# The representation theory of somewhere-to-below shuffles

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**Abstract**

The *somewhere-to-below shuffles* are the elements

$$t_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n}$$

(for $\ell \in \{1, 2, \ldots, n\}$) in the group algebra $\mathbf{k}[S_n]$ of the $n$-th symmetric group $S_n$. Their linear combinations are called the *one-sided cycle shuffles*. We determine the eigenvalues of the action of any one-sided cycle shuffle on any Specht module $\mathcal{S}^\lambda$ of $S_n$.

**Keywords:** symmetric group, permutations, card shuffling, top-to-random shuffle, group algebra, filtration, Specht module, representation theory, substitutional analysis.

**Mathematics Subject Classifications:** 05E99, 20C30, 60J10

## Contents



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# 1 Introduction

This paper is a continuation of [13] with other means. Specifically, our goal here is to answer some natural representation-theoretical questions around the somewhere-to-below shuffles in the symmetric group algebra (including [13, Question 16.12]).

We recall that the *somewhere-to-below shuffles* are $n$ special elements $t_1, t_2, \ldots, t_n$ of the group algebra $\mathbf{k}[S_n]$ of a symmetric group $S_n$ over an arbitrary commutative ring $\mathbf{k}$; they are defined by

$$t_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n],$$

where $\mathrm{cyc}_{\ell,\ell+1,\ldots,k}$ denotes the cycle that sends $\ell \mapsto \ell+1 \mapsto \ell+2 \mapsto \cdots \mapsto k \mapsto \ell$ (and leaves all remaining elements of $[n] = \{1, 2, \ldots, n\}$ unchanged). Together with their linear combinations (called the *one-sided cycle shuffles*), they have been introduced and studied in the paper [13][1] by Grinberg and Lafrenière. One of the main results is [13, Theorem 11.1], which constructs a basis $(a_w)_{w \in S_n}$ of $\mathbf{k}[S_n]$ on which each of the shuffles $t_1, t_2, \ldots, t_n$ acts (by right multiplication) triangularly – i.e., which satisfies

$$a_w t_\ell \in \mathrm{span}\{a_v \mid v \leqslant w\} \qquad \text{for all } w \in S_n \text{ and } \ell \in \{1, 2, \ldots, n\}$$

(for an appropriate total order $<$ on $S_n$). This entails that the shuffles $t_1, t_2, \ldots, t_n$ and their linear combinations have integer eigenvalues; these eigenvalues have indeed been found ([13, §12]) along with their multiplicities ([13, §13]). As a further consequence, the $\mathbf{k}$-subalgebra of $\mathbf{k}[S_n]$ generated by $t_1, t_2, \ldots, t_n$ is isomorphic to an algebra of upper-triangular matrices, and the commutators $[t_i, t_j] := t_i t_j - t_j t_i$ are nilpotent; a followup work [8] proves even stronger claims.

However, like any elements of the group algebra $\mathbf{k}[S_n]$, the shuffles $t_1, t_2, \ldots, t_n$ act not just on the whole algebra $\mathbf{k}[S_n]$, but on any of its modules, i.e., on any representation of $S_n$. Thus, the question about eigenvalues can be asked for each representation of $S_n$, in particular for the *Specht modules* (which are the irreducible representations of $S_n$, at least in characteristic 0).

[1]Note that [13] is a slight abridgement of [11]. The numbering of results in [11] and in [13] is identical except for Section 9, so the reader can consult either version.

The main goal of this paper is to answer this latter question. Let us give a quick outline of the answer (which was announced in [12, §11])[2]:

We shall use some basic notions from the representation theory of $S_n$ and from symmetric functions; the reader can find all prerequisites in [6, Chapters 6 and 7]. For any partition $\lambda$ of $n$, a Specht module $\mathcal{S}^\lambda$ is defined, which is a representation of $S_n$ with a basis indexed by standard tableaux of shape $\lambda$. (In [6], it is called $S^\lambda$.) This $S_n$-module $\mathcal{S}^\lambda$ is irreducible when $\mathbf{k}$ has characteristic 0. Each $u \in \mathbf{k}[S_n]$ acts (on the left) on this Specht module $\mathcal{S}^\lambda$; we let $L_\lambda(u)$ denote this action (viewed as a $\mathbf{k}$-module endomorphism of $\mathcal{S}^\lambda$).

We let $\Lambda$ denote the ring of symmetric functions over $\mathbb{Z}$ (defined in [6, §6.2]). We recall that it has a basis $(s_\lambda)_{\lambda \text{ is a partition}}$ of *Schur functions* $s_\lambda$.

For each $m \in \mathbb{N}$, we let $h_m \in \Lambda$ denote the $m$-th complete homogeneous symmetric function. For each $m > 1$, we let $z_m \in \Lambda$ denote the Schur function

$$z_m := s_{(m-1,1)} = h_{m-1}h_1 - h_m \in \Lambda.$$

A set of integers is called *lacunar* if it contains no two consecutive integers. For each lacunar subset $I$ of $[n-1]$, we define a symmetric function

$$z_I := h_{i_1-1} \prod_{j=2}^{m} z_{i_j - i_{j-1}} \in \Lambda,$$

where $i_1, i_2, \ldots, i_m$ are the elements of $I \cup \{n+1\}$ in increasing order (so that $i_m = n+1$ and $I = \{i_1 < i_2 < \cdots < i_{m-1}\}$). When this symmetric function $z_I$ is expanded in the basis $(s_\lambda)_{\lambda \text{ is a partition}}$ of $\Lambda$, the coefficient of a given Schur function $s_\lambda$ shall be called $c_I^\lambda$. This coefficient $c_I^\lambda$ is actually a Littlewood–Richardson coefficient (since $z_I$ is a skew Schur function), hence a nonnegative integer.

We now claim the following:

**Theorem 1** (part of Theorem 21)**.** *Let $\lambda$ be a partition. Let $\omega_1, \omega_2, \ldots, \omega_n \in \mathbf{k}$. Then, the eigenvalues of the operator $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ on the Specht module $\mathcal{S}^\lambda$ are the linear combinations*

$$\omega_1 m_{I,1} + \omega_2 m_{I,2} + \cdots + \omega_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar satisfying } c_I^\lambda \neq 0,$$

*where the $m_{I,k}$ are certain nonnegative integers defined combinatorially (namely, $m_{I,k}$ is the distance between $k$ and the smallest element of $I \cup \{n+1\}$ that is $\geqslant k$). The algebraic multiplicities of these eigenvalues are the $c_I^\lambda$ in the generic case (i.e., if no two $I$'s produce the same linear combination; otherwise the multiplicities of colliding eigenvalues should be added together). Moreover, if all these linear combinations are distinct, then $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ is diagonalizable.*

[2] The formulation in [12, §11] uses the Frobenius characteristic map, but this has turned out to be a red herring.

The proof of this theorem will rely on the filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$ of $\mathbf{k}[S_n]$ introduced in [13, §8.1]. We call this the *Fibonacci filtration* of $\mathbf{k}[S_n]$, as its length $f_{n+1}$ is the $(n+1)$-st Fibonacci number. We note that this filtration is not completely canonical, as it depends on the choice of a listing $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ of all lacunar subsets of $[n-1]$ in the order of increasing sum of elements (the ties can be broken arbitrarily, whence the non-canonicity). Much about this filtration was said in [13], but we will need some additional information about the action of $S_n$ on its subquotients $F_i / F_{i-1}$:

Let $\mathcal{A}$ be the $\mathbf{k}$-algebra $\mathbf{k}[S_n]$, and let $\mathcal{T}$ be its $\mathbf{k}$-subalgebra generated by $t_1, t_2, \ldots, t_n$. Then, each $F_i$ is a left ideal of $\mathcal{A}$ but is also fixed under right multiplication by each $t_\ell$; therefore, each $F_i$ is an $(\mathcal{A}, \mathcal{T})$-subbimodule of $\mathcal{A}$. Thus, each subquotient $F_i / F_{i-1}$ of the Fibonacci filtration is an $(\mathcal{A}, \mathcal{T})$-bimodule. As a right $\mathcal{T}$-module, it is *scalar* (meaning that each $t_\ell$ acts on it by a scalar, which is in fact the integer $m_{Q_i,\ell}$ from [13, Theorem 8.1 **(c)**]). As a left $\mathcal{A}$-module (i.e., as a representation of $S_n$), we describe it explicitly here:

**Theorem 2** (part of Theorem 8)**.** *Let $i \in [f_{n+1}]$.*

*Consider the lacunar subset $Q_i$ of $[n-1]$ (from the above listing $Q_1, Q_2, \ldots, Q_{f_{n+1}}$). Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Note that $j_1 \geqslant 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = i_m - i_0 = n$.*

*For each $p \in \mathbb{N}$, we let $\mathcal{H}_p$ denote the trivial 1-dimensional representation of $S_p$ (that is, the $\mathbf{k}$-module $\mathbf{k}$ on which $S_p$ acts trivially), and we let $\mathcal{Z}_p$ denote the reflection quotient representation of $S_p$ (that is, the free $\mathbf{k}$-module $\mathbf{k}^p$ on which $S_p$ acts by permuting the coordinates, divided by the submodule consisting of all vectors of the form $(a, a, \ldots, a) \in \mathbf{k}^p$). Then,*

$$F_i / F_{i-1} \cong \operatorname{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \underbrace{\left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right)}_{\substack{\text{the first tensorand is an } \mathcal{H}, \\ \text{while all others are } \mathcal{Z}\text{'s}}}$$

*as $S_n$-representations. Here, we embed $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$ into $S_n$ by the usual parabolic embedding (since $j_1 + j_2 + \cdots + j_m = n$).*

This theorem will be a crucial stepping stone on our way to Theorem 1. Indeed, we will obtain Theorem 1 from it via a rather generalizable argument that requires little of the specifics of our situation; much of this argument will be abstracted in Proposition 26.

We note that neither of our two main results requires any assumption about the characteristic of $\mathbf{k}$. However, in positive characteristic, care must be taken to distinguish between the reflection quotient representation $\mathcal{Z}_p$ in Theorem 2 and the reflection subrepresentation $\mathcal{R}_p$ (which consists of the zero-sum vectors in $\mathbf{k}^p$); the two representations have the same dimension $p-1$ (for $p \geqslant 1$), but are not isomorphic unless $\operatorname{char} \mathbf{k} \neq p$ or $p \leqslant 2$.

We suspect that our results can be generalized ("$q$-deformed") from the symmetric group algebra to the Hecke algebra $\mathcal{H}_q(S_n)$. Most results from [13] can definitely be generalized this way, as will be detailed in forthcoming work.

The first somewhere-to-below shuffle $t_1$ is the illustrious *top-to-random shuffle* studied by Wallach [23, Appendix], Lusztig [17], Diaconis, Fill and Pitman [3], Reizenstein [19, Lemma 29] and several others, for at least three completely different purposes. In a sense, all the somewhere-to-below shuffles $t_\ell$ can be regarded as top-to-random shuffles, evaluated over the parabolic subgroups $S_{\{\ell,\ell+1,\ldots,n\}}$ of $S_n$. However, while each of these shuffles is well-understood, their interaction is nontrivial and novel. In particular, while the eigenvalues of $t_1$ are known since [3, Theorem 4.1][3], and even the eigenvalues on each Specht module can be recovered from the theory of the descent algebra [18, Proposition 4.2], we do not see a way to extend these classical methods to the linear combinations of the $t_\ell$; note that the latter combinations can have as many as $f_{n+1}$ distinct eigenvalues (see [13, Corollary 12.2]) and can fail to act diagonalizably (even over $\mathbb{Q}$), while each single $t_\ell$ is diagonalizable (over $\mathbb{Q}$) with at most $n - \ell + 1$ distinct eigenvalues.

**Note on versions** You are reading the journal version of the present paper. The arXiv version [10] includes more details and some proofs that were omitted from the journal version.

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## 2 Definitions and notations

### 2.1 Basics

We recall some notations from [13].

Let $\mathbf{k}$ be any commutative ring. (We don't require that $\mathbf{k}$ is a field or a $\mathbb{Q}$-algebra, but the reader can think of $\mathbf{k} = \mathbb{Q}$ as a standing example.)

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers.

For any integers $a$ and $b$, we set

$$[a, b] := \{k \in \mathbb{Z} \mid a \leqslant k \leqslant b\} = \{a, a+1, \ldots, b\}.$$

This is an empty set if $a > b$. In general, $[a, b]$ is called an *integer interval.*

For each $n \in \mathbb{Z}$, let $[n] := [1, n] = \{1, 2, \ldots, n\}$.

Fix an integer $n \in \mathbb{N}$. Let $S_n$ be the $n$-th symmetric group, i.e., the group of all permutations of $[n]$. We multiply permutations in the "continental" way: that is, $(\pi\sigma)(i) = \pi(\sigma(i))$ for all $\pi, \sigma \in S_n$ and $i \in [n]$.

For any $k$ distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\operatorname{cyc}_{i_1,i_2,\ldots,i_k}$ be the permutation in $S_n$ that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$, respectively while leaving all remaining

[3]Namely: these eigenvalues are the integers $i \in \{0, 1, \ldots, n\}$ with respective multiplicities equal to the number of permutations in $S_n$ having exactly $i$ fixed points.

elements of $[n]$ unchanged. This permutation is known as a *cycle*. Note that $\mathrm{cyc}_i = \mathrm{id}$ for any single $i \in [n]$.

For any $i \in [n-1]$, we denote the cycle $\mathrm{cyc}_{i,i+1}$ by $s_i$ and call it a *simple transposition*.

### 2.2 Somewhere-to-below shuffles, $\mathcal{A}$ and $\mathcal{T}$

Let $\mathcal{A}$ be the group algebra $\mathbf{k}[S_n]$. In this algebra, define $n$ elements $t_1, t_2, \ldots, t_n$ by setting[4]

$$t_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n]$$

for each $\ell \in [n]$. Thus, in particular, $t_n = \mathrm{cyc}_n = \mathrm{id} = 1$ (where 1 means the unity of $\mathbf{k}[S_n]$). The $n$ elements $t_1, t_2, \ldots, t_n$ are known as the *somewhere-to-below shuffles*.

We let $\mathcal{T}$ be the $\mathbf{k}$-subalgebra of $\mathcal{A}$ generated by these $n$ somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$. Clearly, $\mathcal{A}$ is an $(\mathcal{A}, \mathcal{T})$-bimodule (with $\mathcal{A}$ acting from the left by multiplication, and $\mathcal{T}$ acting from the right by multiplication).

### 2.3 Some $S_n$-representation theory

We recall that the representations of the symmetric group $S_n$ (over $\mathbf{k}$) are precisely the left $\mathbf{k}[S_n]$-modules, i.e., the left $\mathcal{A}$-modules. We will use the following four classes of $S_n$-representations in particular:

1. *The Specht modules $\mathcal{S}^\lambda$:* If $\lambda$ is any partition of $n$, then the *Specht module* $\mathcal{S}^\lambda$ is a representation of $S_n$ constructed using the Young diagram of shape $\lambda$. For its definition, see [9, Definition 5.4.1 **(b)**] (where it is called $\mathcal{S}^{Y(\lambda)}$) or [6, §7.2] (where it is called $S^\lambda$). If $\mathbf{k}$ is a field of characteristic 0, then the Specht module $\mathcal{S}^\lambda$ is irreducible.

2. *The trivial representation $\mathcal{H}_n$:* We let $\mathcal{H}_n$ denote the $\mathbf{k}$-module $\mathbf{k}$, equipped with a trivial $S_n$-action (that is, $\sigma \cdot v = v$ for all $\sigma \in S_n$ and $v \in \mathbf{k}$). This is called the *trivial representation* of $S_n$. It is isomorphic to the Specht module $\mathcal{S}^{(n)}$.

3. *The natural representation $\mathcal{N}_n$:* We let $\mathcal{N}_n$ denote the free $\mathbf{k}$-module $\mathbf{k}^n = \{(v_1, v_2, \ldots, v_n) \mid \text{all } v_i \in \mathbf{k}\}$, on which $S_n$ acts by permuting the coordinates:

$$\sigma \cdot (v_1, v_2, \ldots, v_n) = \left(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots, v_{\sigma^{-1}(n)}\right) \qquad \text{for all } \sigma \in S_n.$$

   This is called the *natural representation* of $S_n$.

4. *The reflection quotient representation $\mathcal{Z}_n$:* If $n > 0$, then the natural representation $\mathcal{N}_n$ has a 1-dimensional subrepresentation

$$\mathcal{D}_n := \{(v_1, v_2, \ldots, v_n) \mid \text{all } v_i \text{ are equal}\} = \{(a, a, \ldots, a) \mid a \in \mathbf{k}\}.$$

   The quotient

$$\mathcal{Z}_n := \mathcal{N}_n / \mathcal{D}_n$$

[4] We view $S_n$ as a subset of $\mathbf{k}[S_n]$ in the obvious way.

is thus another representation of $S_n$. This $\mathcal{Z}_n$ is called the *reflection quotient representation* of $S_n$. As a $\mathbf{k}$-module, it is free of rank $n-1$ (with basis $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_{n-1}})$, where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbf{k}^n$). Here and in the following, the notation $\overline{v}$ denotes the residue class of a vector $v$ modulo some submodule (the submodule is to be inferred from the context).

If $V$ is any $\mathbf{k}$-module, then $V^*$ shall denote its dual $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}}(V, \mathbf{k})$. If $V$ is an $S_n$-representation, then its dual $V^*$ becomes an $S_n$-representation as well (see [9, §5.19.3]). The following proposition situates $\mathcal{Z}_n$ in the world of Specht modules:

**Proposition 3.** *Let $n > 1$ be an integer. Then:*

***(a)*** *The reflection quotient representation $\mathcal{Z}_n$ is isomorphic (as an $S_n$-representation) to the dual $\left(\mathcal{S}^{(n-1,1)}\right)^*$ of the Specht module $\mathcal{S}^{(n-1,1)}$.*

***(b)*** *If $n$ is invertible in $\mathbf{k}$, then $\mathcal{Z}_n$ is isomorphic (as an $S_n$-representation) to the Specht module $\mathcal{S}^{(n-1,1)}$.*

This proposition is clearly part of the folklore, but finding a reference is surprisingly difficult. Thus, we have given a complete proof in the arXiv version of the present note [10, §A.1]. Here, we restrict ourselves to a hint:

*Hint to the proof of Proposition 3.* **(a)** Let $(e_1, e_2, \ldots, e_n)$ be the standard basis of the natural representation $\mathcal{N}_n = \mathbf{k}^n$. The definitions of Young modules and Specht modules (see [9, §5.3 and specifically Example 5.3.16 and Example 5.4.2]) allow us to identify the Young module $\mathcal{M}^{(n-1,1)}$ with the natural representation $\mathcal{N}_n = \mathbf{k}^n$ (where each basis vector $e_i$ of $\mathcal{N}_n$ corresponds to the unique $n$-tabloid of shape $(n-1, 1)$ that has the entry $i$ in its second row). The Specht module $\mathcal{S}^{(n-1,1)}$ then becomes its subrepresentation

$$\mathcal{R}_n := \operatorname{span}_{\mathbf{k}} \{e_k - e_\ell \mid k \neq \ell\} = \{(a_1, a_2, \ldots, a_n) \in \mathbf{k}^n \mid a_1 + a_2 + \cdots + a_n = 0\}.$$

Thus, we must show that $\mathcal{Z}_n \cong \mathcal{R}_n^*$ as $S_n$-representations. For this purpose, we define a $\mathbf{k}$-bilinear form $f : \mathcal{R}_n \times \mathcal{Z}_n \to \mathbf{k}$ by

$$\beta\left((a_1, a_2, \ldots, a_n), \overline{(b_1, b_2, \ldots, b_n)}\right) = \sum_{k=1}^{n} a_k b_k$$

(this is well-defined, since $(a_1, a_2, \ldots, a_n) \in \mathcal{R}_n$ entails $a_1 + a_2 + \cdots + a_n = 0$, and thus the sum $\sum_{k=1}^{n} a_k b_k$ is unchanged if we add a vector in $\mathcal{D}_n$ to $(b_1, b_2, \ldots, b_n)$). This form $f$ is clearly $S_n$-invariant. Now consider the $\mathbf{k}$-linear map

$$\begin{aligned} \beta_R : \mathcal{Z}_n &\to \mathcal{R}_n^*, \\ v &\mapsto \beta(?, v), \end{aligned}$$

where $\beta(?, v)$ denotes the linear map $\mathcal{R}_n \to \mathbf{k}$ sending each $w$ to $\beta(w, v)$. The $\mathbf{k}$-linear map $\beta_R$ is invertible (since it sends the basis vectors $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_{n-1}}$ of $\mathcal{Z}_n$ to the restrictions of the coordinate forms $e_1^*, e_2^*, \ldots, e_{n-1}^* \in \mathcal{N}_n^*$ to $\mathcal{R}_n$, but the latter restrictions form a basis of $\mathcal{R}_n^*$) and $S_n$-equivariant (since the form $\beta$ is $S_n$-invariant). Hence, this map is an isomorphism of $S_n$-representations, and we get $\mathcal{Z}_n \cong \mathcal{R}_n^*$, as desired.

**(b)** If $n$ is invertible in $\mathbf{k}$, then simple linear algebra shows that $\mathcal{N}_n = \mathcal{R}_n \oplus \mathcal{D}_n$ (where $\mathcal{R}_n$ is as in the proof of part **(a)**), so that $\mathcal{R}_n \cong \mathcal{N}_n / \mathcal{D}_n = \mathcal{Z}_n$. Hence, $\mathcal{Z}_n \cong \mathcal{R}_n \cong \mathcal{S}^{(n-1,1)}$ in this case. □

### 2.4 Tensor products, induction and induction products

We shall now discuss certain ways to produce new representations from old.

The symbol "$\otimes$" shall always mean a tensor product over $\mathbf{k}$, unless a different base ring is provided as a subscript.

It is well-known that if $A$ and $B$ are two $\mathbf{k}$-algebras, then the tensor product $U \otimes V$ of any left $A$-module $U$ and any left $B$-module $V$ is canonically a left $A \otimes B$-module. An analogous construction exists for tensor products of $k$ left modules. Thus, if $U$ is a representation of a group $G$, and if $V$ is a representation of a group $H$, then $U \otimes V$ is a representation of $G \times H$, and a similar fact holds for tensor products of $k$ representations.

We recall the notion of an induced representation: If $G$ is a group, and if $H$ is a subgroup of $G$, then any $H$-representation $V$ gives rise to a $G$-representation $\operatorname{Ind}_H^G V$ defined by

$$\operatorname{Ind}_H^G V = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V, \tag{1}$$

where we view $\mathbf{k}[G]$ as a $(\mathbf{k}[G], \mathbf{k}[H])$-bimodule while viewing $V$ as a left $\mathbf{k}[H]$-module (so that the tensor product over $\mathbf{k}[H]$ becomes a left $\mathbf{k}[G]$-module). This $G$-representation $\operatorname{Ind}_H^G V$ is called the *induced representation* of $V$ to $G$.

We furthermore recall the notion of an induction product ([6, §7.3]):

**Definition 4.** Let $n$ and $m$ be two nonnegative integers. Then, the direct product $S_n \times S_m$ can be canonically embedded as a subgroup into $S_{n+m}$, by the group morphism that sends each pair $(\sigma, \tau) \in S_n \times S_m$ to the permutation $\sigma * \tau \in S_{n+m}$ that applies $\sigma$ to the first $n$ elements while applying $\tau$ (appropriately shifted) to the last $m$ elements of $[n+m]$. (To be fully precise: $\sigma * \tau$ is the permutation of $[n+m]$ that sends $1, 2, \ldots, n$ to $\sigma(1), \sigma(2), \ldots, \sigma(n)$ while sending $n+1, n+2, \ldots, n+m$ to $n+\tau(1), n+\tau(2), \ldots, n+\tau(m)$.) This is called the *parabolic embedding* of $S_n \times S_m$ into $S_{n+m}$.

Now, if $U$ is an $S_n$-representation and if $V$ is an $S_m$-representation, then the tensor product $U \otimes V$ is an $S_n \times S_m$-representation, and thus (by the embedding of $S_n \times S_m$ into $S_{n+m}$ we just explained) we can construct the induced representation

$$U * V := \operatorname{Ind}_{S_n \times S_m}^{S_{n+m}} (U \otimes V)$$

of $S_{n+m}$. This induced representation $U * V$ is called the *induction product* of $U$ and $V$.

More generally, if $n_1, n_2, \ldots, n_k$ are any $k$ nonnegative integers, and if $U_i$ is an $S_{n_i}$-representation for each $i \in [k]$, then the *induction product* $U_1 * U_2 * \cdots * U_k$ is defined to be the $S_{n_1+n_2+\cdots+n_k}$-representation

$$U_1 * U_2 * \cdots * U_k := \operatorname{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}}^{S_{n_1+n_2+\cdots+n_k}} (U_1 \otimes U_2 \otimes \cdots \otimes U_k),$$

where we embed $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$ in the obvious way (having each $S_{n_i}$ act on an appropriate interval[5]). The latter embedding is again called the *parabolic embedding* of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$.

These induction products satisfy associativity up to isomorphism: e.g., we have isomorphisms $(U * V) * W \cong U * V * W \cong U * (V * W)$ for all $U, V, W$. More generally:

**Proposition 5.** *Let $n_1, n_2, \ldots, n_k$ be any $k$ nonnegative integers, and let $U_i$ be an $S_{n_i}$-representation for each $i \in [k]$. Let $i \in [0, k]$. Then,*

$$U_1 * U_2 * \cdots * U_k \cong (U_1 * U_2 * \cdots * U_i) * (U_{i+1} * U_{i+2} * \cdots * U_k) .$$

This is again a folklore result. We give a proof in the arXiv version of the present paper [10, §A.2].

### 2.5 Lacunar sets and the submodules $F(I)$

Next, we recall some more concepts from [13].

If $I$ is a finite set of integers, then we let $\operatorname{sum} I$ denote the sum of all elements of $I$. For instance, $\operatorname{sum}\{3, 7\} = 3 + 7 = 10$.

Let $(f_0, f_1, f_2, \ldots)$ be the *Fibonacci sequence*. This is the sequence of integers defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_m = f_{m-1} + f_{m-2} \text{ for all } m \geqslant 2.$$

We shall say that a set $I \subseteq \mathbb{Z}$ is *lacunar* if it contains no two consecutive integers (i.e., there exists no $i \in I$ such that $i + 1 \in I$). For instance, the set $\{1, 4, 6\}$ is lacunar, while the set $\{1, 4, 5\}$ is not.

The number of lacunar subsets of $[n-1]$ is the Fibonacci number $f_{n+1}$. Let $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ be all these $f_{n+1}$ lacunar subsets of $[n-1]$, listed in an order that satisfies

$$\operatorname{sum}(Q_1) \leqslant \operatorname{sum}(Q_2) \leqslant \cdots \leqslant \operatorname{sum}\left(Q_{f_{n+1}}\right) . \tag{2}$$

We fix this order once and for all[6]. Many of our constructions will formally (though rather shallowly) depend on this order.

For any subset $I$ of $[n]$, we define the following:

- We let $I - 1$ denote the set $\{i - 1 \mid i \in I\} = \{j \in \mathbb{Z} \mid j + 1 \in I\}$. For instance, $\{2, 4, 5\} - 1 = \{1, 3, 4\}$. Note that $I$ is lacunar if and only if $I \cap (I - 1) = \varnothing$.

[5] To make this precise: Let $m_i := n_1 + n_2 + \cdots + n_i$ for each $i \in [0, k]$. Then, the integer interval $[n_1 + n_2 + \cdots + n_k]$ is partitioned into the intervals $[m_{i-1} + 1,\ m_i]$ for all $i \in [k]$. The embedding of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$ sends each $k$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ to the permutation $\sigma_1 * \sigma_2 * \cdots * \sigma_k \in S_{n_1+n_2+\cdots+n_k}$ defined by

$$(\sigma_1 * \sigma_2 * \cdots * \sigma_k)(m_{i-1} + x) := m_{i-1} + \sigma_i(x) \qquad \text{for each } i \in [k] \text{ and each } x \in [n_i] .$$

[6] For $n \leqslant 3$, this order is uniquely defined. For $n > 3$, we need to make a choice.

- We let $I'$ be the set $[n-1] \setminus (I \cup (I-1))$. This is the set of all $i \in [n-1]$ satisfying $i \notin I$ and $i+1 \notin I$. We shall refer to $I'$ as the *non-shadow* of $I$.

  For example, if $n = 5$, then $\{2,3\}' = [4] \setminus \{1,2,3\} = \{4\}$.

- We let
$$F(I) := \{q \in \mathbf{k}[S_n] \mid qs_i = q \text{ for all } i \in I'\}.$$
  We can rewrite this equality as
$$\begin{aligned} F(I) &= \{t \in \mathbf{k}[S_n] \mid ts_j = t \text{ for all } j \in I'\} \\ &= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in I'\} \end{aligned} \tag{3}$$
  (since $\mathbf{k}[S_n] = \mathcal{A}$).

# 3 The first main theorem: the Fibonacci filtration

## 3.1 The theorem

For each $i \in [0, f_{n+1}]$, we define a $\mathbf{k}$-submodule
$$F_i := F(Q_1) + F(Q_2) + \cdots + F(Q_i) \qquad \text{of } \mathbf{k}[S_n]$$
(so that $F_0 = 0$). In [13, Theorem 8.1], the following is shown:

**Theorem 6.**

***(a)*** *We have*
$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n].$$
*In other words, the* $\mathbf{k}$*-submodules* $F_0, F_1, \ldots, F_{f_{n+1}}$ *form a* $\mathbf{k}$*-module filtration of* $\mathbf{k}[S_n]$.

***(b)*** *We have* $F_i \cdot t_\ell \subseteq F_i$ *for each* $i \in [0, f_{n+1}]$ *and* $\ell \in [n]$.

***(c)*** *For each* $i \in [f_{n+1}]$ *and* $\ell \in [n]$*, we have*
$$F_i \cdot (t_\ell - m_{Q_i,\ell}) \subseteq F_{i-1}.$$
*Here,* $m_{Q_i,\ell}$ *is a certain integer whose definition we will give in Subsection 4.1 (as we will not use it until then).*

The filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$ will be called the *Fibonacci filtration* of $\mathcal{A}$. We can easily see that it is a filtration of $(\mathcal{A}, \mathcal{T})$-bimodules:

**Proposition 7.** *Let* $i \in [0, f_{n+1}]$*. Then,* $F_i$ *is an* $(\mathcal{A}, \mathcal{T})$*-subbimodule of* $\mathcal{A}$.

*Proof.* For any $I \subseteq [n]$, the set $F(I)$ is closed under addition and left action of $\mathcal{A}$ (by its very definition), hence is a left $\mathcal{A}$-submodule of $\mathcal{A}$. Thus, $F_i$ (being defined as a sum of such sets $F(I)$) is also a left $\mathcal{A}$-submodule of $\mathcal{A}$. Moreover, $F_i$ is also closed under right multiplication by each $t_\ell$ (by Theorem 6 **(b)**), and hence under the right action of $\mathcal{T}$ (since $\mathcal{T}$ is the subalgebra generated by $t_1, t_2, \ldots, t_n$). Thus, $F_i$ is also a right $\mathcal{T}$-submodule of $\mathcal{A}$. Altogether, we conclude that $F_i$ is an $(\mathcal{A}, \mathcal{T})$-subbimodule of $\mathcal{A}$. □

Proposition 7 shows that the subquotients $F_i/F_{i-1}$ are $(\mathcal{A}, \mathcal{T})$-bimodules as well. In particular, they are therefore left $\mathcal{A}$-modules, i.e., representations of $S_n$. Our second main theorem characterizes these representations:

**Theorem 8.** *Let $i \in [f_{n+1}]$. Consider the lacunar subset $Q_i$ of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. (Note that $j_1 \geqslant 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = n$; this follows from Lemma 9 below (applied to $I = Q_i$).) Then,*

$$F_i/F_{i-1} \cong \underbrace{\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}}_{\substack{\text{the first factor is an } \mathcal{H}, \\ \text{while all others are } \mathcal{Z}\text{'s}}}$$

*as $S_n$-representations.*[7]

We will spend the rest of this section proving this theorem, then restating it (in the characteristic-0 case) using Littlewood–Richardson coefficients.

### 3.2 Lemmas on $F(Q_i)$

First, let us show some lemmas about lacunar sets $I$ and the corresponding $\mathbf{k}$-modules $F(I)$:

**Lemma 9.** *Let $I$ be a lacunar subset of $[n-1]$. Write the set $I \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Then, $j_1 \geqslant 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = n$.*

*Proof.* By definition, we have $j_1 = i_1 - i_0 \geqslant 0$, since $i_1 \geqslant 1 = i_0$.

Next, we recall that the set $I$ is lacunar. This lacunarity is preserved even when we insert the new element $n+1$ into this set, since all existing elements of $I$ are $\leqslant n-1$ (since $I \subseteq [n-1]$) and thus cannot be consecutive with $n+1$. That is, the set $I \cup \{n+1\}$ is again lacunar. Since we have written this set as $\{i_1 < i_2 < \cdots < i_m\}$, this yields that any $k \in [2, m]$ satisfies $i_k - i_{k-1} > 1$. In other words, any $k \in [2, m]$ satisfies $j_k > 1$ (since $j_k = i_k - i_{k-1}$). In other words, $j_2, j_3, \ldots, j_m > 1$.

It remains to prove that $j_1 + j_2 + \cdots + j_m = n$. But recall that $j_k = i_k - i_{k-1}$ for each $k \in [m]$. Hence,

$$\sum_{k=1}^{m} j_k = \sum_{k=1}^{m} (i_k - i_{k-1}) = \underbrace{i_m}_{=n+1} - \underbrace{i_0}_{=1} \qquad \text{(by the telescope principle)}$$
$$= n + 1 - 1 = n.$$

[7]Note that the factor $\mathcal{H}_{j_1}$ can be omitted when $j_1 = 0$, since $\mathcal{H}_0 \cong \mathbf{k}$ with the trivial $S_0$-action.

In other words, $j_1 + j_2 + \cdots + j_m = n$. Thus, Lemma 9 is fully proved. $\square$

**Lemma 10.** *Let $i \in [f_{n+1}]$. Consider the lacunar subset $Q_i$ of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Then, the* **k***-module $F_i / F_{i-1}$ is free of rank*

$$\frac{n!}{j_1! j_2! \cdots j_m!} \cdot \prod_{k=2}^{m} (j_k - 1) .$$

*Proof.* We have $Q_i \subseteq [n-1]$. Hence, $n+1$ is the largest element of $Q_i \cup \{n+1\}$. Thus, from $Q_i \cup \{n+1\} = \{i_1 < i_2 < \cdots < i_m\}$, we obtain

$$i_m = n+1 \qquad \text{and} \qquad Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\} .$$

Lemma 9 (applied to $I = Q_i$) shows that $j_1 + j_2 + \cdots + j_m = n$. Let $\binom{n}{j_1, j_2, \ldots, j_m}$ denote the multinomial coefficient $\frac{n!}{j_1! j_2! \cdots j_m!}$. We know from [13, Theorem 13.1 **(a)** and **(c)**] (applied to $p = m-1$) that the **k**-module $F_i / F_{i-1}$ is free of rank

$$\delta_i = \binom{n}{j_1, j_2, \ldots, j_m} \cdot \prod_{k=2}^{m} (j_k - 1) .$$

In view of $\binom{n}{j_1, j_2, \ldots, j_m} = \frac{n!}{j_1! j_2! \cdots j_m!}$, this is precisely the claim of Lemma 10. $\square$

**Lemma 11.** *Let $I$ be a subset of $[n]$. Let $j \in I$. Then, there exists a lacunar subset $J$ of $[n-1]$ such that* $\operatorname{sum} J < \operatorname{sum} I$ *and $J' \subseteq I' \cup \{j\}$.*

*Proof.* Set $K := (I \setminus \{j\}) \cup \{j-1\}$ if $j > 1$, and otherwise set $K := I \setminus \{j\}$. Then, $K$ is a subset of $[n]$ and satisfies $\operatorname{sum} K < \operatorname{sum} I$ (since $K$ is obtained from $I$ by removing the element $j$ and possibly inserting the smaller element $j-1$). Furthermore, [13, Proposition 8.6 **(a)**] says that $K' \subseteq I' \cup \{j\}$.

Now, [13, Corollary 8.8] (applied to $K$ instead of $I$) shows that there exists a lacunar subset $J$ of $[n-1]$ such that $\operatorname{sum} J \leqslant \operatorname{sum} K$ and $J' \subseteq K'$. Consider this $J$.

The set $J$ is a lacunar subset of $[n-1]$ and satisfies $\operatorname{sum} J < \operatorname{sum} I$ (since $\operatorname{sum} J \leqslant \operatorname{sum} K < \operatorname{sum} I$) and $J' \subseteq I' \cup \{j\}$ (since $J' \subseteq K' \subseteq I' \cup \{j\}$). Hence, such a $J$ exists. This proves Lemma 11. $\square$

**Lemma 12.** *Let $i \in [f_{n+1}]$. Consider the lacunar subset $Q_i$ of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Let $k \in [m-1]$. Then,*

$$\{t \in F(Q_i) \ \mid \ t s_{i_k} = t\} \subseteq F_{i-1}.$$

*Proof.* As in the proof of Lemma 10, we find $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$. Thus, $i_k \in Q_i$ (since $k \in [m-1]$). Hence, Lemma 11 (applied to $I = Q_i$ and $j = i_k$) shows that there exists a lacunar subset $J$ of $[n-1]$ such that $\operatorname{sum} J < \operatorname{sum}(Q_i)$ and $J' \subseteq Q_i' \cup \{i_k\}$. Consider this $J$. Since $J$ is lacunar, we have $J = Q_s$ for some $s \in [f_{n+1}]$. Consider this $s$. Thus, $Q_s = J$, so that $\operatorname{sum}(Q_s) = \operatorname{sum} J < \operatorname{sum}(Q_i)$ and therefore $s < i$ (by (2)). Hence, $s \leqslant i-1$, so that $F(Q_s) \subseteq F_{i-1}$ (since the definition of $F_{i-1}$ says that $F_{i-1} = F(Q_1) + F(Q_2) + \cdots + F(Q_{i-1})$). In view of $Q_s = J$, we can rewrite this as

$$F(J) \subseteq F_{i-1}.$$

Now, (3) (applied to $Q_i$ instead of $I$) shows that

$$F(Q_i) = \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i'\},$$

so that

$$\begin{aligned}
&\{t \in F(Q_i) \mid ts_{i_k} = t\} \\
&= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i', \text{ and also } ts_{i_k} = t\} \\
&= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i' \cup \{i_k\}\} \\
&\subseteq \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in J'\} \qquad (\text{since } J' \subseteq Q_i' \cup \{i_k\}) \\
&= F(J) \qquad (\text{by (3), applied to } J \text{ instead of } I) \\
&\subseteq F_{i-1}.
\end{aligned}$$

Thus, Lemma 12 follows. □

### 3.3 The elements $\nabla_{\mathbf{p}}$

**Lemma 13.** *Let $i \in [f_{n+1}]$. Consider the lacunar subset $Q_i$ of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Furthermore, set $i_0 := 1$.*

*For each $k \in [m]$, let $J_k$ denote the integer interval $[i_{k-1},\ i_k - 1]$. Note that the intervals $J_1, J_2, \ldots, J_m$ are disjoint and – except possibly for $J_1$ – nonempty ($J_1$ is empty if and only if $1 \in Q_i$), and their union is $[n]$. Thus, we can view the direct product $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ as a subgroup of $S_n$ in the obvious way (each factor $S_{J_k}$ acts on the elements of $J_k$ while leaving all remaining elements of $[n]$ unchanged).*

*For each $(m-1)$-tuple $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$ (that is, with $p_k \in J_k$ for each $k \in [2, m]$), we define an element*

$$\nabla_{\mathbf{p}} := \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m]}} \sigma \in \mathcal{A}.$$

*(Note that this also depends on $i$, not just on $\mathbf{p}$.)*

*Then:*

***(a)*** *For any* $\tau = (\tau_1, \tau_2, \dots, \tau_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ *and* $\mathbf{p} = (p_2, p_3, \dots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$*, we have*

$$\tau \nabla_{\mathbf{p}} = \nabla_{\tau \mathbf{p}},$$

*where*

$$\tau \mathbf{p} := (\tau_2 (p_2), \tau_3 (p_3), \dots, \tau_m (p_m)) = (\tau (p_2), \tau (p_3), \dots, \tau (p_m)).$$

***(b)*** *The left* $\mathcal{A}$*-module* $F(Q_i)$ *is generated (as a left* $\mathcal{A}$*-module) by any single element of the form* $\nabla_{\mathbf{p}}$ *(with* $\mathbf{p} \in J_2 \times J_3 \times \cdots \times J_m$*).*

***(c)*** *Let* $\ell \in [2, m]$*. For each* $k \in [2, m] \setminus \{\ell\}$*, let* $p_k \in J_k$ *be an element. Then,*

$$\sum_{p_\ell \in J_\ell} \nabla_{(p_2, p_3, \dots, p_m)} \in F_{i-1}.$$

*(Note that the elements* $p_2, p_3, \dots, p_{\ell-1}, p_{\ell+1}, p_{\ell+2}, \dots, p_m$ *in this sum are fixed, whereas* $p_\ell$ *runs through the set* $J_\ell$*.)*

**Example 14.** Let $n = 7$ and $Q_i = \{3, 6\}$ (clearly a lacunar subset of $[n-1] = [6]$). Then, following the notations of Lemma 13, we have $\{i_1 < i_2 < \cdots < i_m\} = Q_i \cup \{n+1\} = \{3, 6, 8\}$, so that $i_1 = 3$ and $i_2 = 6$ and $i_3 = 8$ and $i_0 = 1$. Thus, the integer intervals $J_k = [i_{k-1},\ i_k - 1]$ are

$$J_1 = [1, 2], \qquad J_2 = [3, 5], \qquad J_3 = [6, 7].$$

Taking $\mathbf{p}$ to be the 2-tuple $(p_2, p_3) = (4, 7) \in J_2 \times J_3$, we then have

$$\begin{aligned} \nabla_{\mathbf{p}} &= \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m]}} \sigma = \sum_{\substack{\sigma \in S_n; \\ \sigma([1,2]) = [1,2];\ \sigma([3,5]) = [3,5];\ \sigma([6,7]) = [6,7]; \\ \sigma(3) = 4;\ \sigma(6) = 7}} \sigma \\ &= [1243576] + [1245376] + [2143576] + [2145376] \end{aligned}$$

(writing permutations in one-line notation). Lemma 13 **(b)** says that this element $\nabla_{\mathbf{p}}$ generates the left $\mathcal{A}$-module $F(Q_i)$. Applying Lemma 13 **(c)** to $\ell = 2$ and $p_3 = 7$, we obtain

$$\sum_{p_2 \in J_2} \nabla_{(p_2, 7)} \in F_{i-1}, \qquad \text{that is,} \qquad \nabla_{(3,7)} + \nabla_{(4,7)} + \nabla_{(5,7)} \in F_{i-1}.$$

*Proof of Lemma 13.* As in the proof of Lemma 10, we find $i_m = n + 1$ and $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$.

**(a)** Let $\tau = (\tau_1, \tau_2, \dots, \tau_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ and $\mathbf{p} = (p_2, p_3, \dots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. Recall that

$$\nabla_{\mathbf{p}} = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m]}} \sigma.$$

Multiplying this equality by $\tau$ from the left, we obtain

$$\begin{aligned}
\tau\nabla_{\mathbf{p}} &= \sum_{\substack{\sigma\in S_n;\\ \sigma(J_k)=J_k \text{ for each } k\in[m];\\ \sigma(i_{k-1})=p_k \text{ for each } k\in[2,m]}} \tau\sigma = \sum_{\substack{\sigma\in S_n;\\ \tau(\sigma(J_k))=\tau(J_k) \text{ for each } k\in[m];\\ \tau(\sigma(i_{k-1}))=\tau(p_k) \text{ for each } k\in[2,m]}} \tau\sigma \\
&\qquad \left(\begin{array}{c}\text{here, we have replaced the conditions "}\sigma\left(J_k\right)=J_k\text{"}\\ \text{and "}\sigma\left(i_{k-1}\right)=p_k\text{" under the summation sign}\\ \text{by the conditions "}\tau\left(\sigma\left(J_k\right)\right)=\tau\left(J_k\right)\text{"}\\ \text{and "}\tau\left(\sigma\left(i_{k-1}\right)\right)=\tau\left(p_k\right)\text{" (which are equivalent to}\\ \text{the former two conditions because } \tau \text{ is injective)}\end{array}\right) \\
&= \sum_{\substack{\sigma\in S_n;\\ (\tau\sigma)(J_k)=J_k \text{ for each } k\in[m];\\ (\tau\sigma)(i_{k-1})=\tau(p_k) \text{ for each } k\in[2,m]}} \tau\sigma \\
&\qquad \left(\begin{array}{c}\text{since each } k\in[m] \text{ satisfies } \tau\left(\sigma\left(J_k\right)\right)=\left(\tau\sigma\right)\left(J_k\right)\\ \text{and } \tau\left(\sigma\left(i_{k-1}\right)\right)=\left(\tau\sigma\right)\left(i_{k-1}\right) \text{ (if } k>1\text{)}\\ \text{and } \tau\left(J_k\right)=J_k \text{ (since } \tau\in S_{J_1}\times S_{J_2}\times\cdots\times S_{J_m}\text{)}\end{array}\right) \\
&= \sum_{\substack{\sigma\in S_n;\\ \sigma(J_k)=J_k \text{ for each } k\in[m];\\ \sigma(i_{k-1})=\tau(p_k) \text{ for each } k\in[2,m]}} \sigma \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \sigma \text{ for } \tau\sigma \text{ in the sum,}\\ \text{since the map } S_n\to S_n,\ \sigma\mapsto\tau\sigma \text{ is a bijection}\end{array}\right) \\
&= \nabla_{\tau\mathbf{p}}
\end{aligned}$$

(by the definition of $\nabla_{\tau\mathbf{p}}$, since $\tau\mathbf{p} = \left(\tau\left(p_2\right), \tau\left(p_3\right), \ldots, \tau\left(p_m\right)\right)$). This proves Lemma 13 **(a)**.

**(b)** The definition of the non-shadow $Q_i'$ yields

$$\begin{aligned}
Q_i' &= [n-1] \setminus \left(Q_i \cup \left(Q_i - 1\right)\right) \\
&= [n-1] \setminus \left(\{i_1 < i_2 < \cdots < i_{m-1}\} \cup \{i_1 - 1 < i_2 - 1 < \cdots < i_{m-1} - 1\}\right)
\end{aligned}$$

(since $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$). In other words, $Q_i'$ consists of all elements of $[n-1]$ except for those of the forms $i_k - 1$ and $i_k$ for $k \in [m-1]$.

Let $\Gamma$ be the subgroup of $S_n$ generated by the simple transpositions $s_j$ with $j \in Q_i'$. Thus, $\Gamma$ is generated by all simple transpositions $s_1, s_2, \ldots, s_{n-1}$ except for those of the forms $s_{i_k-1}$ and $s_{i_k}$ for $k \in [m-1]$ (by the description of $Q_i'$ in the previous paragraph). Hence, every permutation $\omega \in \Gamma$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$.

Conversely, if some permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$, then $\omega$ must belong to $\Gamma$ (because such a permutation $\omega$ must preserve the intervals $J_1 = [i_0,\ i_1 - 1]$ as well as $J_k \setminus \{i_{k-1}\} = [i_{k-1} + 1,\ i_k - 1]$ for all $k \in [2, m]$ (since it preserves both $J_k$ and $i_{k-1}$) as well as the length-1 intervals $\{i_{k-1}\}$

for all $k \in [2, m]$, and thus must be a composition of permutations of these intervals; but any such permutation belongs to $\Gamma$ (since any permutation of an integer interval $[a, b]$ can be written as a product of simple transpositions $s_j$ with $j \in [a, b-1]$)).

The subgroup $\Gamma$ of $S_n$ acts from the right on $S_n$ (simply by right multiplication), and thus also acts $\mathbf{k}$-linearly from the right on $\mathcal{A} = \mathbf{k}[S_n]$ (by linear extension), making $\mathcal{A}$ into a permutation module[8] of $\Gamma$. Applying (3) to $I = Q_i$, we see that

$$\begin{aligned} F(Q_i) &= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i'\} \\ &= \{t \in \mathcal{A} \mid t\omega = t \text{ for all } \omega \in \Gamma\} \end{aligned}$$

(since $\Gamma$ is the group generated by the $s_j$ with $j \in Q_i'$, and therefore the condition "$ts_j = t$ for all $j \in Q_i'$" is equivalent to "$t\omega = t$ for all $\omega \in \Gamma$"). Thus, $F(Q_i)$ is the space of fixed points[9] of the right $\Gamma$-action on $\mathcal{A}$.

However, we know from the basic theory of group actions (see, e.g., [16, §3.3.1, "Invariants of Permutation Representations"] or [14, Proposition A.2]) that when a finite group $G$ acts on a set $X$, the space of fixed points of the corresponding permutation module is spanned by the orbit sums[10]. Hence, the $\mathbf{k}$-module $F(Q_i)$ is spanned by the orbit sums of the right $\Gamma$-action on $S_n$ (since $F(Q_i)$ is the set of fixed points of the right $\Gamma$-action on $\mathcal{A}$, which is the permutation module corresponding to the right $\Gamma$-action on $S_n$). In other words, $F(Q_i)$ is spanned by the orbit sums $\sum_{\sigma \in \tau\Gamma} \sigma$ for $\tau \in S_n$ (since each orbit of the right $\Gamma$-action on $S_n$ has the form $\tau\Gamma$ for some $\tau \in S_n$). As a left $\mathcal{A}$-module, $F(Q_i)$ is therefore generated by any **one** of these orbit sums (since any two orbit sums $\sum_{\sigma \in \tau_1\Gamma} \sigma$ and $\sum_{\sigma \in \tau_2\Gamma} \sigma$ can be transformed into each other by left multiplication by $\tau_1\tau_2^{-1} \in S_n \subseteq \mathcal{A}$, and therefore each of them generates the other).

Now, let $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. We shall now show that $\nabla_{\mathbf{p}}$ is one of these orbit sums we just mentioned. Indeed, let $\Omega_{\mathbf{p}}$ be the set of all permutations

[8] Recall the definition of a permutation module:

Let $G$ be a finite group. Let $X$ be a right $G$-set. Let $\mathbf{k}^{(X)}$ be the free $\mathbf{k}$-module with basis $X$. Then, $\mathbf{k}^{(X)}$ becomes a right $\mathbf{k}[G]$-module, where the action of $\mathbf{k}[G]$ on $\mathbf{k}^{(X)}$ is given by bilinearly extending the action of $G$ on $X$ (that is, by the rule $\left(\sum_{g \in G} \alpha_g g\right)\left(\sum_{x \in X} \beta_x x\right) := \sum_{g \in G} \sum_{x \in X} \alpha_g \beta_x gx$). This is called the *permutation module* corresponding to the right $G$-set $X$.

In our present setup, we apply this construction to $G = \Gamma$ and $X = S_n$.

[9] Recall the definition of a space of fixed points: If a $\mathbf{k}$-module $V$ is equipped with a linear right action of a group $G$ (that is, if $V$ is a right $\mathbf{k}[G]$-module), then its *space of fixed points* is defined to be the set $\{a \in V \mid ag = a \text{ for all } g \in G\}$. This is a $\mathbf{k}$-submodule of $V$.

[10] In more details:

Let $G$ be a finite group. Let $X$ be a right $G$-set. Consider the corresponding permutation module $\mathbf{k}^{(X)}$, with its right $G$-action.

For each $G$-orbit $\mathcal{O}$ on $X$, we define the *orbit sum* $z_{\mathcal{O}} := \sum_{x \in \mathcal{O}} x \in \mathbf{k}^{(X)}$. Now, the known fact that we are citing here is saying that these orbit sums $z_{\mathcal{O}}$ (as $\mathcal{O}$ ranges over all $G$-orbits on $X$) form a basis of the space of fixed points of $\mathbf{k}^{(X)}$ (as a $\mathbf{k}$-module).

In [16, §3.3.1, "Invariants of Permutation Representations"] and in [14, Proposition A.2], this is stated for left $G$-actions, but the case of right $G$-actions is analogous.

$\sigma \in S_n$ that satisfy "$\sigma(J_k) = J_k$ for each $k \in [m]$" and "$\sigma(i_{k-1}) = p_k$ for each $k \in [2, m]$". Then, the definition of $\nabla_{\mathbf{p}}$ can be rewritten as

$$\nabla_{\mathbf{p}} = \sum_{\sigma \in \Omega_{\mathbf{p}}} \sigma. \tag{4}$$

We shall now show that $\Omega_{\mathbf{p}}$ is an orbit of the right $\Gamma$-action on $S_n$ (that is, a left coset of $\Gamma$ in $S_n$).

First, we show that the set $\Omega_{\mathbf{p}}$ is nonempty. Indeed, it is easy to construct some permutation $\tau \in \Omega_{\mathbf{p}}$: Namely, we pick a permutation $\tau_1 \in S_{J_1}$ arbitrarily. Furthermore, for each $k \in [2, m]$, we pick a permutation $\tau_k \in S_{J_k}$ that sends $i_{k-1} \in J_k$ to $p_k \in J_k$. The $m$-tuple $(\tau_1, \tau_2, \ldots, \tau_m)$ then belongs to $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ and – viewed as an element of $S_n$ via the embedding $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m} \to S_n$ – belongs to $\Omega_{\mathbf{p}}$.

Hence, $\Omega_{\mathbf{p}}$ is nonempty. Pick any $\tau \in \Omega_{\mathbf{p}}$. Then, $\tau(J_k) = J_k$ for each $k \in [m]$, and $\tau(i_{k-1}) = p_k$ for each $k \in [2, m]$. Moreover, these equalities remain valid if we replace $\tau$ by $\tau\omega$ for any $\omega \in \Gamma$ (because every permutation $\omega \in \Gamma$ preserves the sets $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$). Thus, for each $\omega \in \Gamma$, we have $\tau\omega \in \Omega_{\mathbf{p}}$ as well. In other words, $\tau\Gamma \subseteq \Omega_{\mathbf{p}}$.

Conversely, we claim that $\Omega_{\mathbf{p}} \subseteq \tau\Gamma$. Indeed, let $\sigma \in \Omega_{\mathbf{p}}$ be arbitrary. Then, each $k \in [m]$ satisfies $\sigma(J_k) = J_k = \tau(J_k)$, whereas each $k \in [2, m]$ satisfies $\sigma(i_{k-1}) = p_k = \tau(i_{k-1})$. Set $\omega = \tau^{-1}\sigma \in S_n$; thus, each $k \in [m]$ satisfies $\omega(J_k) = \tau^{-1}(\sigma(J_k)) = J_k$ (since we just saw that $\sigma(J_k) = \tau(J_k)$), and each $k \in [2, m]$ satisfies $\omega(i_{k-1}) = \tau^{-1}(\sigma(i_{k-1})) = i_{k-1}$ (since we just saw that $\sigma(i_{k-1}) = \tau(i_{k-1})$). Thus, the permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$. Hence, $\omega \in \Gamma$ (because if some permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$, then $\omega$ must belong to $\Gamma$). Now, from $\omega = \tau^{-1}\sigma$, we obtain $\sigma = \tau\omega \in \tau\Gamma$ (since $\omega \in \Gamma$). Forget that we fixed $\sigma$. We thus have proved that $\sigma \in \tau\Gamma$ for each $\sigma \in \Omega_{\mathbf{p}}$. In other words, $\Omega_{\mathbf{p}} \subseteq \tau\Gamma$.

Combining this with $\tau\Gamma \subseteq \Omega_{\mathbf{p}}$, we obtain $\Omega_{\mathbf{p}} = \tau\Gamma$. Hence, $\Omega_{\mathbf{p}}$ is an orbit of the right $\Gamma$-action on $S_n$. Thus, $\sum_{\sigma \in \Omega_{\mathbf{p}}} \sigma$ is an orbit sum of this action. In view of (4), this means that $\nabla_{\mathbf{p}}$ is an orbit sum of this action. Hence, as a left $\mathcal{A}$-module, $F(Q_i)$ is generated by $\nabla_{\mathbf{p}}$ (since we have shown that $F(Q_i)$ is generated by any **one** of the orbit sums). This proves Lemma 13 **(b)**.

**(c)** Let

$$\vartheta := \sum_{p_\ell \in J_\ell} \nabla_{(p_2, p_3, \ldots, p_m)}. \tag{5}$$

We then must show that $\vartheta \in F_{i-1}$.

We have

$$\vartheta = \sum_{p_\ell \in J_\ell} \underbrace{\nabla_{(p_2, p_3, \ldots, p_m)}}_{\substack{\in F(Q_i) \\ \text{(since Lemma 13 (b) shows that } \nabla_{\mathbf{q}} \in F(Q_i) \\ \text{for any } \mathbf{q} \in J_2 \times J_3 \times \cdots \times J_m)}} \in F(Q_i)$$

(since $F(Q_i)$ is a $\mathbf{k}$-module). On the other hand,

$$\begin{aligned}\vartheta &= \sum_{p_\ell \in J_\ell} \nabla_{(p_2,p_3,\ldots,p_m)} \\ &= \sum_{p_\ell \in J_\ell} \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k\in[m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k\in[2,m]}} \sigma \\ &\qquad \left(\text{by the definition of } \nabla_{(p_2,p_3,\ldots,p_m)}\right) \\ &= \sum_{p_\ell \in J_\ell} \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k\in[m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k\in[2,m]\setminus\{\ell\}; \\ \sigma(i_{\ell-1})=p_\ell}} \sigma\end{aligned}$$

(here, we have split up the condition "$\sigma(i_{k-1}) = p_k$ for each $k \in [2,m]$" under the second summation sign into two: one for $k \neq \ell$ and one for $k = \ell$). We can rewrite this further as

$$\vartheta = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k\in[m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k\in[2,m]\setminus\{\ell\}; \\ \sigma(i_{\ell-1})\in J_\ell}} \sigma \tag{6}$$

(here, we have subsumed the two summation signs into one by removing the variable $p_\ell$). The condition "$\sigma(i_{\ell-1}) \in J_\ell$" under the summation sign in (6) is redundant, since it follows from the condition "$\sigma(J_k) = J_k$ for each $k \in [m]$" (indeed, the latter condition implies that $\sigma(J_\ell) = J_\ell$ and therefore $\sigma\left(\underbrace{i_{\ell-1}}_{\in J_\ell}\right) \in \sigma(J_\ell) = J_\ell$). Hence, we can remove this condition. Thus, (6) rewrites as

$$\vartheta = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k\in[m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k\in[2,m]\setminus\{\ell\}}} \sigma. \tag{7}$$

However, the two conditions "$\sigma(J_k) = J_k$ for each $k \in [m]$" and "$\sigma(i_{k-1}) = p_k$ for each $k \in [2,m] \setminus \{\ell\}$" under the summation sign in (7) remain unchanged if we replace $\sigma$ by $\sigma s_{i_{\ell-1}}$ (since this replacement merely swaps the values of $\sigma$ on $i_{\ell-1}$ and $i_{\ell-1}+1$, but this does not break any of the two conditions[11]). Hence, the set of the permutations $\sigma$ over which we sum in (7) is fixed under right multiplication by $s_{i_{\ell-1}}$. Therefore, the whole sum is fixed under right multiplication by $s_{i_{\ell-1}}$. Because of (7), this shows that $\vartheta s_{i_{\ell-1}} = \vartheta$. Combining this with $\vartheta \in F(Q_i)$, we obtain

$$\vartheta \in \left\{t \in F(Q_i) \ \mid\ t s_{i_{\ell-1}} = t\right\} \subseteq F_{i-1}$$

(by Lemma 12, applied to $k = \ell - 1$). This proves Lemma 13 **(c)**. $\square$

[11] Here we use the fact that the two elements $i_{\ell-1}$ and $i_{\ell-1}+1$ lie in the same $J_k$ (namely, in $J_\ell = [i_{\ell-1},\ i_\ell - 1]$). This is because $Q_i$ is lacunar, so that $i_{\ell-1} < i_\ell - 1$.

### 3.4 Linear algebra lemmas

We shall furthermore use two facts from linear algebra over any commutative ring $\mathbf{k}$:

**Lemma 15.** *Let $s \in \mathbb{N}$. Let $M$ and $N$ be two free $\mathbf{k}$-modules of rank $s$. Then, any surjective $\mathbf{k}$-linear map $\rho : M \to N$ is an isomorphism.*

*Proof.* The $\mathbf{k}$-modules $M$ and $N$ are isomorphic (being free of the same rank). Thus, we can WLOG assume that $M = N$. Then, the lemma is saying that any surjective endomorphism of a free $\mathbf{k}$-module of finite rank is an isomorphism. But this is a well-known result – one of the simplest forms of Orzech's theorem – and proofs can be found, e.g., in [15, Exercise 2.5.18 (a)], or [7, Corollary 0.2]. Alternatively, a self-contained proof of Lemma 15 is given in the arXiv version of the present paper [10, Lemma 3.9]. □

**Lemma 16.** *Let $V_1, V_2, \ldots, V_m$ be any $\mathbf{k}$-modules. For each $\ell \in [m]$, let $W_\ell$ be a $\mathbf{k}$-submodule of $V_\ell$. For each $\ell \in [m]$, we consider the $\mathbf{k}$-submodule*

$$\underbrace{V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}} \quad \text{of } V_1 \otimes V_2 \otimes \cdots \otimes V_m.$$

*Then, there is a canonical $\mathbf{k}$-module isomorphism*

$$\left(V_1 \otimes V_2 \otimes \cdots \otimes V_m\right) \Big/ \sum_{\ell=1}^{m} \left(V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m\right)$$
$$\cong \left(V_1 / W_1\right) \otimes \left(V_2 / W_2\right) \otimes \cdots \otimes \left(V_m / W_m\right).$$

*Proof.* We construct both the isomorphism and its inverse using the universal properties of tensor products and quotients:

- There is a canonical $\mathbf{k}$-linear map

  $$\Phi : V_1 \otimes V_2 \otimes \cdots \otimes V_m \to \left(V_1 / W_1\right) \otimes \left(V_2 / W_2\right) \otimes \cdots \otimes \left(V_m / W_m\right),$$

  sending each pure tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ to $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$. This $\mathbf{k}$-linear map $\Phi$ is easily seen to vanish on the submodule $\sum_{\ell=1}^{m} \left(V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m\right)$, and thus factors through the quotient module. Hence, we obtain a $\mathbf{k}$-linear map

  $$\overline{\Phi} : \left(V_1 \otimes V_2 \otimes \cdots \otimes V_m\right) \Big/ \sum_{\ell=1}^{m} \left(V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m\right)$$
  $$\to \left(V_1 / W_1\right) \otimes \left(V_2 / W_2\right) \otimes \cdots \otimes \left(V_m / W_m\right)$$

  sending each $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$ to $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$.

- Conversely, there is a canonical $\mathbf{k}$-linear map

$$\begin{aligned}\Psi : (V_1/W_1) \otimes (V_2/W_2) \otimes \cdots \otimes (V_m/W_m) \\ \to (V_1 \otimes V_2 \otimes \cdots \otimes V_m) \diagup \sum_{\ell=1}^{m} (V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)\end{aligned}$$

  sending each $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$ to $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$. To show that this map is well-defined, we need to check that $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$ depends only on the residue classes $\overline{v_i}$ rather than on the $v_i$ themselves (this is easy: replacing $v_i$ by $v_i'$ with $v_i - v_i' \in W_i$ only changes $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ by an element of $V_1 \otimes V_2 \otimes \cdots \otimes W_i \otimes \cdots \otimes V_m$) and that this dependence is multilinear (this is again easy).

Clearly, the maps $\overline{\Phi}$ and $\Psi$ are mutually inverse, hence isomorphisms. Thus, Lemma 16 is proved. □

For our specific needs, we specialize Lemma 16 to the case $W_1 = 0$:

**Lemma 17.** *Let $V_1, V_2, \ldots, V_m$ be any $\mathbf{k}$-modules with $m \geqslant 1$. For each $\ell \in [2, m]$, let $W_\ell$ be a $\mathbf{k}$-submodule of $V_\ell$. For each $\ell \in [2, m]$, we consider the $\mathbf{k}$-submodule*

$$\underbrace{V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}} \quad \text{of } V_1 \otimes V_2 \otimes \cdots \otimes V_m.$$

*Then, there is a canonical $\mathbf{k}$-module isomorphism*

$$\begin{aligned}&(V_1 \otimes V_2 \otimes \cdots \otimes V_m) \diagup \sum_{\ell=2}^{m} (V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m) \\ &\cong V_1 \otimes (V_2/W_2) \otimes (V_3/W_3) \otimes \cdots \otimes (V_m/W_m).\end{aligned}$$

*Proof.* Apply Lemma 16 to $W_1 = 0$, and observe that $V_1/0 \cong V_1$. □

### 3.5 Proof of Theorem 8

We can now prove Theorem 8:

*Proof of Theorem 8.* We shall use the notations of Lemma 13. Note that each $k \in [m]$ satisfies $J_k = [i_{k-1},\ i_k - 1]$ and thus

$$|J_k| = i_k - i_{k-1} = j_k. \tag{8}$$

Explicitly, there is a bijection

$$\begin{aligned}[j_k] &\to J_k, \\ x &\mapsto i_{k-1} - 1 + x\end{aligned} \tag{9}$$

for each $k \in [m]$.

Consider the tensor product $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$. We recall that the trivial representation $\mathcal{H}_{j_1} = \mathbf{k}$ has a 1-element basis $(1)$, while each natural representation $\mathcal{N}_{j_k}$ has basis $(e_p)_{p \in [j_k]} = (e_1, e_2, \ldots, e_{j_k})$. However, by abuse of notation, we shall rename the latter basis of $\mathcal{N}_{j_k}$ as $(e_p)_{p \in J_k} = \left(e_{i_{k-1}}, e_{i_{k-1}+1}, \ldots, e_{i_k - 1}\right)$ instead (by shifting all subscripts up by $i_{k-1} - 1$, that is, renaming each basis vector $e_x$ as $e_{i_{k-1}-1+x}$). Note that this can be done because $j_k = i_k - i_{k-1}$.

Having renamed the basis vectors of the $\mathbf{k}$-module $\mathcal{N}_{j_k}$, let us also replace the symmetric group $S_{j_k}$ acting on this module accordingly. Namely, we reinterpret the symmetric group $S_{j_k}$ acting on $\mathcal{N}_{j_k}$ as the symmetric group $S_{J_k}$ using the bijection (9) between the corresponding sets $[j_k]$ and $J_k$. Thus, the left action of $S_{j_k}$ on $\mathcal{N}_{j_k}$ becomes a left action of $S_{J_k}$ instead; it is still a permutation action (given on our now-renamed basis by the formula $\sigma e_p = e_{\sigma(p)}$ for each $p \in J_k$ and $\sigma \in S_{J_k}$). With these reinterpretations, the parabolic embedding $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m} \to S_n$ becomes the usual embedding $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m} \to S_n$, which simply combines the $m$ permutations without any need for shifting (i.e., any $m$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ is identified with the permutation $\sigma \in S_n$ that sends each element $x \in J_k$ to $\sigma_k(x)$ for each $k \in [m]$).

For each $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$, we have

$$\begin{aligned} \nabla_{\mathbf{p}} &\in F(Q_i) && \text{(by Lemma 13 } \mathbf{(b)}\text{)} \\ &\subseteq F_i && \text{(since } F_i = F(Q_1) + F(Q_2) + \cdots + F(Q_i)\text{)} \end{aligned}$$

and thus $\overline{\nabla_{\mathbf{p}}} \in F_i / F_{i-1}$ (where $\overline{\nabla_{\mathbf{p}}}$ denotes the residue class of $\nabla_{\mathbf{p}} \in F_i$ in the quotient $F_i / F_{i-1}$). Hence, we can define a $\mathbf{k}$-linear map

$$\begin{aligned} \Phi : \mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m} &\to F_i / F_{i-1}, \\ 1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m} &\mapsto \overline{\nabla_{\mathbf{p}}} \\ &\qquad \text{for any } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m. \end{aligned}$$

(This map is defined by linearity, since the pure tensors of the form $1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m}$ with $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$ form a basis of the $\mathbf{k}$-module $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$.) Consider this map $\Phi$.

For each $\ell \in [2, m]$, we can consider the $\mathbf{k}$-submodule $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ of $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$, in which its $\ell$-th factor $\mathcal{N}_{j_\ell}$ is replaced by its submodule $\mathcal{D}_{j_\ell} = \{(a, a, \ldots, a) \mid a \in \mathbf{k}\}$. We claim that the map $\Phi$ sends this submodule to 0. Indeed, this submodule is spanned by sums of the form

$$\sum_{p_\ell \in J_\ell} 1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m}$$

(for fixed $p_2, p_3, \ldots, p_{\ell-1}, p_{\ell+1}, \ldots, p_m$ in the respective intervals $J_k$) [12], and the map $\Phi$

[12] *Proof.* The submodule $\mathcal{D}_{j_\ell}$ is spanned by the single vector

$$(1, 1, \ldots, 1) = e_{i_{\ell-1}} + e_{i_{\ell-1}+1} + \cdots + e_{i_\ell - 1} = \sum_{p_\ell \in J_\ell} e_{p_\ell},$$

sends such sums to

$$\sum_{p_\ell \in J_\ell} \overline{\nabla_{(p_2,p_3,\ldots,p_m)}} = \overline{\sum_{p_\ell \in J_\ell} \nabla_{(p_2,p_3,\ldots,p_m)}} = 0_{F_i/F_{i-1}},$$

since Lemma 13 **(c)** shows that $\sum_{p_\ell \in J_\ell} \nabla_{(p_2,p_3,\ldots,p_m)} \in F_{i-1}$.

Thus, the $\mathbf{k}$-linear map

$$\Phi : \mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m} \to F_i/F_{i-1}$$

sends all the $\mathbf{k}$-submodules $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ for $\ell \in [2,m]$ to 0. By linearity, we can thus conclude that $\Phi$ also sends their sum $\sum_{\ell=2}^{m} (\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m})$ to 0. Therefore, $\Phi$ factors through the quotient $\mathbf{k}$-module

$$\begin{aligned}
&(\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}) \diagup \sum_{\ell=2}^{m} (\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}) \\
&\cong \mathcal{H}_{j_1} \otimes (\mathcal{N}_{j_2}/\mathcal{D}_{j_2}) \otimes (\mathcal{N}_{j_3}/\mathcal{D}_{j_3}) \otimes \cdots \otimes (\mathcal{N}_{j_m}/\mathcal{D}_{j_m}) \\
&\qquad \left( \begin{array}{c} \text{by Lemma 17, applied to } V_1 = \mathcal{H}_{j_1} \text{ and } V_\ell = \mathcal{N}_{j_\ell} \text{ for } \ell > 1 \\ \text{and } W_\ell = \mathcal{D}_{j_\ell} \text{ for } \ell > 1 \end{array} \right) \\
&= \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \qquad (\text{since } \mathcal{N}_p/\mathcal{D}_p = \mathcal{Z}_p \text{ for each } p > 0)\,.
\end{aligned}$$

Thus, we obtain a $\mathbf{k}$-linear map

$$\begin{aligned}
\overline{\Phi} : \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} &\to F_i/F_{i-1}, \\
1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} &\mapsto \overline{\nabla_{\mathbf{p}}} \\
&\qquad \text{for any } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m.
\end{aligned}$$

Consider this map $\overline{\Phi}$. Using Lemma 13 **(a)**, it is easy to see that this map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant[13], and thus is a left $\mathbf{k}\,[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]$-module morphism.

and thus the tensor product $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ is spanned by the pure tensors of the form

$$\begin{aligned}
&1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_{\ell-1}} \otimes \left( \sum_{p_\ell \in J_\ell} e_{p_\ell} \right) \otimes e_{p_{\ell+1}} \otimes \cdots \otimes e_{p_m} \\
&= \sum_{p_\ell \in J_\ell} 1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m} \qquad \text{for fixed } p_2, p_3, \ldots, p_{\ell-1}, p_{\ell+1}, \ldots, p_m.
\end{aligned}$$

[13] *Proof.* Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$ be any $m$-tuple, and let $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. We shall show that

$$\overline{\Phi}\,(\tau \cdot (1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}})) = \tau \cdot \overline{\Phi}\,(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}})\,.$$

But the definition of an induction product yields

$$\begin{aligned}
&\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} \\
&= \operatorname{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right) \\
&= \underbrace{\mathbf{k}\left[S_n\right]}_{=\mathcal{A}} \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right) \qquad \text{(by (1))} \\
&= \mathcal{A} \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right).
\end{aligned}$$

Hence, we can define a left $\mathcal{A}$-module morphism

$$\begin{aligned}
\Psi : \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} &\to F_i / F_{i-1}, \\
a \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} v &\mapsto a \cdot \overline{\Phi}(v)
\end{aligned}$$

(this is well-defined, since $\overline{\Phi} : \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \to F_i / F_{i-1}$ is a left $\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]$-module morphism). Explicitly, $\Psi$ is given by

$$\begin{aligned}
&\Psi\left(a \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)\right) \\
&= a \cdot \overline{\Phi}\left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right) \\
&= a \cdot \overline{\nabla_{\mathbf{p}}} \qquad \text{for any } a \in \mathcal{A} \text{ and } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m
\end{aligned}$$

---

By linearity, this will entail that the map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant (since elements of the form $1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}$ span $\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}$).
Indeed, as we mentioned at the beginning of our proof, we regard each $S_{j_k}$ as $S_{J_k}$, so that the permutations $\tau_1, \tau_2, \ldots, \tau_m$ act not on the sets $[j_1], [j_2], \ldots, [j_m]$ but rather on the sets $J_1, J_2, \ldots, J_m$. The embedding of $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ into $S_n$ is the usual one, so that our $m$-tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is equated with the permutation $\tau \in S_n$ given by

$$\tau(x) = \tau_k(x) \qquad \text{for each } k \in [m] \text{ and } x \in J_k. \tag{10}$$

As in Lemma 13 **(a)**, we set

$$\tau \mathbf{p} := \left(\tau_2(p_2), \tau_3(p_3), \ldots, \tau_m(p_m)\right) = \left(\tau(p_2), \tau(p_3), \ldots, \tau(p_m)\right).$$

Now, we have $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ and thus

$$\begin{aligned}
\tau \cdot \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right) &= \tau_1 1 \otimes \tau_2 \overline{e_{p_2}} \otimes \tau_3 \overline{e_{p_3}} \otimes \cdots \otimes \tau_m \overline{e_{p_m}} \\
&= 1 \otimes \overline{e_{\tau_2(p_2)}} \otimes \overline{e_{\tau_3(p_3)}} \otimes \cdots \otimes \overline{e_{\tau_m(p_m)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\overline{\Phi}\left(\tau \cdot \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)\right) \\
&= \overline{\Phi}\left(1 \otimes \overline{e_{\tau_2(p_2)}} \otimes \overline{e_{\tau_3(p_3)}} \otimes \cdots \otimes \overline{e_{\tau_m(p_m)}}\right) \\
&= \overline{\nabla_{\tau \mathbf{p}}} \qquad \left(\text{by the definition of } \overline{\Phi}, \text{ since } \tau \mathbf{p} = \left(\tau_2(p_2), \tau_3(p_3), \ldots, \tau_m(p_m)\right)\right) \\
&= \overline{\tau \nabla_{\mathbf{p}}} \qquad \left(\text{since Lemma 13 \textbf{(a)} yields } \nabla_{\tau \mathbf{p}} = \tau \nabla_{\mathbf{p}}\right) \\
&= \tau \cdot \overline{\nabla_{\mathbf{p}}} = \tau \cdot \overline{\Phi}\left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)
\end{aligned}$$

(since the definition of $\overline{\Phi}$ yields $\overline{\nabla_{\mathbf{p}}} = \overline{\Phi}\left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)$). This is precisely what we wanted to show. Hence, we have proved that the map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant.

(by the definition of $\overline{\Phi}$). Hence, using Lemma 13 **(b)**, it is easy to see that this map $\Psi$ is surjective[14].

We now know that $\Psi$ is a surjective left $\mathcal{A}$-module morphism from $\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ to $F_i/F_{i-1}$. We shall now show that $\Psi$ is an isomorphism.

From Lemma 10, we know that the **k**-module $F_i/F_{i-1}$ is free of rank

$$\frac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod_{k=2}^{m} (j_k - 1) .$$

But the **k**-module

$$\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} = \operatorname{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right)$$

is also free of rank[15]

$$\underbrace{\frac{|S_n|}{|S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}|}}_{= \frac{n!}{j_1!j_2!\cdots j_m!}} \cdot \underbrace{\dim\left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right)}_{\substack{= \dim\left(\mathcal{H}_{j_1}\right) \cdot \prod_{k=2}^{m} \dim\left(\mathcal{Z}_{j_k}\right) \\ = 1 \cdot \prod_{k=2}^{m} (j_k - 1) \\ \text{(since } \mathcal{H}_{j_1} \text{ is free of rank 1,} \\ \text{whereas each } \mathcal{Z}_{j_k} \text{ is free of rank } j_k - 1\text{)}}}$$
$$= \frac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod_{k=2}^{m} (j_k - 1) .$$

---

[14] *Proof.* The map $\Psi$ is left $\mathcal{A}$-linear. Hence, its image is a left $\mathcal{A}$-submodule of $F_i/F_{i-1}$.
By the definition of $F_i$, we have

$$\begin{aligned} F_i &= F(Q_1) + F(Q_2) + \cdots + F(Q_i) \\ &= \underbrace{F(Q_1) + F(Q_2) + \cdots + F(Q_{i-1})}_{= F_{i-1}} + F(Q_i) = F_{i-1} + F(Q_i) . \end{aligned}$$

Hence, the composition of canonical maps

$$F(Q_i) \overset{\text{inclusion}}{\longrightarrow} F_i \overset{\text{projection}}{\longrightarrow} F_i/F_{i-1} \tag{11}$$

is surjective.
But Lemma 13 **(b)** shows that the left $\mathcal{A}$-module $F(Q_i)$ is generated by a single element of the form $\nabla_{\mathbf{p}}$. Hence, the quotient $\mathcal{A}$-module $F_i/F_{i-1}$ is generated by a single element of the form $\overline{\nabla_{\mathbf{p}}}$ (since the map (11) is surjective). But any such element of the form $\overline{\nabla_{\mathbf{p}}}$ lies in the image of $\Psi$ (since we have $\overline{\nabla_{\mathbf{p}}} = \Psi\left(1 \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)\right)$ when $\mathbf{p} = (p_2, p_3, \ldots, p_m)$). Thus, the image of $\Psi$ must contain a generator of $F_i/F_{i-1}$, and thus must be the entire $\mathcal{A}$-module $F_i/F_{i-1}$ (since this image is a left $\mathcal{A}$-submodule of $F_i/F_{i-1}$). In other words, $\Psi$ is surjective.

[15] Here, we are denoting the rank of a free **k**-module $V$ by $\dim V$, and we are using the fact that an induced representation $\operatorname{Ind}_H^G V$ is free of rank $\frac{|G|}{|H|} \cdot \dim V$ (as a **k**-module) whenever $V$ is free (as a **k**-module). (The latter fact is an easy consequence of the fact that $\mathbf{k}[G]$ is a free right $\mathbf{k}[H]$-module of rank $\frac{|G|}{|H|}$.)

Thus, Lemma 15 (applied to $M = \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ and $N = F_i/F_{i-1}$ and $s = \dfrac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod_{k=2}^{m} (j_k - 1)$ and $\rho = \Psi$) shows that the surjective $\mathbf{k}$-linear map $\Psi : \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} \to F_i/F_{i-1}$ must be an isomorphism. Since $\Psi$ is a left $\mathcal{A}$-module morphism, we thus conclude that $\Psi$ is a left $\mathcal{A}$-module isomorphism. Therefore, $F_i/F_{i-1} \cong \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ as left $\mathcal{A}$-modules, i.e., as $S_n$-representations. Hence, Theorem 8 is proved. □

### 3.6 In terms of Littlewood–Richardson coefficients

In the characteristic-0 case, we can restate the claim of Theorem 8 in terms of Littlewood–Richardson coefficients. Let us first recount the bare minimum of symmetric function theory needed to state this.

We will use standard notations for (integer) partitions; in particular, the size of a partition $\lambda$ will be denoted by $|\lambda|$. We let Par denote the set of all partitions. We let $\Lambda$ be the ring of symmetric functions over $\mathbb{Z}$ (not over $\mathbf{k}$); we refer to [15, §2.1] or [21, §4.3] for its definition[16]. To each partition $\lambda$ corresponds a special symmetric function $s_\lambda \in \Lambda$ called the *Schur function*; see [15, (2.2.4)] or [21, §4.4] or [4, Definition 5.3] for its definition. It is well-known (see [21, (4.26) and Theorem 4.9.4] or [15, Definition 2.5.8 and Corollary 2.6.12] or [4, Theorem 10.40]) that a product $s_\mu s_\nu$ of two Schur functions (for $\mu, \nu \in \text{Par}$) can always be written as an $\mathbb{N}$-linear combination of Schur functions – i.e., there exist coefficients $c^\lambda_{\mu,\nu} \in \mathbb{N}$ for all $\lambda, \mu, \nu \in \text{Par}$ such that every two partitions $\mu$ and $\nu$ satisfy

$$s_\mu s_\nu = \sum_{\lambda \in \text{Par}} c^\lambda_{\mu,\nu} s_\lambda. \tag{12}$$

These coefficients $c^\lambda_{\mu,\nu}$ are known as the *Littlewood–Richardson coefficients*. More generally, if $\mu_1, \mu_2, \ldots, \mu_k$ are any $k$ partitions, then we can write the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ in the form

$$s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = \sum_{\lambda \in \text{Par}} c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k} s_\lambda \tag{13}$$

with coefficients $c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k} \in \mathbb{N}$. These "*$k$-Littlewood–Richardson coefficients*" $c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k}$ are, in fact, easily computed by recursion using the standard Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}$: Namely, for $k = 0$, we have

$$c^\lambda = \delta_{\lambda,\varnothing} \qquad \text{(Kronecker delta)};$$

for $k = 1$, we have

$$c^\lambda_\mu = \delta_{\lambda,\mu} \qquad \text{(Kronecker delta)};$$

and for any higher $k$, we have

$$c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k} = \sum_{\nu \in \text{Par}} c^\nu_{\mu_1,\mu_2,\ldots,\mu_{k-1}} c^\lambda_{\nu,\mu_k}$$

[16]Note that [21, §4.3] uses $\mathbb{C}$ as the base ring, but everything works for any base ring.

(since the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ can be computed as $\left(s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{k-1}}\right) s_{\mu_k}$).

Note that any Schur function $s_\lambda$ is homogeneous of degree $|\lambda|$. Hence, a Littlewood–Richardson coefficient $c^\lambda_{\mu,\nu}$ is always 0 unless $|\lambda| = |\mu| + |\nu|$. Thus, we can rewrite the equality (12) as

$$s_\mu s_\nu = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = |\mu| + |\nu|}} c^\lambda_{\mu,\nu} s_\lambda. \tag{14}$$

Likewise, we can rewrite (13) as

$$s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = |\mu_1| + |\mu_2| + \cdots + |\mu_k|}} c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k} s_\lambda. \tag{15}$$

We note that there is a second Littlewood–Richardson rule ([21, Theorem 4.9.2], [15, (2.6.4)], [4, Theorem 10.40]) that decomposes a skew Schur function $s_{\lambda/\mu}$ into an $\mathbb{N}$-linear combination of (straight) Schur functions $s_\nu$ as follows:

$$s_{\lambda/\mu} = \sum_{\nu \in \mathrm{Par}} c^\lambda_{\mu,\nu} s_\nu. \tag{16}$$

The formula (13) can also be viewed as a particular case of that second rule, since the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ can be written as the skew Schur function $s_{\mu_1 * \mu_2 * \cdots * \mu_k}$ corresponding to the skew shape $\mu_1 * \mu_2 * \cdots * \mu_k$ obtained by attaching the Young diagrams of $\mu_1, \mu_2, \ldots, \mu_k$ to each other along their northeastern/southwestern corners (see [20, §1] for the precise definition; this claim follows from [4, Proposition 5.9]; cf. also [22, Figure 7.2]). Thus, a $k$-Littlewood–Richardson coefficient $c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k}$ can actually be rewritten as a (regular) Littlewood–Richardson coefficient using (16): If we write the skew shape $\mu_1 * \mu_2 * \cdots * \mu_k$ as $\alpha/\beta$, then

$$c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k} = c^\alpha_{\beta,\lambda}. \tag{17}$$

The same Littlewood–Richardson coefficients govern the decomposition of induction products of Specht modules into Specht modules in characteristic 0. Namely, if $\mathbf{k}$ is a field of characteristic 0, and if $\mu$ and $\nu$ are two partitions of respective sizes $i$ and $j$, then

$$\mathcal{S}^\mu * \mathcal{S}^\nu \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = i + j}} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{\mu,\nu}} \tag{18}$$

as $S_{i+j}$-modules[17]. Indeed, this follows from the Schur function equality (14) using the Frobenius characteristic map [21, Theorem 4.7.4] (in fact, this map – or, rather, its inverse – sends Schur functions $s_\lambda$ to Specht modules $\mathcal{S}^\lambda$, while sending products of symmetric functions to induction products of representations[18]). Likewise, if $\mathbf{k}$ is a field of characteristic 0, and if $\mu_1, \mu_2, \ldots, \mu_k$ are $k$ partitions of respective sizes $i_1, i_2, \ldots, i_k$, then

$$\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k} \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = i_1 + i_2 + \cdots + i_k}} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{\mu_1,\mu_2,\ldots,\mu_k}} \tag{19}$$

[17] The notation $V^{\oplus k}$ means the direct sum $V \oplus V \oplus \cdots \oplus V$ of $k$ copies of $V$.

[18] For products with two factors, this is proved in [21, Theorem 4.7.4] (using characters and Frobenius reciprocity). For products with $k$ factors, it follows from the two-factor case using Proposition 5.

as $S_{i_1+i_2+\cdots+i_k}$-modules. When $\mathbf{k}$ is a field of characteristic $\neq 0$, or, more generally, just any commutative ring, then the isomorphisms (18) and (19) still hold in a weaker form, where the direct sums are replaced by filtrations whose subquotients are Specht modules of the form $\mathcal{S}^\lambda$ (see [2, Theorem 9.7] for the more general problem of finding such filtrations of skew Specht modules[19]).

Now, Theorem 8 gives rise to the following decomposition of $F_i/F_{i-1}$:

**Corollary 18.** *Assume that* $\mathbf{k}$ *is a field of characteristic* 0.

*Let* $i \in [f_{n+1}]$. *Consider the lacunar subset* $Q_i$ *of* $[n-1]$. *Write the set* $Q_i \cup \{n+1\}$ *as* $\{i_1 < i_2 < \cdots < i_m\}$, *so that* $i_m = n+1$. *Furthermore, set* $i_0 := 1$. *Set* $j_k := i_k - i_{k-1}$ *for each* $k \in [m]$. *(Note that* $j_1 \geqslant 0$ *and* $j_2, j_3, \ldots, j_m > 1$ *and* $j_1 + j_2 + \cdots + j_m = n$; *this follows from Lemma 9 (applied to* $I = Q_i$*).)*

*Let* $\operatorname{Par}_n$ *be the set of all partitions of* $n$. *Then,*

$$F_i/F_{i-1} \cong \bigoplus_{\lambda \in \operatorname{Par}_n} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}}$$

*(where the subscripts in* $c^\lambda_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}$ *are the partition* $(j_1)$ *followed by the partitions* $(j_k - 1, 1)$ *for all* $k \in [2, m]$*).*

*Proof.* The set $Q_i \cup \{n+1\}$ is lacunar (since $Q_i$ is lacunar, and since $Q_i \subseteq [n-1]$ ensures that $n+1$ is larger than any element of $Q_i$ by at least 2). In other words, every $k > 1$ satisfies $i_k > i_{k-1} + 1$. Hence, every $k > 1$ satisfies $j_k > 1$ (since $j_k = i_k - i_{k-1}$) and therefore

$$\mathcal{Z}_{j_k} \cong \mathcal{S}^{(j_k-1,1)} \tag{20}$$

(by Proposition 3 **(b)**, since $j_k$ is invertible in $\mathbf{k}$).

Also, we have $\mathcal{H}_p \cong \mathcal{S}^{(p)}$ for each $p \in \mathbb{N}$ (since both $\mathcal{H}_p$ and $\mathcal{S}^{(p)}$ are trivial 1-dimensional representations of $S_p$). Thus, $\mathcal{H}_{j_1} \cong \mathcal{S}^{(j_1)}$.

Now, Theorem 8 yields

$$\begin{aligned}
F_i/F_{i-1} &\cong \underbrace{\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}}_{\substack{\text{the first factor is an } \mathcal{H},\\ \text{while all others are } \mathcal{Z}\text{'s}}} \\
&\cong \mathcal{S}^{(j_1)} * \mathcal{S}^{(j_2-1,1)} * \mathcal{S}^{(j_3-1,1)} * \cdots * \mathcal{S}^{(j_m-1,1)} \\
&\qquad \left(\begin{array}{c}\text{since } \mathcal{H}_{j_1} \cong \mathcal{S}^{(j_1)}, \text{ and since } \mathcal{Z}_{j_k} \cong \mathcal{S}^{(j_k-1,1)} \text{ for each } k > 1 \\ \text{(by (20))}\end{array}\right) \\
&\cong \bigoplus_{\substack{\lambda \in \operatorname{Par};\\ |\lambda| = j_1 + j_2 + \cdots + j_m}} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}} \qquad \text{(by (19))} \\
&= \bigoplus_{\lambda \in \operatorname{Par}_n} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}}
\end{aligned}$$

[19] Indeed, the induction product $\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k}$ is easily seen to be isomorphic to the skew Specht module $\mathcal{S}^{\alpha/\beta}$, where $\alpha/\beta = \mu_1 * \mu_2 * \cdots * \mu_k$. Thus, [2, Theorem 9.7] (applied to this skew diagram $\alpha/\beta$) yields a Specht series (i.e., a filtration whose subquotients are Specht modules of the form $\mathcal{S}^\lambda$) for $\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k}$.

(since the set of all $\lambda \in \text{Par}$ satisfying $|\lambda| = j_1 + j_2 + \cdots + j_m$ is precisely $\text{Par}_n$ (because $j_1 + j_2 + \cdots + j_m = n$)). This proves Corollary 18. $\square$

# 4 The Specht module spectrum

## 4.1 The theorem

We need a few more notations from [13]. For any subset $I$ of $[n]$, we define the following:

- We let $\widehat{I}$ be the set $\{0\} \cup I \cup \{n+1\}$. We shall refer to $\widehat{I}$ as the *enclosure* of $I$.

  For example, if $n = 5$, then $\widehat{\{2,3\}} = \{0, 2, 3, 6\}$.

- For any $\ell \in [n]$, we let $m_{I,\ell}$ be the number

$$\left(\text{smallest element of } \widehat{I} \text{ that is } \geqslant \ell\right) - \ell \in [0, n+1-\ell] \subseteq [0, n].$$

  For example, if $n = 6$ and $I = \{2, 5\}$, then

$$(m_{I,1},\ m_{I,2},\ m_{I,3},\ m_{I,4},\ m_{I,5},\ m_{I,6}) = (1,\ 0,\ 2,\ 1,\ 0,\ 1).$$

  We note that an $\ell \in [n]$ satisfies $m_{I,\ell} = 0$ if and only if $\ell \in \widehat{I}$ (or, equivalently, $\ell \in I$).

We recall that any partition $\lambda$ of $n$ gives rise to an $S_n$-representation called the Specht module $\mathcal{S}^\lambda$. If $\lambda$ is a partition of $n$, and if $a \in \mathcal{A}$, then the action of $a$ on $\mathcal{S}^\lambda$ (that is, the $\mathbf{k}$-linear map $\mathcal{S}^\lambda \to \mathcal{S}^\lambda,\ w \mapsto aw$) will be denoted by $L_\lambda(a)$.

**Definition 19.** Let $\lambda$ be a partition of $n$. Let $I$ be a lacunar subset of $[n-1]$. Write the set $I \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. (Note that Lemma 9 shows that $j_1 \geqslant 0$ and $j_2, j_3, \ldots, j_m > 1$, hence $j_2, j_3, \ldots, j_m \geqslant 2$.)

The $m$-Littlewood–Richardson coefficient $c^\lambda_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}$ (as defined in (13), where the subscripts are the partition $(j_1)$ followed by the partitions $(j_k - 1, 1)$ for all $k \in [2, m]$) will then be denoted by $c^\lambda_I$.

As we recall from Corollary 18, if $I = Q_i$ for some $i \in [f_{n+1}]$, then this coefficient $c^\lambda_I$ is the multiplicity of the Specht module $\mathcal{S}^\lambda$ in the left $\mathcal{A}$-module $F_i/F_{i-1}$ when $\mathbf{k}$ is a field of characteristic 0. Indeed, we can rewrite Corollary 18 as follows using Definition 19:

**Corollary 20.** *Assume that* $\mathbf{k}$ *is a field of characteristic* 0.

*Let* $i \in [f_{n+1}]$. *Let* $\text{Par}_n$ *be the set of all partitions of* $n$. *Then,*

$$F_i/F_{i-1} \cong \bigoplus_{\lambda \in \text{Par}_n} \left(\mathcal{S}^\lambda\right)^{\oplus c^\lambda_{Q_i}}.$$

We shall now state our first main theorem:

**Theorem 21.** *Let* $\mathbf{k}$ *be any field. Let* $\lambda$ *be a partition of* $n$*. Let* $\omega_1, \omega_2, \ldots, \omega_n \in \mathbf{k}$*. For each subset* $I$ *of* $[n]$*, we set*

$$\omega_I := \omega_1 m_{I,1} + \omega_2 m_{I,2} + \cdots + \omega_n m_{I,n} \in \mathbf{k}.$$

*Then:*

***(a)*** *The eigenvalues of the operator* $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ *on the Specht module* $\mathcal{S}^\lambda$ *are the elements*

$$\omega_I \text{ for all lacunar subsets } I \subseteq [n-1] \text{ satisfying } c_I^\lambda \neq 0,$$

*and their respective algebraic multiplicities are the* $c_I^\lambda$ *in the generic case (i.e., if no two* $I$*'s produce the same* $\omega_I$*; otherwise the multiplicities of colliding eigenvalues should be added together).*

***(b)*** *If all these* $\omega_I$ *(for all lacunar subsets* $I \subseteq [n-1]$ *satisfying* $c_I^\lambda \neq 0$*) are distinct, then* $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ *is diagonalizable.*

***(c)*** *We have*

$$\prod_{\substack{I \subseteq [n-1] \text{ is lacunar;} \\ c_I^\lambda \neq 0}} \left(L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_I \operatorname{id}_{\mathcal{S}^\lambda}\right) = 0.$$

To prove this theorem, we will need a further theorem, which "maps" the Fibonacci filtration from $\mathcal{A}$ to a given Specht module $\mathcal{S}^\lambda$:

**Theorem 22.** *Let* $\mathbf{k}$ *be any field of characteristic* 0*. Let* $\lambda$ *be a partition of* $n$*. Then, there exists a filtration*

$$0 = F^\lambda_{f_{n+1}} \subseteq F^\lambda_{f_{n+1}-1} \subseteq F^\lambda_{f_{n+1}-2} \subseteq \cdots \subseteq F^\lambda_2 \subseteq F^\lambda_1 \subseteq F^\lambda_0 = \mathcal{S}^\lambda$$

*(note the "backward" indexing!) of the Specht module* $\mathcal{S}^\lambda$ *by left* $\mathcal{T}$*-submodules with the following four properties:*

1. *Each subquotient* $F^\lambda_{i-1}/F^\lambda_i$ *(for* $i \in [f_{n+1}]$*) has dimension* $c^\lambda_{Q_i}$ *as a* $\mathbf{k}$*-vector space (see Definition 19 for the meaning of* $c_I^\lambda$*).*
2. *In particular, an* $i \in [f_{n+1}]$ *satisfies* $F^\lambda_{i-1} = F^\lambda_i$ *if and only if* $c^\lambda_{Q_i} = 0$*.*
3. *On each subquotient* $F^\lambda_{i-1}/F^\lambda_i$ *(for* $i \in [f_{n+1}]$*), each element* $t_\ell \in \mathcal{T}$ *(for* $\ell \in [n]$*) acts as multiplication by the scalar* $m_{Q_i,\ell}$*.*
4. *More generally, on each subquotient* $F^\lambda_{i-1}/F^\lambda_i$ *(for* $i \in [f_{n+1}]$*), each element* $P(t_1, t_2, \ldots, t_n) \in \mathcal{T}$ *(where* $P$ *is a polynomial in* $n$ *noncommuting indeterminates over* $\mathbf{k}$*) acts as multiplication by the scalar* $P(m_{Q_i,1}, m_{Q_i,2}, \ldots, m_{Q_i,n})$*.*

### 4.2 Lemmas about $S_n$-representations

In order to prove Theorem 21 and Theorem 22, we need a few lemmas about left $\mathcal{A}$-modules (i.e., representations of $S_n$). We begin with something basic and well-known (see, e.g., [5, last paragraph of §5.13] or the arXiv version of the present paper [10, §A.3]):

**Proposition 23.** *Assume that* $\mathbf{k}$ *is a field of characteristic* $0$*. Let* $\lambda$ *and* $\mu$ *be two partitions of* $n$*. Then, the Specht modules* $\mathcal{S}^\lambda$ *and* $\mathcal{S}^\mu$ *satisfy*

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \begin{cases} \mathbf{k}, & \text{if } \lambda = \mu; \\ 0, & \text{if } \lambda \neq \mu \end{cases} \qquad \text{as } \mathbf{k}\text{-vector spaces.}$$

*(Here and in the following, "*$\operatorname{Hom}_{\mathcal{A}}$*" always stands for the set of left* $\mathcal{A}$*-module morphisms. This is always a* $\mathbf{k}$*-vector space, but usually not an* $\mathcal{A}$*-module on any side.)*

The next lemma is again an easy consequence of known facts:

**Lemma 24.** *Assume that* $\mathbf{k}$ *is a field of characteristic* $0$*. Let* $\lambda$ *be a partition of* $n$*. Define the contravariant functor* $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^\lambda\right)$ *from the category of left* $\mathcal{A}$*-modules to the category of* $\mathbf{k}$*-vector spaces that is given by*

$$X \mapsto \operatorname{Hom}_{\mathcal{A}}\left(X, \mathcal{S}^\lambda\right) \qquad \text{on objects}$$

*and likewise on morphisms. (This is a contravariant Hom functor.)*

*This contravariant functor* $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^\lambda\right)$ *is exact (i.e., respects exact sequences).*

*Proof.* The $\mathbf{k}$-algebra $\mathcal{A} = \mathbf{k}\left[S_n\right]$ is semisimple (by Maschke's theorem, since $\mathbf{k}$ is a field of characteristic 0). Hence, every short exact sequence of left $\mathcal{A}$-modules is split. Consequently, any Hom functor from the category of left $\mathcal{A}$-modules is exact (since Hom functors respect finite direct sums and thus are exact on split exact sequences; see the arXiv version of the present paper [10, proof of Lemma 4.6] for details). Thus, the contravariant Hom functor $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^\lambda\right)$ is exact. This proves Lemma 24. □

Note that the Specht module $\mathcal{S}^\lambda$ in Lemma 24 could be replaced by any left $\mathcal{A}$-module, but we will use $\mathcal{S}^\lambda$ only. The same applies to the following lemma:

**Lemma 25.** *Let* $\lambda$ *be a partition of* $n$*. Let* $J$ *be a left* $\mathcal{A}$*-submodule of* $\mathcal{A}$ *(that is, a left ideal of* $\mathcal{A}$*). Then, there is a canonical* $\mathbf{k}$*-vector space isomorphism*

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}/J,\ \mathcal{S}^\lambda\right) &\to \left\{v \in \mathcal{S}^\lambda \ \mid \ Jv = 0\right\}, \\ f &\mapsto f\left(\overline{1_{\mathcal{A}}}\right). \end{aligned}$$

*Proof.* The left $\mathcal{A}$-module morphisms from $\mathcal{A}/J$ to $\mathcal{S}^\lambda$ can be identified with the left $\mathcal{A}$-module morphisms from $\mathcal{A}$ to $\mathcal{S}^\lambda$ that vanish on $J$. Thus, we obtain a $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} \Phi : \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}/J,\ \mathcal{S}^\lambda\right) &\to \left\{g \in \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A},\ \mathcal{S}^\lambda\right) \ \mid \ g\left(J\right) = 0\right\}, \\ f &\mapsto \left(\mathcal{A} \to \mathcal{S}^\lambda,\ a \mapsto f\left(\overline{a}\right)\right). \end{aligned}$$

However, recall the well-known $\mathbf{k}$-vector space isomorphism $\operatorname{Hom}_A(A, M) \cong M$ that holds for any $\mathbf{k}$-algebra $A$ and any left $A$-module $M$. Thus, in particular, the left $\mathcal{A}$-module morphisms from $\mathcal{A}$ to $\mathcal{S}^\lambda$ can be identified with the elements of $\mathcal{S}^\lambda$ via the $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} \Psi : \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A},\ \mathcal{S}^\lambda\right) &\to \mathcal{S}^\lambda, \\ g &\mapsto g\left(1_{\mathcal{A}}\right). \end{aligned}$$

This latter isomorphism $\Psi$ has the property that an arbitrary left $\mathcal{A}$-module morphism $g \in \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A},\ \mathcal{S}^\lambda\right)$ satisfies $g(J) = 0$ if and only if its image $\Psi(g)$ satisfies $J \cdot \Psi(g) = 0$ (since $g(J) = g(J \cdot 1_{\mathcal{A}}) = J \cdot \underbrace{g(1_{\mathcal{A}})}_{=\Psi(g)} = J \cdot \Psi(g)$). Thus, $\Psi$ can be restricted to a $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} \Psi' : \left\{g \in \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A},\ \mathcal{S}^\lambda\right) \ \mid\ g(J) = 0\right\} &\to \left\{v \in \mathcal{S}^\lambda \ \mid\ Jv = 0\right\}, \\ g &\mapsto g\left(1_{\mathcal{A}}\right). \end{aligned}$$

The composition $\Psi' \circ \Phi$ is thus a $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}/J,\ \mathcal{S}^\lambda\right) &\to \left\{v \in \mathcal{S}^\lambda \ \mid\ Jv = 0\right\}, \\ f &\mapsto f\left(\overline{1_{\mathcal{A}}}\right). \end{aligned}$$

This is clearly canonical in $J$, so that Lemma 25 is proved. □

### 4.3 The proofs

We are now ready to prove Theorem 22 and Theorem 21, in this order.

*Proof of Theorem 22.* For each $i \in [0, f_{n+1}]$, we define a subset $F_i^\lambda$ of $\mathcal{S}^\lambda$ by

$$F_i^\lambda := \left\{v \in \mathcal{S}^\lambda \ \mid\ F_i v = 0\right\}.$$

This subset $F_i^\lambda$ is actually a left $\mathcal{T}$-submodule of $\mathcal{S}^\lambda$ (since Proposition 7 shows that $F_i$ is a right $\mathcal{T}$-submodule of $\mathcal{A}$, and therefore any $t \in \mathcal{T}$ and $v \in F_i^\lambda$ satisfy $\underbrace{F_i t}_{\subseteq F_i} v \subseteq F_i v = 0$ and thus $F_i t v = 0$, so that $tv \in F_i^\lambda$). Moreover, because of

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n] = \mathcal{A},$$

we have

$$\mathcal{S}^\lambda = F_0^\lambda \supseteq F_1^\lambda \supseteq F_2^\lambda \supseteq \cdots \supseteq F_{f_{n+1}}^\lambda = 0.$$

This is a left $\mathcal{T}$-module filtration of $\mathcal{S}^\lambda$, albeit written backwards. We can rewrite it as

$$0 = F_{f_{n+1}}^\lambda \subseteq F_{f_{n+1}-1}^\lambda \subseteq F_{f_{n+1}-2}^\lambda \subseteq \cdots \subseteq F_2^\lambda \subseteq F_1^\lambda \subseteq F_0^\lambda = \mathcal{S}^\lambda.$$

Our goal is to show that this filtration satisfies the four properties 1, 2, 3 and 4 claimed in Theorem 22.

For this purpose, we fix $i \in [f_{n+1}]$. First, we shall show property 3. We must show that each element $t_\ell \in \mathcal{T}$ acts on $F_{i-1}^\lambda / F_i^\lambda$ as multiplication by the scalar $m_{Q_i,\ell}$. So we let $\ell \in [n]$ and $\overline{v} \in F_{i-1}^\lambda / F_i^\lambda$ (with $v \in F_{i-1}^\lambda$) be arbitrary. We must show that $t_\ell \overline{v} = m_{Q_i,\ell} \overline{v}$ in $F_{i-1}^\lambda / F_i^\lambda$.

We have $v \in F_{i-1}^\lambda$. In other words, $v \in \mathcal{S}^\lambda$ and $F_{i-1} v = 0$ (by the definition of $F_{i-1}^\lambda$).

Theorem 6 **(c)** yields $F_i \cdot (t_\ell - m_{Q_i,\ell}) \subseteq F_{i-1}$. Hence,

$$\underbrace{F_i \cdot (t_\ell - m_{Q_i,\ell})}_{\subseteq F_{i-1}} v \subseteq F_{i-1} v = 0,$$

so that $F_i \cdot (t_\ell - m_{Q_i,\ell})\, v = 0$. In other words, $(t_\ell - m_{Q_i,\ell})\, v \in F_i^\lambda$ (by the definition of $F_i^\lambda$). In other words, $t_\ell v - m_{Q_i,\ell} v \in F_i^\lambda$. In other words, $\overline{t_\ell v} = \overline{m_{Q_i,\ell} v}$ in $F_{i-1}^\lambda / F_i^\lambda$. In other words, $t_\ell \overline{v} = m_{Q_i,\ell} \overline{v}$ in $F_{i-1}^\lambda / F_i^\lambda$. Thus, the proof of property 3 is complete.

Property 4 follows immediately from property 3.

Let us now prove property 1.

Note that $F_i$ is a left $\mathcal{A}$-submodule of $\mathcal{A}$ (by Proposition 7). Lemma 25 shows that whenever $J$ is a left $\mathcal{A}$-submodule of $\mathcal{A}$ (that is, a left ideal of $\mathcal{A}$), there is a canonical $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/J,\ \mathcal{S}^\lambda \right) &\to \left\{ v \in \mathcal{S}^\lambda \ \mid\ Jv = 0 \right\}, \\ f &\mapsto f\left( \overline{1_{\mathcal{A}}} \right). \end{aligned}$$

Thus, we have

$$\left\{ v \in \mathcal{S}^\lambda \ \mid\ Jv = 0 \right\} \cong \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/J,\ \mathcal{S}^\lambda \right) \tag{21}$$

canonically for each left $\mathcal{A}$-submodule $J$ of $\mathcal{A}$. Now, the definition of $F_i^\lambda$ yields

$$F_i^\lambda = \left\{ v \in \mathcal{S}^\lambda \ \mid\ F_i v = 0 \right\} \cong \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/F_i,\ \mathcal{S}^\lambda \right) \tag{22}$$

canonically (by (21)) and similarly

$$F_{i-1}^\lambda \cong \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/F_{i-1},\ \mathcal{S}^\lambda \right). \tag{23}$$

However, Lemma 24 shows that the contravariant functor $\operatorname{Hom}_{\mathcal{A}} \left( -, \mathcal{S}^\lambda \right)$ from the category of left $\mathcal{A}$-modules to the category of $\mathbf{k}$-vector spaces is exact. Hence, applying this contravariant functor to the exact sequence

$$0 \to F_i / F_{i-1} \to \mathcal{A}/F_{i-1} \to \mathcal{A}/F_i \to 0$$

of left $\mathcal{A}$-modules, we obtain an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/F_i,\ \mathcal{S}^\lambda \right) \to \operatorname{Hom}_{\mathcal{A}} \left( \mathcal{A}/F_{i-1},\ \mathcal{S}^\lambda \right) \to \operatorname{Hom}_{\mathcal{A}} \left( F_i/F_{i-1},\ \mathcal{S}^\lambda \right) \to 0$$

of $\mathbf{k}$-vector spaces. In view of (23) and (22), we can rewrite this latter exact sequence as

$$0 \to F_i^\lambda \to F_{i-1}^\lambda \to \operatorname{Hom}_{\mathcal{A}} \left( F_i/F_{i-1},\ \mathcal{S}^\lambda \right) \to 0.$$

The arrow $F_i^\lambda \to F_{i-1}^\lambda$ here is the canonical inclusion (since the isomorphisms in (23) and (22) are the canonical ones), and thus we obtain

$$\operatorname{Hom}_{\mathcal{A}}\left(F_i/F_{i-1},\ \mathcal{S}^\lambda\right) \cong F_{i-1}^\lambda / F_i^\lambda \tag{24}$$

from the exactness of our sequence.

However, Corollary 20 says that

$$F_i/F_{i-1} \cong \bigoplus_{\nu \in \operatorname{Par}_n} \left(\mathcal{S}^\nu\right)^{\oplus c_{Q_i}^\nu}. \tag{25}$$

Hence,

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}\left(F_i/F_{i-1},\ \mathcal{S}^\lambda\right) &\cong \operatorname{Hom}_{\mathcal{A}}\left(\bigoplus_{\nu \in \operatorname{Par}_n} \left(\mathcal{S}^\nu\right)^{\oplus c_{Q_i}^\nu},\ \mathcal{S}^\lambda\right) \\ &\cong \bigoplus_{\nu \in \operatorname{Par}_n} \left(\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\nu, \mathcal{S}^\lambda\right)\right)^{\oplus c_{Q_i}^\nu} \end{aligned} \tag{26}$$

(since Hom functors respect finite direct sums).

But each $\nu \in \operatorname{Par}_n$ satisfies

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\nu, \mathcal{S}^\lambda\right) \cong \begin{cases} \mathbf{k}, & \text{if } \nu = \lambda; \\ 0, & \text{if } \nu \neq \lambda \end{cases} \qquad \text{(by Proposition 23)}.$$

Thus, we can rewrite (26) as

$$\operatorname{Hom}_{\mathcal{A}}\left(F_i/F_{i-1},\ \mathcal{S}^\lambda\right) \cong \mathbf{k}^{\oplus c_{Q_i}^\lambda}.$$

Comparing this with (24), we see that

$$F_{i-1}^\lambda / F_i^\lambda \cong \mathbf{k}^{\oplus c_{Q_i}^\lambda}.$$

Thus, the $\mathbf{k}$-vector space $F_{i-1}^\lambda / F_i^\lambda$ has dimension $c_{Q_i}^\lambda$. This proves property 1.

Property 2 follows immediately from property 1 (since $F_{i-1}^\lambda = F_i^\lambda$ is equivalent to $\dim\left(F_{i-1}^\lambda / F_i^\lambda\right) = 0$). Hence, our proof of Theorem 22 is complete. □

*Proof of Theorem 21.* In the following, the symbol dim will always refer to the dimension of a $\mathbf{k}$-vector space, even if some other module structures are present. Thus, in particular, if $X$ is a $\mathcal{T}$-module, then $\dim X$ will mean the dimension of $X$ as a $\mathbf{k}$-vector space.

**(a)** Let us first assume that $\mathbf{k}$ is a field of characteristic 0. We shall later extend this to the general case.

Theorem 22 shows that there exists a filtration

$$0 = F_{f_{n+1}}^\lambda \subseteq F_{f_{n+1}-1}^\lambda \subseteq F_{f_{n+1}-2}^\lambda \subseteq \cdots \subseteq F_2^\lambda \subseteq F_1^\lambda \subseteq F_0^\lambda = \mathcal{S}^\lambda \tag{27}$$

of the Specht module $\mathcal{S}^\lambda$ by left $\mathcal{T}$-submodules with the four properties 1, 2, 3 and 4 stated in Theorem 22. Consider this filtration. Fix any basis $(v_1, v_2, \ldots, v_s)$ of $\mathcal{S}^\lambda$ that conforms with this filtration (i.e., a basis that begins with a basis of $F^\lambda_{f_{n+1}-1}$, then extends it to a basis of $F^\lambda_{f_{n+1}-2}$, then extends it to a basis of $F^\lambda_{f_{n+1}-3}$, and so on), so that each $F^\lambda_i$ is spanned by $v_1, v_2, \ldots, v_{j(i)}$ for some $j(i) \in [0, s]$. Note that the inclusions in (27) yield

$$0 = j(f_{n+1}) \leqslant j(f_{n+1} - 1) \leqslant j(f_{n+1} - 2) \leqslant \cdots \leqslant j(2) \leqslant j(1) \leqslant j(0) = s.$$

Note that each $i \in [f_{n+1}]$ satisfies $j(i) = \dim\left(F^\lambda_i\right)$ and $j(i-1) = \dim\left(F^\lambda_{i-1}\right)$ and thus

$$\begin{aligned} j(i-1) - j(i) &= \dim\left(F^\lambda_{i-1}\right) - \dim\left(F^\lambda_i\right) = \dim\left(F^\lambda_{i-1}/F^\lambda_i\right) \\ &= c^\lambda_{Q_i} \end{aligned} \tag{28}$$

(by property 1 of our filtration).

The operator $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ preserves the filtration (27) (since this filtration is a filtration by left $\mathcal{T}$-submodules, but $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$). Hence, the matrix $M$ that represents this operator with respect to the basis $(v_1, v_2, \ldots, v_s)$ is block-upper-triangular with blocks of sizes

$$j(i-1) - j(i) \qquad \text{for all } i \in [f_{n+1}]$$

(because, e.g., the fact that the operator preserves $F^\lambda_i$ means that the first $j(i)$ columns of the matrix $M$ have zeroes everywhere below the $j(i)$-th row). Moreover, property 4 of our filtration shows that the element $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$ acts as multiplication by the scalar

$$\omega_1 m_{Q_i,1} + \omega_2 m_{Q_i,2} + \cdots + \omega_n m_{Q_i,n} = \omega_{Q_i} \qquad \text{(by the definition of } \omega_{Q_i}\text{)}$$

on each subquotient $F^\lambda_{i-1}/F^\lambda_i$. In other words, for each $v \in F^\lambda_{i-1}$, the vector $\overline{v} \in F^\lambda_{i-1}/F^\lambda_i$ satisfies

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \cdot \overline{v} = \omega_{Q_i} \overline{v}, \tag{29}$$

and therefore

$$\begin{aligned} &(L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n))(v) \\ &= (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \cdot v \\ &= \omega_{Q_i} v + \left(\text{some element of } F^\lambda_i\right) \qquad \text{(by (29))} \\ &= \omega_{Q_i} v + \left(\text{some linear combination of } v_1, v_2, \ldots, v_{j(i)}\right) \end{aligned}$$

(since $F^\lambda_i$ is spanned by $v_1, v_2, \ldots, v_{j(i)}$). We can apply this in particular to $v = v_k$ for each $k \in [j(i-1)]$ (since $F^\lambda_{i-1}$ is spanned by $v_1, v_2, \ldots, v_{j(i-1)}$), and conclude that

$$\begin{aligned} &(L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n))(v_k) \\ &= \omega_{Q_i} v_k + \left(\text{some linear combination of } v_1, v_2, \ldots, v_{j(i)}\right) \end{aligned}$$

for each $k \in [j(i-1)]$.

Thus, the matrix $M$ that represents the operator $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ with respect to the basis $(v_1, v_2, \ldots, v_s)$ is not only block-upper-triangular, but also has the property that its $i$-th diagonal block (for each $i \in [f_{n+1}]$, counted from the end) is the scalar matrix $\omega_{Q_i} \cdot I_{j(i-1)-j(i)} = \omega_{Q_i} \cdot I_{c^\lambda_{Q_i}}$ (by (28)). Consequently, the matrix $M$ is upper-triangular, and its diagonal entries are the elements $\omega_{Q_i}$ for all $i \in [f_{n+1}]$, with each $\omega_{Q_i}$ appearing $c^\lambda_{Q_i}$ times (this means that if $c^\lambda_{Q_i} = 0$, then $\omega_{Q_i}$ does not appear at all).

Of course, this allows us to read off the eigenvalues of this matrix $M$, and thus of the operator $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ (since the eigenvalues of a triangular matrix are just its diagonal entries). We conclude that the eigenvalues of $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ are the elements $\omega_{Q_i}$ for all $i \in [f_{n+1}]$, with each $\omega_{Q_i}$ appearing with algebraic multiplicity $c^\lambda_{Q_i}$. Since $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ are just the lacunar subsets of $[n-1]$, we can rewrite this as follows: The eigenvalues of $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ are the elements $\omega_I$ for all lacunar subsets $I \subseteq [n-1]$, with each $\omega_I$ appearing with algebraic multiplicity $c^\lambda_I$. We can restrict this list to those lacunar subsets $I \subseteq [n-1]$ that satisfy $c^\lambda_I \neq 0$ (since an eigenvalue $\omega_I$ that appears with algebraic multiplicity $c^\lambda_I = 0$ simply does not appear at all).

This proves Theorem 21 **(a)** in the case when $\mathbf{k}$ is a field of characteristic 0. It remains to extend the proof to the case when $\mathbf{k}$ is an arbitrary field. But there is a standard trick for this: We recast our result as a polynomial identity. Namely, Theorem 21 **(a)** is saying that

$$\underbrace{\det\left(x\,\mathrm{id}_{\mathcal{S}^\lambda} - L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)\right)}_{\substack{\text{This is the characteristic polynomial of the} \\ \text{endomorphism } L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \text{ of } \mathcal{S}^\lambda}} = \prod_{I \subseteq [n-1] \text{ lacunar}} (x - \omega_I)^{c^\lambda_I}$$

in the polynomial ring $\mathbf{k}[x]$. This is a polynomial identity in the $n+1$ indeterminates $x, \omega_1, \omega_2, \ldots, \omega_n$ (since the Specht module $\mathcal{S}^\lambda$ has a basis consisting of the standard polytabloids, and the action of $S_n$ on this basis is independent of the base field $\mathbf{k}$). Thus, knowing that this identity holds whenever $\mathbf{k}$ is a field of characteristic 0, we can immediately conclude that it holds for all fields $\mathbf{k}$ (and even all commutative rings $\mathbf{k}$). This proves Theorem 21 **(a)** in the general case.

**(c)** Again, let us first assume that $\mathbf{k}$ is a field of characteristic 0. Recall the filtration (27) constructed in the proof of part **(a)**. As we saw in that proof, the element $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$ acts as multiplication by the scalar $\omega_{Q_i}$ on each subquotient $F^\lambda_{i-1}/F^\lambda_i$ of that filtration. In other words, for each $i \in [f_{n+1}]$, we have

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)\,\overline{v} = \omega_{Q_i}\overline{v} \qquad \text{for each } \overline{v} \in F^\lambda_{i-1}/F^\lambda_i,$$

that is,

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)\, v - \omega_{Q_i} v \in F^\lambda_i \qquad \text{for each } v \in F^\lambda_{i-1}.$$

In other words, for each $i \in [f_{n+1}]$, we have

$$\left(L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_{Q_i}\,\mathrm{id}_{\mathcal{S}^\lambda}\right) F^\lambda_{i-1} \subseteq F^\lambda_i.$$

Hence, the operator

$$\prod_{i\in[f_{n+1}]}\left(L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)-\omega_{Q_i}\operatorname{id}_{\mathcal{S}^\lambda}\right)\in\operatorname{End}_{\mathbf{k}}\left(\mathcal{S}^\lambda\right)$$

[20] sends the whole $\mathcal{S}^\lambda$ to 0 (because its first factor sends $\mathcal{S}^\lambda = F_0^\lambda$ down to $F_1^\lambda$, then its second factor sends $F_1^\lambda$ further down to $F_2^\lambda$, then its third factor sends $F_2^\lambda$ onward to $F_3^\lambda$, and so on, until the last factor sends $F_{f_{n+1}-1}^\lambda$ down to $F_{f_{n+1}}^\lambda = 0$). Moreover, for this to hold, we do not actually need all the $f_{n+1}$ factors of this product, but rather only those factors that correspond to the numbers $i \in [f_{n+1}]$ satisfying $c_{Q_i}^\lambda \neq 0$ (because if $c_{Q_i}^\lambda = 0$, then property 2 of our filtration shows that $F_{i-1}^\lambda = F_i^\lambda$, and thus we don't need to apply the $L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)-\omega_{Q_i}\operatorname{id}_{\mathcal{S}^\lambda}$ factor to send us from $F_{i-1}^\lambda$ down into $F_i^\lambda$). Hence, the operator

$$\prod_{\substack{i\in[f_{n+1}];\\ c_{Q_i}^\lambda\neq 0}}\left(L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)-\omega_{Q_i}\operatorname{id}_{\mathcal{S}^\lambda}\right)\in\operatorname{End}_{\mathbf{k}}\left(\mathcal{S}^\lambda\right)$$

sends the whole $\mathcal{S}^\lambda$ to 0 as well. In other words,

$$\prod_{\substack{i\in[f_{n+1}];\\ c_{Q_i}^\lambda\neq 0}}\left(L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)-\omega_{Q_i}\operatorname{id}_{\mathcal{S}^\lambda}\right)=0.$$

Since $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ are just the lacunar subsets of $[n-1]$, we can rewrite this as

$$\prod_{\substack{I\subseteq[n-1]\text{ is lacunar;}\\ c_I^\lambda\neq 0}}\left(L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)-\omega_I\operatorname{id}_{\mathcal{S}^\lambda}\right)=0.$$

This proves Theorem 21 **(c)** in the case when $\mathbf{k}$ is a field of characteristic 0. Just as in our proof of part **(a)**, we can derive the general case from this case by a polynomial identity argument (treating $\omega_1, \omega_2, \ldots, \omega_n$ as indeterminates, and now considering polynomials with values in $\operatorname{End}_{\mathbf{k}}\left(\mathcal{S}^\lambda\right)$, which can be encoded as tuples of usual polynomials).

**(b)** Theorem 21 **(c)** shows that the endomorphism $L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right) \in \operatorname{End}_{\mathbf{k}}\left(\mathcal{S}^\lambda\right)$ is annihilated by the polynomial $\prod_{\substack{I\subseteq[n-1]\text{ is lacunar;}\\ c_I^\lambda\neq 0}}(x-\omega_I) \in \mathbf{k}[x]$ (meaning that the polynomial vanishes when we substitute the endomorphism for $x$). But it is well-known that a linear endomorphism (of a finite-dimensional $\mathbf{k}$-vector space) that is annihilated by a polynomial of the form $\prod(x-r)$ with pairwise distinct scalars $r$ is always diagonalizable. Hence, if the $\omega_I$ in the above polynomial are pairwise distinct, then the endomorphism $L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)$ is diagonalizable. This proves Theorem 21 **(b)**. ☐

[20] This product is well-defined (and does not depend on the order of its factors), since all its factors (being polynomials in $L_\lambda\left(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n\right)$) commute.

## 5 Final remarks

Thus we have computed the eigenvalues – and their algebraic multiplicities – for the action of any one-sided cycle shuffle $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n$ on any Specht module $\mathcal{S}^\lambda$. With a trivial amount of work, we could extend this analysis to the action of any element of $\mathcal{T}$ (that is, of any noncommutative polynomial in $t_1, t_2, \ldots, t_n$). This automatically allows us to identify the eigenvalues of such elements on any $S_n$-representation $V$, as long as the decomposition of $V$ into Specht modules is known.

The proof of our result was achieved in a rather roundabout way: We did no work in the Specht modules $\mathcal{S}^\lambda$ themselves. Instead, we used a filtration of $\mathcal{A}$ (the Fibonacci filtration) whose subquotients $F_i/F_{i-1}$ we were able to decompose into Specht modules (Theorem 8). Then, we "projected" this filtration onto each Specht module $\mathcal{S}^\lambda$ (Theorem 22) and used the semisimplicity of $\mathcal{A}$ (actually, the complete reducibility of $\mathcal{S}^\lambda$ would have sufficed) to triangularize the action of $\mathcal{T}$ on $\mathcal{S}^\lambda$. This is in contrast to other instances of similar questions, such as the recent [1], where the solution requires significant exploration of the inner life of $\mathcal{S}^\lambda$.

Our above method for proving Theorems 22 and 21 – in which we used the Fibonacci filtration to triangularize $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ – is partly generalizable:

**Proposition 26.** *Let $A$ be a $\mathbf{k}$-algebra[21], and let $T$ be a $\mathbf{k}$-subalgebra of $A$. Let*

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m = A \tag{30}$$

*be a filtration of $A$ by $(A,T)$-subbimodules. Let $V$ be any left $A$-module. If $B$ is any $(A,T)$-subbimodule of $A$, then we can define a left $T$-submodule*

$$V^B := \{v \in V \mid Bv = 0\}$$

*of $V$. Then, we have a filtration*

$$0 = V^{F_m} \subseteq V^{F_{m-1}} \subseteq V^{F_{m-2}} \subseteq \cdots \subseteq V^{F_0} = V \tag{31}$$

*of $V$ by left $T$-submodules.*

- ***(a)*** *Its subquotients $V^{F_{i-1}}/V^{F_i}$ can be canonically embedded into $\operatorname{Hom}_A(F_i/F_{i-1},\ V)$ as left $T$-modules. Thus, if some element $t \in T$ acts triangularly from the right on the filtration (30) (meaning that it acts as a scalar on each subquotient $F_i/F_{i-1}$), then it also acts triangularly from the left on the filtration (31).*

- ***(b)*** *If $\mathbf{k}$ is a field and the algebra $A$ is semisimple, then these embeddings $V^{F_{i-1}}/V^{F_i} \to \operatorname{Hom}_A(F_i/F_{i-1},\ V)$ are isomorphisms. Thus, in this case, knowing the dimensions of the Hom-spaces $\operatorname{Hom}_A(F_i,\ V)$ allows us to compute the multiplicities of eigenvalues for a triangular $t \in T$ acting on $V$.*

*Proof.* This is implicit in our above proofs of Theorems 22 and 21 (where $A$, $T$ and $V$ were taken to be $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{S}^\lambda$, and where the submodules $V^{F_i}$ were called $F_i^\lambda$). □

[21] Recall that $\mathbf{k}$ is an arbitrary commutative ring.

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