Reduced Words for Clans

Brian Burks Brendan Pawlowski[a]

Submitted: Aug 3, 2025; Accepted: Apr 24, 2026; Published: Jun 19, 2026

Abstract

Clans are combinatorial objects indexing the orbits of $\mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$ on the variety of flags in $\mathbb{C}^{p+q}$. This geometry leads to a partial order on the set of clans analogous to weak Bruhat order on the symmetric group, and we study the saturated chains in this order. We prove an analogue of Tits' theorem on reduced words in a Coxeter group. We also obtain enumerations of reduced word sets for particular clans in terms of standard tableaux and shifted standard tableaux.

Mathematics Subject Classifications: 14N15, 05A05, 05E05

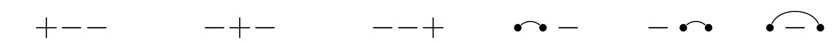
1 Introduction

For $p, q \in \mathbb{N}$, a (p,q)-*clan* is an involution in the symmetric group S_{p+q}, each of whose fixed points is labeled either $+$ or $-$, for which

$$(\# \text{ of fixed points labeled } +) - (\# \text{ of fixed points labeled } -) = p - q.$$

We draw clans as partial matchings of $[n] := \{1, 2, \ldots, n\}$ where $n = p + q$:

Example 1. The $(1,2)$-clans are



$(1^+)(2^-)(3^-)$	$(1^-)(2^+)(3^-)$	$(1^-)(2^-)(3^+)$	$(1\,2)(3^-)$	$(1^-)(2\,3)$	$(1\,3)(2^-)$
$1^+2^-3^-$	$1^-2^+3^-$	$1^-2^-3^+$	213^-	1^-32	32^-1

where the last two lines are cycle notation and one-line notation, respectively. When a clan consists entirely of fixed points, we simplify the one-line notation: $--+$ instead of $1^-2^-3^+$.

Our treatment of clans will be combinatorial and algebraic, but their origins are in geometry. A *complete flag* $F_\bullet$ in a vector space V is a chain of subspaces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = V$ where $\dim F_i = i$. Let $\mathrm{Fl}(V)$ be the set of complete flags in V. The (left) action

[a] br.pawlowski@gmail.com

 https://doi.org/10.37236/14451

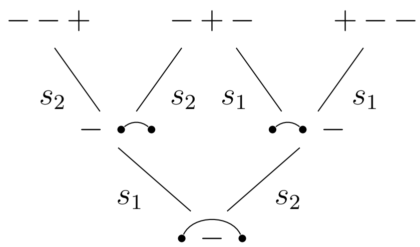
of $\mathrm{GL}(V)$ on V induces an action on $\mathrm{Fl}(V)$, hence an action of any subgroup. Identify $\mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$ with the subgroup of $\mathrm{GL}(\mathbb{C}^{p+q})$ consisting of block diagonal matrices with a $p \times p$ block in the upper left and a $q \times q$ block in the lower right. There are then finitely many $\mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$-orbits on $\mathrm{Fl}(\mathbb{C}^{p+q})$, and Matsuki and Ōshima introduced (p, q)-clans (in a somewhat different form) as combinatorial objects in bijection with these orbits [18]; see also [25].

A closed subgroup $K \subseteq \mathrm{GL}(\mathbb{C}^n)$ is *spherical* if it acts on $\mathrm{Fl}(\mathbb{C}^n)$ with finitely many orbits (more generally, one can replace $\mathrm{GL}(\mathbb{C}^n)$ with a reductive algebraic group G and $\mathrm{Fl}(\mathbb{C}^n)$ with the generalized flag variety of G). From the geometry arises a natural partial order on the set of K-orbits called *weak order* [20, 5]. This poset is graded by codimension and has a unique minimal element. The central objects of this paper are the saturated chains containing the minimal element in the case $K = \mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$.

The covering relations in weak order are labelled by integers in $[n-1]$, so a saturated chain from the minimal element to a clan γ can be identified with a word on the alphabet $[n-1]$, and we call such a word a *reduced word of* γ. This is by analogy with the more familiar case where K is the subgroup of lower triangular matrices, in which the K-orbits on $\mathrm{Fl}(\mathbb{C}^n)$ are in bijection with permutations of n, and their closures are the Schubert varieties in $\mathrm{Fl}(\mathbb{C}^n)$. There, weak order is defined by the covering relations $ws_i < w$ whenever ws_i has fewer inversions than w, where s_i is the adjacent transposition $(i, i{+}1) \in S_n$. The saturated chains from the minimal element to w are then labeled by the *reduced words of* w: the minimal-length words $a_1 \cdots a_\ell$ such that $w = s_{a_1} \cdots s_{a_\ell}$.

Remark 2. We have tried to ensure that it will always be clear from context whether "reduced word" refers to a reduced word of a clan or of a permutation. In fact, any reduced word of a clan is also a reduced word of some permutation (Lemma 15), so the reuse of terminology is less troublesome than it may seem.

Example 3. Here is the weak order on the set of $(1, 2)$-clans (we have labelled the edges by the transpositions $s_1, \dots, s_{n-1}$ rather than the integers $1, \dots, n-1$); see §2 for the general definition of weak order on clans:



The reduced words of $- + -$ are **12** and **21**, while the only reduced word of $- - +$ is **12**.

Let $\mathcal{R}(w)$ be the set of reduced words of $w \in S_n$. The adjacent transpositions s_i satisfy the *Coxeter relations*

$$s_i^2 = 1 \quad \text{and} \quad s_i s_k = s_k s_i \text{ if } |i-k| > 1 \quad \text{and} \quad s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| = 1 \tag{1}$$

and S_n is the group generated by symbols $s_1, \ldots, s_n$ subject to these relations; see [3] for an introduction to the theory of Coxeter groups. The next theorem is a special case of a well-known general result of Tits on reduced words in Coxeter groups, asserting that the relations (1) preserve each set $\mathcal{R}(w)$, and can be used to transform any reduced word of w into any other. Let $\overset{S}{\equiv}$ be the equivalence relation on the set of words on the alphabet $\mathbb{N}$ defined as the transitive closure of the relations

$$\cdots ik \cdots \overset{S}{\equiv} \cdots ki \cdots \text{ if } |i-k|>1 \qquad \text{and} \qquad \cdots iji \cdots \overset{S}{\equiv} \cdots jij \cdots \text{ if } |i-j|=1. \quad (2)$$

Theorem 4 ([3], Theorem 3.3.1)**.** *Each set $\mathcal{R}(w)$ for $w \in S_n$ is an equivalence class of $\overset{S}{\equiv}$.*

Example 3 shows that an exact analogue of this theorem cannot hold for clans, because different clans can share the same reduced word. What we will prove in Section 2 is a clan version of the following rephrasing of Theorem 4: if a and b are both reduced words for permutations, then $a \overset{S}{\equiv} b$ if and only if $\{w \in S_n : a \in \mathcal{R}(w)\} = \{w \in S_n : b \in \mathcal{R}(w)\}$. Let $\mathcal{R}(\gamma)$ be the set of reduced words for a clan γ, and $\mathrm{Clan}_{p,q}$ the set of (p,q)-clans. Let $\equiv$ be the equivalence relation on the set of words on $\mathbb{N}$ defined as the transitive closure of the relations $a_1a_2\cdots a_\ell \equiv (n-a_1)a_2\cdots a_\ell$ together with the relations (2).

Theorem 5. *If a and b are both reduced words for (p,q)-clans, then $a \equiv b$ if and only if $\{\gamma \in \mathrm{Clan}_{p,q} : a \in \mathcal{R}(\gamma)\} = \{\gamma \in \mathrm{Clan}_{p,q} : b \in \mathcal{R}(\gamma)\}$.*

In [21], Stanley defined a symmetric function F_w associated to a permutation w in which the coefficient of a squarefree monomial is the number of reduced words of w. For many w of interest (e.g. the reverse permutation $n\cdots 21$), the Schur expansion of F_w is simple enough that one obtains enumerations of reduced words in terms of standard tableaux. A formula of Billey-Jockusch-Stanley [2] shows that F_w is a certain limit of Schubert polynomials, which represent the cohomology classes of Schubert varieties in $\mathrm{Fl}(\mathbb{C}^n)$.

We follow a similar approach to prove some enumerative results for reduced words of clans in Section 3. Wyser and Yong [26] defined polynomials which represent the cohomology classes of the $\mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$-orbit closures on $\mathrm{Fl}(\mathbb{C}^n)$, and a result of Brion [4] implies an analogue of the Billey-Jockusch-Stanley formula. We define the Stanley symmetric function F_γ of a clan γ as a limit of the Wyser-Yong polynomials. In particular, the maximal clans in weak order are the *matchless* clans, those whose underlying involution is the identity permutation, and we show in this case that F_γ is the product of two Schur polynomials. This gives a simple product formula for the number of reduced words:

Theorem 6. *Suppose $\gamma \in \mathrm{Clan}_{p,q}$ is matchless with $+$'s in positions $\phi^+ \subseteq [n]$ and $-$'s in positions $\phi^- = [n] \setminus \phi^+$. Then*

$$\#\mathcal{R}(\gamma) = (pq)! \prod_{\substack{i\in\phi^+ \\ j\in\phi^-}} \frac{1}{|i-j|}.$$

The permutation $w \in S_n$ with the most reduced words is the reverse permutation $n \cdots 21$, the unique maximal element in weak order. Similarly, a clan $\gamma \in \mathrm{Clan}_{p,q}$ maximizing $\#\mathcal{R}(\gamma)$ must be matchless, but otherwise it is not obvious what these clans are. We investigate this question in Section 4, including connections to work of Romik and Pittel on random Young tableaux of rectangular shape [19] suggested by Theorem 6.

The orbits of the orthogonal group $\mathrm{O}(\mathbb{C}^n)$ on $\mathrm{Fl}(\mathbb{C}^n)$ are indexed by the involutions in S_n, and the resulting weak order on involutions has been studied by various authors [6, 7, 12, 11, 13, 20]. If one forgets the signs of fixed points, clan weak order becomes the poset dual to involution weak order. We explore this relationship in Section 5, and use it to deduce the following enumeration from known enumerations of reduced words in involution weak order [10]:

Theorem 7. *The number of maximal chains in* $\mathrm{Clan}_{p,q}$ *is*

$$2^{pq}\binom{pq}{\lambda}\prod_{i=1}^{\min(p,q)}\binom{p+q-2i}{p-i,q-i}^{-1},$$

where $\lambda = (p+q-1, p+q-3, \ldots, p-q+1)$ *and* $\binom{pq}{\lambda}$ *is the multinomial coefficient* $\binom{pq}{\lambda_1,\ldots,\lambda_\ell}$*. This is* 2^q *times the number of marked shifted standard tableaux of shifted shape* λ *(cf. Definition 71).*

Acknowledgements

We thank Eric Marberg for useful suggestions, including the question motivating Section 4, and Zach Hamaker for pointing out the relevance of the work of Romik and Pittel. This work was done as part of the University of Michigan REU program, and we thank David Speyer and everyone else involved in the program. Brian Burks was supported by NSF grant DMS-1600223.

2 Reduced words for clans

Let $\mathrm{Clan}_{p,q}$ be the set of (p,q)-clans. We usually write n to mean $p+q$ without comment. Let s_i be the adjacent transposition $(i, i+1)$, and write $\iota(\gamma)$ for the underlying involution of a clan γ. We define conjugation of γ by s_i as follows: take the underlying involution of $s_i\gamma s_i$ to be $s_i\iota(\gamma)s_i$, and give the fixed points of $s_i\gamma s_i$ the same signs that they have in γ except that the signs of i and $i+1$ (if any) become the respective signs of $i+1$ and i.

Example 8.

$$s_2(1\,2)(3^-)s_2 = (1\,3)(2^-) \quad \text{and} \quad s_1(1\,2)(3^-)s_1 = (1\,2)(3^-);$$
$$s_2(1^-)(2^-)(3^+)s_2 = (1^-)(2^+)(3^-).$$

Conjugation preserves the number of $+$'s and $-$'s, hence the set of (p,q)-clans. Imagining a clan as an ordered row of unlabeled nodes, each of which has a strand or a sign attached to it (as in Example 1), conjugation by s_i simply swaps the i^{th} and $(i+1)^{\text{th}}$ node, with any attached strand or sign being carried along.

Using conjugation we now define a different, partial action of the s_i on clans; this action is closely related to the monoid action defined in [20, §4.6].

- If i and $i+1$ are fixed points of γ of opposite sign, then $\gamma * s_i$ is γ except that i and $i+1$ are now matched.
- If i and $i+1$ are matched in γ, or are fixed points of equal sign, we leave $\gamma * s_i$ undefined.
- If i and $i+1$ are not fixed points and are not matched with each other, $\gamma * s_i = s_i \gamma s_i$.

Let $\ell(w)$ be the Coxeter length of a permutation w (number of inversions).

Definition 9. The *weak order* on $\mathrm{Clan}_{p,q}$ is the transitive closure of the relation $\gamma * s_i < \gamma$ if $\ell(\iota(\gamma * s_i)) > \ell(\iota(\gamma))$.

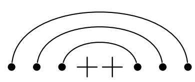
See Example 3 for the Hasse diagram of weak order on $\mathrm{Clan}_{1,2}$, and Figure 2 for the $(p,q) = (2,2)$ case. This definition is due to Yamamoto [27], who gave a correspondence between this combinatorial weak order on clans and the geometric weak order on $\mathrm{GL}_p \times \mathrm{GL}_q$-orbits on $\mathrm{Fl}(\mathbb{C}^n)$ (cf. Remark 14). One should mark the reversal here compared to weak Bruhat order on the symmetric group, which has covering relations $ws_i < w$ whenever $\ell(ws_i) < \ell(w)$. By contrast, the largest elements of $\mathrm{Clan}_{p,q}$ have the *fewest* inversions when viewed as permutations.

When passing from γ to $\gamma * s_i < \gamma$, only the i^{th} and $(i+1)^{\text{th}}$ nodes in the matching diagrams change, and it is helpful to have a list of the possible local moves. In Figure 1, we have drawn the i^{th} and $(i+1)^{\text{th}}$ nodes of γ on the left, and those of $\gamma * s_i$ on the right, assuming $\gamma * s_i < \gamma$.

Definition 10. A clan γ is *matchless* if $\iota(\gamma)$ is the identity permutation.

There are $\binom{p+q}{p,q}$ matchless clans in $\mathrm{Clan}_{p,q}$, and they are exactly the maximal elements in weak order. There is a unique minimal element in weak order on $\mathrm{Clan}_{p,q}$, which we will call $\gamma_{p,q}$: its underlying involution is $(1,n)(2,n-1)\cdots(m,n-m+1)$ where $m = \min(p,q)$, and the fixed points $m+1, m+2, \ldots, n-m$ are are all labeled with the sign of $p-q$.

Example 11. The minimal element $\gamma_{5,3} \in \mathrm{Clan}_{5,3}$ has $|p-q| = 2$ fixed points, labeled $+$ since $p - q > 0$, and $\min(p,q) = 3$ arcs:



Definition 12. A word $a_1 \cdots a_\ell$ with letters in $\mathbb{N}$ is a *reduced word* for $\gamma \in \mathrm{Clan}_{p,q}$ if there is a saturated chain from the minimal element $\gamma_{p,q} \in \mathrm{Clan}_{p,q}$ to γ with edge labels $s_{a_1}, \ldots, s_{a_\ell}$ (in that order, beginning at $\gamma_{p,q}$ and ending at γ). Let $\mathcal{R}(\gamma)$ be the set of reduced words of γ.

Figure 1: Possible local changes in a covering relation in weak order (left side > right side)

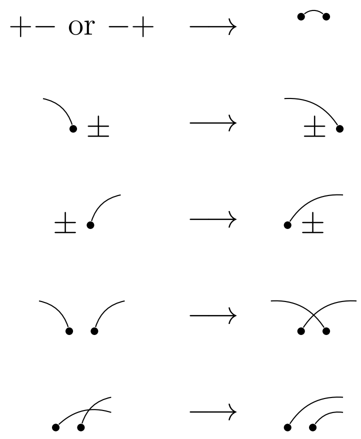
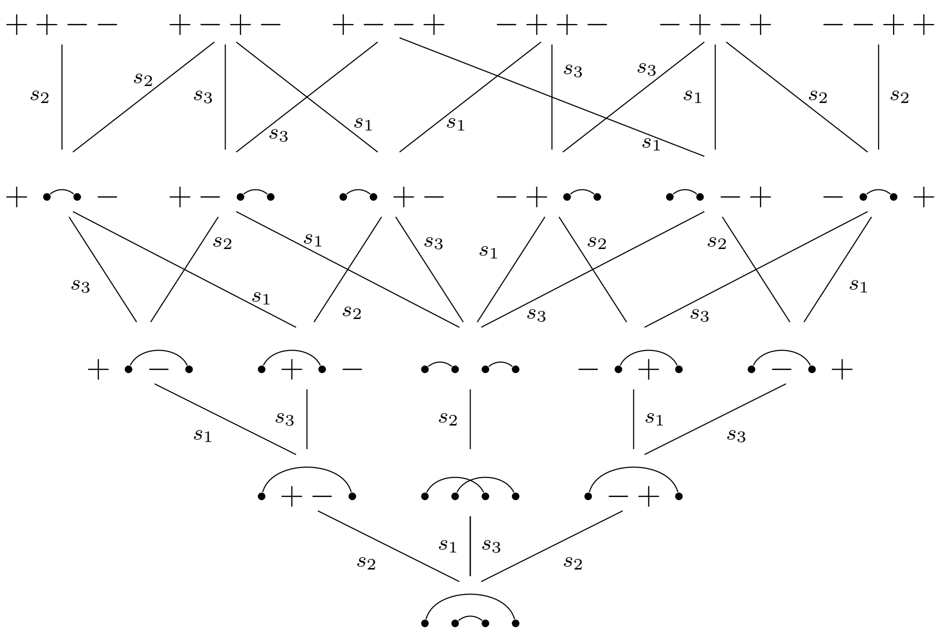


Figure 2: Weak order on $\mathrm{Clan}_{2,2}$.



We will use bold for reduced words to distinguish them from permutations.

Example 13. From Example 3 one can see that

$\mathcal{R}($$) = \{\varepsilon\}$

$\mathcal{R}(-$ $) = \{\mathbf{1}\}$

$\mathcal{R}($ $-) = \{\mathbf{2}\}$

$$\mathcal{R}(--+) = \{\mathbf{12}\}$$
$$\mathcal{R}(-+-) = \{\mathbf{12}, \mathbf{21}\}$$
$$\mathcal{R}(+--) = \{\mathbf{21}\}$$

where ε is the empty word. Unlike reduced words in Coxeter groups, a word can be a reduced word for more than one clan.

Warning. We write reduced words starting at the minimal element $\gamma_{p,q} \in \mathrm{Clan}_{p,q}$ by analogy with reduced words for Coxeter groups. However, if $a_1 \cdots a_\ell$ is a reduced word for $\gamma \in \mathrm{Clan}_{p,q}$, then $(\cdots((\gamma_{p,q} * s_{a_1}) * s_{a_2}) * \cdots) * s_{a_\ell}$ need not be defined (although if it is, then it equals γ). Rather, one must say that $a_1 \cdots a_\ell$ is a reduced word for γ if $(\cdots((\gamma * s_{a_\ell}) * s_{a_{\ell-1}}) * \cdots) * s_{a_1} = \gamma_{p,q}$ and ℓ is minimal.

Remark 14. The motivation for this definition of weak order on clans comes from geometry. Given any subset $Y \subseteq \mathrm{Fl}(\mathbb{C}^n)$ and $1 \leqslant i < n$, let $Y * s_i$ be the subset

$$\{F_\bullet : F_1 \subseteq \cdots \subseteq F_{i-1} \subseteq F' \subseteq F_{i+1} \subseteq \cdots \subseteq F_n \text{ is in } Y \text{ for some } i\text{-dimensional } F'\}.$$

In particular, $Y * s_i$ contains Y. Recall from the introduction that the $\mathrm{GL}(\mathbb{C}^p) \times \mathrm{GL}(\mathbb{C}^q)$-orbits on $\mathrm{Fl}(\mathbb{C}^n)$ can be labeled by (p,q)-clans. Letting Y_γ denote the orbit labeled by γ, one has $Y_\gamma * s_i = Y_{\gamma * s_i}$ if $\gamma * s_i < \gamma$. This operation is important in Schubert calculus: the Zariski closures $\overline{Y}_\gamma$ and $\overline{Y}_{\gamma * s_i}$ have associated cohomology classes $[\overline{Y}_\gamma]$ and $[\overline{Y}_{\gamma * s_i}]$, and under the Borel isomorphism identifying the cohomology ring $H^*(\mathrm{Fl}(\mathbb{C}^n), \mathbb{Z})$ with a quotient of $\mathbb{Z}[x_1, \ldots, x_n]$, these two classes are related by a divided difference operator; see Section 3.

Lemma 15 ([20], Lemma 3.16)**.** *The reduced word set* $\mathcal{R}(\gamma)$ *of any* $\gamma \in \mathrm{Clan}_{p,q}$ *is closed under the braid relations* (2)*, i.e. under the following operations on words:*

$$\cdots ik \cdots \rightsquigarrow \cdots ki \cdots \text{ if } |i-k| > 1 \qquad \text{and} \qquad \cdots iji \cdots \rightsquigarrow \cdots jij \cdots \text{ if } |i-j| = 1.$$

Moreover, any $a \in \mathcal{R}(\gamma)$ *is a reduced word for some permutation.*

Definition 16. The set of *atoms* of a (p,q)-clan γ is the set of permutations $\mathcal{A}(\gamma) \subseteq S_n$ such that $\mathcal{R}(\gamma) = \bigcup_{w \in \mathcal{A}(\gamma)} \mathcal{R}(w)$.

The set $\mathcal{A}(\gamma)$ is guaranteed to exist by Lemma 15.

Example 17. Example 13 shows that $\mathcal{A}(--+) = \{s_1s_2\} = \{231\}$ and $\mathcal{A}(-+-) = \{s_1s_2, s_2s_1\} = \{231, 312\}$. A more interesting example: $\mathcal{A}(+--+) = \{4132, 3241\}$, because

$$\begin{aligned}\mathcal{R}(+--+) &= \{\mathbf{2321}, \mathbf{3231}, \mathbf{3213}, \mathbf{1231}, \mathbf{1213}, \mathbf{2123}\}\\ &= \{\mathbf{2321}, \mathbf{3231}, \mathbf{3213}\} \cup \{\mathbf{1231}, \mathbf{1213}, \mathbf{2123}\}\\ &= \mathcal{R}(4132) \cup \mathcal{R}(3241).\end{aligned}$$

Given a word a, define $\mathrm{Clan}_{p,q}(a) = \{\gamma \in \mathrm{Clan}_{p,q} : a \in \mathcal{R}(\gamma)\}$. If w is a permutation, we also define $\mathrm{Clan}_{p,q}(w)$ to be $\{\gamma \in \mathrm{Clan}_{p,q} : w \in \mathcal{A}(\gamma)\}$.

Definition 18. For fixed p, q, let $\overset{\mathcal{R}}{\sim}$ be the equivalence relation on words on $\mathbb{N}$ defined by $a \overset{\mathcal{R}}{\sim} b$ if and only if $\mathrm{Clan}_{p,q}(a) = \mathrm{Clan}_{p,q}(b)$. Let $\overset{\mathcal{A}}{\sim}$ be the equivalence relation on S_n defined by $v \overset{\mathcal{A}}{\sim} w$ if and only if $\mathrm{Clan}_{p,q}(v) = \mathrm{Clan}_{p,q}(w)$.

Note that $\overset{\mathcal{R}}{\sim}$ is the strongest equivalence relation on words for which each $\mathcal{R}(\gamma)$ for $\gamma \in \mathrm{Clan}_{p,q}$ is a union of equivalence classes. Tits' lemma (Theorem 4) and Lemma 15 imply that $a \overset{\mathcal{R}}{\sim} b$ if and only if $a \in \mathcal{R}(v), b \in \mathcal{R}(w)$ for some $v, w \in S_n$ with $v \overset{\mathcal{A}}{\sim} w$.

We first study $\overset{\mathcal{A}}{\sim}$, for which we need a more explicit description of the sets $\mathcal{A}(\gamma)$ due to Can, Joyce, and Wyser [7]. Given a subset $S \subseteq [n]$ and a clan $\gamma \in \mathrm{Clan}_{p,q}$, call a pair $(i < j) \in S$ *valid* if either i and j are matched by γ, or if they are fixed points of opposite sign which are adjacent in the sense that there is no $i' \in S$ with $i < i' < j$. Consider the following algorithm which (nondeterministically) builds a permutation $w \in S_n$ by removing one pair of points from $[n]$ at a time and correspondingly deciding upon two entries of w. Set $S := [n]$ to start.

Algorithm 19.

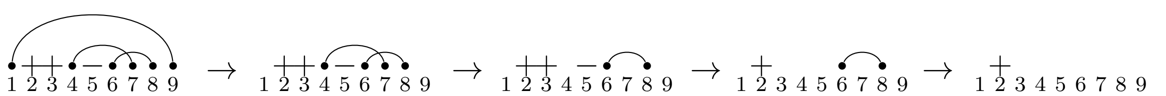
(a) Choose a valid pair $(i < j) \in S$ such that γ has no *matched* pair $i', j' \in S$ with $i' < i < j < j'$.

- If $i < j$ are matched by γ, set $w(i) = s+1$ and $w(j) = n-s$, where $s = (n-|S|)/2$ (this is the number of pairs deleted from $[n]$ in step (c) so far).
- If $i < j$ are fixed by γ, set $w(i) = n - s$ and $w(j) = s + 1$, with s as above.

(b) If S consists entirely of fixed points of γ of the same sign, fill in the remaining undefined entries of w with the unused entries of $[n]$ in increasing order, and return w.

(c) If we did not finish in step (b), then replace S with $S \setminus \{i, j\}$ and go back to (a).

Example 20. Here is one way this algorithm can run when $\gamma = (1\,9)(2^+)(3^+)(4\,7)(5^-)(6\,8)$:



$w = \_\_\_\_\_\_\_\_\_$ $\quad w = 1\_\_\_\_\_\_\_9$ $\quad w = 1\_\_2\_\_8\_9$ $\quad w = 1\_723\_8\_9$ $\quad w = 1\_7236849$

At this point no more pairs can be selected in step (a), so the algorithm returns 157236849.

Theorem 21 ([7]). *$\mathcal{A}(\gamma)$ is the set of permutations which can be generated by Algorithm 19.*

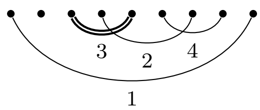
We note that [7] works with the set $\mathcal{W}(\gamma) := \{w^{-1} : w \in \mathcal{A}(\gamma)\}$ rather than our $\mathcal{A}(\gamma)$.

The possible outcomes of Algorithm 19 can be also encoded by recording, for each i which is removed in the course of the algorithm, which step it was removed at.

Definition 22. A *labelled shape* for γ is a pair (ω, F) where ω is the partial function $[n] \dashrightarrow \mathbb{N}$ obtained from an instance of Algorithm 19 by setting $\omega(i) = \omega(j) = k$ if $\{i, j\}$ is the k^{th} pair deleted from $[n]$ in step (c) of the algorithm, and F is the subset of the domain of ω consisting of fixed points of γ.

We think of a labelled shape (ω, F) as the edge-labelled partial matching on $[n]$ with an arc labeled k matching i and j for each $\omega^{-1}(k) = \{i, j\}$, where the arc is marked if $i, j \in F$. We draw these marked arcs as doubled edges. Given this marking, we will omit F from the notation since it can be recovered as the set of endpoints of the marked arcs.

Example 23. The instance of Algorithm 19 in Example 20 gives the labelled shape



We have drawn the arcs below the baseline to avoid confusion with the matchings in a clan. These diagrams help explain why Theorem 21 is true. When following a maximal chain up from $\gamma_{p,q}$ to γ, each matching $(k, n-k+1)$ in $\gamma_{p,q}$ eventually becomes either a matching in γ or a pair of opposite-sign fixed points, which we record as an arc labelled k in the labelled shape.

It is not hard to give a more direct characterization of the labelled shapes of a clan.

Proposition 24. *Let ω be a partial matching on $[n]$ with its e arcs labelled $1, 2, \ldots, e$, where arcs may be marked or unmarked. Then ω is a labelled shape for $\gamma \in \mathrm{Clan}_{p,q}$ if and only if $e = \min(p, q)$, and for all arcs $\{i < j\}$ of ω,*

(i) i and j are either matched by γ or are a pair of fixed points of opposite sign, according to whether the arc $\{i < j\}$ is unmarked or marked respectively.

(ii) If $\{i < j\}$ is marked and $i < i' < j$, then $\omega(i')$ is defined and $\omega(i') < \omega(i) = \omega(j)$.

(iii) If $\{i' < j'\}$ is an unmarked arc of ω with $i' < i < j < j'$, then $\omega(i') = \omega(j') < \omega(i) = \omega(j)$.

Proof. The number of fixed points remaining in step (b) of Algorithm 19 after all possible pairs $\{i, j\}$ have been removed is

$$|(\#\text{ of }+\text{'s in }\gamma) - (\#\text{ of }-\text{'s in }\gamma)| = |p - q|.$$

The number of pairs which were removed is therefore $m = \min(p, q)$, so the image of ω is $[m]$ and every $k \in [m]$ has $|\omega^{-1}(k)| = 2$.

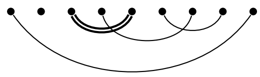
If the algorithm removes a pair i, j then it must have already removed all pairs i', j' matched by γ with $i' < i < j < j'$, so (iii) is necessary, and if i, j were matched by γ then this is the only condition needed for i, j to be removable. To remove a pair i, j fixed by γ (so $\{i, j\}$ is marked), one also needs that every i' with $i < i' < j$ has already been removed, meaning $\omega(i') < \omega(i) = \omega(j)$ as demanded by (ii). ☐

Given an atom $w \in \mathcal{A}(\gamma)$, let $\mathrm{lsh}(w)$ be the corresponding labelled shape. Explicitly, the arcs of $\mathrm{lsh}(w)$ are $\{w^{-1}(k), w^{-1}(n-k+1)\}$ for $k = 1, 2, \ldots, \min(p, q)$, each arc being marked or unmarked according to whether $w^{-1}(k) > w^{-1}(n-k+1)$ or $w^{-1}(k) < w^{-1}(n-k+1)$. Recall that we are trying to characterize the equivalence relation on permutations where $v \overset{\mathcal{A}}{\sim} w$ if $\mathrm{Clan}_{p,q}(v) = \mathrm{Clan}_{p,q}(w)$. The set $\mathrm{Clan}_{p,q}(w)$ is easy to compute from $\mathrm{lsh}(w)$, and in fact the edge labelling on $\mathrm{lsh}(w)$ is not even necessary for this.

Definition 25. The *unlabelled shape* $\mathrm{ush}(w)$ of w is the pair (π, F) where π is the partial matching obtained from $\mathrm{lsh}(w)$ by removing the arc labels, and F is the set of endpoints of marked arcs in $\mathrm{lsh}(w)$.

As before, we consider $\mathrm{ush}(w)$ to be a partial matching with some arcs marked and omit mention of F.

Example 26. Drawing marked arcs as doubled edges, the unlabelled shape of $w = 157236849$ (whose labelled shape is shown in Example 23) is



Theorem 27. *Let v and w be atoms for some members of* $\mathrm{Clan}_{p,q}$. *Then $v \overset{\mathcal{A}}{\sim} w$ if and only if* $\mathrm{ush}(v) = \mathrm{ush}(w)$.

Proof. First, $\mathrm{Clan}_{p,q}(v)$ depends only on $\mathrm{ush}(v)$. Indeed, if $\mathrm{ush}(v)$ has e marked arcs then $\mathrm{Clan}_{p,q}(v)$ consists of the 2^e clans obtained by:

- Replacing each marked arc $\{i, j\}$ by fixed points i^+, j^- or i^-, j^+;
- Leaving each unmarked arc as a matching;
- Leaving each unmatched point as a fixed point whose sign is the sign of $p - q$.

This shows that if $\mathrm{ush}(v) = \mathrm{ush}(w)$ then $v \overset{\mathcal{A}}{\sim} w$.

Conversely, suppose $\mathrm{ush}(v) \neq \mathrm{ush}(w)$. If the unmarked arcs of $\mathrm{ush}(v)$ are different from those of $\mathrm{ush}(w)$, then by the previous paragraph every clan in $\mathrm{Clan}_{p,q}(v)$ has different arcs than every clan in $\mathrm{Clan}_{p,q}(w)$, so assume all unmarked arcs are the same. Then there must be, say, a marked arc $\{i, j\}$ in $\mathrm{ush}(v)$ such that i, j are not connected by a marked arc in $\mathrm{ush}(w)$. But then there are clans in $\mathrm{Clan}_{p,q}(w)$ which give the same sign to i and j, while every clan in $\mathrm{Clan}_{p,q}(v)$ gives them opposite signs. In any case, $\mathrm{Clan}_{p,q}(v) \neq \mathrm{Clan}_{p,q}(w)$ so $v \overset{\mathcal{A}}{\not\sim} w$. ☐

An adjacent transposition s_k where $k < \min(p, q)$ acts on a labelled shape ω by swapping the labels k and $k{+}1$, giving a new partial matching $s_k\omega$ with labelled and possibly marked arcs, although $s_k\omega$ may not be a valid labelled shape for a clan.

Lemma 28. *Let $w \in \mathcal{A}(\gamma)$ and $k < \min(p, q)$. Then $s_k \operatorname{lsh}(w)$ is a labelled shape for γ if and only if $\ell(s_k s_{n-k} w) = \ell(w)$, and if that holds then $s_k \operatorname{lsh}(w) = \operatorname{lsh}(s_k s_{n-k} w)$.*

Proof. The map lsh^{-1} sending a labelled shape to its associated atom makes sense when applied to any partial matching with labelled and marked arcs, though the result may not be an atom. In particular, it sends $s_k \operatorname{lsh}(w)$ to the permutation $s_k s_{n-k} w$ regardless of whether the former is a valid labelled shape; here and below, it is helpful to note here that $s_k s_{n-k} = s_{n-k} s_k$ since $k < \min(p, q)$.

There are two cases in which $s_k \operatorname{lsh}(w)$ is not a valid labelled shape:

- Suppose the arc $\{i < j\}$ of $\operatorname{lsh}(w)$ labeled $k{+}1$ is nested inside the arc $\{i' < j'\}$ labeled k, meaning that $i' < i < j < j'$, and that the arc labeled k is unmarked. Then w has the form

  $$\cdots k \cdots k{+}1 \cdots n{-}k \cdots n{-}k{+}1 \cdots \quad \text{or} \quad \cdots k \cdots n{-}k \cdots k{+}1 \cdots n{-}k{+}1 \cdots$$

  and $\ell(s_k s_{n-k} w) = \ell(w) + 2$.

- Suppose the arc $\{i < j\}$ of $\operatorname{lsh}(w)$ labeled $k{+}1$ is marked, and that the arc $\{i' < j'\}$ labeled k has $i < i' < j$ or $i < j' < j$. If $\{i' < j'\}$ is marked, the definition of labeled shape forces $i < i' < j' < j$, so w has the form

  $$\cdots n{-}k \cdots n{-}k{+}1 \cdots k \cdots k{+}1 \cdots$$

  If $\{i' < j'\}$ is unmarked, then depending on exactly where i', j' are positioned with respect to i, j, the permutation w has one of the forms

  $$\begin{gathered} \cdots n{-}k \cdots k \cdots n{-}k{+}1 \cdots k{+}1 \cdots \\ \cdots n{-}k \cdots k \cdots k{+}1 \cdots n{-}k + 1 \cdots \\ \cdots k \cdots n{-}k \cdots n{-}k + 1 \cdots k{+}1 \cdots \end{gathered}$$

  In all of these cases, $\ell(s_k s_{n-k} w) = \ell(w) + 2$ again.

Conversely, suppose $\ell(s_k s_{n-k} w) \neq \ell(w)$, so $\ell(s_k s_{n-k} w) = \ell(w) \pm 2$. If $\ell(s_k s_{n-k} w) = \ell(w) + 2$, then k precedes $k + 1$ in the one-line notation of w and $n - k$ precedes $n - k + 1$. There are 6 permutations of $k, k{+}1, n{-}k, n{-}k{+}1$ for which this holds, and they are exactly the 6 cases we considered above in which $s_k \operatorname{lsh}(w)$ is not a valid labelled shape.

So, suppose $\ell(s_k s_{n-k} w) = \ell(w) - 2$. We claim that in this case, $\operatorname{lsh}(w)$ could not have been a valid labelled shape to begin with. Now $k + 1$ precedes k in w and $n - k + 1$ precedes $n - k$, and the 6 possibilities can be checked directly. If w has one of the forms

$$\begin{gathered} \cdots n{-}k{+}1 \cdots k{+}1 \cdots k \cdots n{-}k \cdots \\ \cdots n{-}k{+}1 \cdots k{+}1 \cdots n{-}k \cdots k \cdots \\ \cdots n{-}k{+}1 \cdots n{-}k \cdots k{+}1 \cdots k \cdots \\ \cdots k{+}1 \cdots n{-}k{+}1 \cdots k \cdots n{-}k \cdots \end{gathered}$$

then the arc $\{i < j\}$ in $\mathrm{lsh}(w)$ labelled k is marked, yet there is $i < i' < j$ such that i' is labelled $k{+}1$, contradicting Proposition 24(ii). If w has the form

$$\cdots k{+}1 \cdots k \cdots n{-}k{+}1 \cdots n{-}k \cdots \quad \text{or} \quad \cdots k{+}1 \cdots n{-}k{+}1 \cdots k \cdots n{-}k \cdots$$

then the arc in $\mathrm{lsh}(w)$ labelled k is nested inside the unmarked arc labelled $k{+}1$, contradicting Proposition 24(iii). □

Theorem 29. *When restricted to the set of atoms for members of* $\mathrm{Clan}_{p,q}$*, the relation* $\overset{\mathcal{A}}{\sim}$ *agrees with the transitive closure of the relations* $u \sim s_k s_{n-k} u$ *where* $\ell(s_k s_{n-k} u) = \ell(u)$ *and* $k < \min(p, q)$*.*

Proof. Lemma 28 implies that if $\ell(s_k s_{n-k} u) = \ell(u)$ where $k < \min(p, q)$, then u and $s_k s_{n-k} u$ have the same unlabelled shape, so $u \overset{\mathcal{A}}{\sim} s_k s_{n-k} u$ by Theorem 27.

Conversely, suppose $v \overset{\mathcal{A}}{\sim} w$, so $\mathrm{ush}(v) = \mathrm{ush}(w)$. The labelled shapes $\mathrm{lsh}(v)$ and $\mathrm{lsh}(w)$ are certainly connected by a series of applications of adjacent transpositions, so v and w are connected by transformations $u \mapsto s_k s_{n-k} u$ by Lemma 28, but we must see that this can be done in such a way that all of the intermediate steps are valid labelled shapes.

Proposition 24 shows that the valid labellings of the unlabelled shape $\mathrm{ush}(w)$ can be thought of as the linear extensions of a poset. The elements of the poset are the arcs of $\mathrm{ush}(w)$, and $\{i' < j'\} \leqslant \{i < j\}$ if either:

- $\{i' < j'\}$ is unmarked and $i' < i < j < j'$; or,
- $\{i < j\}$ is marked and $i < i' < j$ or $i < j' < j$.

The theorem now holds by the following general fact: if P is a finite poset and G is the graph whose vertices are the linear extensions $f : P \to [\#P]$ with an edge (f, g) whenever $g = s_i \circ f$, then G is connected. To prove this, fix a single linear extension f_0 and use it to identify P with $[\#P]$, so that any linear extension can be identified with a permutation $\pi = \pi_1 \cdots \pi_{\#P}$ in which i appears before j whenever $i \leqslant_P j$. With this identification, there is an edge (π, π') in G if $\pi' = \pi \circ s_i$. Let us see by induction on inversion number that every vertex π of G is connected to the identity permutation. The base case that π has no inversions is trivial. Otherwise, there is i such that $\pi_i > \pi_{i+1}$. Then πs_i is again a linear extension, is connected by π by definition, and is connected to the identity by induction. □

We can now prove Theorem 5, which we restate here. Let $\equiv$ be the equivalence relation on the set of words on $\mathbb{N}$ defined as the transitive closure of the relations $a_1 a_2 \cdots a_\ell \equiv (n - a_1) a_2 \cdots a_\ell$ together with the braid relations (2).

Theorem 30 (Theorem 5)**.** *If* a *and* b *are both reduced words for* (p, q)*-clans, then* $a \equiv b$ *if and only if* $\mathrm{Clan}_{p,q}(a) = \mathrm{Clan}_{p,q}(b)$*.*

Proof. The theorem asserts that the equivalence relations $\equiv$ and $\overset{\mathcal{R}}{\sim}$ agree when restricted to the set of reduced words of (p,q)-clans. Suppose $a = a_1a_2\cdots a_\ell \in \mathcal{R}(\gamma)$ and $a \equiv b$. By Lemma 15, Coxeter relations preserve $\mathcal{R}(\gamma)$, so we can assume $b = (n-a_1)a_2\cdots a_\ell$. We claim $b \in \mathcal{R}(\gamma)$ as well. Since $a_1 < \min(p,q)$, it holds that $\gamma_{p,q} * s_{a_1}$ is well-defined and equal to $(\cdots(\gamma * s_{a_\ell}) * \cdots) * s_{a_2}$. But also $\gamma_{p,q} * s_{a_1} = \gamma_{p,q} * s_{n-a_1}$, so

$$\begin{aligned}((\cdots(\gamma * s_{a_\ell}) * \cdots) * s_{a_2}) * s_{n-a_1} &= (\gamma_{p,q} * s_{a_1}) * s_{n-a_1} \\ &= (\gamma_{p,q} * s_{n-a_1}) * s_{n-a_1} \\ &= \gamma_{p,q},\end{aligned}$$

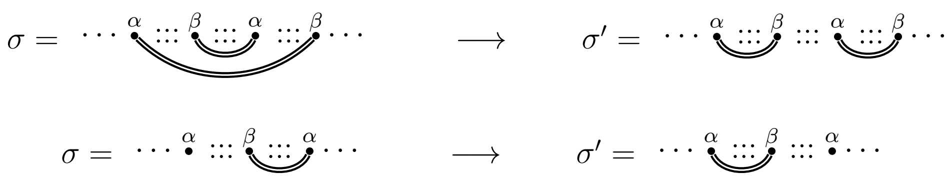
which implies that $b \in \mathcal{R}(\gamma)$. We have $a \in \mathcal{R}(u)$ and $b \in \mathcal{R}(s_{n-a_1}s_{a_1}u)$ for some $u \in \mathcal{A}(\gamma)$, and $\ell(s_{n-a_1}s_{a_1}u) = \ell(u)$ as a and b have equal length. Theorem 29 shows $u \sim s_{n-a_1}s_{a_1}u$, so $a \overset{\mathcal{R}}{\sim} b$.

Conversely, suppose $a \overset{\mathcal{R}}{\sim} b$ where a, b are reduced words for some (p,q)-clans. Then there are atoms $v \sim w$ with $a \in \mathcal{R}(v)$ and $b \in \mathcal{R}(w)$, and applying Theorem 29, we can assume that $w = s_{n-k}s_kv$ where $k < \min(p,q)$. Since $s_ks_{n-k} = s_{n-k}s_k$ and $\ell(v) = \ell(s_{n-k}s_kv)$, exactly one of s_k and s_{n-k} is a left descent of v, say s_k. Then there is $a' \in \mathcal{R}(v)$ with $a'_1 = k$, and the equality $\ell(v) = \ell(s_{n-k}s_kv)$ implies that $b' := (n-a'_1)a'_2\cdots a'_\ell$ is a reduced word of $s_{n-k}s_kv = w$. Since a is related to a' and b to b' via Coxeter relations by Tits' lemma, we see that $a \equiv a' \equiv b' \equiv b$. $\square$

Remark 31. Theorem 5 actually holds assuming only that *one* of a and b is known to be a reduced word of some (p,q)-clan. Indeed, if a is a reduced word and $a \equiv b$, then b is also a reduced word by the first paragraph of the proof above. If a is a reduced word and $a \overset{\mathcal{R}}{\sim} b$, then $\mathrm{Clan}_{p,q}(b) = \mathrm{Clan}_{p,q}(a) \neq \emptyset$, so b is also a reduced word.

Theorem 5 can be interpreted as giving a simple prescription for generating each equivalence class making up $\mathcal{R}(\gamma)$ beginning with one reduced word. It is also natural to ask for simple transformations relating the equivalence classes to each other. We will think about transformations of unlabelled shapes, since these index the equivalence classes of $\overset{\mathcal{A}}{\sim}$ by Theorem 27. Let $\mathrm{ush}(\mathcal{A}(\gamma))$ be the set of unlabelled shapes for γ.

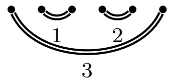
Lemma 32. *Let γ be a (p,q)-clan. Suppose σ and σ' are partial matchings of $[n] = [p+q]$ in which arcs can be unmarked or marked, and assume that $\sigma \in \mathrm{ush}(\mathcal{A}(\gamma))$. Write $\sigma \to \sigma'$ if σ and σ' are related by a transformation of the form*



where $\cdots$ conceals an arbitrary partial matching (with marked/unmarked arcs), ::: conceals only a complete *matching (no unpaired fixed points allowed), and $\{\alpha,\beta\} = \{+,-\}$; here α, β are the signs assigned by γ to the points above which they are shown. Then $\sigma' \in \mathrm{ush}(\mathcal{A}(\gamma))$, and the directed graph with vertices $\mathrm{ush}(\mathcal{A}(\gamma))$ and edges $\sigma \to \sigma'$ is acyclic.*

Proof. Proposition 24 implies that the unlabelled shapes of γ are the partial matchings of $[n]$ with $\min(p,q)$ arcs, each arc having a pair of endpoints which are either matched by γ or opposite-sign fixed points, and such that no two marked arcs cross and no unpaired fixed point is underneath a marked arc. From this description it is clear that the transformations in the theorem do preserve $\operatorname{ush}(\mathcal{A}(\gamma))$.

Given $\sigma \in \operatorname{ush}(\mathcal{A}(\gamma))$, label the *right* endpoints of the marked arcs $1, 2, \ldots$ from left to right, and then label each marked arc according to its right endpoint. This is a labelled shape of γ; write $\operatorname{st}(\sigma) \in \mathcal{A}(\gamma)$ for the associated atom. For instance,

$\sigma =$  $\rightsquigarrow$ 

so $\operatorname{st}(\sigma) = 461523$.

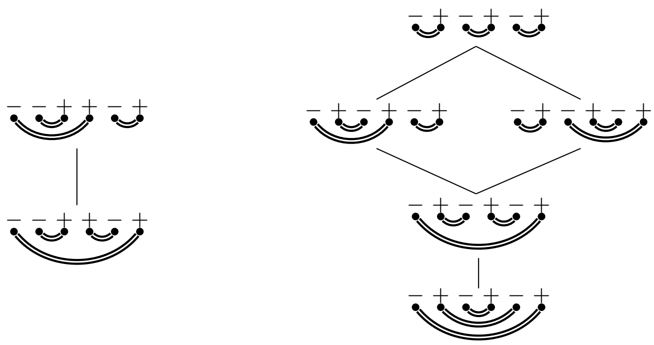
If $\sigma \to \sigma'$, then $\operatorname{st}(\sigma)$ and $\operatorname{st}(\sigma')$ are related by transformations of the form

$$\cdots n{-}\ell{+}1 \cdots n{-}k{+}1 \cdots k \cdots \ell \cdots \to \cdots n{-}k'{+}1 \cdots k' \cdots n{-}\ell{+}1 \cdots \ell \cdots \qquad (k' \leqslant k < \ell)$$
$$\cdots j \cdots n{-}k{+}1 \cdots k \cdots \to \cdots n{-}k'{+}1 \cdots k' \cdots j \cdots \qquad (k' \leqslant \min(p,q) < j < \max(p,q))$$

In both cases, $\operatorname{st}(\sigma')$ is lexicographically larger than $\operatorname{st}(\sigma)$, which shows $\operatorname{ush}(\mathcal{A}(\gamma))$ is acyclic. $\square$

Lemma 32 gives $\operatorname{ush}(\mathcal{A}(\gamma))$ a poset structure, with a covering relation $\sigma \lessdot \sigma'$ when $\sigma \to \sigma'$.

Example 33. Here are the posets $\operatorname{ush}(\mathcal{A}(\gamma))$ for $\gamma = --++-+$ and $\gamma = -+-+-+-$:

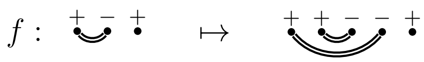


Theorem 34. *The poset* $\operatorname{ush}(\mathcal{A}(\gamma))$ *has a unique maximal element* $\sigma_{\max}$*, which can be constructed as follows. First,* $\sigma_{\max}$ *has an unmarked arc for every matching of* γ*. Next, find the minimal fixed point* i *of* γ *such that the minimal fixed point* $j > i$ *has opposite sign, and connect* i *and* j *by a marked arc in* $\sigma_{\max}$*; repeat this process, ignoring fixed points that have already been connected, until all remaining fixed points have the same sign.*

Proof. The unmarked arcs in $\sigma \in \operatorname{ush}(\mathcal{A}(\gamma))$ are determined by the matchings of γ, and play no role in the poset structure. We may therefore ignore them, and assume that γ is matchless. If γ has no pair of fixed points of opposite sign, then $\operatorname{ush}(\mathcal{A}(\gamma))$ has one element, so the theorem is trivially true. Otherwise, let i be minimal such that $\gamma(i)$ and

$\gamma(i+1)$ have opposite sign, and define $\bar{\gamma} \in \mathrm{Clan}_{p-1,q-1}$ by removing i and $i+1$ from the matching diagram of γ.

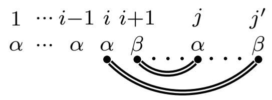
There is an injection $f : \mathrm{ush}(\mathcal{A}(\bar{\gamma})) \to \mathrm{ush}(\mathcal{A}(\gamma))$ which adds the fixed points $i, i{+}1$ back and connects them with a marked arc. For example, if $\gamma = {+}{+}{-}{-}{+}$ then $\bar{\gamma} = {+}{-}{+}$, and



By induction, $\mathrm{ush}(\mathcal{A}(\bar{\gamma}))$ has a unique maximal element $\sigma'_{\max}$ constructed as described in the theorem. Its image $f(\sigma'_{\max})$ equals the unlabeled shape $\sigma_{\max}$, which we must now see is actually the unique maximal element of $\mathrm{ush}(\mathcal{A}(\gamma))$. First, suppose $\sigma \in \mathrm{ush}(\mathcal{A}(\gamma))$ is in the image of f, or equivalently that σ has $\{i < i+1\}$ as a marked arc. Since f does not add any unpaired fixed points, any transformation which can be performed in $\mathrm{ush}(\mathcal{A}(\bar{\gamma}))$ can also be performed in $\mathrm{ush}(\mathcal{A}(\gamma))$, so f is a poset homomorphism. This implies $\sigma \leqslant f(\sigma'_{\max}) = \sigma_{\max}$.

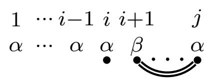
Now suppose σ is not in the image of f; we claim σ cannot be maximal. Consider two cases:

- Suppose σ pairs i with j' and $i{+}1$ with j. Then $i+1 < j < j'$, for otherwise there would be an unpaired fixed point of σ below a marked arc, or else two marked arcs would cross. That is, σ has the form



  where $\{\alpha, \beta\} = \{+, -\}$ and there are no unpaired fixed points in $[i, j']$. But now we can apply the transformation replacing the marked arcs $\{i < j'\}, \{i{+}1 < j\}$ by $\{i < i{+}1\}, \{j < j'\}$, so σ is not maximal.

- Suppose one of $i, i{+}1$ is unpaired in σ (they cannot both be unpaired). Then in fact i must be unpaired, because otherwise it would have to be paired with some $j > i{+}1$, but then the unpaired fixed point $i{+}1$ would be below the marked arc $\{i < j\}$. So, say $i{+}1$ is paired with j. We must have $j > i{+}1$, because otherwise the unpaired fixed point i would be below the marked arc $\{j < i{+}1\}$. That is, σ has the form



  Now we can apply the transformation replacing the marked arc $\{i{+}1 < j\}$ by $\{i < i{+}1\}$, so σ is not maximal. □

Theorem 34 gives a prescription for generating all of $\mathrm{ush}(\mathcal{A}(\gamma))$ from one element $\sigma_{\max}$ by applying simple transformations. It would be interesting to be able to do this at the level of reduced words: that is, to give a uniform way of beginning with a relation $\mathrm{ush}(v) \to \mathrm{ush}(w)$ and producing $a \in \mathcal{R}(v)$ and $b \in \mathcal{R}(w)$ which are related in some simple way.

3 Enumerating reduced words for clans

Definition 35 ([2]). Let $a = a_1 \cdots a_\ell$ be a word on the alphabet $\mathbb{N}$. A *compatible sequence* for a is a word b of length ℓ such that

- $1 \leqslant b_1 \leqslant \cdots \leqslant b_\ell$
- $b_i \leqslant a_i$ for each i
- For each i, if $a_i < a_{i+1}$, then $b_i < b_{i+1}$.

We use bold for compatible sequences just as for reduced words.

Let $\mathrm{comp}(a)$ be the set of compatible sequences for a. For instance, $\mathrm{comp}(\mathbf{3213}) = \{\mathbf{1112}, \mathbf{1113}\}$ while $\mathrm{comp}(\mathbf{3231})$ is empty.

Definition 36. The *Schubert polynomial* of a permutation $w \in S_n$ is

$$\mathfrak{S}_w = \sum_{a \in \mathcal{R}(w)} \sum_{b \in \mathrm{comp}(a)} x_{b_1} \cdots x_{b_\ell}.$$

The *Stanley symmetric function* of w is the formal power series $F_w = \lim_{m\to\infty} \mathfrak{S}_{w^{+m}}$, where w^{+m} is the permutation defined inductively by $w^{+m} = (w^{+(m-1)})^{+1}$ and $w^{+1} = 1(w_1+1)\cdots(w_n+1)$ in one-line notation.

Following work of Bernstein, Gelfand, and Gelfand [1], Lascoux and Schützenberger originally defined Schubert polynomials recursively using divided difference operators (cf. (4) below). It is a theorem of Billey, Jockusch, and Stanley [2] that Definition 36 yields the same polynomials. It is not hard to check that $\lim_{m\to\infty} \mathfrak{S}_{w^{+m}}$ does exist as a formal power series, so that F_w is well-defined. The fact that it is actually a symmetric function is rather less obvious, at least from Definition 36, and was proved by Stanley [21].

Definition 37. The *Schubert polynomial* of a (p, q)-clan γ is

$$\mathfrak{S}_\gamma = \sum_{a \in \mathcal{R}(\gamma)} \sum_{b \in \mathrm{comp}(a)} x_{b_1} \cdots x_{b_\ell}.$$

Note that since $\mathcal{R}(\gamma) = \bigsqcup_{w \in \mathcal{A}(\gamma)} \mathcal{R}(w)$, we have $\mathfrak{S}_\gamma = \sum_{w\in\mathcal{A}(\gamma)} \mathfrak{S}_w$.

Example 38. As per Example 17,

$$\mathcal{R}(+--+) = \{\mathbf{3213}, \mathbf{3231}, \mathbf{2321}, \mathbf{1213}, \mathbf{1231}, \mathbf{2123}\}.$$

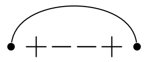
$\mathrm{comp}(a)$ is empty for all $a \in \mathcal{R}(+--+)$ except $\mathbf{3213}$ and $\mathbf{2123}$, while $\mathrm{comp}(\mathbf{3213}) = \{\mathbf{1112}, \mathbf{1113}\}$ and $\mathrm{comp}(\mathbf{2123}) = \{\mathbf{1123}\}$. Thus $\mathfrak{S}_{+--+} = x_1^3x_2 + x_1^3x_3 + x_1^2x_2x_3$. Alternatively, $\mathcal{A}(+--+) = \{4132, 3241\}$ and

$$\mathfrak{S}_{+--+} = \mathfrak{S}_{4132} + \mathfrak{S}_{3241} = (x_1^3x_2 + x_1^3x_3) + (x_1^2x_2x_3).$$

Definition 39. The *Stanley symmetric function* of γ is

$$F_\gamma = \lim_{m\to\infty} \mathfrak{S}_{\gamma^{+m}},$$

where γ^{+m} is the $(p+m, q+m)$-clan defined inductively by $\gamma^{+m} = (\gamma^{+(m-1)})^{+1}$ and where γ^{+1} is obtained from γ by shifting all of $1, 2, \ldots, n$ up by one and then multiplying by the cycle $(1, n{+}2)$.

For example, $(+--+)^{+1} =$ . Of course, Definition 39 means nothing until we check that the polynomials $\mathfrak{S}_{\gamma^{+m}}$ actually converge to a formal power series as $m \to \infty$.

Proposition 40. *$\mathcal{A}(\gamma^{+1}) = \{w^{+1} : w \in \mathcal{A}(\gamma)\}$, and F_γ is a well-defined symmetric function equal to $\sum_{w\in\mathcal{A}(\gamma)} F_w$.*

Proof. Given a word $a = a_1 \cdots a_\ell$, let $a^{+1} = (a_1{+}1)\cdots(a_\ell{+}1)$. It is clear that if $a \in \mathcal{R}(\gamma)$, then $a^{+1} \in \mathcal{R}(\gamma^{+1})$.

For the converse, let $\operatorname{maxcyc}(\alpha)$ denote the size of the largest cycle in a clan α, i.e. the maximum of $|j-i|$ over all 2-cycles $(i\,j)$ in α, or 0 if α is matchless. We claim that if $\alpha \in \mathrm{Clan}_{p+1,q+1}$ has a reduced word b containing 1 or $n+1 = p+q+1$, then $\operatorname{maxcyc}(\alpha) < n+1$. First, the minimal element $\gamma_{p+1,q+1}$ contains the cycle $(1, n+2)$, and it is easy to see that if there is a saturated chain from $\gamma_{p+1,q+1}$ to some α' only involving edge labels in $[2, n]$, then α' must still contain $(1, n+2)$. On the other hand, if $\alpha' = \alpha'' * s_1$ or $\alpha'' * s_n$, then α'' does *not* contain $(1, n+2)$, and so $\operatorname{maxcyc}(\alpha'') < n+1$. Finally, considering each possibility in Figure 1 shows that if $\beta_1 \leqslant \beta_2$ in weak order, then $\operatorname{maxcyc}(\beta_1) \geqslant \operatorname{maxcyc}(\beta_2)$. If α has a reduced word containing 1 or $n+1$, then there exists a chain $\gamma_{p+1,q+1} \leqslant \alpha' \lessdot \alpha'' \leqslant \alpha$ of the form just described, and so $\operatorname{maxcyc}(\alpha) < n+1$ as claimed.

In particular, $\operatorname{maxcyc}(\gamma^{+1}) = n+1$, so all reduced words of γ^{+1} are supported on $[2, n]$. This means $a \mapsto a^+$ is a bijection $\mathcal{R}(\gamma) \to \mathcal{R}(\gamma^+)$, hence $\mathcal{A}(\gamma^{+1}) = \{w^{+1} : w \in \mathcal{A}(\gamma)\}$. Now

$$F_\gamma = \lim_{m\to\infty} \sum_{w\in\mathcal{A}(\gamma^{+m})} \mathfrak{S}_w = \lim_{m\to\infty} \sum_{w\in\mathcal{A}(\gamma)} \mathfrak{S}_{w^{+m}} = \sum_{w\in\mathcal{A}(\gamma)} F_w. \qquad \square$$

Proposition 41. *Letting ℓ be the degree of F_γ, the coefficient of $x_1x_2\cdots x_\ell$ in F_γ is $\#\mathcal{R}(\gamma)$.*

Proof. If $m \geqslant \ell - 1$, then every letter of a^{+m} for $a \in \mathcal{R}(\gamma)$ is at least ℓ, and so $\operatorname{comp}(a^{+m})$ contains $\mathbf{12}\cdots\ell$. Proposition 40 therefore shows that the coefficient of $x_1x_2\cdots x_\ell$ in $\mathfrak{S}_{\gamma^{+m}}$ is $\#\mathcal{R}(\gamma^{+m}) = \#\mathcal{R}(\gamma)$ for all $m \geqslant \ell - 1$. $\square$

This proposition holds equally well for Stanley symmetric functions of permutations, which was Stanley's motivation for defining F_w. One can then use symmetric function techniques to extract coefficients of F_w and enumerate $\mathcal{R}(w)$. For instance, Stanley showed that $F_{n\cdots21}$ is the Schur function $s_{(n-1,n-2,\ldots,1)}$, and comparing coefficients of $x_1x_2\cdots$ shows that $\#\mathcal{R}(n\cdots21)$ equals the number of standard tableaux of shape $(n-1, n-2, \ldots, 1)$ [21]. Our intent is to do the same for clans.

Definition 42. Let X_k be the alphabet $\{x_1, \ldots, x_k\}$. Let λ be a partition and ϕ a sequence of natural numbers of the same length. The *flagged Schur polynomial* of shape λ with flag ϕ is the polynomial $\sum_T x^T$ where T runs over all semistandard tableaux of shape T whose entries in each row i come from $\{1, 2, \ldots, \phi_i\}$, and $x^T = \prod_i x_i^{\#\text{ of } i\text{'s in } T}$. We write $s_\lambda(X_{\phi_1}, \ldots, X_{\phi_\ell})$ or just $s_\lambda(X_\phi)$ for this polynomial.

Example 43.

- $s_{21}(X_1, X_2) = x_1^2 x_2$ is a sum over the single tableau $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$.
- $s_{21}(X_2, X_2) = x_1^2 x_2 + x_1 x_2^2$ is a sum over the two tableaux $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$.
- $s_{21}(X_1, X_1) = 0$.

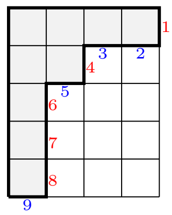
Definition 44. Given a matchless (p, q)-clan γ, let $\phi^+(\gamma)$ be the list of positions of the $+$'s in increasing order, and likewise $\phi^-(\gamma)$ the list of positions of the $-$'s. Also define two partitions $\lambda^+(\gamma)$ and $\lambda^-(\gamma)$ by

$$\begin{aligned} \lambda^+(\gamma)_i &= \#\{j : \phi^-(\gamma)_j > \phi^+(\gamma)_i\} \qquad &&\text{for } i = 1, \ldots, p \\ \lambda^-(\gamma)_i &= \#\{j : \phi^+(\gamma)_j > \phi^-(\gamma)_i\} \qquad &&\text{for } i = 1, \ldots, q. \end{aligned}$$

The map $\phi^+(\gamma) \mapsto \lambda^+(\gamma)$ is a bijection between p-subsets of $[p+q]$ and partitions λ whose Young diagram is contained in the $p \times q$ rectangle $[p] \times [q]$. Graphically, if one labels the $p + q$ segments of the southwest boundary of the (upper left justified) Young diagram of $\lambda^+(\gamma)$ with $1, 2, \ldots, p+q$ from top to bottom, the set of vertical segments is $\phi^+(\gamma)$ and the set of horizontal segments is $\phi^-(\gamma)$.

Example 45. Letting $\gamma = + - - + - + + + -$, we have

$$\begin{aligned} \phi^- &= (2, 3, 5, 9) \\ \lambda^- &= (4, 4, 3, 0) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi^+ &= (1, 4, 6, 7, 8) \\ \lambda^+ &= (4, 2, 1, 1, 1) \end{aligned} \qquad \text{and} \qquad \lambda^+ =$$



If $\lambda \subseteq [p] \times [q]$, write $\lambda^\vee$ for the partition whose Young diagram is the complement of λ in $[p] \times [q]$ (rotated 180°). Also let λ^t denote the partition conjugate to λ. For a matchless (p, q)-clan γ, let $\operatorname{rev}(\gamma)$ be the clan obtained by reversing the one-line notation of γ. Let $\operatorname{neg}(\gamma)$ be the (q, p)-clan obtained by switching the signs of all fixed points in γ. The next proposition is clear from the description above of the map $\gamma \mapsto \lambda^+(\gamma)$ in terms of lattice paths.

Proposition 46. $\lambda^+(\gamma)^\vee = \lambda^+(\operatorname{rev}(\gamma))$ *and* $\lambda^+(\gamma)^t = \lambda^+(\operatorname{neg}(\operatorname{rev}(\gamma))$, *and therefore*

$$\lambda^-(\gamma) = \lambda^+(\operatorname{neg}(\gamma)) = (\lambda^+(\gamma)^t)^\vee.$$

Wyser and Yong [26] defined clan Schubert polynomials by induction on weak order, with flagged Schur polynomials as base cases, and using the following operators on polynomials.

Definition 47. For $1 \leqslant i < n$, the *divided difference operator* ∂_i acting on $R[x_1, \ldots, x_n]$ for a commutative ring R sends f to $\partial_i f = (f - s_i f)/(x_i - x_{i+1})$, where s_i acts on $R[x_1, \ldots, x_n]$ by swapping x_i and x_{i+1}. The *isobaric divided difference operator* π_i is defined by $\pi_i(f) = \partial_i(x_i f)$.

Definition 48. The *Wyser-Yong Schubert polynomials* labeled by the members of $\mathrm{Clan}_{p,q}$ are defined by induction on clan weak order using the recurrence

$$\begin{aligned} \mathfrak{S}^{\mathrm{WY}}_{\gamma} &= s_{\lambda^+(\gamma)}(X_{\phi^+(\gamma)})\, s_{\lambda^-(\gamma)}(X_{\phi^-(\gamma)}) && \text{if } \gamma \text{ is matchless} \\ \mathfrak{S}^{\mathrm{WY}}_{\gamma * s_i} &= \partial_i \mathfrak{S}^{\mathrm{WY}}_{\gamma} && \text{if } \gamma * s_i < \gamma. \end{aligned}$$

Theorem 49. *Definition 48 makes sense: given a fixed γ, the polynomial $\mathfrak{S}^{\mathrm{WY}}_{\gamma}$ is independent of the choice of matchless clan γ' and saturated chain*

$$\gamma < \cdots < (\gamma' * s_{a_1}) * s_{a_2} < \gamma' * s_{a_1} < \gamma'.$$

used to compute it. Also, if $\gamma * s_i \not< \gamma$, then $\partial_i \mathfrak{S}^{\mathrm{WY}}_{\gamma} = 0$ (this includes the case where $\gamma * s_i$ is not defined).

Proof. The first sentence is [26, Theorem 2.10], where it is also shown that $\mathfrak{S}^{\mathrm{WY}}_{\gamma}$ represents the cohomology class $[\overline{Y}_{\gamma}]$. As for the second claim, suppose that $\partial_i \mathfrak{S}^{\mathrm{WY}}_{\gamma} \neq 0$. Comparing the description of ∂_i in [9, §10.3] with the geometric description of weak order in [26, §2.1], one sees that if $\partial_i[\overline{Y}_{\gamma}]$ is nonzero, then it equals some $[\overline{Y}_{\gamma'}]$ where $\gamma' < \gamma$ is a covering in weak order labeled by s_i. □

Brion [4] gives a formula for $[\overline{Y}_{\gamma}]$ as a sum of Schubert classes from which one can deduce a formula for $\mathfrak{S}^{\mathrm{WY}}_{\gamma}$ as a sum of Schubert polynomials. This formula is simply Definition 37, so the next lemma is not really new, but we include a self-contained proof because it is not entirely obvious that the summands in Brion's formula are indeed the $\mathfrak{S}_w$ for $w \in \mathcal{A}(\gamma)$.

Lemma 50. $\mathfrak{S}_{\gamma} = \mathfrak{S}^{\mathrm{WY}}_{\gamma}$ *for any clan* $\gamma \in \mathrm{Clan}_{p,q}$.

Proof. We claim that $\mathfrak{S}_{\gamma}$ satisfies the same recurrence as $\mathfrak{S}^{\mathrm{WY}}_{\gamma}$, namely, if $1 \leqslant i < n$, then

$$\partial_i \mathfrak{S}_{\gamma} = \begin{cases} \mathfrak{S}_{\gamma * s_i} & \text{if } \gamma * s_i < \gamma \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

Given that $\mathfrak{S}_{\gamma} = \sum_{w \in \mathcal{A}(\gamma)} \mathfrak{S}_w$ and that ordinary Schubert polynomials satisfy the recurrence

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w s_i} & \text{if } \ell(w s_i) < \ell(w) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

by [2, §1], this claim follows from two simple facts about atoms:

(i) If $\gamma * s_i < \gamma$, then $\mathcal{A}(\gamma * s_i) = \{ws_i : w \in \mathcal{A}(\gamma) \text{ and } \ell(ws_i) < \ell(w)\}$.

(ii) If $\gamma * s_i \not< \gamma$, then $\ell(ws_i) > \ell(w)$ for all $w \in \mathcal{A}(\gamma)$.

If $w \in \mathcal{A}(\gamma)$ and $\ell(ws_i) < \ell(w)$, then w has a reduced word ending in $\mathbf{i}$, so $\gamma * s_i < \gamma$; this proves (ii). As for (i), if $\gamma * s_i < \gamma$ then $\mathcal{R}(\gamma * s_i) = \{a_1 \cdots a_\ell : a_1 \cdots a_\ell \mathbf{i} \in \mathcal{R}(\gamma)\}$, which is equivalent to (i).

Now we show that $\mathfrak{S}_\gamma = \mathfrak{S}_\gamma^{\mathrm{WY}}$ for all γ by induction on the rank of γ in weak order. The base case is $\mathfrak{S}_{\gamma_{p,q}} = \mathfrak{S}^{\mathrm{WY}}_{\gamma_{p,q}} = 1$ (the second equality is clear from the geometry, if not from Definition 48). Equation (3) and Theorem 49 show that for any $i < n$,

$$\partial_i(\mathfrak{S}_\gamma - \mathfrak{S}_\gamma^{\mathrm{WY}}) = 0,$$

using that $\mathfrak{S}_{\gamma * s_i} = \mathfrak{S}^{\mathrm{WY}}_{\gamma * s_i}$ by induction. The kernel of ∂_i on $\mathbb{Z}[x_1, \ldots, x_n]$ consists of those polynomials symmetric in x_i and x_{i+1}, so this shows $\mathfrak{S}_\gamma - \mathfrak{S}_\gamma^{\mathrm{WY}}$ is symmetric in $x_1, \ldots, x_n$. By [26, Proposition 2.9], $\mathfrak{S}_\gamma^{\mathrm{WY}}$ is a linear combination of Schubert polynomials $\mathfrak{S}_w$ for $w \in S_n$, so the same is true of $\mathfrak{S}_\gamma - \mathfrak{S}_\gamma^{\mathrm{WY}}$. But the Schubert polynomials $\mathfrak{S}_w$ for $w \in S_n$ are linearly independent modulo the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by symmetric polynomials [17, §2.5.2], so $\mathfrak{S}_\gamma - \mathfrak{S}_\gamma^{\mathrm{WY}} = 0$. $\square$

Our next goal is to leverage the formulas of Wyser and Yong to prove enumerative results for clan words via the Stanley symmetric functions F_γ. To do this, we must better understand the procedure of passing from $\mathfrak{S}_\gamma$ to F_γ. An important fact about the divided difference operators ∂_i is that they satisfy the braid relations for S_n: that is, $\partial_i \partial_k = \partial_k \partial_i$ if $|i - k| > 1$ and $\partial_i \partial_j \partial_i = \partial_j \partial_i \partial_j$ if $|i - j| = 1$. As a consequence, we can define ∂_w as the composition $\partial_{a_1} \cdots \partial_{a_\ell}$ for a reduced word $a \in \mathcal{R}(w)$, and the resulting operator is independent of the choice of a. The same holds for the π_i.

Lemma 51 ([12], Theorem 3.40; [15], equation (4.25))**.** *Let $w_n = n(n-1) \cdots 21 \in S_n$. If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and $N \geqslant n$, then $\pi_{w_N} f$ is a symmetric polynomial in $x_1, \ldots, x_N$. Moreover, $\lim_{N \to \infty} \pi_{w_N} \mathfrak{S}_w = F_w$ for any permutation w.*

It follows by linearity that $\pi_{w_n} \mathfrak{S}_\gamma = F_\gamma$ for $\gamma \in \mathrm{Clan}_{p,q}$. The result we are working towards is that, if γ is matchless, then $F_\gamma = s_{\lambda^+(\gamma)} s_{\lambda^-(\gamma)}$. While it is true that the two Schur functions here are the images under $\lim_{N \to \infty} \pi_{w_N}$ of the two factors in Definition 48, in general π_{w_N} is not a ring homomorphism, so we must work a little harder.

Lemma 52 ([15], equation (3.10))**.** *Suppose k is such that $\phi_k \neq \phi_{k'}$ for all $k' \neq k$. Then*

$$\pi_{\phi_k} s_\lambda(X_{\phi_1}, \ldots, X_{\phi_k}, \ldots, X_{\phi_\ell}) = s_\lambda(X_{\phi_1}, \ldots, X_{\phi_k + 1}, \ldots, X_{\phi_\ell}).$$

If $i \notin \{\phi_1, \ldots \phi_\ell\}$, then $\pi_i s_\lambda(X_\phi) = s_\lambda(X_\phi)$.

Proof. Let us first verify this when $\lambda = (d)$ has length 1, so $s_\lambda(X_r)$ is the homogeneous symmetric polynomial $h_d(X_r) = h_d(x_1, \ldots, x_r)$, and we must see that $\pi_r h_d(X_r) = h_d(X_{r+1})$. This is easy using the generating function

$$\prod_{i=1}^{r} \frac{1}{1 - x_i t} = \sum_{d=0}^{\infty} h_d(X_r) t^i.$$

The first $r-1$ factors on the left are symmetric in x_r and x_{r+1}, so commute with π_r, so one only needs to verify by direct computation that $\pi_r(1-x_rt)^{-1} = (1-x_rt)^{-1}(1-x_{r+1}t)^{-1}$.

For general λ, we use the Jacobi-Trudi identity for flagged Schur functions [23]:

$$s_\lambda(X_\phi) = \det\left(h_{\lambda_i-i+j}(X_{\phi_i})\right)_{1\leqslant i,j\leqslant \ell(\lambda)}.$$

This determinant expands as a sum of terms of the form

$$\pm\, h_{d_1}(X_{\phi_1})\cdots h_{d_\ell}(X_{\phi_\ell}). \tag{5}$$

If $i \neq r$, then $h_d(X_r)$ is symmetric in x_i and x_{i+1}. In particular, the hypothesis $\phi_{k'} \neq \phi_k$ for $k' \neq k$ ensures that every factor in the term (5) is symmetric in x_{ϕ_k} and x_{ϕ_k+1} except for $h_{d_k}(X_{\phi_k})$. The effect of applying π_{ϕ_k} to the term (5) is therefore the same as the effect of applying it only to the factor $h_{d_k}(X_{\phi_k})$, and the previous paragraph shows that this is the same as replacing ϕ_k by ϕ_k+1. This argument also shows that if $i \notin \{\phi_1,\ldots,\phi_\ell\}$, then $s_\lambda(X_\phi)$ is symmetric in x_i and x_{i+1}, hence fixed by π_i. $\square$

Theorem 53. *$F_\gamma = s_{\lambda^+(\gamma)}s_{\lambda^-(\gamma)}$ for a matchless clan γ.*

Proof. Abbreviate $\lambda^\pm(\gamma)$ and $\phi^\pm(\gamma)$ as $\lambda^\pm$ and $\phi^\pm$. By Lemma 51 and the formulas of Definition 48,

$$F_\gamma = \lim_{N\to\infty} \pi_{w_N}(s_{\lambda^+}(X_{\phi^+})\, s_{\lambda^-}(X_{\phi^-})).$$

Fix $N \geqslant n = p+q$. Let a^i be the word $\mathbf{i}(\mathbf{i{+}1})\cdots(\mathbf{N{-}1})$ for $i < N$. It is not hard to check that $a^{N-1}\cdots a^2a^1$ is a reduced word for w_N, and we will take π_{w_N} to be the specific composition $\pi_{a^{N-1}}\cdots\pi_{a^1}$. Let $f = s_{\lambda^+}(X_{\phi^+})\, s_{\lambda^-}(X_{\phi^-})$.

First consider $\pi_{N-1}(f)$. If $N-1 > n$, then f is symmetric in x_{N-1} and x_N (since these variables do not even appear), so f is fixed by π_{N-1}. Otherwise, $N-1$ appears in exactly one of ϕ^- and ϕ^+; say $(\phi^-)_k = N-1$. The sequences ϕ^+ and ϕ^- are disjoint and have no repeated entries, so it follows from Lemma 52 that

$$\pi_{N-1}(f) = s_{\lambda^+}(X_{\phi^+})\pi_{N-1}(s_{\lambda^-}(X_{\phi^-})) = s_{\lambda^+}(X_{\phi^+})s_{\lambda^-}(X_{\phi_1^-},\ldots,X_{\phi_k^-+1},\ldots,X_{\phi_q^-});$$

in words, $\pi_{n-1}(f)$ is obtained from f by incrementing the entry $N-1$ of ϕ^- to N. This does not alter any entries of $\phi^\pm$ which are less than $N-1$, and so the same argument shows that subsequently applying $\pi_{N-2},\pi_{N-3},\ldots,\pi_1$ (in that order) has the effect of incrementing every value in ϕ^- and ϕ^+ which is less than N. That is, $\pi_{a^1}(f) = s_{\lambda^+}(X_{\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\phi^-})$ where for a sequence ϕ we define $\uparrow\phi$ as the sequence with

$$(\uparrow\phi)_i = \begin{cases}\phi_i+1 & \text{if } \phi_i < N\\ \phi_i & \text{if } \phi_i \geqslant N.\end{cases}$$

Similarly, consider the action of π_{a^2}. Ignoring entries equal to N, the flags $\uparrow\phi^+$ and $\uparrow\phi^-$ are still disjoint with no repeated entries, and so the argument of the last paragraph shows that

$$\pi_{a^2}(\pi_{a^1}f) = \pi_{a^2}(s_{\lambda^+}(X_{\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\phi^-})) = s_{\lambda^+}(X_{\uparrow\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\uparrow\phi^-}).$$

Continuing in this way, we see that

$$\pi_{w_N} f = s_{\lambda^+}(X_{\uparrow^{N-1}\phi^+}) s_{\lambda^-}(X_{\uparrow^{N-1}\phi^-})) = s_{\lambda^+}(X_N, \ldots, X_N) s_{\lambda^-}(X_N, \ldots, X_N).$$

Since λ^- and λ^+ have length at most $n \leqslant N$ by definition, $s_{\lambda^\pm}(X_N, \ldots, X_N)$ is simply the ordinary Schur polynomial $s_{\lambda^\pm}(x_1, \ldots, x_N)$. Thus, $\lim_{N\to\infty} \pi_{w_N} f = s_{\lambda^+} s_{\lambda^-}$. □

Corollary 54. *Let f^λ be the number of standard tableaux of shape λ. Then for a matchless clan $\gamma \in \mathrm{Clan}_{p,q}$,*

$$\#\mathcal{R}(\gamma) = \binom{|\lambda^+| + |\lambda^-|}{|\lambda^+|, |\lambda^-|} f^{\lambda^+} f^{\lambda^-} = (pq)! \prod_{\substack{i\in\phi^+ \\ j\in\phi^-}} \frac{1}{|i-j|}.$$

Proof. The first equality follows from Theorem 53 by comparing coefficients of $x_1 x_2 \cdots$, as per Proposition 41. For the second, apply the hook length formula. The hook lengths of λ^+ are exactly the distances from each $+$ in γ to some following $-$. To be precise, the hook with corner (i,j) in λ^+ has size $\phi^+_i - \phi^-_{q-j+1}$. This statement and the corresponding statement for λ^- imply via the hook length formula that

$$f^{\lambda^+} f^{\lambda^-} = |\lambda^+|! \prod_{\substack{i\in\phi^+ \\ j\in\phi^- \\ i<j}} \frac{1}{j-i} \; |\lambda^-|! \prod_{\substack{i\in\phi^+ \\ j\in\phi^- \\ i>j}} \frac{1}{i-j}.$$

Since $|\lambda^+| + |\lambda^-|$ is the total number of pairs of a $+$ and following $-$ or vice versa, i.e. pq, the second equality follows. □

4 Maximizing the number of reduced words

In this section we investigate the question of which $\gamma \in \mathrm{Clan}_{p,q}$ has the most reduced words. Let $\mathrm{R}(\gamma) = \#\mathcal{R}(\gamma)$ be the number of reduced words of a clan γ.

Proposition 55. *If $\gamma \in \mathrm{Clan}_{p,q}$ maximizes R, then γ is matchless.*

Proof. If γ is not matchless, there is a saturated chain $\gamma = \gamma_0 < \cdots < \gamma_k = \delta$ in weak order where δ is matchless and the edges are labeled $i_1, \ldots, i_k$. Then the function appending $i_1, \ldots, i_k$ to a word is an injection $f : \mathcal{R}(\gamma) \to \mathcal{R}(\delta)$. Suppose first that δ does not have the form $+^p-^q = ++\cdots+--\cdots-$ or $-^q+^p$. Then there exist indices $j \neq k$ such that $\{\delta_j, \delta_{j+1}\} = \{+,-\} = \{\delta_k, \delta_{k+1}\}$. In this case $\delta > \delta * s_j, \delta * s_k$, so δ has reduced words ending with both j and k. In particular, the injection $f : \mathcal{R}(\gamma) \to \mathcal{R}(\delta)$ cannot be surjective.

Now suppose $\delta = +^p-^q$. Then the only clan covered by δ is $\delta * s_p$, so our saturated chain $\gamma = \gamma_0 < \cdots < \gamma_k = \delta$ must have $\gamma_{k-1} = \delta * s_p$. Since $\delta * s_p$ is also covered by $\delta' = +^{p-1}-+-^{q-1}$, we may replace δ by δ' in our saturated chain, in which case $\mathrm{R}(\gamma) < \mathrm{R}(\delta')$ by the previous paragraph. Obviously the same argument works if $\delta = -^q+^p$. □

In the smallest case $q \geqslant p = 1$, Corollary 54 says that $\mathrm{R}(\gamma) = \binom{q}{\phi_1^+ - 1}$, so R is maximized by the $(1, q)$-clan(s)

$$\underbrace{- - \cdots -}_{r} + \underbrace{- - \cdots -}_{q-r}$$

where r is $\lfloor \frac{q}{2} \rfloor$ or $\lceil \frac{q}{2} \rceil$. This example already shows that $\#\mathcal{R}(\gamma)$ need not have a unique maximizer.

Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ for $x > 0$ be the usual Euler gamma function, so $\Gamma(m) = (m-1)!$ for any positive integer m.

Proposition 56. *For real numbers* $1 \leqslant \phi_1 < \cdots < \phi_m \leqslant p + q = n$ *let*

$$f(\phi_1, \ldots, \phi_m) = \prod_{1 \leqslant k < \ell \leqslant m} \frac{1}{(\phi_k - \phi_\ell)^2} \prod_{k=1}^{m} \Gamma(\phi_k) \Gamma(n + 1 - \phi_k).$$

If $\gamma \in \mathrm{Clan}_{p,q}$ *is matchless, then* $\#\mathcal{R}(\gamma) = (pq)!/f(\phi^+(\gamma)) = (pq)!/f(\phi^-(\gamma))$.

Proof. Regroup the factors in Corollary 54:

$$\mathrm{R}(\gamma) = (pq)! \prod_{i \in \phi^+} \left[\prod_{\substack{j \in \phi^- \\ j < i}} \frac{1}{i - j} \prod_{\substack{j \in \phi^- \\ j > i}} \frac{1}{j - i} \right].$$

Rewriting

$$\prod_{\substack{j \in \phi^- \\ j < i}} \frac{1}{i - j} = \frac{1}{(i-1)!} \prod_{\substack{k \in \phi^+ \\ k < i}} (i - k) \qquad \text{and} \qquad \prod_{\substack{j \in \phi^- \\ j > i}} \frac{1}{j - i} = \frac{1}{(n-i)!} \prod_{\substack{k \in \phi^+ \\ k > i}} (k - i)$$

gives

$$\#\mathcal{R}(\gamma) = (pq)! \prod_{i \in \phi^+} \left[\frac{1}{(i-1)!(n-i)!} \prod_{\substack{k \in \phi^+ \\ k \neq i}} (k - i)^2 \right] = \frac{(pq)!}{f(\phi^+)}.$$

The same argument works with the roles of ϕ^+ and ϕ^- reversed. □

Although R need not have a unique maximizer, the next lemma provides a weaker uniqueness statement.

Lemma 57. *On the domain* $1 \leqslant \phi_1 < \cdots < \phi_p \leqslant p + q = n$*, the function* $\log f(\phi_1, \ldots, \phi_p)$ *is strictly convex in each variable. In particular,* f *has a unique global minimum* $\phi^* = (\phi_1^*, \ldots, \phi_p^*)$*, and any minimizer of* f *restricted to the integer lattice* $\mathbb{Z}^p$ *is one of the* 2^p *points obtained from* ϕ^* *by rounding each coordinate either up or down.*

Proof. Fixing $\phi_2, \ldots, \phi_p$, we have

$$\log f(\phi_1, \ldots, \phi_p) = \log \Gamma(\phi_1) + \log \Gamma(n+1-\phi_1) - 2\sum_{k=1} \log(\phi_k - \phi_1) + C$$

for some constant C. By the Bohr-Mollerup theorem, $\log \Gamma(\phi_1)$ is a convex function of ϕ_1, and taking second derivatives shows the same is true of $\log \Gamma(n+1-\phi_1)$ and $-\log(\phi_k - \phi_1)$. In fact, $-\log(\phi_k - \phi_1)$ is strictly convex, so the sum $\log f(\phi_1, \ldots, \phi_p)$ is also strictly convex in ϕ_1, and in every ϕ_i by symmetry.

Strict convexity implies that $\log f$ (hence f) has at most one global minimum. To see that it does have one, observe that $f(\phi) \to \infty$ as ϕ approaches the boundary of the domain of f where $\phi_i = \phi_{i+1}$ for some i, so that for sufficiently small $\varepsilon > 0$, the global minimum of f on the compact set where $1 \leqslant \phi_i \leqslant \phi_{i+1} - \varepsilon \leqslant n$ for each i will also be a global minimum of f on its whole domain.

Finally, using the convexity of $\log f$ in each variable individually, the claim about the minimizer of $\log f$ restricted to $\mathbb{Z}^n$ reduces to the fact that if $g : [a, b] \to \mathbb{R}$ is a convex function with global minimum x^*, then g is decreasing on $[a, x^*]$ and increasing on $[x^*, b]$. ☐

Lemma 58. *Let* $\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. *Then* $\log(x - \frac{1}{2}) < \psi(x) < \log(x)$ *for* $x > 1$.

Proof. The expression

$$\psi(x) = \log(x) - \frac{1}{2x} - 2\int_0^\infty \frac{t}{(t^2 + x^2)(e^{2\pi t} - 1)} \, \mathrm{dt} \quad (\text{for } x > 0), \tag{6}$$

obtained by differentiating an integral formula for $\log \Gamma(x)$ due to Binet [16, §12.32], shows that $\psi(x) < \log(x)$ for $x > 1$. Define $g(x) = \psi(x) - \log(x - \frac{1}{2})$. From the recursion $\Gamma(x+1) = x\Gamma(x)$ we deduce $g(x) - g(x+1) = \log \frac{x+1/2}{x-1/2} - \frac{1}{x}$. The Laurent expansion

$$\log \frac{x + \frac{1}{2}}{x - \frac{1}{2}} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k} x^{2k+1}} \quad (\text{for } |x| > 1)$$

about $x = \infty$ makes clear that $g(x) - g(x+1) > 0$ for $x > 1$. But (6) implies $\lim_{x\to\infty} \psi(x) - \log(x) = 0$, hence also $\lim_{x\to\infty} g(x) = 0$, so we conclude that $g(x) > 0$ for all $x > 1$. ☐

We now consider the case $p = 2$. The next theorem comes close to completely determining a maximizer of R on $\mathrm{Clan}_{2,q}$: up to a small error, it must be invariant under reversal with its two + signs separated by distance $\sqrt{n} = \sqrt{q+2}$. More precisely, we determine a set of at most 16 clans which contain all maximizers of R.

Theorem 59. *Set* $\alpha_1 = \frac{n+1}{2} - \frac{1}{2}\sqrt{n}$ *and* $\alpha_2 = \frac{n+1}{2} + \frac{1}{2}\sqrt{n}$. *Then the clans* $\gamma \in \mathrm{Clan}_{2,q}$ *maximizing* R *have*

$$\phi^+(\gamma)_1 \in \{\lfloor \alpha_1 \rfloor - 1, \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \lceil \alpha_1 \rceil + 1\}$$
$$\phi^+(\gamma)_2 \in \{\lfloor \alpha_2 \rfloor - 1, \lfloor \alpha_2 \rfloor, \lceil \alpha_2 \rceil, \lceil \alpha_2 \rceil + 1\}.$$

Proof. Let $\phi^* = (\phi_1^*, \phi_2^*)$ be the unique minimizer of f from Lemma 57. That lemma also shows that $\phi^+(\gamma)_1$ is one of the two closest integers to ϕ_1^*, so if it is known that $|\phi_1^* - \alpha_1| < 1$, then $\phi^+(\gamma)_1$ must be one of $\lfloor \alpha_1 \rfloor - 1, \lfloor \alpha_1 \rfloor, \lceil \alpha_1 \rceil, \lceil \alpha_1 \rceil + 1$. The analogous fact for $\phi^+(\gamma)_2$ holds by symmetry of ϕ^*. Thus, it suffices to prove that $|\phi_1^* - \alpha_1| < 1$.

The uniqueness in Lemma 57 means that ϕ^* is invariant under the transformation

$$(\phi_1^*, \dots, \phi_p^*) \mapsto (n+1-\phi_p^*, \dots, n+1-\phi_1^*),$$

since f itself is. Thus we must minimize the single variable function $f(\phi_1, n+1-\phi_1)$ on the domain $\phi_1 \in [1, \frac{n}{2}]$. It is helpful to let $m = \frac{n+1}{2}$ and use the new coordinate $x = m - \phi_1$:

$$\begin{aligned}\log f(\phi_1, n+1-\phi_1) &= 2\log\Gamma(n+1-\phi_1) + 2\log\Gamma(\phi_1) - 2\log(n+1-2\phi_1) \\ &= 2\log\Gamma(m+x) + 2\log\Gamma(m-x) - 2\log(2x).\end{aligned}$$

Set $g(x) = \log\Gamma(m+x) + \log\Gamma(m-x) - \log(2x)$. Then

$$g'(x) = \Psi(m+x) - \Psi(m-x) - \frac{1}{x},$$

where $\Psi(y) = \frac{d}{dy}\log\Gamma(y)$. The inequalities $\log(y - \frac{1}{2}) < \Psi(y) < \log(y)$ for $y > \frac{1}{2}$ and $\log(1+y) \leqslant y$ for $y > -1$ imply

$$\begin{aligned} &-\log\frac{m-x}{m+x-\frac{1}{2}} - \frac{1}{x} < g'(x) < \log\frac{m+x}{m-x-\frac{1}{2}} - \frac{1}{x} \\ \implies\quad &\frac{2x-\frac{1}{2}}{m+x-\frac{1}{2}} - \frac{1}{x} < g'(x) < \frac{2x+\frac{1}{2}}{m-x-\frac{1}{2}} - \frac{1}{x}.\end{aligned}$$

The positive zeros of the lower and upper bounds here are, respectively,

$$\frac{3}{8} + \frac{1}{2}\sqrt{2m - \frac{7}{16}} \quad \text{and} \quad -\frac{3}{8} + \frac{1}{2}\sqrt{2m - \frac{7}{16}},$$

or $\pm\frac{3}{8} + \frac{1}{2}\sqrt{n + \frac{9}{16}}$. It follows that g has a critical point x^* in $(-\frac{3}{8} + \frac{1}{2}\sqrt{n+\frac{9}{16}}, \frac{3}{8} + \frac{1}{2}\sqrt{n+\frac{9}{16}})$, and strict convexity of g forces x^* to be its unique global minimizer.

Using these bounds on x^* and the inequality $\sqrt{n+9/16} - \sqrt{n} = \frac{9/16}{\sqrt{n+9/16}+\sqrt{n}} \leqslant 9/32$ for $n \geqslant 1$ gives

$$-\frac{3}{8} < x^* - \frac{1}{2}\sqrt{n} = \alpha_1^* - \phi_1^* < \frac{3}{8} + \frac{9}{64} = \frac{33}{64}.$$

Thus the desired bound $|\phi_1^* - \alpha_1| < 1$ holds. ☐

Although we do not have an exact description of the clans maximizing R in general, we can prove a sort of continuity result showing that a maximizer of R in $\mathrm{Clan}_{p,q}$ cannot be very different from a maximizer in $\mathrm{Clan}_{p,q+1}$. Define a partial order $\preceq$ on matchless

(p,q)-clans by declaring $\gamma' \preceq \gamma$ if $\phi^+(\gamma')_i \leqslant \phi^+(\gamma)_i$ for $i = 1, \dots, p$. This partial order is a lattice, with

$$\phi^+(\gamma \vee \gamma')_i = \max(\phi^+(\gamma)_i, \phi^+(\gamma)_i) \quad \text{and} \quad \phi^+(\gamma \wedge \gamma')_i = \min(\phi^+(\gamma)_i, \phi^+(\gamma)_i).$$

Write $\gamma-$ and $-\gamma$ for the clans obtained by appending or prepending a $-$ to the one-line notation of γ.

Lemma 60. *If* $\gamma' \prec \gamma$ *and* $\mathrm{R}(\gamma') \leqslant \mathrm{R}(\gamma)$*, then* $\mathrm{R}(\gamma'-) < \mathrm{R}(\gamma-)$*.*

Proof. Abbreviate $\phi^+(\gamma)$ and $\phi^+(\gamma')$ as ϕ and ϕ'. Proposition 56 shows

$$\mathrm{R}(\gamma) = (pq)! \prod_{1 \leqslant i < j \leqslant p} (\phi_j - \phi_i)^2 \prod_{i=1}^{p} \frac{1}{(\phi_i - 1)!(n - \phi_i)!}.$$

Thus,

$$\frac{\mathrm{R}(\gamma-)}{\mathrm{R}(\gamma'-)} = \prod_{i=1}^{p} \frac{(n+1-\phi_i')!}{(n+1-\phi_i)!} = \frac{\mathrm{R}(\gamma)}{\mathrm{R}(\gamma')} \prod_{i=1}^{p} \frac{n - \phi_i' + 1}{n - \phi_i + 1},$$

and the last expression strictly exceeds $\mathrm{R}(\gamma)/\mathrm{R}(\gamma') \geqslant 1$ because $\gamma' \prec \gamma$. □

Theorem 61. *Suppose* $\gamma \in \mathrm{Clan}_{p,q}$ *and* $\delta \in \mathrm{Clan}_{p,q+1}$ *are maximizers of R. Then all entries of the vector* $\phi^+(\delta) - \phi^+(\gamma)$ *are either* 0 *or* 1*.*

Proof. Suppose $\varepsilon \in \mathrm{Clan}_{p,q+1}$ is such that $\gamma- \not\preceq \varepsilon$. We will show that ε then does not maximize R, so that necessarily $\gamma- \preceq \delta$ and (by a symmetric argument) $-\gamma \preceq \delta$, which together imply the theorem.

It is clear that the one-line notation of $\varepsilon \wedge \gamma-$ ends in $-$, so let $\zeta \in \mathrm{Clan}_{p,q}$ be such that $\zeta- = \varepsilon \wedge \gamma-$. Then $\mathrm{R}(\zeta) \leqslant \mathrm{R}(\gamma)$ by the choice of γ, and the (strict!) inequality $\varepsilon \wedge \gamma- \prec \gamma-$ implies $\zeta \prec \gamma$. Therefore $\mathrm{R}(\zeta-) = \mathrm{R}(\varepsilon \wedge \gamma-) < \mathrm{R}(\gamma-)$ by Lemma 60.

Now, using the formula of Proposition 56,

$$\frac{\mathrm{R}(\varepsilon \wedge \gamma-)\mathrm{R}(\varepsilon \vee \gamma-)}{\mathrm{R}(\gamma-)\mathrm{R}(\varepsilon)} = \left[\prod_{1 \leqslant i < j \leqslant p} \frac{\phi_j^+(\varepsilon \wedge \gamma-) - \phi_i^+(\varepsilon \wedge \gamma-)}{\phi_j^+(\gamma-) - \phi_i^+(\gamma-)} \frac{\phi_j^+(\varepsilon \vee \gamma-) - \phi_i^+(\varepsilon \vee \gamma-)}{\phi_j^+(\varepsilon) - \phi_i^+(\varepsilon)} \right]^2,$$

which is at least 1 by the general inequality

$$(a_2 - a_1)(b_2 - b_1) \leqslant (\min(a_2, b_2) - \min(a_1, b_1))(\max(a_2, b_2) - \max(a_1, b_1))$$

when $a_1 < a_2$ and $b_1 < b_2$. Having shown $\mathrm{R}(\varepsilon \wedge \gamma-) < \mathrm{R}(\gamma-)$ in the previous paragraph, this implies $\mathrm{R}(\varepsilon \vee \gamma-) > \mathrm{R}(\varepsilon)$, so ε does not maximize R. □

We conclude this section with an asymptotic description of an element of $\mathrm{Clan}_{p,q}$ which we expect to be near a maximizer of R assuming p/q approaches a fixed θ as $p, q \to \infty$, although we do not attempt to state any precise results. The uniqueness in Lemma 57 shows that if $\gamma^* \in \mathrm{Clan}_{p,q}$ maximizes $\mathrm{R}(\gamma^*)$, then γ^* and $\mathrm{rev}(\gamma^*)$ should be effectively equal (to be precise, $\phi^+(\gamma^*)$ and $\phi^+(\mathrm{rev}(\gamma^*))$ differ by a vector with entries from $\{0, 1, -1\}$). Proposition 46 then implies $\lambda^+(\gamma^*) \approx \lambda^+(\gamma^*)^\vee = \lambda^-(\gamma^*)^t$, so

$$\mathrm{R}(\gamma^*) \approx \binom{pq}{pq/2} f^{\lambda^+(\gamma^*)} f^{\lambda^-(\gamma^*)} = \binom{pq}{pq/2} f^{\lambda^+(\gamma^*)} f^{(\lambda^+(\gamma^*)^\vee)^t} = \binom{pq}{pq/2} f^{\lambda^+(\gamma^*)} f^{\lambda^+(\gamma^*)^\vee}.$$

Thus, maximizing R is equivalent to maximizing $f^\lambda f^{\lambda^\vee}$ over $\lambda \subseteq [p] \times [q]$ with $|\lambda| \approx \lfloor pq/2 \rfloor$.

Let $\mathrm{SYT}(\lambda)$ be the set of standard tableaux of shape λ, and write $(q^p) = (\overbrace{q, \ldots, q}^{p})$ for the partition whose Young diagram is a $p \times q$ rectangle. For any fixed $0 \leqslant k \leqslant pq$, there is a bijection

$$\mathrm{SYT}(q^p) \to \bigcup_{\substack{\lambda \vdash k \\ \lambda \subseteq [p] \times [q]}} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda^\vee),$$

which sends $T \in \mathrm{SYT}(q^p)$ to (T_1, T_2) where T_1 is the subtableau of T containing $1, 2, \ldots, k$, and T_2 is the complement of T_1 in T rotated 180° and with the entries $pq, pq-1, \ldots, k+1$ replaced by $1, 2, \ldots, pq-k$. It follows that $f^\lambda f^{\lambda^\vee} / f^{(q^p)}$ is the probability that the entries $1, 2, \ldots, |\lambda|$ of a uniformly random member of $\mathrm{SYT}(q^p)$ form a subtableau of shape λ. By the previous paragraph we would like to maximize this probability over λ with $|\lambda| = \lfloor pq/2 \rfloor$.

In [19], Romik and Pittel describe a "typical" random standard tableau of shape (q^p) when p, q are large (and in a fixed ratio). To be precise, given $T \in \mathrm{SYT}(q^p)$ let $S_T : [0,1) \times [0, p/q)$ be the function

$$S_T(x, y) = \frac{1}{pq} T(\lfloor qy \rfloor + 1, \lfloor qx \rfloor + 1),$$

where $T(i, j)$ is the entry of T in row i and column j. That is, we think of T as a surface whose height above the xy-plane is given by the entries of T, rescaled so that the maximum height is 1 and the surface lies above the rectangle $[0, 1) \times [0, p/q)$. It is helpful to picture T in the French style here, so that 1 is in its lower-left corner at $(0, 0)$ and pq is in its upper-right corner.

Theorem 62 ([19], Theorem 5). *Fix $\theta \in (0, 1]$, and suppose $p_1, p_2, \ldots$ is a sequence of integers such that $\lim_{q\to\infty} p_q/q = \theta$. There is an explicit function $L_\theta : [0, 1] \times [0, \theta] \to [0, 1]$ such that for all $\varepsilon > 0$ and all $(x, y) \in [0, 1) \times [0, \theta)$,*

$$\lim_{q\to\infty} \mathbf{P}(|S_T(x, y) - L_\theta(x, y)| > \varepsilon \,:\, T \in \mathrm{SYT}(q^{p_q}) \text{ uniformly random}) = 0.$$

In particular, for large q, a random $T \in \mathrm{SYT}(q^{p_q})$ has its entries $1, 2, \ldots, \lfloor p_q q/2 \rfloor$ contained in a subtableau whose shape resembles the region in $[0, 1) \times [0, \theta)$ below the level curve $\{(x, y) : L_\theta(x, y) = \frac{1}{2}\}$. We expect that if a matchless clan γ is chosen as the top element of a uniformly random maximal chain in $\mathrm{Clan}_{p_q,q}$ with q large, then $\lambda^+(\gamma)$ should resemble this same limiting shape, and γ should be close to a maximizer of R with high probability.

It is natural to describe the resulting "limit clan" by a density function $f : [0,1] \to \mathbb{R}$, so that for $t \in [0,1]$,

$$\text{\# of +'s among } \gamma_1, \gamma_2, \ldots, \gamma_{\lfloor t(p+q) \rfloor} \approx p \int_0^t f(t')\, dt'. \tag{7}$$

Write $C(t) = \int_0^t f(t')\, dt'$. If (7) holds, then

$$C\left(\frac{\phi_i^+}{p+q}\right) = C\left(\frac{q - \lambda_i^+ + i}{p+q}\right) \approx \frac{i}{p} \tag{8}$$

for $i \in [p]$, by definition of $\phi^+(\gamma)$. Letting $p, q \to \infty$ (with $p/q \to \theta$) and replacing i/p with $t \in [0,1]$, equation (8) becomes

$$C\left(\frac{1 - x(t) + \theta t}{1 + \theta}\right) = t.$$

where $x(t)$ is such that $L_\theta(x(t), \theta t) = \frac{1}{2}$. Using the explicit formulas from [19], one finds

$$C'(t) = f(t) = \begin{cases} \frac{1+\theta}{2\theta}\left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{1-\theta}{1+\theta}\frac{1}{2\sqrt{t(1-t)}}\right)\right] & \text{if } |t - \frac{1}{2}| < \frac{\sqrt{\theta}}{\theta+1} \\ 0 & \text{otherwise} \end{cases}$$

5 Connections to involution words

Let $\mathcal{I}_n$ be the set of involutions in S_n. Given $z \in \mathcal{I}_n$ and an adjacent transposition s_i, define

$$z * s_i = \begin{cases} zs_i & \text{if } s_i z = z s_i \\ s_i z s_i & \text{otherwise} \end{cases}$$

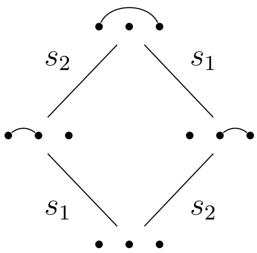
Note that $z * s_i$ is again an involution. The *weak order* on $\mathcal{I}_n$ is the transitive closure of the relations $z * s_i < z$ when $\ell(z * s_i) < \ell(z)$ [7, 12, 11, 13, 20].

Definition 63. A *reduced involution word* for $z \in \mathcal{I}_n$ is the sequence of labels along a saturated chain in weak order from the identity involution to z. Equivalently, it is a minimal-length word $a_1 \cdots a_\ell$ such that

$$z = (\cdots((1 * s_{a_1}) * s_{a_2}) * \cdots) * s_{a_\ell}.$$

To avoid confusion with usual reduced words for z, we write $\hat{\mathcal{R}}(z)$ for the set of reduced involution words of z.

Example 64. The weak order on $\mathcal{I}_3$, with involutions drawn as partial matchings of $\{1, 2, 3\}$:



The reduced involution words of the maximal element $(1\,3)$ are **12** and **21**.

Just as in weak Bruhat order on S_n, the operation $z \mapsto z * s_i$ moves up or down in involution order according to whether i is an ascent or descent of z.

Proposition 65 ([13], Lemma 3.8)**.** *If $z(i) > z(i+1)$ then $z * s_i < z$, and if $z(i) < z(i+1)$ then $z * s_i > z$.*

Write $\kappa(z)$ for the number of 2-cycles in an involution z, and define $\mathcal{I}_{p,q} = \{z \in \mathcal{I}_{p+q} : \kappa(z) \leqslant \min(p, q)\}$. Let $w_{p,q}$ be the involution $(1, n)(2, n{-}1) \cdots (m, n{-}m{+}1)$ where $m = \min(p, q)$, so $w_{p,q} = \iota(\gamma_{p,q})$.

Lemma 66. *The set $\mathcal{I}_{p,q}$ has $w_{p,q}$ as its unique maximal element in involution weak order.*

Proof. Set $m = \min(p, q)$. First we prove the lemma with $\mathcal{I}_{p,q} = \{z \in \mathcal{I}_n : \kappa(z) \leqslant m\}$ replaced by $\{z \in \mathcal{I}_n : \kappa(z) = m\}$; call the latter set J. Suppose z is maximal in J. By Proposition 65 this is equivalent to the condition that if $z(i) < z(i+1)$, then $z * s_i \notin J$, which can only happen if $z * s_i$ has $\kappa(z) + 1$ cycles, i.e. if $z(i) = i$ and $z(i+1) = i+1$. Letting i and j be such that

$$z(1) > \cdots > z(i-1) > z(i) < z(i+1) < \cdots < z(j-1) > z(j),$$

it follows that $i, i{+}1, \ldots, j{-}1$ are all fixed points. We have $z(j) < z(j{-}1) = j{-}1$, and $z(j)$ is none of $\{z(j{-}1), \ldots, z(i)\} = \{j{-}1, \ldots, i\}$, so it must be one of $i-1, \ldots, 2, 1$ (assuming $z(j)$ exists). But the one-line notation of z must end with $(i-1)\cdots 21$: otherwise, z would have an ascent beginning with one of $1, 2, \ldots, i-1$, which would contradict maximality of z since those are not fixed points. This completely determines z:

$$\begin{aligned} z &= n(n{-}1)\cdots(n{-}i{+}2)i(i{+}1)\cdots(n{-}i{+}1)(i{-}1)\cdots 21 \\ &= (1, n)(2, n{-}1)\cdots(i{-}1, n{-}i{+}2) \\ &= (1, n)(2, n{-}1)\cdots(m, n{-}m{+}1) = w_{p,q} \qquad \text{(given that } \kappa(z) = m) \end{aligned}$$

Now let us see that $w_{p,q}$ is also the unique maximal element of $\{z \in \mathcal{I}_n : \kappa(z) \leqslant m\}$. By induction on m, we can assume $y := (1, n)(2, n-1)\cdots(m{-}1, n{-}m{+}2)$ is the unique maximal element of $\{z \in \mathcal{I}_n : \kappa(z) \leqslant m-1\}$, and it is enough to show that $y < w_{p,q}$. But $y * s_m = (1, n)(2, n-1)\cdots(m{-}1, n{-}m{+}2)(m, m{+}1)$ is in J, so $y < y * s_m \leqslant w_{p,q}$ by the previous paragraph. $\square$

Proposition 67.

(a) If $\gamma * s_i$ is defined, then $\iota(\gamma * s_i) = \iota(\gamma) * s_i$.

(b) If $\gamma * s_i < \gamma$, then $\iota(\gamma * s_i) > \iota(\gamma)$.

(c) Let $\gamma \in \mathrm{Clan}_{p,q}$ and suppose i is a descent of $\iota(\gamma)$.

- *If $\iota(\gamma) * s_i = s_i\iota(\gamma)s_i$, there is a unique $\gamma' > \gamma$ in $\mathrm{Clan}_{p,q}$ such that $\gamma' * s_i = \gamma$;*
- *If $\iota(\gamma) * s_i = \iota(\gamma)s_i$, there are exactly two such γ'.*

(d) $\iota : \mathrm{Clan}_{p,q} \to \mathcal{I}_n$ is an order-reversing map with image $\mathcal{I}_{p,q}$.

Proof.

(a) The cases in which $\gamma * s_i$ is defined are: (1) if i and $i+1$ are fixed points of γ of opposite sign, then $\gamma * s_i$ is obtained from γ by making i and $i+1$ matched; (2) if $\{i, i+1\}$ is not $\iota(\gamma)$-invariant, then $\gamma * s_i = s_i\gamma s_i$. In case (1) $\iota(\gamma * s_i) = \iota(\gamma)s_i$ and s_i commutes with $\iota(\gamma)$, while in case (2) $\iota(\gamma * s_i) = s_i\iota(\gamma)s_i \neq \iota(\gamma)$, so in either case we get $\iota(\gamma * s_i) = \iota(\gamma) * s_i$.

(b) The relation $\gamma * s_i < \gamma$ implies $\ell(\iota(\gamma * s_i)) > \ell(\iota(\gamma))$, and by part (a) this is the same as $\ell(\iota(\gamma) * s_i) > \ell(\iota(\gamma))$, which means $\iota(\gamma) * s_i > \iota(\gamma)$ in weak order on $\mathcal{I}_n$.

(c) Suppose γ' is such that $\gamma' * s_i = \gamma$. Because i is a descent of $\iota(\gamma)$, Proposition 65 implies that $\ell(\iota(\gamma')) = \ell(\iota(\gamma) * s_i) < \ell(\iota(\gamma))$ (using part (a)), so that $\gamma' > \gamma$.

If $\iota(\gamma) * s_i = s_i\iota(\gamma)s_i$ then $i, i+1$ are not matched by γ and they are not both fixed points, so the same is true of γ'. In that case, $\gamma' * s_i$ is defined as $s_i\gamma' s_i$, forcing $\gamma' = s_i\gamma s_i$.

If $\iota(\gamma) * s_i = \iota(\gamma)s_i$, then i and $i+1$ are matched by γ (they cannot be fixed points since i is a descent of $\iota(\gamma)$). Thus γ' and γ agree on $[n] \setminus \{i, i+1\}$, and $i, i+1$ must be fixed points of γ' labeled $-+$ or $+-$ in order to have $\gamma' * s_i = \gamma$.

(d) Part (b) shows that ι is order-reversing. An involution $z \in S_n$ has $n - 2\kappa(z)$ fixed points, and constructing $\gamma \in \mathrm{Clan}_{p,q}$ with $\iota(\gamma) = z$ is equivalent to choosing a of those fixed points to label $+$ and b of them to label $-$, subject to the constraints $a + b = n - 2\kappa(z)$ and $a - b = p - q$. This gives $a = p - \kappa(z)$ and $b = q - \kappa(z)$, so $z \in \iota(\mathrm{Clan}_{p,q})$ if and only if $\kappa(z) \leqslant \min(p, q)$. In fact, we get the stronger result that

$$\#\{\gamma \in \mathrm{Clan}_{p,q} : \iota(\gamma) = z\} = \binom{n - 2\kappa(z)}{p - \kappa(z)} = \binom{n - 2\kappa(z)}{q - \kappa(z)}.$$

□

Lemma 68. *Suppose C is a saturated chain $1 = z^0 < z^1 < z^2 < \cdots < z^r = z$ in $\mathcal{I}_{p,q}$.*

(a) For a fixed $\gamma \in \mathrm{Clan}_{p,q}$ with $\iota(\gamma) = z$, there are exactly $2^{\kappa(z)}$ saturated chains in $\mathrm{Clan}_{p,q}$ with minimal element γ whose image under ι is C.

(b) *The total number of saturated chains in* $\mathrm{Clan}_{p,q}$ *with image* C *is*

$$\binom{n-2\kappa(z)}{p-\kappa(z)}2^{\kappa(z)} = \binom{n-2\kappa(z)}{q-\kappa(z)}2^{\kappa(z)}.$$

Proof. Part (b) follows from (a) because the number of $\gamma \in \mathrm{Clan}_{p,q}$ such that $\iota(\gamma) = z$ is $\binom{n-2\kappa(z)}{p-\kappa(z)}$, as per the proof of Proposition 67(d).

As for part (a), let k be the number of covering relations $z^j < z^{j+1}$ in the chain $z^0 < z^1 < \cdots < z^r = z$ for which $z^{j+1} = z^j s_i$ for some i (as opposed to $z^{j+1} = s_i z^j s_i$). Proposition 67(c,a) show that the number of saturated chains in $\mathrm{Clan}_{p,q}$ with image C and minimal element γ is 2^k. But the number k is $\kappa(z)$ for any saturated chain from 1 to z, because

$$\kappa(z * s_i) = \begin{cases} \kappa(z) & \text{if } z * s_i = s_i z s_i \\ \kappa(z)+1 & \text{if } z * s_i = z s_i \text{ and } \ell(z s_i) > \ell(z) \end{cases}. \qquad \square$$

Because ι is order-reversing, Lemma 68 does not in general relate reduced words for $\gamma \in \mathrm{Clan}_{p,q}$ to reduced involution words for $\iota(\gamma)$. However, it does when $z = w_{p,q}$ is maximal in $\mathcal{I}_{p,q}$.

Corollary 69. *The number of maximal chains in the poset* $\mathrm{Clan}_{p,q}$ *is* $2^{\min(p,q)} \# \hat{\mathcal{R}}(w_{p,q})$.

We can go further using known results for involution words.

Definition 70. The *involution Stanley symmetric function* of $z \in \mathcal{I}_n$ is

$$\hat{F}_z = \lim_{m\to\infty} \sum_{a \in \hat{\mathcal{R}}(\mathbf{z}^{+\mathbf{m}})} \sum_{b \in \mathrm{comp}(a)} x_{b_1} \cdots x_{b_\ell}.$$

Just as for clans, the set $\hat{\mathcal{R}}(z)$ is closed under the Coxeter relations for S_n [20, 3.16], so can be written as a disjoint union $\bigcup_{w\in\mathcal{A}(z)} \mathcal{R}(w)$ over some set $\mathcal{A}(z) \subseteq S_n$. This implies that $\hat{F}_z = \sum_{w\in\mathcal{A}(z)} F_w$, so $\hat{F}_z$ is indeed a symmetric function.

Definition 71. A partition λ is *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, and the *shifted shape* of a strict λ is the set of boxes $\{(i,j) : 1 \leqslant i \leqslant \ell(\lambda) \text{ and } i \leqslant j \leqslant i + \lambda_i - 1\}$ in matrix coordinates. A filling of a shifted shape by the alphabet $\{1' < 1 < 2' < 2 < \cdots\}$ is a *marked shifted semistandard tableau* if:

- Its entries are weakly increasing across rows and down columns;
- No unprimed (resp. primed) number appears twice in a column (resp. row);
- There are no primed numbers on the main diagonal.

The *Schur P-function* of shifted shape λ is $P_\lambda = \sum_T x^T$ where T runs over marked shifted semistandard tableaux of shape λ. Here x^T is the monomial in which the power of x_i is the number of entries i and i' in T. The *Schur Q-function* Q_λ is then defined to be $2^{\ell(\lambda)} P_\lambda$. These are both symmetric functions [14, III §8].

Example 72. Here is a marked shifted semistandard tableau of shifted shape $(5, 4, 1)$:

1	2′	2	2	6′
	2	4	5	6′
		5		

Lemma 73. *Suppose* $z = (1, b_1)(2, b_2) \cdots (k, b_k)$ *where* $b_1 > \cdots > b_k > k$*. Then* $\hat{F}_z = P_\lambda$ *where* $\lambda = (b_1 - 1, b_2 - 2, \ldots, b_k - k)$*.*

Proof. For $y \in \mathcal{I}_n$, let $D(y) = \{(i, j) : j > i,\ z(j) < z(i), \text{ and } z(j) \leqslant i\}$, thought of as a subset of $[n] \times [n]$ in matrix coordinates. Let μ be the partition whose parts are the row lengths of $D(y)$. By [10, Corollary 4.42], $\hat{F}_y$ is a nonnegative integer combination of Schur P-functions, whose leading term in dominance order is P_{μ^t}, where μ^t is the partition conjugate to μ. For z as defined above, one checks that $D(z)$ is the transpose of the shifted Young diagram of $\lambda = (b_1 - 1, \ldots, b_k - k)$, so the leading term of $\hat{F}_z$ is P_λ.

Equivalently, the leading term of the Schur Q expansion of $2^{\kappa(z)} \hat{F}_z$ is $2^{\kappa(z)} P_\lambda = 2^k P_\lambda = Q_\lambda$. By [10, Theorem 4.67], $2^{\kappa(y)} \hat{F}_y$ equals a single Schur Q-function with coefficient 1 if and only if y is a 2143-avoiding permutation, i.e. there are no $a < b < c < d$ such that $y(b) < y(a) < y(d) < y(c)$. This condition holds for z, so $2^{\kappa(z)} \hat{F}_z = Q_\lambda$, or $\hat{F}_z = P_\lambda$. $\square$

Theorem 74. *Assume* $p \geqslant q$ *without loss of generality. Then*

$$\sum_{\substack{\gamma \in \mathrm{Clan}_{p,q} \\ \gamma \textit{ matchless}}} F_\gamma = 2^q P_{(n-1, n-3, \ldots, n-2q+1)} = Q_{(n-1, n-3, \ldots, n-2q+1)}.$$

The number of maximal chains in $\mathrm{Clan}_{p,q}$ *is*

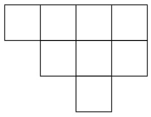
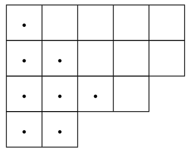
$$2^{pq}(pq)! \left[\prod_{i=1}^{q} \frac{(p+q-2i)!(p+q-2i+1)!}{(p-i)!(q-i)!} \right]^{-1} = 2^{pq} \binom{pq}{n-1, n-3, \ldots, n-2q+1} \prod_{i=1}^{q} \binom{p+q-2i}{p-i,\, q-i}^{-1}.$$

Proof. Lemma 68 gives a 2^q-to-1 correspondence between maximal chains in $\mathrm{Clan}_{p,q}$ and reduced involution words of $w_{p,q}$ which preserves the labeling of covering relations,

$$\sum_{\substack{\gamma \in \mathrm{Clan}_{p,q} \\ \gamma \text{ matchless}}} F_\gamma = 2^q \hat{F}_{w_{p,q}} = 2^q P_{(n-1, n-3, \ldots, n-2q+1)}, \tag{9}$$

where the second equality holds by Lemma 73.

Let g^λ denote the number of *unmarked* standard shifted tableaux of shape λ: fillings of the shifted shape of λ by $1, 2, \ldots, |\lambda|$ which are strictly increasing across rows and down columns. As in Proposition 41, the coefficient of $x_1 x_2 \cdots x_{pq}$ on the lefthand side of (9) is the number of maximal chains in $\mathrm{Clan}_{p,q}$, while the coefficient on the right side is $2^q 2^{|\lambda| - \ell(\lambda)} g^\lambda = 2^{pq} g^\lambda$. The *shifted hook length formula* [22] computes g^λ, as follows. The *doubled shape* $\tilde{\mu}$ of a strict partition μ is obtained by placing a copy of the shifted shape of μ to the right of its transpose so that their main diagonals are adjacent (but are not identified):

$\mu =$  $\rightsquigarrow$ $\tilde{\mu} =$ 

where $\cdot$ marks the new boxes. The shifted hook length formula is then $g^\mu = |\mu|!/\prod_{(i,j)\in\mu} h_{ij}$, where h_{ij} is the usual hook length of box (i,j) in $\tilde{\mu}$, but (i,j) only runs over those boxes corresponding to the original shifted shape μ. In the example above, $g^\mu = 8!/(7\cdot 5\cdot 4\cdot 2\cdot 4\cdot 3\cdot 1\cdot 1)$.

When $\lambda = (p+q-1, p+q-3, \ldots, p-q+1)$, this formula gives

$$2^{pq} g^\lambda = 2^{pq}(pq)! \left[\prod_{i=1}^{q-1} 2^{q-i} \frac{(p+q-2i)!}{(p-i)!} \prod_{i=1}^{q} \frac{(p+q-2i+1)!}{2^{q-i}(q-i)!} \right]^{-1},$$

where the i^{th} factor in the first product is the product of the hook lengths in row i and columns $1,\ldots,q-1$, and the i^{th} factor in the second product is the product of the remaining hook lengths in row i. □

As a corollary of Theorem 74 we obtain an interesting symmetric function identity, which also appears in [8, §4.6] and [24, §7] in a slightly different form.

Corollary 75.

$$\sum_{\lambda\subseteq[p]\times[q]} s_\lambda s_{(\lambda^\vee)^t} = Q_{(p+q-1,p+q-3,\ldots,p-q+1)}.$$

Proof. When γ is matchless, $F_\gamma = s_{\lambda^+(\gamma)} s_{\lambda^-(\gamma)}$ by Theorem 53, where $\lambda^+(\gamma)_i$ is the number of $-$'s following the i^{th} $+$ in γ, and $\lambda^-(\gamma)_i$ is the number of $+$'s following the i^{th} $-$. Now,

$$\begin{aligned} \lambda^-(\gamma)^t_j &= \#\{i : \lambda^-(\gamma)_i \geqslant j\} \\ &= \text{number of } -\text{'s followed by at least } j\ +\text{'s} \\ &= q - (\text{number of } -\text{'s followed by at most } (j-1)\ +\text{'s}) \\ &= q - (\text{number of } -\text{'s following the } (p-j+1)^{\text{th}}\ +) \\ &= q - \lambda^+(\gamma)_{p-j+1} = \lambda^+(\gamma)^\vee_j. \end{aligned}$$

That is, $\lambda^-(\gamma)^t = \lambda^+(\gamma)^\vee$. The map $\gamma \mapsto \lambda^+(\gamma)$ is a bijection between matchless (p,q)-clans and partitions contained in $[p]\times[q]$, so the corollary follows from Theorem 74. □

References

[1] I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand. Schubert cells and cohomology of the space G/P. *Russian Math. Surveys*, 28(3):1–26, 1973.

[2] S. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2:345–374, 1993.

[3] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups.* Springer, 2005.

[4] M. Brion. The behaviour at infinity of the Bruhat decomposition. *Comment. Math. Helv.*, 73:137–174, 1998.

[5] M. Brion. On orbit closures of spherical subgroups in flag varieties. *Comment. Math. Helv.*, 76:263–299, 2001.

[6] M. Can and M. Joyce. Weak order on complete quadrics. *Trans. Amer. Math. Soc.*, 365:6269–6282, 2013.

[7] M. Can, M. Joyce, and B. Wyser. Chains in weak order posets associated to involutions. *J. Combin. Theory Ser. A*, 137:207–225, 2016.

[8] E. DeWitt. *Identities Relating Schur s-Functions and Q-Functions.* PhD thesis, University of Michigan, 2012.

[9] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry.* Cambridge University Press, 1997.

[10] Z. Hamaker, E. Marberg, and B. Pawlowski. Schur P-positivity and involution Stanley symmetric functions. *Int. Math. Res. Notices*, rnx274.

[11] Z. Hamaker, E. Marberg, and B. Pawlowski. Involution words II: braid relations and atomic structures. *J. Algebraic Combin.*, 45:701–743, 2017.

[12] Z. Hamaker, E. Marberg, and B. Pawlowski. Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures. *J. Combin. Theory Ser. A*, 160:217–260, 2018.

[13] A. Hultman. The combinatorics of twisted involutions in Coxeter groups. *Trans. Amer. Math. Soc.*, 359(6):2787–2798, 2007.

[14] I. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 1995.

[15] Ian G. Macdonald. *Notes on Schubert Polynomials*, volume 6. Publications du LACIM, Université du Québec à Montréal, 1991.

[16] Ian G. Macdonald. *Notes on Schubert Polynomials*, volume 6. Publications du LACIM, Université du Québec à Montréal, 1991.

[17] L. Manivel. *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence.* Société Mathématique de France, 1998.

[18] T. Matsuki and T. Ōshima. Embeddings of discrete series into principal series. In *The Orbit Method in Representation Theory*, volume 82 of *Progress in Mathematics*, 1990.

[19] B. Pittel and D. Romik. Limit shapes for random square Young tableaux. *Adv. in Applied Math.*, 38:164–209, 2007.

[20] R. W. Richardson and T. A. Springer. The Bruhat order on symmetric varieties. *Geom. Dedicata*, 35:389–436, 1990.

[21] R. P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5:359–372, 1984.

[22] R. M. Thrall. A combinatorial problem. *Michigan Math. J.*, 1:81–88, 1952.

[23] M. Wachs. Flagged Schur functions, Schubert polynomials, and symmetrizing operators. *J. Combin. Theory Ser. A*, 40:276–289, 1985.

[24] D. Worley. *A theory of shifted Young tableaux*. PhD thesis, Massachusetts Institute of Technology, 1984.

[25] B. J. Wyser. Schubert calculus of Richardson varieties stable under spherical Levi subgroups. *J. Algebraic Combin.*, 38:829–850, 2013.

[26] B. J. Wyser and A. Yong. Polynomials for $\mathrm{GL}_p \times \mathrm{GL}_q$ orbit closures in the flag variety. *Selecta Math.*, 20:1038–1110, 2014.

[27] A. Yamamoto. Orbits in the flag variety and images of the moment map for classical groups I. *Represent. Theory*, 1:329–404, 1997.