid_{0 \leqslant x \leqslant \lambda_1}$.*

Proof. Let h be the y-coordinate of $\mathcal{LP}(\lambda/\mu) \mid_{x=m}$. We had shown that $m \geqslant 2$ during the construction of $\Phi_{d,\lambda}$. Note that $\mathcal{LP}(\lambda/\nu)$ arises from $\mathcal{LP}(\lambda/\mu)$ by turning the $(m-1)$-th and m-th steps upwards (see Remark 14). This "turning upwards" operation moves all the nodes of $\mathcal{LP}(\lambda/\mu) \mid_{x \geqslant m}$ upwards by 4 units. Consequently, all the nodes of $\mathcal{LP}(\lambda/\nu) \mid_{m \leqslant x \leqslant \lambda_1}$ is weakly higher than $y = h + 4$. In addition, the "turning" operation moves nodes $\mathcal{LP}(\lambda/\mu) \mid_{x=m-1}$ and $\mathcal{LP}(\lambda/\mu) \mid_{x=m}$ strongly above $y = h + 2$. Therefore, $\mathcal{LP}(\lambda/\nu) \mid_{m-2<x \leqslant \lambda_1}$ is strongly higher than $y = h + 2$, the height of $\mathcal{LP}(\lambda/\nu) \mid_{x=m-2}$.

Furthermore, since μ is an even partition, all the minimal-height (*NOT "minimum"!*) points of $\mathcal{LP}(\lambda/\mu)$ possess even heights. Recall that $x = m$ is the leftmost lowest point of $\mathcal{LP}(\lambda/\mu) \mid_{0 \leqslant x \leqslant \lambda_1}$, indicating that all the minimal-height points of $\mathcal{LP}(\lambda/\nu) \mid_{0 \leqslant x < m-2} = \mathcal{LP}(\lambda/\mu) \mid_{0 \leqslant x < m-2}$ are strongly higher than $y = h$ and thus weakly higher than $y = h+2$, the height of $\mathcal{LP}(\lambda/\nu) \mid_{x=m-2}$.

To sum up, we have shown that $x = m - 2$ is the rightmost lowest point of the lattice path $\mathcal{LP}(\lambda/\nu) \mid_{0 \leqslant x \leqslant \lambda_1}$. □

Lemma 16. *The map $\Phi_{d,\lambda}$ in (5) is bijective.*

Proof. **Injectivity:** Suppose that $\Phi_{d,\lambda}(\lambda/\mu^{(1)}) = \Phi_{d,\lambda}(\lambda/\mu^{(2)}) = \lambda/\nu$. We show that $\lambda/\mu^{(1)} = \lambda/\mu^{(2)}$ as means to deduce that $\Phi_{d,\lambda}$ is injective. Let $x = m_i$ be the leftmost minimum of $\mathcal{LP}(\lambda/\mu^{(i)}) \mid_{0 \leqslant x \leqslant \lambda_1}$ $(i = 1, 2)$. Then Lemma 15 indicates that both $x = m_i - 2$ $(i = 1, 2)$ are the rightmost minimum of $\mathcal{LP}(\lambda/\nu) \mid_{0 \leqslant x \leqslant \lambda_1}$ and hence $m_1 = m_2$. Therefore, since the same "turning upwards" operations (see Remark 14) on $\mathcal{LP}(\lambda/\mu^{(1)})$ and $\mathcal{LP}(\lambda/\mu^{(2)})$ generate the same result $\mathcal{LP}(\lambda/\nu)$, we have that $\mathcal{LP}(\lambda/\mu^{(1)}) = \mathcal{LP}(\lambda/\mu^{(2)})$ and thus $\lambda/\mu^{(1)} = \lambda/\mu^{(2)}$ (because $\mathcal{LP}(\lambda/\mu^{(i)})$ tells us which columns of λ intersect the horizontal strip $\lambda/\mu^{(i)}$, providing enough information to reconstruct $\lambda/\mu^{(i)}$). As a result, $\Phi_{d,\lambda}$ is injective.

Surjectivity: Given $\lambda/\nu \in \mathrm{HS}(d-1, \lambda)$, note that $\lambda_1 \leqslant n - 2d + a$ and $\mathcal{LP}(\lambda/\nu)$ goes downwards after the λ_1-th step. Therefore, let H be the height of $\mathcal{LP}(\lambda/\nu)\mid_{x=\lambda_1}$. Then

$$\begin{aligned}H &\geqslant \mathcal{LP}(\lambda/\nu)\mid_{x=n-2d+a} = |\lambda/\nu| - (n-2d+a-|\lambda/\nu|)\\ &= n-2d+2-(n-2d+a-(n-2d+2)) = n-2d-a+4\\ &\geqslant n-2\cdot\frac{n-a}{2}-a+4 = 4 > 0.\end{aligned}$$

However, the height of the starting point $(0,0)$ of $\mathcal{LP}(\lambda/\nu)$ is 0, which is less than H. Consequently, $x=\lambda_1$ is not the lowest point of $\mathcal{LP}(\lambda/\nu)\mid_{0\leqslant x\leqslant\lambda_1}$. Now let $x=m'<\lambda_1$ be the rightmost lowest point of $\mathcal{LP}(\lambda/\nu)\mid_{0\leqslant x\leqslant\lambda_1}$. We further deduce that $m'\leqslant\lambda_1-4$ from the inequality $H\geqslant 4$ shown above. Since $x=m'$ is the rightmost lowest point of $\mathcal{LP}(\lambda/\nu)\mid_{0\leqslant x\leqslant\lambda_1}$, both the $(m'+1)$-th and $(m'+2)$-th steps of $\mathcal{LP}(\lambda/\nu)\mid_{0\leqslant x\leqslant\lambda_1}$ must be NE steps. Turn these two steps downwards, obtaining a new lattice path $\mathcal{LP}(\lambda/\mu)$ for some horizontal strip $\lambda/\mu\in\mathrm{HS}(d,\lambda)\setminus\mathrm{PHS}(d,\lambda)$ (we can reconstruct λ/μ from λ and $\mathcal{LP}(\lambda/\mu)$, and $\lambda/\mu\notin\mathrm{PHS}(d,\lambda)$ since the lowest point of $\mathcal{LP}(\lambda/\nu)$ must be weakly lower than the origin). Then Remark 14 implies that $\Phi_{d,\lambda}(\lambda/\mu)=\lambda/\nu$. In conclusion, $\Phi_{d,\lambda}$ is surjective. □

We are ready to state and prove our first positive combinatorial formula for the graded Frobenius image $\mathrm{grFrob}(R(\mathcal{M}_{n,a});q)$.

Proposition 17. *We have that*

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \sum_{\lambda/\mu} s_\lambda$$

summing over all the horizontal strips

$$\lambda/\mu \in \bigsqcup_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}} \mathrm{PHS}(d,\lambda)$$

where $\mathrm{PHS}(d,\lambda)$ *is given by Equation* (4).

Proof. Theorem 4 indicates that

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \{h_d[h_2]\cdot h_{n-2d} - h_{d-1}[h_2]\cdot h_{n-2d+2}\}_{\lambda_1\leqslant n-2d+a}$$

where we interpret $h_{-1}=0$. Therefore, the $d=0$ case is easily verified. For $0<d\leqslant\frac{n-a}{2}$, applying Pieri's rule to the formula above yields that

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}}\ \sum_{\lambda/\mu\in\mathrm{HS}(d,\lambda)} s_\lambda - \sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}}\ \sum_{\lambda/\mu\in\mathrm{HS}(d-1,\lambda)} s_\lambda$$

Then Lemma 16 helps us replace the index set of the last summation with $\mathrm{HS}(d,\lambda) \setminus \mathrm{PHS}(d,\lambda)$, indicating that

$$\begin{aligned}\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) &= \sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}} \sum_{\lambda/\mu\in\mathrm{HS}(d,\lambda)} s_\lambda - \sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}} \sum_{\lambda/\mu\in\mathrm{HS}(d,\lambda)\setminus\mathrm{PHS}(d,\lambda)} s_\lambda \\ &= \sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant n-2d+a}} \sum_{\lambda/\mu\in\mathrm{PHS}(d,\lambda)} s_\lambda\end{aligned}$$

which completes the proof. □

4 Compact version: a positive combinatorial formula as an explicit graded refinement of $h_{(n-a)/2}[h_2] \cdot h_a$

In this section, we will prove Theorem 1 using Proposition 17. Note that Theorem 1 is more concise than Proposition 17, providing an explicit graded refinement of the Frobenius image $\mathrm{Frob}(\mathbb{C}[\mathcal{M}_{n,a}]) = \mathrm{Frob}(R(\mathcal{M}_{n,a})) = h_{(n-a)/2}[h_2] \cdot h_a$ in the form of a positive combinatorial formula.

In order to convert Proposition 17 into Theorem 1, we need to construct bijections between their index sets. We introduce some basic settings and notations as follows.

In the rest of this section, fix two integers n and a such that $n > 0$, $0 \leqslant a \leqslant n$ and $a \equiv n \mod 2$. Let $0 \leqslant d \leqslant \frac{n-a}{2}$ be an integer and $\lambda \vdash n$ be a partition such that $\lambda_1 \leqslant n - 2d + a$. Then keep the notations in Section 3. In addition, define the set (see Definition 9 for the definition of the statistic width wid())

$$\mathrm{WHS}(d,\lambda) := \{\lambda/\mu \in \mathrm{HS}((n-a)/2, \lambda) \,:\, \mathrm{wid}(\lambda/\mu) = n - 2d + a\}. \tag{6}$$

Then define the *left shadow map*

$$\mathcal{LS}_{d,\lambda} \,:\, \mathrm{PHS}(d,\lambda) \longrightarrow \mathrm{WHS}(d,\lambda) \tag{7}$$

through the following steps in order:

- Given $\lambda/\mu \in \mathrm{PHS}(d,\lambda)$, construct a set

  $S = \{1 \leqslant i \leqslant \lambda_1 \,:\, (i,j)$ is a reflection pair of $\mathcal{LP}(\lambda/\mu)$ for some $1 \leqslant j \leqslant n-2d+a\}$.

  (Recall the term "reflection pair" given by Definition 7.)

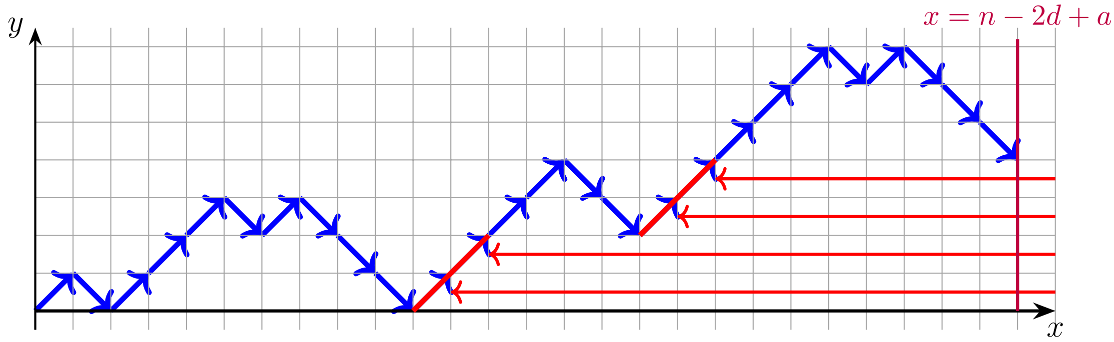
- Let λ/ν be the unique horizontal strip such that all the columns of λ intersecting λ/ν are exactly indexed by the set S.

- Let $\mathcal{LS}_{d,\lambda}(\lambda/\mu) := \lambda/\nu$.

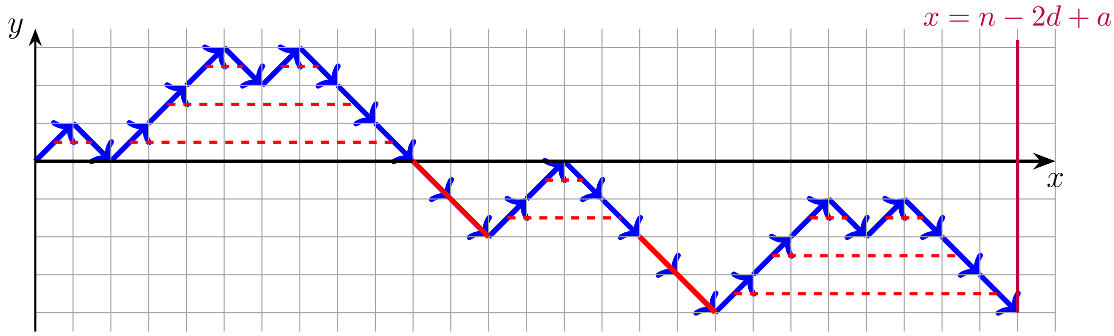
Roughly speaking, the left shadow map is reflecting down all NE steps that do not belong to reflection pairs (i,j) such that $j \leqslant n - 2d + a$.

Example 18. To visualize the effect of $\mathcal{LS}_{d,\lambda}$ on lattice paths, we can intuitively construct $\mathcal{LP}(\lambda/\nu)$ according to $\mathcal{LP}(\lambda/\mu)$ sequentially as follows:

- Put a large light source on the right of $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant n-2d+a}$. It emits horizontal light rays leftwards. Then the heights of all the NE steps touched by these rays must be consecutive, and their minimum is 0. Other steps perfectly match each other, forming reflection pairs.



- Whenever the light rays touch an NE step, replace it with an SE step, obtaining a new lattice path $\mathcal{LP}(\lambda/\nu)\ |_{0\leqslant x\leqslant n-2d+a}$. Then the heights of these new SE steps are consecutive as well, and their maximum is 0. In addition, these new SE steps (red in the figure below) are reflections of the original NE steps (red in the figure above) across the x-axis. Other steps perfectly match each other, forming reflection pairs (Definition 7).



Lemma 19. *$\mathcal{LS}_{d,\lambda}$ is well-defined, i.e. $\lambda/\nu \in \mathrm{WHS}(d,\lambda)$.*

Proof. First, we show that $\lambda/\nu \in \mathrm{HS}((n-a)/2,\lambda)$. Note that there are exactly $n-2d+a-|\lambda/\mu| = n-2d+a-(n-2d) = a$ SE steps in $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant n-2d+a}$. Recall that $\lambda/\mu \in \mathrm{PHS}(d,\lambda)$ implies that $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant \lambda_1}$ is weakly higher than the x-axis. Moreover, the height of $\mathcal{LP}(\lambda/\mu)\ |_{x=n-2d+a}$ is $|\lambda/\mu|-(n-2d+a-|\lambda/\mu|) = 2|\lambda/\mu|-(n-2d+a) = 2(n-2d)-(n-2d+a) = n-2d-a \geqslant n-2\cdot\frac{n-a}{2}-a = 0$, and $\mathcal{LP}(\lambda/\mu)\ |_{\lambda_1\leqslant x\leqslant n-2d+a}$ only contains SE steps. Consequently, $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant n-2d+a}$ is weakly higher than the x-axis. It follows that each horizontal leftward ray starting at all of the a SE steps of $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant n-2d+a}$ must touch an NE step of $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant \lambda_1}$, so any SE step of $\mathcal{LP}(\lambda/\mu)\ |_{0\leqslant x\leqslant n-2d+a}$ belongs to a reflection pair. It follows that $|S|$

equals the number of SE steps of $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant n-2d+a}$, which equals a as shown before. As a result, $|S| = a$ and thus $|\lambda/\nu| = a$, indicating that $\nu \vdash n-a$. Furthermore, μ is an even partition, revealing that all the minimal-height (NOT "minimum") points of $\mathcal{LP}(\lambda/\mu)$ must have even heights, so the leftmost element of any interval I_k that occurs in the interval decomposition $S = \bigsqcup I_k$ (i.e., the I_k are non-adjacent sets of consecutive integers) is odd; hence, ν is also an even partition. Therefore, $\lambda/\nu \in \mathrm{HS}((n-a)/2, \lambda)$.

It remains to show that $\mathrm{wid}(\lambda/\nu) = n-2d+a$. Define

$$A \coloneqq \{(i,j) \,:\, (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\mu) \text{ such that } j \leqslant n-2d+a\},$$
$$B \coloneqq \{(i,j) \,:\, (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\nu)\}.$$

We claim that $A = B$. In fact, for any $(i,j) \in A$ and any NE step k such that $i \leqslant k < j$, there exists $i < \ell \leqslant j$ such that $(k,\ell) \in A$, which is immediate from the connectedness of $\mathcal{LP}(\lambda/\mu)$. In addition, note that note that $S = \bigsqcup_{(k,\ell)\in A}\{k\}$. Therefore, $\mathcal{LS}_{d,\lambda}$ does not change any step of $\mathcal{LP}(\lambda/\mu)$ on the interval $[i,j]$, indicating that $(i,j) \in B$. The arbitrariness of (i,j) means that $A \subseteq B$. Furthermore, by construction, all the steps of $\mathcal{LP}(\lambda/\nu)$ not in $\bigsqcup_{(k,\ell)\in A}\{k\}$ are SE steps, forcing the containment $A \subseteq B$ to be an equality. Now the claim $A = B$ has been proved.

Thanks to this claim, we have that

$$n-2d+a \geqslant \max\{j \,:\, (i,j) \in B \text{ for some } i\}. \tag{8}$$

Recall that $\lambda_1 \leqslant n-2d+a$. Then we have that

$$n-2d+a \geqslant \max\{\lambda_1, \max\{j \,:\, (i,j) \in B \text{ for some } i\}\}.$$

Moreover, if $\lambda_1 < n-2d+a$, then the $(n-2d+a)$-th step of $\mathcal{LP}(\lambda/\mu)$ is an SE step, which implies that $\max\{j \,:\, (i,j) \in A \text{ for some } i\} = n-2d+a$ (because we have shown that any SE step of $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant n-2d+a}$ must belong to a reflection pair in the first paragraph of this proof). Then, from the claim $A = B$, we deduce that $\max\{j \,:\, (i,j) \in B \text{ for some } i\} = n-2d+a$. Thus, the equality of (8) holds. If $\lambda_1 = n-2d+a$, it is clear that the equality of (8) holds. To sum up, we deduce that

$$n-2d+a = \max\{\lambda_1, \max\{j \,:\, (i,j) \in B \text{ for some } i\}\},$$

which means that $\mathrm{wid}(\lambda/\nu) = n-2d+a$ according to Definition 9.

In one word, we have shown that $\lambda/\nu \in \mathrm{HS}((n-a)/2, \lambda)$ and $\mathrm{wid}(\lambda/\nu) = n-2d+a$, indicating that $\lambda/\nu \in \mathrm{WHS}(d,\lambda)$. □

Now we want to construct the inverse map of $\mathcal{LS}_{d,\lambda}$ given by (7), called the *right shadow map*.

$$\mathcal{RS}_{d,\lambda} \,:\, \mathrm{WHS}(d,\lambda) \longrightarrow \mathrm{PHS}(d,\lambda). \tag{9}$$

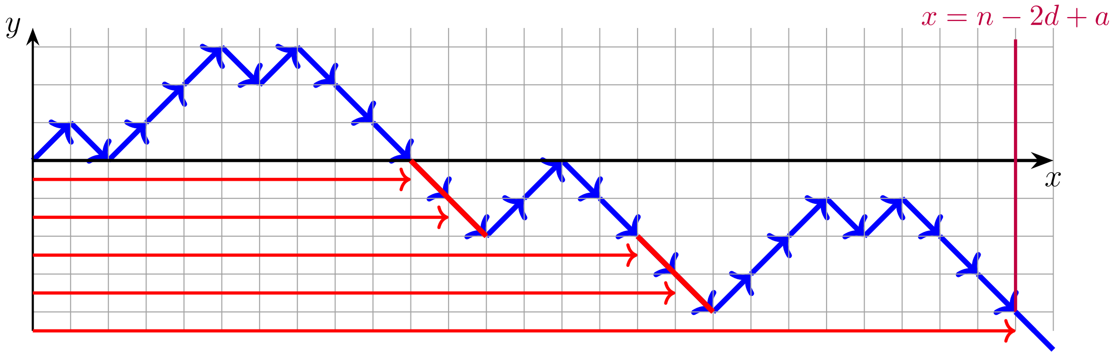
Given $\lambda/\nu \in \mathrm{WHS}(d,\lambda)$, we associate $\mathcal{RS}_{d,\lambda}(\lambda/\nu)$ using the following steps:

- Let $T = \{j \in \mathbb{N} : (i, j)$ is a reflection pair of $\mathcal{LP}(\lambda/\nu)$ for some $i\}$.
- $\lambda/\nu \in \mathrm{WHS}(d, \lambda)$ implies that $\mathrm{wid}(\lambda/\nu) = n - 2d + a$, so we have that $\max T \leqslant n - 2d + a$. Let λ/μ be the unique horizontal strip such that all the columns of λ intersecting λ/μ are exactly indexed by $[n - 2d + a] \setminus T$.
- Let $\mathcal{RS}_{d,\lambda}(\lambda/\nu) \coloneqq \lambda/\mu$.

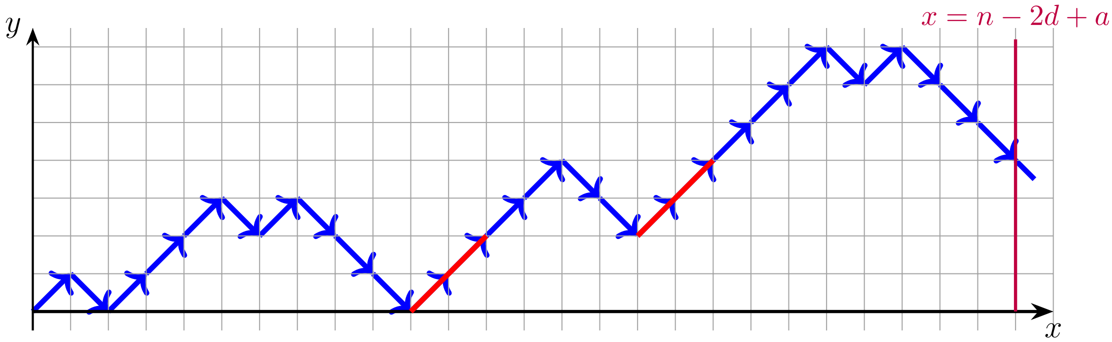
Roughly speaking, the right shadow map is reflecting up all the SE steps that do not belong to reflection pairs (i, j) such that $j \leqslant n - 2d + a$.

Example 20. To visualize the effect of $\mathcal{RS}_{d,\lambda}$ on lattice paths, we describe how to construct $\mathcal{LP}(\lambda/\mu)$ according to $\mathcal{LP}(\lambda/\nu)$ intuitively. The construction needs two steps:

- Put a large light source on the left of $\mathcal{LP}(\lambda/\nu)$ and let it emit horizontal light rays rightwards. Then the heights of all the SE steps touched by these rays must be consecutive, and their maximum is 0.



- Replace all the SE steps of $\mathcal{LP}(\lambda/\nu)\ |_{0 \leqslant x \leqslant n-2d+a}$ touched by light rays with NE steps to obtain $\mathcal{LP}(\lambda/\mu)$. Clearly, these new NE steps (red in the figure below) are reflections of the old SE steps (red in the figure above) across the x-axis, so these new NE steps are weakly higher than the x-axis, indicating that $\mathcal{LP}(\lambda/\mu)\ |_{0 \leqslant x \leqslant n-2d+a}$ is weakly higher than the x-axis.



Lemma 21. *$\mathcal{RS}_{d,\lambda}$ is well-defined, i.e. $\lambda/\mu \in \mathrm{PHS}(d, \lambda)$.*

Proof. First, we show that $\mu \vdash 2d$. Since $\nu \vdash \frac{n-a}{2}$ implies that $|\lambda/\nu| = a$, $\mathcal{LP}(\lambda/\nu)$ totally has a NE steps and hence $|T| = a$. Therefore, $|\lambda/\mu| = |[n-2d+a] \setminus T| = n-2d+a-a = n-2d$, revealing that $\mu \vdash 2d$. Furthermore, since ν is an even partition, all the minimal-height points of $\mathcal{LP}(\lambda/\nu)$ have even heights. Therefore, the rightmost element of any interval J_k that occurs in the interval decomposition $T = \bigsqcup J_k$ (i.e., the J_k are non-adjacent sets of consecutive integers) is even, which means that μ is even.

It remains to show that $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant n-2d+a}$ is weakly higher than the x-axis. Let B be the set of reflection pairs of $\mathcal{LP}(\lambda/\nu)$. Note that $T = \bigsqcup_{(i,j)\in B}\{j\}$. Consequently, for any integer $1 \leqslant k \leqslant n-2d+a$, we have the injection

$$\begin{aligned} [k] \cap T &\hookrightarrow [k] \setminus T \\ j &\mapsto i, \quad \text{if } (i,j) \in B. \end{aligned}$$

It follows that $|[k] \setminus T| \geqslant |[k] \cap T|$. By construction, the height of $\mathcal{LP}(\lambda/\mu)\mid_{x=k}$ equals

$$|[k] \setminus T| - |[k] \cap T| \geqslant 0.$$

The arbitrariness of k yields that $\mathcal{LP}(\lambda/\mu)\mid_{0\leqslant x\leqslant n-2d+a}$ is weakly higher than the x-axis, concluding our proof. □

Now we mention the last technical result before the proof of Theorem 1.

Lemma 22. *$\mathcal{LS}_{d,\lambda}$ and $\mathcal{RS}_{d,\lambda}$ are the inverse maps of each other.*

Remark 23. Before the rigorous proof, we want to intuitively understand why Lemma 22 is true. Examples 18 and 20 respectively illustrate how to convert the lattice path $\mathcal{LP}(\lambda/\mu)$ into $\mathcal{LP}(\mathcal{LS}_{d,\lambda}(\lambda/\mu))$ and how to convert $\mathcal{LP}(\lambda/\nu)$ into $\mathcal{LP}(\mathcal{RS}_{d,\lambda}(\lambda/\nu))$ using "horizontal rays" intuitively. Note that these two operations on lattice paths are the inverse of each other. Furthermore, we can reconstruct a horizontal strip using the lattice path associated with it. Therefore, $\mathcal{LS}_{d,\lambda}$ and $\mathcal{RS}_{d,\lambda}$ are the inverse of each other.

Proof of Lemma 22. We first show that $\mathcal{RS}_{d,\lambda} \circ \mathcal{LS}_{d,\lambda} = \mathrm{id}_{\mathrm{PHS}(d,\lambda)}$. For $\lambda/\mu \in \mathrm{PHS}(d,\lambda)$, we write $\lambda/\nu = \mathcal{LS}_{d,\lambda}(\lambda/\mu)$. It suffices to show that $\mathcal{RS}_{d,\lambda}(\lambda/\nu) = \lambda/\mu$. Recall two sets S, T used to define $\mathcal{LS}_{d,\lambda}, \mathcal{RS}_{d,\lambda}$ respectively:

$$\begin{aligned} S &= \{1 \leqslant i \leqslant \lambda_1 : (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\mu) \text{ for some } 1 \leqslant j \leqslant n-2d+a\}, \\ T &= \{j \in \mathbb{N} : (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\nu) \text{ for some } i\}. \end{aligned}$$

Recall two sets A, B in the proof of Lemma 19:

$$\begin{aligned} A &= \{(i,j) : (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\mu) \text{ such that } j \leqslant n-2d+a\}, \\ B &= \{(i,j) : (i,j) \text{ is a reflection pair of } \mathcal{LP}(\lambda/\nu)\}. \end{aligned}$$

In the proof of Lemma 19, we have shown the claim $A = B$. Therefore, we have that

$$T = \bigsqcup_{(i,j)\in B}\{j\} = \bigsqcup_{(i,j)\in A}\{j\} = \{j \leqslant n-2d+a : j \text{ is an SE step of } \mathcal{LP}(\lambda/\mu)\}$$

where the last equal sign arises from the fact $\lambda/\mu \in \mathrm{PHS}(d,\lambda)$. Therefore, $[n-2d+a]\backslash T$ is exactly the set of NE steps of $\mathcal{LP}(\lambda/\mu)$, i.e., the set of indices of columns of λ intersecting λ/μ. By construction, we have that $\mathcal{RS}_{d,\lambda}(\lambda/\nu) = \lambda/\mu$.

Then, we show that $\mathcal{LS}_{d,\lambda} \circ \mathcal{RS}_{d,\lambda} = \mathrm{id}_{\mathrm{WHS}(d,\lambda)}$. For $\lambda/\nu \in \mathrm{WHS}(d,\lambda)$, write $\lambda/\mu = \mathcal{RS}_{d,\lambda}(\lambda/\nu)$. It suffices to show that $\mathcal{LS}_{d,\lambda}(\lambda/\mu) = \lambda/\nu$. Keep the notations S, T, A, B above. We still claim that $A = B$. In fact, for any $(i,j) \in B$ and any $i < \ell \leqslant j$, there exists $i \leqslant k < j$ such that $(k,\ell) \in B$, which is immediate from the connectedness of $\mathcal{LP}(\lambda/\nu)$. In acction, note that $T = \bigsqcup_{(i,j)\in B}\{j\}$. Therefore, $\mathcal{RS}_{d,\lambda}$ does not change any steps of $\mathcal{LP}(\lambda/\nu)$ on the interval $[i,j]$, indicating that $(i,j) \in A$. The arbitrariness of (i,j) means that $B \subseteq A$. Furthermore, by construction, all the steps of $\mathcal{LP}(\lambda/\mu) \mid_{0\leqslant x\leqslant n-2d+a}$ not in $\bigsqcup_{(i,j)\in B}\{j\}$ are NE steps, forcing the containment $B \subseteq A$ to be an equality. It follows that

$$\begin{aligned} S &= \bigsqcup_{(i,j)\in A}\{i\} = \bigsqcup_{(i,j)\in B}\{i\} \\ &= \{\text{NE steps of } \mathcal{LP}(\lambda/\nu)\} = \{1 \leqslant i \leqslant \lambda_1 \,:\, \text{the } i\text{-th column meets } \lambda/\nu\}, \end{aligned}$$

which, by construction, yields that $\mathcal{LS}_{d,\lambda}(\lambda/\mu) = \lambda/\nu$. □

Finally, we are ready to prove Theorem 1.

Proof of Theorem 1. By Proposition 17, we have that

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \sum_{\substack{\lambda\vdash n \\ \lambda_1\leqslant n-2d+a}} \sum_{\lambda/\mu\in\mathrm{PHS}(d,\lambda)} s_\lambda.$$

The one-to-one correspondence between $\mathrm{PHS}(d,\lambda)$ and $\mathrm{WHS}(d,\lambda)$ arising from Lemma 22 enables us to replace the index set of the last summation above with $\mathrm{WHS}(d,\lambda)$. That is,

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \sum_{\substack{\lambda\vdash n \\ \lambda_1\leqslant n-2d+a}} \sum_{\lambda/\mu\in\mathrm{WHS}(d,\lambda)} s_\lambda,$$

which is equivalent to Theorem 1. (Recall the definition of $\mathrm{WHS}(d,\lambda)$ in Equation (6).) □

5 Application

Recall that Proposition 17 assigns a restriction $\lambda_1 \leqslant n-2d+a$ to the index range of the summation. In fact, we can remove this restriction in some cases.

Corollary 24. *For integers n, a, d as in Proposition 17, if further $d \leqslant a$, then*

$$\mathrm{Frob}(R(\mathcal{M}_{n,a})_d) = \sum_{\lambda/\mu} s_\lambda$$

where λ/μ ranges over $\bigsqcup_{\lambda\vdash n}\mathrm{PHS}(d,\lambda)$.

Proof. By concentrating on the summation index sets of Proposition 17 and Corollary 24, it suffices to show that $\mathrm{PHS}(d,\lambda)=\varnothing$ whenever $\lambda_1>n-2d+a$. Assume, for the sake of contradiction, that $\lambda/\mu\in\mathrm{PHS}(d,\lambda)$. Among columns $1,2,\ldots,\mu_1$, write c for the number of columns intersecting λ/μ and μ_1-c for the number of columns not intersecting λ/μ. Then $c\geqslant\mu_1-c$ since $\mathcal{LP}(\lambda/\mu)\,|_{0\leqslant x\leqslant\mu_1}$ is weakly higher than the x-axis. Consequently, we deduce that $c\geqslant\frac{\mu_1}{2}$ and hence

$$\begin{aligned}\lambda_1&=\mu_1+|\{\text{cells of }\lambda/\mu\text{ with column indices greater than }\mu_1\}|=\mu_1+|\lambda/\mu|-c\\&=\mu_1+n-2d-c\leqslant\mu_1+n-2d-\frac{\mu_1}{2}=n-2d+\frac{\mu_1}{2}\leqslant n-2d+d\leqslant n-2d+a\end{aligned}$$

where the second-to-last sign of inequality arises from $\mu\vdash 2d$. However, the inequality above contradicts $\lambda_1>n-2d+a$. □

Note that the summation in Corollary 24 does not depend on a. We thus immediately obtain some $\mathfrak{S}_n$-module isomorphisms of the form $R(\mathcal{M}_{n,a})_d\cong R(\mathcal{M}_{n,a'})_d$ with $a\neq a'$ as follows.

Corollary 25. *For nonnegative integers n,a,a',d such that $n>0$, $\max\{a,a'\}\leqslant n$, $a\equiv a'\equiv n \mod 2$, $d\leqslant\min\{a,a',\frac{n-a}{2},\frac{n-a'}{2}\}$, we have isomorphisms of $\mathfrak{S}_n$-modules*

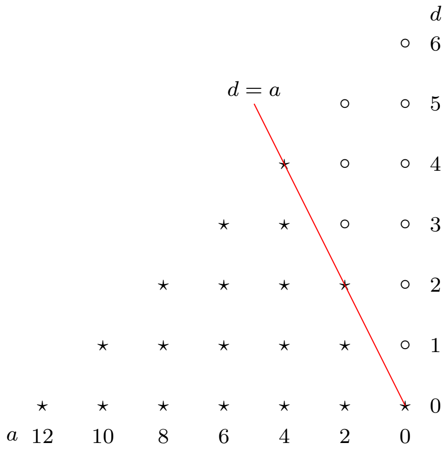
$$R(\mathcal{M}_{n,a})_d\cong R(\mathcal{M}_{n,a'})_d.$$

Remark 26. The isomorphisms in Corollary 25 indicates that $\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a})_d=\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a'})_d$. In fact, we suppose that $a\geqslant a'$, then [14, Proposition A.7] implies that $\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a})_d\subseteq\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a'})_d$ and hence the natural $\mathfrak{S}_n$-module surjection $R(\mathcal{M}_{n,a})_d\twoheadrightarrow R(\mathcal{M}_{n,a'})_d$. Corollary 25 forces this surjection to be an isomorphism, so we have that $\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a})_d=\mathrm{gr}\mathbf{I}(\mathcal{M}_{n,a'})_d$.

Remark 27. Theorem 4 implies chains of $\mathfrak{S}_n$-module surjections where $\delta=\begin{cases}0, & \text{if } n \text{ is even}\\ 1, & \text{if } n \text{ is odd}\end{cases}$ ([6, Lemma 5.16])

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & R_n(\mathcal{M}_{n,\delta})_{\frac{n-\delta}{2}}\\
 & & & & & & & & R(\mathcal{M}_{n,\delta+2})_{\frac{n-\delta-2}{2}} & \twoheadrightarrow & R(\mathcal{M}_{n,\delta})_{\frac{n-\delta-2}{2}}\\
 & & & & & & R(\mathcal{M}_{n,\delta+4})_{\frac{n-\delta-4}{2}} & \twoheadrightarrow & R(\mathcal{M}_{n,\delta+2})_{\frac{n-\delta-4}{2}} & \twoheadrightarrow & R(\mathcal{M}_{n,\delta})_{\frac{n-\delta-4}{2}}\\
 & & & & \iddots & & \vdots & & \vdots & & \vdots\\
 & & R(\mathcal{M}_{n,n-2})_1 & \twoheadrightarrow & \cdots & \twoheadrightarrow & R(\mathcal{M}_{n,\delta+4})_1 & \twoheadrightarrow & R(\mathcal{M}_{n,\delta+2})_1 & \twoheadrightarrow & R(\mathcal{M}_{n,\delta})_1\\
R(\mathcal{M}_{n,n})_0 & \twoheadrightarrow & R(\mathcal{M}_{n,n-2})_0 & \twoheadrightarrow & \cdots & \twoheadrightarrow & R(\mathcal{M}_{n,\delta+4})_0 & \twoheadrightarrow & R(\mathcal{M}_{n,\delta+2})_0 & \twoheadrightarrow & R(\mathcal{M}_{n,\delta})_0
\end{array}$$

Interestingly, Corollary 25 replaces all the surjections "$\twoheadrightarrow$" between these $\mathfrak{S}_n$-modules $R(\mathcal{M}_{n,a})_d$ weakly under the straight line $d=a$ with isomorphisms "$\cong$". That is, if we use circles and stars to represent each graded component $R(\mathcal{M}_{n,a})_d$ above and thus generate the following diagram, then all the stars on the same row are isomorphic. In this example, $n = 12$.



6 Future directions

In [6], Liu et al. proposed some interesting open problems. Our positive combinatorial formulae may yield potential ways to address two of them.

Problem 28. Find a statistic stat : $\mathcal{M}_{n,a} \to \mathbb{Z}_{\geqslant 0}$ such that

$$\operatorname{Hilb}(R(\mathcal{M}_{n,a}); q) = \sum_{w \in \mathcal{M}_{n,a}} q^{\operatorname{stat}(w)}.$$

Here are some ideas about Problem 28. Theorem 1 immediately yields a formula for the Hilbert series of $R(\mathcal{M}_{n,a})$:

$$\begin{aligned}\operatorname{Hilb}(R(\mathcal{M}_{n,a}); q) &= \sum_{\lambda \vdash n} \sum_{\substack{\mu \vdash n-a, \\ \mu \text{ is even,} \\ \lambda/\mu \text{ is a} \\ \text{horizontal stripe}}} q^{\frac{n+a-\operatorname{wid}(\lambda/\mu)}{2}} \cdot |\operatorname{SYT}(\lambda)| \\ &= \sum_{\lambda \vdash n} \sum_{\substack{\mu \vdash n-a, \\ \mu \text{ is even,} \\ \lambda/\mu \text{ is a} \\ \text{horizontal stripe}}} \sum_{P \in \operatorname{SYT}(\lambda)} q^{\frac{n+a-\operatorname{wid}(\lambda/\mu)}{2}} \\ &= \sum_{(P, \lambda/\mu)} q^{\frac{n+a-\operatorname{wid}(\lambda/\mu)}{2}} \end{aligned} \tag{10}$$

where the last sum runs over the set

$$\operatorname{TH}_{n,a} \coloneqq \{(P, \lambda/\mu) \,:\, \lambda \vdash n,\, P \in \operatorname{SYT}(\lambda),\, \lambda/\mu \text{ is a horizontal stripe of size } a,\, \mu \text{ is even}\}.$$

By Propositions 2 and 3, there is a bijection $\operatorname{bij}_{n,a} : \mathcal{M}_{n,a} \xrightarrow{1:1} \operatorname{TH}_{n,a}$ given by the following steps in order:

- For $w \in \mathcal{M}_{n,a}$, let M be the symmetric matrix given by

$$M_{i,j} = \begin{cases} 1, & \text{if } i \neq j \text{ and } w(i) = j \\ 0, & \text{otherwise.} \end{cases}$$

- Write $\mathrm{RSK}(M) = (T, T)$ and let $Q = T'$ be the conjugate of T.
- Insert all fixed points of w into Q in increasing order using the row insertion algorithm, obtaining $P \in \mathrm{SYT}(n)$. Let $\lambda(w) \coloneqq \mathrm{sh}(P)$ and $\mu(w) \coloneqq \mathrm{sh}(Q)$.
- Let $\mathrm{bij}_{n,a}(w) \coloneqq (P, \lambda(w)/\mu(w)) \in \mathrm{TH}_{n,a}$.

Combining Expression (10) with this bijection, we have that

$$\mathrm{Hilb}(R(\mathcal{M}_{n,a}); q) = \sum_{w \in \mathcal{M}_{n,a}} q^{\frac{n+a-\mathrm{wid}(\lambda(w)/\mu(w))}{2}},$$

so we may take $\mathrm{stat}(w) \coloneqq \frac{n+a-\mathrm{wid}(\lambda(w)/\mu(w))}{2}$ to solve Problem 28. However, it is difficult to compute this statistic $\mathrm{stat}(w)$ only using the information of w itself, as we have to first apply $\mathrm{bij}_{n,a}$ to w to convert w into a pair $(P, \lambda(w)/\mu(w))$.

Problem 29. Find an explicit monomial basis of $R(\mathcal{M}_{n,a})$.

Here are some ideas for Problem 29. Like what we did for Problem 28, we use the row insertion operation. Proposition 17 plays a crucial role now. Fix $0 \leqslant d \leqslant \frac{n-a}{2}$ and consider the index set in Proposition 17. Given a horizontal strip

$$\lambda/\mu \in \bigsqcup_{\substack{\lambda \vdash n \\ \lambda_1 \leqslant n-2d+a}} \mathrm{PHS}(d, \lambda)$$

and $P \in \mathrm{SYT}(\lambda)$, we want to convert the pair $(P, \lambda/\mu)$ into a monomial $m \in \mathbb{C}[\mathbf{x}_{n\times n}]$ of degree d so that we can construct a basis of $R(\mathcal{M}_{n,a})_d$ using all of such monomials m.

The inverse of the row insertion operation enables us to push the horizontal strip λ/μ out of P, yielding a new tableau Q of shape μ. Let $M = \mathrm{RSK}^{-1}(Q', Q')$. Since $\mathrm{sh}(Q) = \mu \vdash 2d$ is an even partition, M must be a symmetric 0-1 matrix with totally $2d$ 1's on distinct rows and columns and avoiding the diagonal (see Proposition 2). Write

$$m = \prod_{\substack{i<j \\ M_{i,j}=1}} x_{i,j}.$$

The tableau version of Pieri's rule (Proposition 3) indicates that the map $(P, \lambda/\mu) \mapsto m$ is injective. Therefore, we exactly obtain $\dim(R(\mathcal{M}_{n,a})_d)$ distinct monomials m, so we guess that such monomials m form a basis of $R(\mathcal{M}_{n,a})_d$. This conjecture has been verified for $a = 0$ and $n \leqslant 8$ through computer coding.

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