

Degree-Similar Graphs and Cospectral Graphs

Yi-Zheng Fan^a Ruo-Jie Xing^a Yi-Liu Zhang^a Wei Wang^b

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Abstract

Let G be a graph with adjacency matrix $A(G)$ and degree matrix $D(G)$, and let $L_\mu(G) := A(G) - \mu D(G)$. Two graphs G_1 and G_2 are called *degree-similar* if there exists an invertible matrix M such that $M^{-1}A(G_1)M = A(G_2)$ and $M^{-1}D(G_1)M = D(G_2)$. In this paper, we address three problems concerning degree-similar graphs proposed by Godsil and Sun. First, we present a new characterization of degree-similar graphs using degree partition, from which we derive methods and examples for constructing cospectral graphs and degree-similar graphs. Second, we construct infinite pairs of non-degree-similar trees G_1 and G_2 such that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal form over $\mathbb{Q}(\mu)[t]$, which provides a negative answer to a problem posed by Godsil and Sun. Third, we establish several invariants of degree-similar graphs and obtain results on unicyclic graphs that are degree-similar determined. Lastly we prove that for a strongly regular graph G and any two edges e and f of G , $G \setminus e$ and $G \setminus f$ have identical μ -polynomial, i.e., $\det(tI - L_\mu(G \setminus e)) = \det(tI - L_\mu(G \setminus f))$, which enables the construction of pairs of non-isomorphic graphs with the same μ -polynomial, where $G \setminus e$ denotes the graph obtained from G by deleting the edge e .

Mathematics Subject Classifications: 05C50

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $A(G)$ and $D(G)$ be respectively the adjacency matrix and degree matrix of G . Godsil and Sun [6] introduced the notion of degree similar graphs. Two graphs G_1 and G_2 are called *degree-similar* if there exists an invertible matrix M such that

$$M^{-1}A(G_1)M = A(G_2), \quad M^{-1}D(G_1)M = D(G_2). \quad (1)$$

^aCenter for Pure Mathematics, School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China (fanyz@ahu.edu.cn, xingrj@stu.ahu.edu.cn, zhangyl@stu.ahu.edu.cn).

^bCorresponding author. School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China (wang_weiw@xjtu.edu.cn).

Clearly, if G_1 and G_2 are degree-similar, then their adjacency matrices A , Laplacian matrices $L := D - A$, signless Laplacians $Q := D + A$, and normalized Laplacians $N := D^{-1/2}LD^{-1/2}$ are all similar, and hence cospectral with respect to the above matrices. As noted in [6] or [16], if G_1 and G_2 are degree-similar and have no isolated vertices, then their Ihara zeta functions are equal. For more on Ihara zeta functions, see [12].

Degree-similar graphs have a stronger condition than some earlier versions of cospectral graphs. The *generalized α -adjacency matrix* of a graph G is defined to be

$$A_\alpha(G) := A(G) + \alpha J,$$

where $\alpha \in \mathbb{R}$ and J is an all-one matrix. The *generalized α -characteristic polynomial* (or *α -polynomial* for short) of G is defined to be

$$\phi(G, t, \alpha) = \det(tI - A_\alpha(G)).$$

Here, we use the prefix ‘ α -’ to distinguish it from another generalized adjacency matrix or polynomial to be introduced below. Johnson and Newman proved the following interesting theorem (see [2, 3]). For further details on the generalized α -adjacency matrix or the generalized α -characteristic polynomial, refer to [13, 9, 2, 8, 3].

Theorem 1. *The following statements are equivalent.*

- (1) *Two graphs G_1 and G_2 are cospectral with respect to generalized α -adjacency matrix for all α .*
- (2) *$A_\alpha(G_1)$ and $A_\alpha(G_2)$ are cospectral for two distinct values of α .*
- (3) *G_1 and G_2 are cospectral with respect to the adjacency matrix, and so are their complements.*
- (4) *There exists an orthogonal matrix Q such that $Q^\top A(G_1)Q = A(G_2)$ and $Q\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes the all-one vector.*

Wang and Xu [17] called the union of the spectrum of $A(G)$ of a graph G and the spectrum of $A(G^c)$ the *generalized spectrum* of G , where G^c denotes the complement of the graph G . Wang and his coauthors investigated the problem of graphs determined by generalized spectrum (or equivalently, determined by α -polynomial) in a series of papers [17, 18, 14, 15] by using walk-matrices and Smith normal forms over the ring of integers.

The *generalized μ -adjacency matrix* of a graph G is defined by

$$L_\mu(G) := A(G) - \mu D(G),$$

and the *generalized μ -characteristic polynomial* (or *μ -polynomial* for short) of G is defined by Wang et al. [16] as follows:

$$\psi(G, t, \mu) := \det(tI - L_\mu(G)).$$

If G_1 and G_2 have the same μ -polynomial, then they are cospectral with respect to the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix and the normalized Laplacian matrix. Wang et al. [16] proved that if G_1 and G_2 have the same

μ -polynomial, then they have the same degree sequence. The authors also constructed two non-isomorphic degree-similar graphs which are surely cospectral graphs with respect to generalized μ -adjacency matrix for all μ . There is no similar result for generalized μ -adjacency matrices as Theorem 1 for generalized α -adjacency matrices. For example, there exist two cospectral graphs with respect to A and L but not with respect to Q ([2, Fig. 4]), also two cospectral graphs with respect to A and Q but not with respect to D (namely they have different degree sequences) ([4, Table 4, third pair]).

By Lemma 4.4 of [6] (see Lemma 6), if G_1 and G_2 are degree-similar, and one of them is connected, then their complements are also degree similar. So, in this case, G_1 and G_2 have the same generalized spectra, and hence have the same α -polynomials by Theorem 1, which are called A_α -cospectral.

In general, if G_1 and G_2 are degree-similar over \mathbb{R} , surely $L_\mu(G_1)$ and $L_\mu(G_2)$ are similar over $\mathbb{R}(\mu)$, the latter of which is equivalent to that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal forms (abbreviated as SNFs) over $\mathbb{R}(\mu)[t]$. By Lemma 9.2 of [6], $L_\mu(G_1)$ and $L_\mu(G_2)$ are similar over $\mathbb{R}(\mu)$ if and only if they are similar over $\mathbb{Q}(\mu)$, which implies that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same SNF over $\mathbb{Q}(\mu)[t]$ if G_1 and G_2 are degree-similar. Clearly, if $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same SNF over $\mathbb{Q}(\mu)[t]$, then G_1 and G_2 have the same μ -polynomials by considering the last invariant divisors, which are called L_μ -cospectral.

By the above discussion, we have the following implication relations listed in Fig. 1, where the implication under $*$ means an additional condition of ‘connectedness’, and (A, A^c) -cospectral means cospectral with respect to the adjacency matrix A of a graph and the adjacency matrix of the complement of the graph, and (A, L, Q, N) -cospectral means cospectral with respect to the adjacency matrix A , the Laplacian L , the signless Laplacian Q , and the normalized Laplacian N .

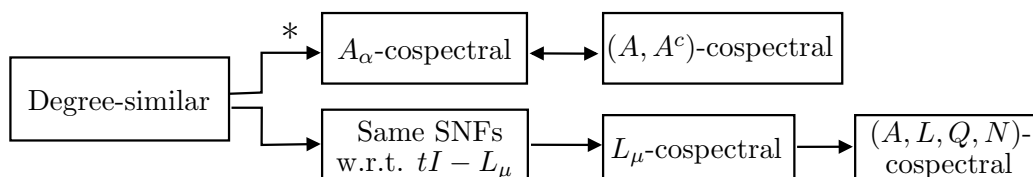


Figure 1: The implication relations among degree-similarity, SNFs and different versions of cospectral properties

Wang et al. [16] proposed the following problem: *Suppose that two graphs G_1 and G_2 have the same μ -polynomials, i.e., they are L_μ -cospectral. Does there exist an orthogonal matrix Q such that*

$$Q^\top A(G_1)Q = A(G_2), \quad Q^\top D(G_1)Q = D(G_2)? \quad (2)$$

Godsil and Sun [6] give an example of infinite pairs of graphs that share the common μ -polynomials but are not degree-similar, which gives a negative answer to the above problem. In the same paper [6], Godsil and Sun presented three interesting problems on degree-similar graphs as follows.

Problem 2. [6] Find more degree-similar graphs. In particular, are there non-isomorphic degree-similar unicyclic graphs?

We give a new characterization of degree-similar graphs by using degree partition, from which we derive some methods for constructing new pairs of degree-similar graphs from known ones. It is known that two trees are degree-similar if and only if they are isomorphic. Therefore, unicyclic graphs are the first candidates for finding non-isomorphic degree-similar graphs. By using SAGEMATH, we could not find non-isomorphic degree-similar unicyclic graphs with at most 20 vertices. A graph G is called *degree-similar determined* if any graph that is degree-similar to G must be isomorphic to G . We give some invariants for degree-similar graphs, and prove some classes of unicyclic graphs are degree-similar determined.

Problem 3. [6] Let G_1 and G_2 be two graphs such that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same SNF over $\mathbb{Q}(\mu)[t]$. Are G_1 and G_2 degree similar?

Godsil and Sun [6] show that if $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same SNF over $\mathbb{Q}(\mu)[t]$, then $A(G_1)$ and $A(G_2)$ are similar over \mathbb{Q} , as are $D(G_1)$ and $D(G_2)$. We give a negative answer to Problem 3 by constructing an infinite family of tree pairs.

For a graph G and an edge e of G , denote by $G \setminus e$ the graph obtained from G by deleting the edge e . In [7] the authors proved that if G is a strongly regular graph, then for any two edges e and f of G , the graphs $G \setminus e$ and $G \setminus f$ are (A, L, Q, N) -cospectral. In this paper, we prove that $G \setminus e$ and $G \setminus f$ have the same μ -polynomials, or they are L_μ -cospectral, which generalizes Godsil-Sun's result and pushes Problem 4 a step forward if the answer to Problem 4 is positive.

Problem 4. Let G be a strongly regular graph with two different edges e and f . Are $G \setminus e$ and $G \setminus f$ degree similar?

The paper is organized as follows. In Section 2, we present a new characterization of degree-similar graphs by using degree partition, from which we derive some methods and examples for constructing degree-similar graphs or cospectral graphs. In Section 3, we construct an infinite family of pairs of non-degree-similar trees G_1 and G_2 such that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ share the same SNF, and hence give a negative answer to Problem 3. In Section 4, we give some invariants for degree-similar graphs, and prove some classes of unicyclic graphs are degree-similar determined, pushing the study of Problem 2. In Section 5, we prove that for a strongly regular graph G and any two edges e, f of G , $G \setminus e$ and $G \setminus f$ have the same μ -polynomials, or they are L_μ -cospectral, pushing the study of Problem 4. Finally we introduce orthogonally degree-similar graphs and some remarks for the notion.

2 Degree partitions

In this section we will use degree partition to give a new characterization of degree-similar graphs, from which we present some methods for constructing cospectral graphs

and degree similar graphs. We also give some examples of constructions at the end of this section.

We first introduce some concepts and notations. Let G be a graph with vertex set $V(G)$, and let $u \in V(G)$. We use $N_G(u)$ to denote the set of neighbors of u in G . The *degree* of u , denoted by $\deg_G(u)$, is defined to be the cardinality of the set $N_G(u)$. Suppose that G has t distinct degrees d_1, \dots, d_t . The *degree partition* of G , denoted by $\pi(G)$, is a partition of the vertex set $V(G)$ of G , which consists of subsets

$$V_i = \{v \in V(G) : \deg_G(v) = d_i\}$$

for $i \in [t] := \{1, \dots, t\}$, namely, $\pi(G) = \{V_1, \dots, V_t\}$.

Let M be a matrix with rows and columns indexed by the vertices of G . Let U_1, U_2 be the subsets of $V(G)$. Denote by $M[U_1|U_2]$ the submatrix of M with rows indexed by U_1 and columns indexed by U_2 , and by $M(U_1|U_2)$ the submatrix of M with rows indexed by $V(G) \setminus U_1$ and columns indexed by $V(G) \setminus U_2$. We simply write $M[U_1|U_1]$ as $M[U_1]$ and $M(U_1|U_1)$ as $M(U_1)$.

By Lemma 4.1 of [6], the invertible matrix M in Definition 1 of degree-similar graphs is block diagonal. Here we give a more detailed statement by using degree partition.

Lemma 5. *Let G_1, G_2 be two graphs with the same vertex set. Then G_1 and G_2 are degree similar if and only if, by reordering the vertices of G_1 and G_2 , G_1 and G_2 have the same degree partition, say $\pi = \{V_1, \dots, V_t\}$, and there exist invertible matrices M_1, \dots, M_t with rows and columns indexed by V_1, \dots, V_t respectively, such that*

$$M_i^{-1}A(G_1)_{ij}M_j = A(G_2)_{ij}, \quad i, j = 1, \dots, t, \quad (3)$$

where

$$A(G_k)_{ij} := A(G_k)[V_i|V_j], \quad k = 1, 2; \quad i, j = 1, \dots, t. \quad (4)$$

Proof. Suppose that G_1, G_2 are degree-similar graphs with the same vertex set V . Then there exists an invertible matrix M such that

$$M^{-1}D(G_1)M = D(G_2), \quad M^{-1}A(G_1)M = A(G_2).$$

Let d_1, \dots, d_t be all distinct degrees of G_1 , and let $U_i = \{v \in V(G_1) : \deg_{G_1}(v) = d_i\}$ for $i \in [t]$. Since $M^{-1}D(G_1)M = D(G_2)$, the graph G_2 has the same degree sequences as G_1 . Let $W_i = \{v \in V(G_2) : \deg_{G_2}(v) = d_i\}$ for $i \in [t]$. Note that $|U_i| = |W_i|$ for $i \in [t]$.

Now, by reordering the vertices of G_1 , for some permutation matrix P , we have

$$PD(G_1)P^\top = \text{diag}(d_1I_{|U_1|}, \dots, d_tI_{|U_t|}) =: \tilde{D}.$$

Similarly, by reordering the vertices of G_2 , for some permutation matrix P' ,

$$P'D(G_2)P'^\top = \text{diag}(d_1I_{|W_1|}, \dots, d_tI_{|W_t|}) = \tilde{D}.$$

So, after the above reordering of the vertices, G_1 and G_2 have the same degree partition, say $\pi = \{V_1, \dots, V_t\}$. The matrices $\tilde{A}(G_1) = PA(G_1)P^\top$ and $\tilde{A}(G_2) = P'A(G_2)P'^\top$ are respectively the adjacency matrices of G_1 and G_2 after the reordering of vertices.

Let $\tilde{M} = PMP^\top$. We have

$$\tilde{M}^{-1}\tilde{D}\tilde{M} = \tilde{D}, \quad \tilde{M}^{-1}\tilde{A}(G_1)\tilde{M} = \tilde{A}(G_2).$$

Partition \tilde{M} conformable with π , and let $\tilde{M}_{ij} := \tilde{M}[V_i|V_j]$ for $i, j \in [t]$. Since $\tilde{D}\tilde{M} = \tilde{M}\tilde{D}$, we have

$$d_i\tilde{M}_{ij} = \tilde{M}_{ij}d_j, \quad i, j \in [t],$$

which implies that $\tilde{M}_{ij} = 0$ for $i \neq j$. Hence $\tilde{M} = \text{diag}\{\tilde{M}_{ii} : i \in [t]\}$, a block diagonal compatible with π . Let $\tilde{A}(G_k)_{ij} = \tilde{A}(G_k)[V_i|V_j]$, $k = 1, 2, i, j = 1, \dots, t$. From the fact $\tilde{M}^{-1}\tilde{A}(G_1)\tilde{M} = \tilde{A}(G_2)$, we have

$$\tilde{M}_{ii}^{-1}\tilde{A}(G_1)_{ij}\tilde{M}_{jj} = \tilde{A}(G_2)_{ij}, \quad i, j \in [t].$$

The necessity now follows by taking $M_i = \tilde{M}_{ii}$ for $i \in [t]$ and noting $\tilde{A}(G_k)$ is the adjacency matrix of G_k after reordering of vertices for $k = 1, 2$.

Conversely, if G_1 and G_2 have the same degree partition π , then by reordering the vertices, we can write the degree matrices $D(G_1)$ and $D(G_2)$ in the following form:

$$D(G_1) = D(G_2) = \text{diag}(d_1I_{|V_1|}, \dots, d_tI_{|V_t|}).$$

Let $M = \text{diag}(M_1, \dots, M_t)$. It is easy to verify that

$$M^{-1}D(G_1)M = D(G_2), \quad M^{-1}A(G_1)M = A(G_2).$$

So, G_1 and G_2 are degree-similar. □

Lemma 6. [6] *If G_1, G_2 are degree-similar and one of them is connected, then their complements are degree-similar.*

In Lemma 5, if replacing all M_i 's by kM_i 's for any nonzero k , or equivalently replacing M by kM , the Eq. (3) still holds, where $M = \text{diag}\{M_i : i \in [t]\}$. If, in addition, one of G_1 and G_2 is connected, from the proof of Lemma 6, the matrix M satisfies

$$M^{-1}JM = J.$$

So M has constant row sum and constant column sum, implying that

$$MJ = JM = cJ$$

for some nonzero c . By taking $c = 1$ we have the following result.

Corollary 7. *Let G_1 and G_2 be two graphs on the same vertex set, where G_1 is connected. Then G_1 and G_2 are degree-similar if and only if, by reordering the vertices of G_1 and G_2 , G_1 and G_2 have the same degree partition, say $\pi = \{V_1, \dots, V_t\}$, and there exist invertible matrices M_1, \dots, M_t with rows and columns indexed by V_1, \dots, V_t respectively, such that*

$$M_i^{-1}A(G_1)_{ij}M_j = A(G_2)_{ij}, \quad M_i^\top \mathbf{1} = M_i \mathbf{1} = \mathbf{1}, \quad i, j = 1, \dots, t, \quad (5)$$

where $A(G_k)_{ij}$ is defined as in (4).

We give the following result for construction of degree-similar graphs from a known pair of degree-similar graphs. Let G be a graph with vertex set $V(G)$, and let U, W be two disjoint subsets of $V(G)$. Denote by $G[U]$ the subgraph of G induced by the vertices of U , and $G[U, W]$ the bipartite subgraph of G with vertex sets $U \cup W$ whose edges are those of G connecting U and W .

Theorem 8. *Let G_1, G_2 be degree-similar graphs on the same vertex set, which have the same degree partition $\pi = \{V_1, \dots, V_t\}$, where G_1 is connected. Let \tilde{G}_1, \tilde{G}_2 be obtained from G_1, G_2 respectively by applying some of following operations simultaneously:*

- (1) replacing some $G_k[V_i]$'s with their complements,
- (2) replacing some $G_k[V_i]$'s with empty graphs,
- (3) replacing some $G_k[V_i, V_j]$'s for $i \neq j$ with their complements in complete bipartite graph with bipartition $\{V_i, V_j\}$,
- (4) replacing some $G_k[V_i, V_j]$'s with empty graphs,

Then, with respect to adjacency matrix, \tilde{G}_1 is cospectral with \tilde{G}_2 with cospectral complements.

Furthermore, if both \tilde{G}_1 and \tilde{G}_2 have the same degree partition as π , then \tilde{G}_1 is degree similar to \tilde{G}_2 . In particular, taking operation (1) if each vertex of V_i has degree $(|V_i| - 1)/2$ in the graph $G_k[V_i]$, and (or) taking operation (3) if each vertex of V_i has degree $|V_j|/2$ and each vertex of V_j has degree $|V_i|/2$ in the graph $G_k[V_i, V_j]$, then \tilde{G}_1 is degree-similar to \tilde{G}_2 .

Proof. By Corollary 7, there exist invertible matrices M_i with rows and columns indexed by V_i for $i \in [t]$, such that

$$M_i^{-1}A(G_1)_{ij}M_j = A(G_2)_{ij}, \quad M_i^\top \mathbf{1} = M_i \mathbf{1} = \mathbf{1}, \quad i, j = 1, \dots, t,$$

where $(A_k)_{ij}$ is defined as in (4).

Let $A(\tilde{G}_k)_{ij} := A(\tilde{G}_k)[V_i|V_j]$ for $k = 1, 2$ and $i, j = 1, 2, \dots, t$. Let $M = \text{diag}\{M_i : i \in [t]\}$. To verify \tilde{G}_1 is cospectral with \tilde{G}_2 , it suffices to prove $M^{-1}A(\tilde{G}_1)M = A(\tilde{G}_2)$, or equivalently,

$$M_i^{-1}A(\tilde{G}_1)_{ij}M_j = A(\tilde{G}_2)_{ij}, \quad i, j = 1, \dots, t. \quad (6)$$

Observe that if taking operation (1), $A(\tilde{G}_k)_{ii} = J - I - A(G_k)_{ii}$; and if taking operation (3), $A(\tilde{G}_k)_{ij} = J - A(G_k)_{ij}$. Since $M_i^\top \mathbf{1} = M_i \mathbf{1} = \mathbf{1}$, we have

$$M_i^{-1}A(\tilde{G}_1)_{ii}M_i = M_i^{-1}(J - I - A(G_1)_{ii})M_i = J - I - A(G_2)_{ii} = A(\tilde{G}_2)_{ii},$$

and

$$M_i^{-1}A(\tilde{G}_1)_{ij}M_j = M_i^{-1}(J - A(G_k)_{ij})M_j = J - A(G_k)_{ij} = A(\tilde{G}_2)_{ij}.$$

Similarly, if taking operation (2), $A(\tilde{G}_k)_{ii} = O$; and if taking operation (3), $A(\tilde{G}_k)_{ij} = O$, where O denotes a zero matrix of appropriate size. Obviously, $M_i^{-1}OM_i = O$, $M_i^{-1}OM_j = O$. So, Eq. (6) holds, and \tilde{G}_1 is cospectral with \tilde{G}_2 . Using the fact $M^\top \mathbf{1} = M \mathbf{1} = \mathbf{1}$ and noting $A(G^c) = J - I - A(G)$ for a graph G , we have

$$M^{-1}A(\tilde{G}_1^c)M = A(\tilde{G}_2^c),$$

implying that \tilde{G}_1 and \tilde{G}_2 have cospectral complements.

If \tilde{G}_1 and \tilde{G}_2 have the same degree partition as π , surely $M^{-1}D(\tilde{G}_1)M = D(\tilde{G}_2)$. Combing with the proved Eq. (6), we get \tilde{G}_1 is degree-similar to \tilde{G}_2 . If each vertex of V_i has degree $(|V_i| - 1)/2$ in the graph $G_k[V_i]$, taking the operation (1) in G_k will preserve the degree of each vertex of G_k . Similarly, if each vertex of V_i has degree $|V_j|/2$ and each vertex of V_j has degree $|V_i|/2$ in the graph $G_k[V_i, V_j]$, taking operation (3) in G_k also preserves the degree of each vertex of G_k . So, \tilde{G}_1 and \tilde{G}_2 have the same degree partition as π , and hence they are degree-similar. \square

By Theorem 8, we will produce $3^t \cdot 3^{\binom{t}{2}}$ pairs of cospectral graphs from a pair of degree-similar graphs G_1 and G_2 , where t is the number of parts in the degree partition of G_1 or G_2 . It is possible that some of these pairs of graphs are isomorphic. Next we give some examples of cospectral graphs and degree-similar graphs by using Theorem 8.

Example 9. The first pair of non-isomorphic degree-similar graphs $X_{1,1}$ and $X_{1,2}$ in Fig. 2 were introduced by Wang et al. [16]. We use three kinds of colored vertices for degree partition, and denote by V_r, V_g, V_b the set of red vertices of degree 4, the set of green vertices of degree 3 and the set of blue vertices of degree 2, respectively.

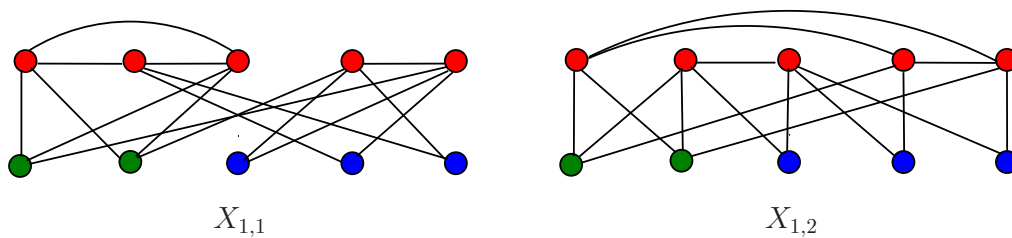


Figure 2: Degree-similar graphs $X_{1,1}$ and $X_{1,2}$ ([18])

By taking the complements of $X_{1,k}[V_r]$, we get a pair of cospectral graphs $X_{2,k}$ with cospectral complements for $k = 1, 2$; see Fig. 3.

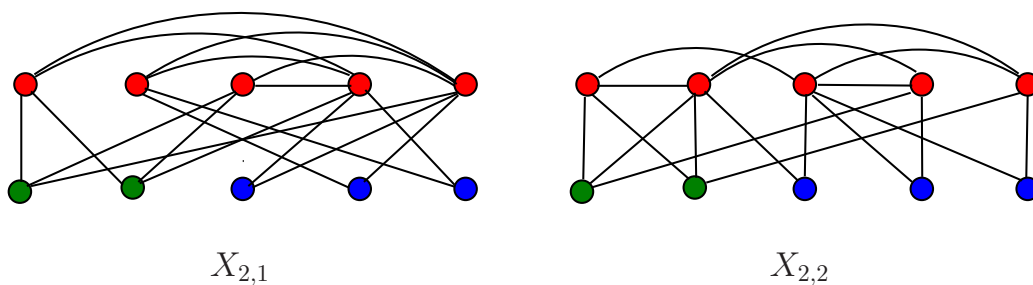


Figure 3: Cospectral graphs $X_{2,1}$ and $X_{2,2}$ with cospectral complements

By replacing $X_{1,k}[V_r]$ with empty graphs, we get a pair of cospectral graphs $X_{3,k}$ with cospectral complements for $k = 1, 2$; see Fig. 4.

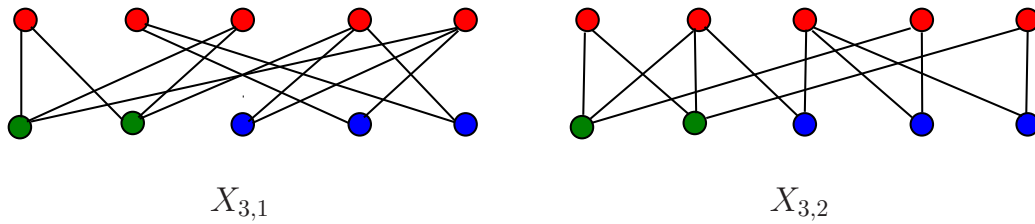


Figure 4: Cospectral graphs $X_{3,1}$ and $X_{3,2}$ with cospectral complements

By taking complements of $X_{1,k}[V_r, V_g]$ in the complete bipartite graph with two parts V_r and V_g , we get a pair of cospectral graphs $X_{4,k}$ with cospectral complements for $k = 1, 2$; see Fig. 5.

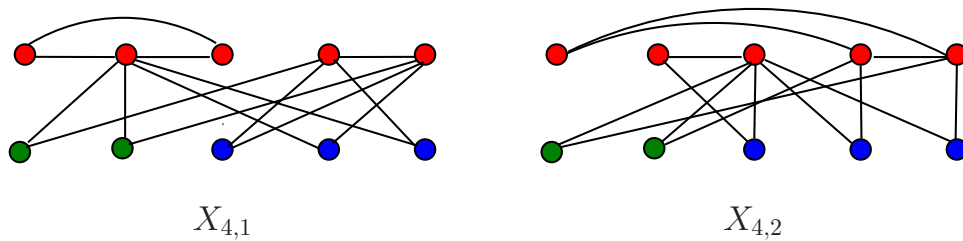


Figure 5: Cospectral graphs $X_{4,1}$ and $X_{4,2}$ with cospectral complements

If replacing $X_{1,k}[V_r, V_g]$ by empty graphs, we get a pair of cospectral graphs $X_{5,k}$ with cospectral complements for $k = 1, 2$; see Fig. 6. By deleting the isolated green vertices, we have two cospectral tricyclic graphs which are isomorphic. This means that the operations in Theorem 8 may yield pairs of isomorphic graphs.

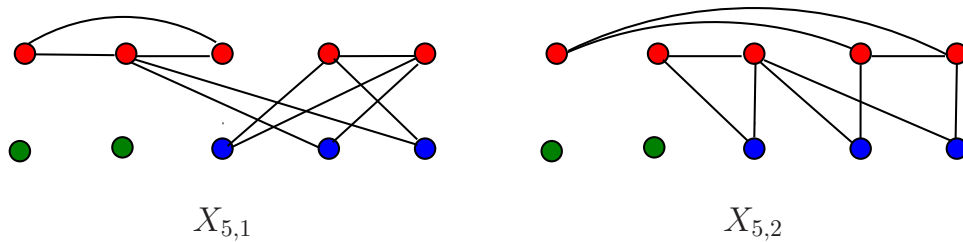


Figure 6: Cospectral graphs $X_{5,1}$ and $X_{5,2}$ with cospectral complements

If replacing $X_{4,k}[V_r, V_b]$ by empty graphs, we will get two cospectral graphs $X_{6,k}$ with cospectral complements; see Fig. 7. By deleting the blue vertices, we get two non-isomorphic cospectral bicyclic graphs.

Example 10. The second pair of non-isomorphic degree-similar graphs $Y_{1,1}$ and $Y_{1,2}$ in Fig. 8 were introduced by Godsil and Sun [6].

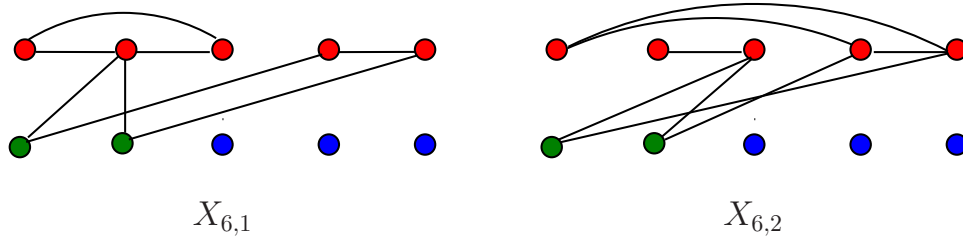


Figure 7: Cospectral graphs $X_{6,1}$ and $X_{6,2}$ with cospectral complements

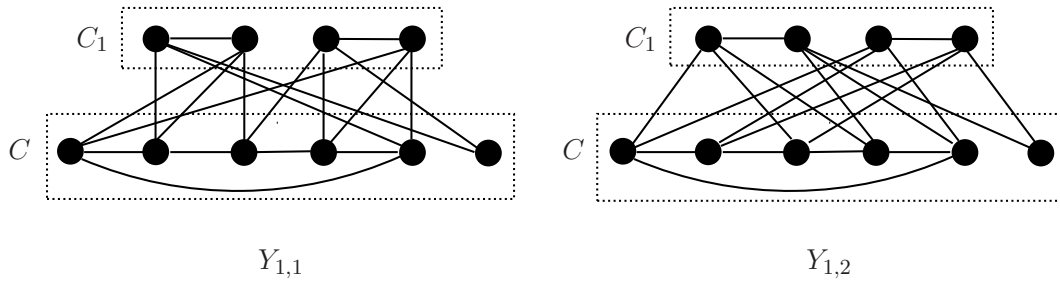


Figure 8: Degree-similar graphs $Y_{1,1}$ and $Y_{1,2}$ ([6])

By Lemma 6.2 of [6], for any graph Y , adding all possible edges between Y and C_1 (or Y and C , or Y and $C \cup C_1$) in Fig. 8, the resulting two graphs are also degree-similar. If letting $Y = P_3$, we get a pair of degree similar graphs $Y_{2,1}$ and $Y_{2,2}$ in Fig. 9; see Example 6.3 of [6].

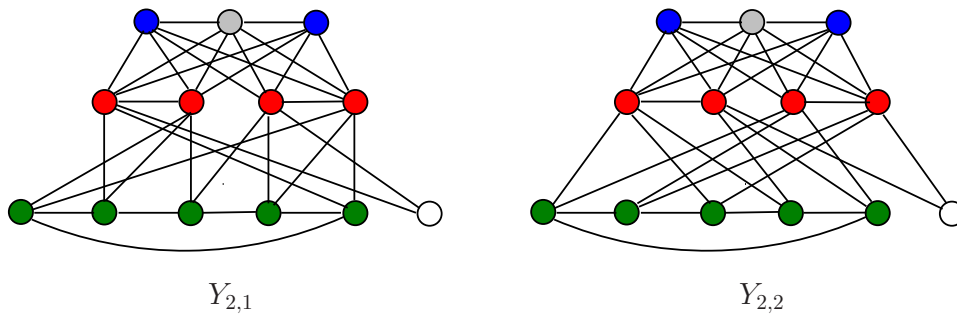


Figure 9: Degree-similar graphs $Y_{2,1}$ and $Y_{2,2}$ ([6])

By taking complements of the subgraphs induced on green vertices, we get a pair of degree-similar graphs $Y_{3,1}$ and $Y_{3,2}$ by Theorem 8; see Fig. 10. In fact, replacing the path P_3 in $Y_{3,k}$ (for $k = 1, 2$) by any nontrivial connected graph Y and adding all possible edges between Y and C_1 (the red vertices), the resulting two graphs are still degree-similar.

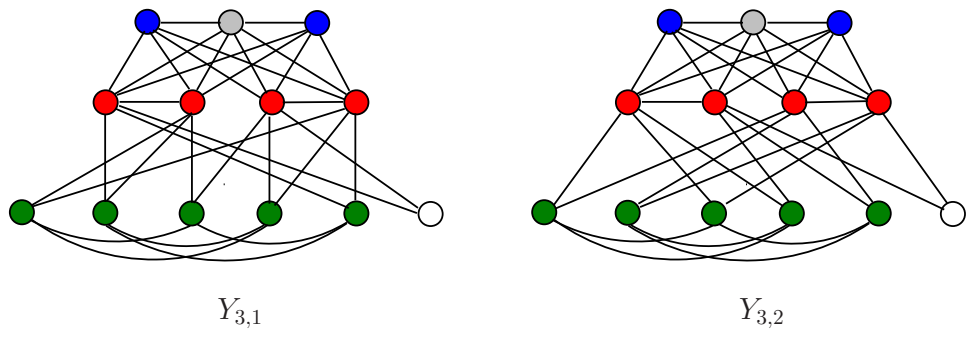


Figure 10: Degree-similar graphs $Y_{3,1}$ and $Y_{3,2}$

3 Trees

In this section, we will construct an infinite family of tree pairs G_1 and G_2 such that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal form over $\mathbb{Q}(\mu)[t]$ but G_1 and G_2 are not degree-similar, and hence give a negative answer to Problem 3 asked by Godsil and Sun [6].

Let G_1 and G_2 be two graphs with roots u and v respectively. The *coalescence* of G_1 and G_2 , denoted by $G_1(u) \odot G_2(v)$, is the graph formed by identifying the root u of G_1 and the root v of G_2 . The following tree T in Fig. 11 was appeared in [10] for constructing non-isomorphism cospectral graphs. McKay [10] showed that for any nontrivial tree T with root r , $T(r) \odot T(4)$ is not isomorphic to $T(r) \odot T(7)$, but they are cospectral with respect to adjacency matrix, Laplacian matrix and signless Laplacian matrix, and also normalized Laplacian matrix [11].

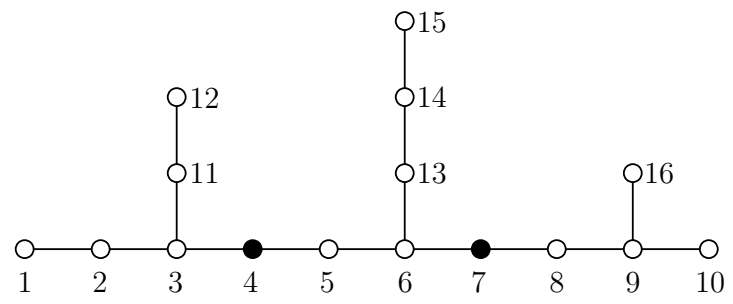


Figure 11: A tree T on 16 vertices

Let G be a general nontrivial graph with root r . Let $G_1 := G(r) \odot T(4)$ and $G_2 := G(r) \odot T(7)$. Godsil and Sun [6] proved that G_1 and G_2 have the same μ -polynomial, namely, $\psi(G_1, t, \mu) = \psi(G_2, t, \mu)$, but G_1 is not degree-similar to G_2 when G is any nontrivial tree, which answered a problem proposed by Wang et al. [16]. In the following we will prove that $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal form when G is a path.

Lemma 11. *let \mathbb{T} be the tree in Fig. 11, and let P_{m+1} be a path on $m + 1$ vertices with one endpoint r , where $m \geq 0$. Let $G_1 := P_{m+1}(r) \odot \mathbb{T}(4)$ and $G_2 := P_{m+1}(r) \odot \mathbb{T}(7)$. Then $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal form over $\mathbb{Q}(\mu)[t]$.*

Proof. Let $n := m + 16$ be the number of vertices of G_1 or G_2 . Along the path P_{m+1} starting from the root r , label other vertices of P_{m+1} as $17, 18, \dots, n$, where n is the another endpoint of P_{m+1} . Denote $d_{k,l}(G_i) := \det(tI - L_\mu(G_i)(k|l))$ and $d_k(G_i) := \det(tI - L_\mu(G_i)(k))$ for $i = 1, 2$.

We first investigate the $(n - 1)$ th determinant divisor of $tI - L_\mu(G_1)$, denoted by $D_{n-1}(G_1)$. By direct calculation,

$$d_{n,17}(G_1) = \det(tI - L_\mu(G_1)[V(\mathbb{T})]), \quad d_{n,4}(G_1) = \det(tI - L_\mu(G_1)[V(\mathbb{T}) \setminus \{4\}]),$$

and

$$\gcd(d_{n,17}(G_1), d_{n,4}(G_1)) = \alpha(t, \mu)\beta(t, \mu)\gamma(t, \mu), \tag{7}$$

where

$$\begin{aligned} \alpha(t, \mu) &:= (t + \mu), \quad \beta(t, \mu) := (t^2 + 3\mu t + 2\mu^2 - 1), \\ \gamma(t, \mu) &:= (t^3 + 6\mu t^2 + (11\mu^2 - 3)t + 6\mu^3 - 5\mu). \end{aligned} \tag{8}$$

So

$$D_{n-1}(G_1) \mid \alpha(t, \mu)\beta(t, \mu)\gamma(t, \mu).$$

Also by direct calculation,

$$d_{n,17}(G_2) = \det(tI - L_\mu(G_2)[V(\mathbb{T})]) = \det(tI - L_\mu(G_1)[V(\mathbb{T})]) = d_{n,17}(G_1),$$

$$d_{n,7}(G_2) = \det(tI - L_\mu(G_2)[V(\mathbb{T}) \setminus \{7\}]) = \det(tI - L_\mu(G_1)[V(\mathbb{T}) \setminus \{4\}]) = d_{n,4}(G_1).$$

So, by (7) we have

$$\gcd(d_{n,17}(G_2), d_{n,7}(G_2)) = \alpha(t, \mu)\beta(t, \mu)\gamma(t, \mu),$$

which implies that the $(n - 1)$ th determinant divisor of $tI - L_\mu(G_2)$, denoted by $D_{n-1}(G_2)$, also satisfies that

$$D_{n-1}(G_2) \mid \alpha(t, \mu)\beta(t, \mu)\gamma(t, \mu).$$

In the following, by Claims 1-3, we will prove that for $i = 1, 2$, none of $\alpha(t, \mu)$, $\beta(t, \mu)$ and $\gamma(t, \mu)$ divides $D_{n-1}(G_i)$, which implies that $D_{n-1}(G_i) = 1$.

Claim 12. $\alpha(t, \mu) \nmid D_{n-1}(G_i)$ for $i = 1, 2$.

Proof. We first show that $\alpha(t, \mu) \nmid D_{n-1}(G_1)$. Otherwise, $\alpha(t, \mu) \mid d_{10}(G_1)$. Expanding $d_{10}(G_1)$ at the vertex 16, we have

$$d_{10}(G_1) = (t + \mu) \det(tI - L_\mu(G_1)(10, 16)) - \det(tI - L_\mu(G_1)(10, 16, 9)).$$

Noting that $\alpha(t, \mu) = t + \mu$, we have

$$(t + \mu) \mid \det(tI - L_\mu(G_1)(10, 16, 9)).$$

Similarly, expanding the above determinant at the vertices 15, 12, 1, n successively, if $n \geq 18$,

$$(t + \mu) \mid \det(tI - L_\mu(G_1)(10, 16, 9, 15, 14, 12, 11, 1, 2, n, n - 1));$$

and if $n = 17$,

$$(t + \mu) \mid \det(tI - L_\mu(G_1)(10, 16, 9, 15, 14, 12, 11, 1, 2, 17, 4)).$$

Let

$$U = \begin{cases} \{10, 16, 9, 15, 14, 12, 11, 1, 2, n, n - 1\}, & \text{if } n \geq 18, \\ \{10, 16, 9, 15, 14, 12, 11, 1, 2, 17, 4\}, & \text{if } n = 17. \end{cases} \quad (9)$$

So we get

$$(t + \mu) \mid \det(tI - L_\mu(G_1)(U)).$$

Now taking $t = -\mu$, we have

$$\det(-\mu I - L_\mu(G_1)(U)) = \det(\mu(D' - I) - A') = 0,$$

where D', A' are the principal submatrices of $D(G_1)$ and $A(G_1)$ indexed by the vertices of $V(G_1) \setminus U$, respectively. As all the vertices of $V(G_1) \setminus U$ have degree greater than 1, each diagonal entry of $D' - I$ is positive. So, for sufficiently large μ , $\mu(D' - I) - A'$ is strictly diagonally dominant, and hence $\mu(D' - I) - A'$ is nonsingular, which yields a contradiction.

Next we show that $\alpha(t, \mu) \nmid D_{n-1}(G_2)$. Otherwise, by expanding $d_{10}(G_2)$ at the vertex 16, we have

$$\alpha(t, \mu) \mid \det(tI - L_\mu(G_2)(10, 16, 9)),$$

and successively expanding determinants at the vertex 15, 12, 1, n , we have

$$(t + \mu) \mid \det(tI - L_\mu(G_2)(U)),$$

where U is defined as in (9). We will get a contradiction by taking $t = -\mu$ and applying a similar discussion as above. \square

Claim 13. $\beta(t, \mu) \nmid D_{n-1}(G_i)$ for $i = 1, 2$.

Proof. We first prove $\beta(t, \mu) \nmid D_{n-1}(G_1)$. Otherwise, $\beta(t, \mu) \mid d_1(G_1)$. Expanding $\det(tI - L_\mu(G_1))$ at the vertex 1, we have

$$\det(tI - L_\mu(G_1)) = (t + \mu)d_1(G_1) - \det(tI - L_\mu(G_1)(1, 2)). \quad (10)$$

As $\beta(t, \mu) \mid \det(tI - L_\mu(G_1))$, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_1)(1, 2)).$$

Again, expanding $\det(tI - L_\mu(G_1)(1, 2))$ at the vertex 3, we have

$$\begin{aligned} \det(tI - L_\mu(G_1)(1, 2)) &= (t + 3\mu) \det(tI - L_\mu(T)[11, 12]) \\ &\quad \times \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12)) \\ &\quad - (t + \mu) \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12)) \\ &\quad - \det(tI - L_\mu(T)[11, 12]) \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12, 4)). \end{aligned} \tag{11}$$

As $\beta(t, \mu) = \det(tI - L_\mu(T)[11, 12])$ by direct calculation, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12)).$$

Expanding $\det(tI - L_\mu(G_1)(1, 2, 3, 11, 12))$ at the vertex 13, we have

$$\begin{aligned} \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12)) &= (t + 2\mu) \det(tI - L_\mu(T)[14, 15]) \\ &\quad \times \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12, 13, 14, 15)) \\ &\quad - (t + \mu) \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12, 13, 14, 15)) \\ &\quad - \det(tI - L_\mu(T)[14, 15]) \\ &\quad \times \det(tI - L_\mu(G_1)(1, 2, 3, 11, 12, 13, 14, 15, 6)). \end{aligned} \tag{12}$$

As $\beta(t, \mu) = \det(tI - L_\mu(T)[14, 15])$ also by direct calculation, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_1)(W)),$$

where

$$W := \{1, 2, 3, 11, 12, 13, 14, 15\}. \tag{13}$$

If taking $\mu = 0$, then $\beta(t, 0) = t^2 - 1$ is a factor of $\det(tI - A(G_1)(W))$. So, 1 is an eigenvalue of $A(G_1)(W)$. Note that $A(G_1)(W) = A(G_1(W))$, the adjacency matrix of the subgraph $G_1(W)$ which is obtained from G_1 by deleting all vertices of W together with their incident edges. Let x be an eigenvector of $A(G_1(W))$ corresponding to the eigenvalue 1. By eigenvector equation, for each vertex $u \in V(G_1) \setminus W$,

$$x_u = \sum_{v \in N_{G_1(W)}(u)} x_v. \tag{14}$$

So, if letting $x_n = a$, then $x_{n-1} = a$ and $x_{n-2} = 0$, and along the path P from the vertex n to the vertex 9, the values of some vertices of $G_1(W)$ given by x are listed in Fig. 12.

Therefore, x_9 has one of the following values: $a, 0, -a$. If $x_9 = a$, then $x_{10} = x_{16} = a$ by eigenvector equation, and hence $x_8 = x_9 - x_{10} - x_{16} = -a$, which yields a contradiction as a vertex of P with value a can not be adjacent to a vertex of P with value $-a$. If $x_9 = 0$, then $x_{10} = x_{16} = 0$, and hence $x_8 = 0$, also yielding a contradiction. Similarly, if $x_9 = -a$, we also get a contradiction as discussed in the case of $x_9 = a$.

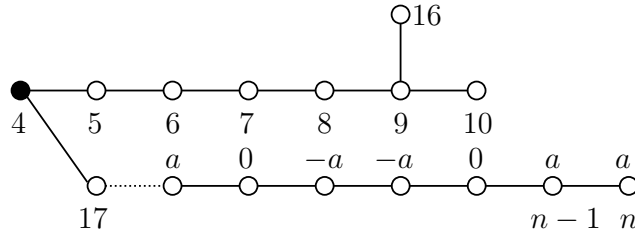


Figure 12: The graph $G_1(W)$ and some entries of eigenvector x

Next we prove $\beta(t, \mu) \nmid D_{n-1}(G_2)$. Otherwise, expanding $\det(tI - L_\mu(G_2))$ at the vertex 1, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_2))(1, 2),$$

expanding $\det(tI - L_\mu(G_2))(1, 2)$ at the vertex 3, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_2))(1, 2, 3, 11, 12),$$

and expanding $\det(tI - L_\mu(G_2))(1, 2, 3, 11, 12)$ at the vertex 13, we have

$$\beta(t, \mu) \mid \det(tI - L_\mu(G_2)(W)),$$

where W is defined as in (13).

Now taking $\mu = 0$, $\beta(t, 0) = t^2 - 1$ is a factor of the determinant $\det(tI - A(G_2(W)))$, which implies that the adjacency matrix $A(G_2(W))$ has an eigenvalue 1. Let x be an eigenvector of $A(G_2(W))$ corresponding to the eigenvalue 1. If $x_{10} := a$, then $x_9 = x_{16} = a$, $x_8 = -a$, and $x_7 = -2a$. If $x_4 := b$, then $x_5 = b$, $x_6 = 0$ and $x_7 = -b$. So $2a = b$, and hence $x_{17} = -a$. The values of x of some vertices of $G_2(W)$ are listed in Fig. 13. However, $x_{n-1} = x_n$ by eigenvector equation, implying $a = 0$ and hence $x = 0$; a contradiction. \square

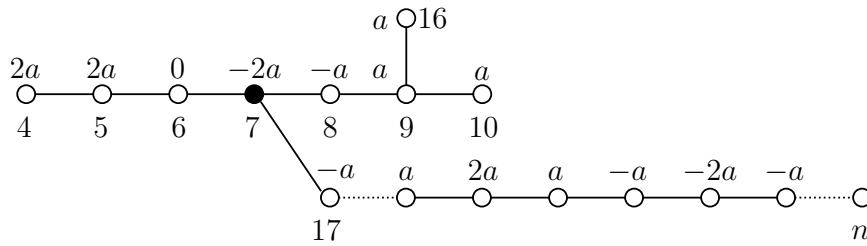


Figure 13: The graph $G_2(W)$ and some entries of the eigenvector x

Claim 14. $\gamma(t, \mu) \nmid D_{n-1}(G_i)$ for $i = 1, 2$.

Proof. We first prove $\gamma(t, \mu) \nmid D_{n-1}(G_1)$. Otherwise, $\gamma(t, \mu) \mid d_{10,16}(G_1)$. By direct calculation,

$$d_{10,16}(G_1) = \det(tI - L_\mu(G_1))(9, 10, 16).$$

So,

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1))(9, 10, 16).$$

Expanding $\det(tI - L_\mu(G_1))$ at the vertex 9,

$$\begin{aligned} \det(tI - L_\mu(G_1)) &= (t + 3\mu)(t + \mu)^2 \det(tI - L_\mu(G_1))(9, 10, 16) \\ &\quad - 2(t + \mu) \det(tI - L_\mu(G_1))(9, 10, 16) \\ &\quad - (t + \mu)^2 \det(tI - L_\mu(G_1))(8, 9, 10, 16), \end{aligned} \tag{15}$$

which implies

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1))(8, 9, 10, 16).$$

Again, expanding $\det(tI - L_\mu(G_1))(9, 10, 16)$ at the vertex 8, we have

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1))(7, 8, 9, 10, 16);$$

and expanding $\det(tI - L_\mu(G_1))(8, 9, 10, 16)$ at the vertex 7, we have

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1))(6, 7, 8, 9, 10, 16).$$

Note that

$$\begin{aligned} \det(tI - L_\mu(G_1))(6, 7, 8, 9, 10, 16) &= \det(tI - L_\mu(T))[13, 14, 15] \\ &\quad \times \det(tI - L_\mu(G_1)[\{1, 2, 3, 4, 5, 11, 12\} \cup V(P_m)]), \end{aligned}$$

and $\gamma(t, \mu)$ is coprime to $\det(tI - L_\mu(T))[13, 14, 15]$, where P_m is a subpath of P_{m+1} obtained by removing the root r . So

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1)[\{1, 2, 3, 4, 5, 11, 12\} \cup V(P_m)]).$$

Expanding the above determinant at the vertex 4, we have

$$\begin{aligned} &\det(tI - L_\mu(G_1)[\{1, 2, 3, 4, 5, 11, 12\} \cup V(P_m)]) \\ &= (t + 3\mu)(t + 2\mu) \det(tI - L_\mu(T)[1, 2, 3, 11, 12]) \det(tI - L_\mu(G_1)[V(P_{m-1})]) \\ &\quad - \det(tI - L_\mu(T)[1, 2, 3, 11, 12]) \det(tI - L_\mu(G_1)[V(P_m)]) \\ &\quad - (t + 2\mu) \det(tI - L_\mu(T)[1, 2])^2 \det(tI - L_\mu(G_1)[V(P_m)]) \\ &\quad - (t + 2\mu) \det(tI - L_\mu(T)[1, 2, 3, 11, 12]) \det(tI - L_\mu(G_1)[V(P_{m-1})]), \end{aligned}$$

where P_{m-1} is the subpath of P_m by removing the endpoint 17. By direct calculation, $\gamma(t, \mu)$ divides $\det(tI - L_\mu(T)[1, 2, 3, 11, 12])$ and is coprime to $(t + 2\mu) \det(tI - L_\mu(T)[1, 2])^2$. We have

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_1)[V(P_m)]). \tag{16}$$

As $\gamma(t, \mu)$ has degree 3 in t , we can assume $m \geq 3$; otherwise we would have a contradiction.

If taking $t = \mu$, then $\gamma(\mu, \mu) = 8\mu(3\mu^2 - 1)$ will divide

$$\delta_m(\mu) := \det(\mu I - L_\mu(G_1)[V(P_m)]) = \det(\mu(D' + I) - A'),$$

where D' is a degree diagonal matrix on the vertices P_m with entries $D'_{uu} = 2$ for all $u \neq n$ and $D'_{nn} = 1$, and A' is the adjacency matrix of P_m . So

$$\delta_m(1/\sqrt{3}) = \det(1/\sqrt{3} \cdot (D' + I) - A') = 0.$$

This implies that the matrix $1/\sqrt{3} \cdot (D' + I) - A'$ has an eigenvector x corresponding to the zero eigenvalue. By eigenvector equation, for all the vertices u of P_m other than n ,

$$\sqrt{3}x_u = \sum_{v \in N_{P_m}(u)} x_v,$$

and for the last vertex n ,

$$2/\sqrt{3} \cdot x_n = x_{n-1}.$$

So, if letting $x_{17} = a$, then $x_{18} = \sqrt{3}a$ and $x_{19} = 2a$. Along the path P_m starting from the vertex 17, the values of some vertices of P_m given by x are listed in Fig. 14.

Therefore, the value x_{n-1} belongs to the set $S := \{\pm a, \pm\sqrt{3}a, \pm 2a, 0\}$. It suffices to consider the cases of x_{n-1} having values among $a, \sqrt{3}a, 2a, 0$. If $x_{n-1} = a$, then $x_n = \sqrt{3}a/2$, and hence $x_{n-2} = \sqrt{3}a/2$, yielding a contradiction as $x_{n-2} \in S$. Similarly, if $x_{n-1} = \sqrt{3}a$, then $x_n = x_{n-2} = 3a/2$; and if $x_{n-1} = 0$, then $x_n = x_{n-2} = 0$ and then $x = 0$; which also yields a contradiction. For the last case, if $x_{n-1} = 2a$, then $x_n = x_{n-2} = \sqrt{3}a$, and

$$x_{n-3} = a, x_{n-4} = 0, x_{n-5} = -a.$$

So, in this case, we have

$$m \equiv 4 \pmod{12}. \tag{17}$$

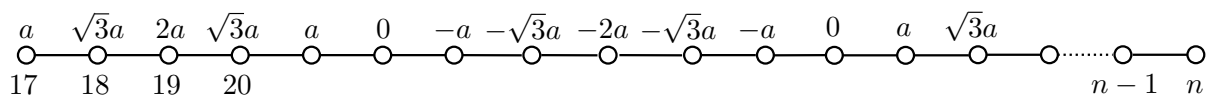


Figure 14: The path P_m and some entries of the eigenvector x

If taking $t = -\mu$, then $\gamma(-\mu, \mu) = -2\mu$ divides the following determinant

$$\det(-\mu I - L_\mu(G_1)[V(P_m)]) = \det(\mu(D' - I) - A') =: \eta_m(\mu),$$

where D' and A' are defined as in the above. By recursive formula,

$$\eta_m(\mu) = \mu\eta_{m-1}(\mu) - \eta_{m-2}(\mu).$$

So, if $\mu \mid \eta_m(\mu)$, so does $\eta_{m-2}(\mu)$. As m is even by Eq. (17), then $\mu \mid \eta_2(\mu)$. However, $\eta_2(\mu) = -1$, which yields a contradiction.

Next we prove $\gamma(t, \mu) \nmid D_{n-1}(G_2)$. Otherwise, $\gamma(t, \mu) \mid d_{10,16}(G_2)$. Note that

$$d_{10,16}(G_2) = \det(tI - L_\mu(G_2)(9, 10, 16)),$$

implying that $\gamma(t, \mu) \mid \det(tI - L_\mu(G_2)(9, 10, 16))$. Now, expanding $\det(tI - L_\mu(G_2))$ at the vertex 9 in a similar way as (15), we have

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_2)(8, 9, 10, 16)).$$

Expanding $\det(tI - L_\mu(G_2)(8, 9, 10, 16))$ at the vertex 4, we have

$$\begin{aligned} & \det(tI - L_\mu(G_2)(8, 9, 10, 16)) \\ &= (t + 2\mu) \det(tI - L_\mu(T)[1, 2, 3, 11, 12]) \det(tI - L_\mu(G_2)[\{5, 6, 7, 13, 14, 15\} \cup V(P_m)]) \\ & \quad - \det(tI - L_\mu(T)[1, 2])^2 \det(tI - L_\mu(G_2)[\{5, 6, 7, 13, 14, 15\} \cup V(P_m)]) \\ & \quad - \det(tI - L_\mu(T)[1, 2, 3, 11, 12]) \det(tI - L_\mu(G_2)[\{6, 7, 13, 14, 15\} \cup V(P_m)]). \end{aligned}$$

As $\gamma(t, \mu)$ divides $\det(tI - L_\mu(T)[1, 2, 3, 11, 12])$ and is coprime to $\det(tI - L_\mu(T)[1, 2])^2$,

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_2)[\{5, 6, 7, 13, 14, 15\} \cup V(P_m)]).$$

Expanding the above determinant at the vertex 7, we have

$$\begin{aligned} & \det(tI - L_\mu(G_2)[\{5, 6, 7, 13, 14, 15\} \cup V(P_m)]) \\ &= (t + 3\mu) \det(tI - L_\mu(T)[5, 6, 13, 14, 15]) \det(tI - L_\mu(G_2)[V(P_m)]) \\ & \quad - \det(tI - L_\mu(T)[5, 13, 14, 15]) \det(tI - L_\mu(G_2)[V(P_m)]) \\ & \quad - \det(tI - L_\mu(T)[5, 6, 13, 14, 15]) \det(tI - L_\mu(G_2)[V(P_{m-1})]). \end{aligned}$$

By direct calculation, $\gamma(t, \mu)$ divides $\det(tI - L_\mu(T)[5, 6, 13, 14, 15])$ and is coprime to $\det(tI - L_\mu(T)[5, 13, 14, 15])$. We have

$$\gamma(t, \mu) \mid \det(tI - L_\mu(G_2)[V(P_m)]).$$

We will get a contradiction by the same discussion to (16). □

By Claims 12,13 and 14, we have $D_{n-1}(G_1) = D_{n-1}(G_2) = 1$. By [6, Lemma 3.1],

$$\det(tI - L_\mu(G_1)) = \det(tI - L_\mu(G_2)) =: \psi(t, \mu).$$

So $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal form over $\mathbb{Q}(\mu)$ as follows:

$$1, \dots, 1, \psi(t, \mu),$$

with 1 appears $n - 1$ times. So the lemma follows. □

By a similar discussion, we can show that Lemma 11 also holds if P_{m+1} is replaced by a star with its center as root. However, due to the length of paper, we omit the result and its proof here. We believe that Lemma 11 holds when P_{m+1} is replaced by any nontrivial tree.

Conjecture 15. let \mathbb{T} be the tree in Fig. 11, and let T be any nontrivial tree with root r . Let $G_1 = T(r) \odot \mathbb{T}(4)$ and $G_2 = T(r) \odot \mathbb{T}(7)$. Then $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal forms over $\mathbb{Q}(\mu)[t]$.

We give a negative answer to Problem 3 asked by Godsil and Sun [6] by the fact that two trees are degree similar if and only if they are isomorphic [10].

Corollary 16. let \mathbb{T} be the tree in Fig. 11, and let P_{m+1} be a path on at least 2 vertices with an endpoint r as root. Let $G_1 = P_{m+1}(r) \odot \mathbb{T}(4)$ and $G_2 = P_{m+1}(r) \odot \mathbb{T}(7)$. Then $tI - L_\mu(G_1)$ and $tI - L_\mu(G_2)$ have the same Smith normal forms over $\mathbb{Q}(\mu)[t]$, but G_1 and G_2 are not degree similar.

4 Unicyclic graphs

Recall that a graph is called *unicyclic* if it is connected and contains only one cycle. In this section we first give some invariants for degree-similar graphs, and then prove some results on degree-similar determined unicyclic graphs. If two graphs are degree-similar, then they have the same spectra with respect to adjacency matrix, Laplacian matrix, signless Laplacian matrix and normalized Laplacian matrix (if there exist no isolated vertices), respectively. So we have many invariants for degree-similar graphs, some of which are listed below.

Lemma 17. Let G_1 and G_2 be a pair of degree-similar graphs. Then the following statements hold.

- (1) G_1 and G_2 have the same numbers of vertices, isolated vertices, edges, connected components, and bipartite connected components, respectively.
- (2) If G_1 and G_2 are connected, then they have the same number of spanning trees.
- (3) If G_1 and G_2 are connected, then they have the same number of walks of any given length.

Proof. By definition, G_1 and G_2 have the same number of vertices. Surely, they have the same degree sequence, implying that they have the same number of isolated vertices, and also the same number of edges. Also by definition, G_1 and G_2 have the same spectra with respect to Laplacian matrix and signless Laplacian matrix, respectively. It is well known that the multiplicity of zero as a Laplacian eigenvalue (respectively, as a signless Laplacian eigenvalue) of a graph equals the number of its connected components (respectively, the number of bipartite connected components); see Propositions 1.3.7 and 1.3.9 in [1]. Therefore, G_1 and G_2 have the same number of connected components and bipartite connected components, respectively.

By Matrix-Tree Theorem (or see Propositions 1.3.4 in [1]), the number of spanning trees of a graph equals the product of nontrivial Laplacian eigenvalues divided by the number of vertices of the graph. So, G_1 and G_2 have the same number of spanning trees if they are connected.

By Corollary 7, there exists an invertible matrix M such that

$$M^{-1}A(G_1)M = A(G_2), \quad M^T \mathbf{1} = M\mathbf{1} = \mathbf{1}.$$

Thus, for any positive integer k ,

$$\mathbf{1}^\top A(G_2)^k \mathbf{1} = \mathbf{1}^\top M^{-1} A(G_1)^k M \mathbf{1} = \mathbf{1}^\top A(G_1)^k \mathbf{1},$$

which implies that G_1 and G_2 have the same number of walks of length k . □

Recall that the *girth* of a graph is the minimum length of the cycles in the graph.

Lemma 18. *Let U_1 and U_2 be two degree-similar unicyclic graphs. Then they have the same girth.*

Proof. The result follows by Lemma 17 (2), since the girth of a unicyclic graph is exactly the number of its spanning trees. □

Theorem 19. *Let U be a unicyclic graph on n vertices with girth $g \in \{n, n - 1, n - 2\}$. Then U is degree-similar determined, namely, any graph G that is degree-similar to U must be isomorphic to U .*

Proof. Let G be a graph that is degree-similar to U . By Lemma 17 (1), G is connected with n vertices and n edges, which implies that G is also unicyclic. By Lemma 18, G is a unicyclic graph with the same girth g as U . Thus, if $g = n$ or $g = n - 1$, G is isomorphic to U obviously.

Now we consider the case of g equal to $n - 2$. In this case, U has exactly 2 vertices outside its cycle C of length $n - 2$. Thus, U is one of the following graphs: $C(r) \odot P_3(u)$, $C(r) \odot P_3(w)$, and $C(r_1, r_2, d)$, where P_3 is a path on 3 vertices with an endpoint u and a non-endpoint w , $C(r_1, r_2, d)$ is obtained from C by attaching one pendent edge at the vertex r_1 of C and another pendent edge at r_2 of C , and the distance between r_1 and r_2 is $d \geq 1$. Since G shares the same degree sequence with U , if $U = C(r) \odot P_3(u)$ with only one vertex of degree 3, surely $G \cong U$. Similarly, if $U = C(r) \odot P_3(w)$ with one vertex of maximum degree 4, we also have $G \cong U$.

If $U = C(r_1, r_2, d)$, then $G = C(r'_1, r'_2, d')$ for some vertices r'_1, r'_2 of C with distance d' by considering the degree sequence. We assert $d = d'$ and then $G \cong U$. Otherwise, without loss of generality, assume that $d < d'$. Let $\omega_{d+2}(U)$ and $\omega_{d+2}(G)$ be the numbers of walks of length $d + 2$ in the graph U and G , respectively, and let $\omega_{d+2}^{(i)}(U)$ and $\omega_{d+2}^{(i)}(G)$ be the numbers of walks of length $d + 2$ in the graph U and G that contain exactly i distinct pendent vertices, respectively, where $i = 0, 1, 2$. It is easily verified that

$$\omega_{d+2}^{(0)}(U) = \omega_{d+2}^{(0)}(G), \quad \omega_{d+2}^{(1)}(U) = \omega_{d+2}^{(1)}(G).$$

Observe that the distance between two pendent vertices of U is exactly $d + 2$, while the distance between two pendent vertices of G is $d' + 2$. Since $d < d' \leq (n - 2)/2$, we have

$$\omega_{d+2}^{(2)}(U) = 1 > \omega_{d+2}^{(2)}(G) = 0.$$

Therefore,

$$\omega_{d+2}(U) = \sum_{i=0}^2 \omega_{d+2}^{(i)}(U) > \sum_{i=0}^2 \omega_{d+2}^{(i)}(G) = \omega_{d+2}(G),$$

which yields a contradiction to Lemma 17 (3). □

Finally in this section we give another class of degree-similar determined unicyclic graphs by using Lemma 5.

Theorem 20. *Let T be a tree with root v , where T contains no vertices of degree 2 and v is the unique vertex of T with maximum degree. Let C_g be a cycle of length g with root r . Then the unicyclic graph $U = C_g(r) \odot T(v)$ is degree-similar determined.*

Proof. Let G be a graph that is degree-similar to U . By Lemma 17(1) and Lemma 18, G is a unicyclic graph with girth g . By Corollary 7, we can assume that G and U have the same vertex set, and the same degree partition, say $\pi = \{V_1, V_2, \dots, V_t\}$. By the assumption on U , we can assume $V_1 = V(C_g) \setminus \{r\}$, the set of vertices of U with degree 2; and $V_2 = \{r\}$ (or $\{v\}$), the set of the unique vertex of U with maximum degree $2 + \deg_T(v)$. Also, we can write $G = C_g(r) \odot T'(w)$. By Lemma 5, there exist invertible matrices M_1, \dots, M_t such that

$$M_i^{-1}A(U)_{ij}M_j = A(G)_{ij}, \quad i, j \in [t], \quad (18)$$

where $A(U)_{ij} = A(U)[V_i|V_j]$ and $A(G)_{ij} = A(G)[V_i|V_j]$ for $i, j \in [t]$.

Observe that T and T' share the same degree partition $\pi' = \{V_2, \dots, V_t\}$, which is obtained from π only by removing V_1 . By (18) for $i, j = 2, \dots, t$ and using Lemma 5, T is degree-similar to T' . By Lemma 17(1), T' is also a tree, and hence $T \cong T'$ ([4]). As v is the unique vertex of T with the maximum degree, w is the unique vertex of T' with the same maximum degree, and then we have $G \cong U$. \square

5 Strongly regular graphs

Recall that a graph G is called *strongly regular* with parameters $(n, d; a, c)$ if it has n vertices and is regular of degree d , any two adjacent vertices share exactly a common neighbors, and any non-adjacency vertices share exactly c common vertices. Godsil, Sun and Zhang [7] proved that if G is a strongly regular graph, then for any two edges e and f of G , the graphs $G \setminus e$ and $G \setminus f$ are (A, L, Q, N) -cospectral. In [6] the authors proposed Problem 4, namely, *are $G \setminus e$ and $G \setminus f$ degree similar?* In this section, we will prove that $G \setminus e$ and $G \setminus f$ are L_μ -cospectral, which generalized Godsil-Sun-Zhang's result ([7, Theorem 1]) and push the Problem 4 a step forward.

A graph G is called *walk regular* if for any positive integer k , the number of closed walks of length k is the same at all vertices. If further, the number of walks from vertex u to v of length k is the same for all adjacent vertex pairs u, v , then we say G is *1-walk regular*. Surely, a 1-walk regular graph is regular and also strongly regular.

Let G be a 1-walk regular graph. By Lemma 2.2 of [7], for any function f defined on the eigenvalues of $A := A(G)$, there exist α_f and β_f such that

$$f(A) \circ I = \alpha_f I, \quad f(A) \circ A = \beta_f A, \quad (19)$$

where \circ denotes Schur product. Let G be a graph and let u, v be vertices of G . Denote by $\delta_{u,v}$ the Kronecker notation, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise, and denote

by e_u the column vector with rows indexed by the vertices of G whose entries are given by $e_u(v) = \delta_{u,v}$. Denote $\epsilon_{u,v} = -\mu\delta_{u,v} + (1 - \delta_{u,v})$.

We first give a general result by using a similar technique in [7]. We need the following matrix results for preparation.

Theorem 21 (Sherman-Morrison). *Suppose B is an $n \times n$ invertible real matrix and u, v are n -dimensional real vectors. Then $B + uv^\top$ is invertible if and only if $1 + v^\top B^{-1}u \neq 0$. In this case,*

$$(B + uv^\top)^{-1} = B^{-1} - \frac{B^{-1}uv^\top B^{-1}}{1 + v^\top B^{-1}u}.$$

Lemma 22. [5] *Assume that C and D^\top are both matrices of size $m \times n$. Then*

$$\det(I_m - CD) = \det(I_n - DC).$$

Lemma 23. *Let G be a 1-walk regular graph with adjacency matrix A and degree matrix D . Let $u_1, v_1, \dots, u_r, v_r$ be vertices in the same clique of G . Then the value of*

$$e_{u_r}^\top (tI - A + \mu D \pm (\epsilon_{u_1, v_1} e_{u_1} e_{v_1}^\top + \dots + \epsilon_{u_{r-1}, v_{r-1}} e_{u_{r-1}} e_{v_{r-1}}^\top))^{-1} e_{v_r} \quad (20)$$

is independent on the choice of the clique and on the ordering of vertices of the chosen clique.

Proof. Suppose that G is d -regular. Then $tI - A + \mu D = (t + \mu d)I - A$. Let

$$f(x) = (t + \mu d - x)^{-1},$$

which is defined on all eigenvalues of A . As G is 1-walk regular, by (19), there exist $\alpha(t, \mu)$ and $\beta(t, \mu)$ such that

$$f(A) \circ I = \alpha(t, \mu)I, \quad f(A) \circ A = \beta(t, \mu)A.$$

We prove the result by induction. When $r = 1$, as u_1, v_1 are in the same clique of G ,

$$e_{u_1}^\top f(A) e_{v_1} = \delta_{u_1, v_1} \alpha(t, \mu) + (1 - \delta_{u_1, v_1}) \beta(t, \mu),$$

which only depends on whether u_1 and v_1 are the same or not.

Let $M_0 = (t + \mu d)I - A$ and for $s = 1, \dots, r - 1$,

$$M_s = tI - A + \mu D \pm (\epsilon_{u_1, v_1} e_{u_1} e_{v_1}^\top + \dots + \epsilon_{u_s, v_s} e_{u_s} e_{v_s}^\top). \quad (21)$$

Then (20) can be written as $e_{u_r}^\top M_{r-1}^{-1} e_{v_r}$. Assume that the result holds for $r = k$, where $k \geq 1$. Now, by Theorem 21,

$$\begin{aligned} e_{u_{k+1}}^\top M_k^{-1} e_{v_{k+1}} &= e_{u_{k+1}}^\top (M_{k-1} \pm \epsilon_{u_k, v_k} e_{u_k} e_{v_k}^\top)^{-1} e_{v_{k+1}} \\ &= e_{u_{k+1}}^\top \left(M_{k-1}^{-1} \mp \frac{\epsilon_{u_k, v_k} M_{k-1}^{-1} e_{u_k} e_{v_k}^\top M_{k-1}^{-1}}{1 + \epsilon_{u_k, v_k} e_{v_k}^\top M_{k-1}^{-1} e_{u_k}} \right) e_{v_{k+1}} \quad (\text{by Theorem 21}) \\ &= e_{u_{k+1}}^\top M_{k-1}^{-1} e_{v_{k+1}} \mp \frac{\epsilon_{u_k, v_k} (e_{u_{k+1}}^\top M_{k-1}^{-1} e_{u_k}) (e_{v_k}^\top M_{k-1}^{-1} e_{v_{k+1}})}{1 + \epsilon_{u_k, v_k} e_{v_k}^\top M_{k-1}^{-1} e_{u_k}}, \end{aligned}$$

whose value does not depend on which clique the vertices are in, and remains unchanged if we reorder the vertices inside the clique, since each term satisfies this condition by the induction hypothesis. \square

Theorem 24. *Let G be a 1-walk regular graph with clique number ω . Then for any graph H on at most ω vertices, removing edges of H from cliques of G results in graphs with the same μ -polynomials.*

Proof. Let \bar{H} be the graph obtained from H by adding $|V(G)| - |V(H)|$ isolated vertices. Order the vertices of \bar{H} so that the vertices of H correspond to a clique in G . Assume H has m edges labelled as $e_i = \{u_i, v_i\}$ for $i \in [m]$. The μ -polynomial of $\hat{G} := G - E(H)$ is

$$\begin{aligned} \psi(\hat{G}, t, \mu) &:= \det(tI - A + \mu D + (e_{u_1}e_{v_1}^\top + e_{v_1}e_{u_1}^\top + \cdots + e_{u_m}e_{v_m}^\top + e_{v_m}e_{u_m}^\top)) \\ &\quad - \mu(e_{u_1}e_{u_1}^\top + e_{v_1}e_{v_1}^\top + \cdots + e_{u_m}e_{u_m}^\top + e_{v_m}e_{v_m}^\top). \end{aligned} \quad (22)$$

We will prove that $\psi(\hat{G}, t, \mu)$ is independent of which clique of G the vertex set of H corresponds to or how the vertices of H are ordered. When $m = 1$,

$$\begin{aligned} &\psi(\hat{G}, t, \mu) \\ &= \det(tI - A + \mu D + (e_{u_1}e_{v_1}^\top + e_{v_1}e_{u_1}^\top) - \mu(e_{u_1}e_{u_1}^\top + e_{v_1}e_{v_1}^\top)) \\ &= \det(tI - A + \mu D + (e_{u_1}, e_{v_1})(e_{v_1} - \mu e_{u_1}, e_{u_1} - \mu e_{v_1})^\top) \\ &= \det(tI - A + \mu D) \det(I + (tI - A + \mu D)^{-1}(e_{u_1}, e_{v_1})(e_{v_1} - \mu e_{u_1}, e_{u_1} - \mu e_{v_1})^\top) \\ &= \det(tI - A + \mu D) \det(I_2 + (e_{v_1} - \mu e_{u_1}, e_{u_1} - \mu e_{v_1})^\top (tI - A + \mu D)^{-1}(e_{u_1}, e_{v_1})) \\ &= \det M_0 \det \begin{pmatrix} 1 + e_{v_1}^\top M_0^{-1} e_{u_1} - \mu e_{u_1}^\top M_0^{-1} e_{u_1} & e_{v_1}^\top M_0^{-1} e_{v_1} - \mu e_{u_1}^\top M_0^{-1} e_{v_1} \\ e_{u_1}^\top M_0^{-1} e_{u_1} - \mu e_{v_1}^\top M_0^{-1} e_{u_1} & 1 + e_{u_1}^\top M_0^{-1} e_{v_1} - \mu e_{v_1}^\top M_0^{-1} e_{v_1} \end{pmatrix}, \end{aligned}$$

where the fourth equality follows from Lemma 22 and $M_0 := tI - A + \mu D$ in the last equality. In this case, the μ -polynomial $\psi(\hat{G}, t, \mu)$ is independent of the choice of the edge $\{u_1, v_1\}$, since each entry in the 2×2 matrix does not by Lemma 23.

Define M_s as in (21), we have

$$\begin{aligned} M_{4m} &= tI - A + \mu D + (e_{u_1}e_{v_1}^\top + e_{v_1}e_{u_1}^\top + \cdots + e_{u_m}e_{v_m}^\top + e_{v_m}e_{u_m}^\top) \\ &\quad - \mu(e_{u_1}e_{u_1}^\top + e_{v_1}e_{v_1}^\top + \cdots + e_{u_m}e_{u_m}^\top + e_{v_m}e_{v_m}^\top). \end{aligned}$$

Then

$$\begin{aligned} &\psi(\hat{G}, t, \mu) \\ &= \det(M_{4(m-1)} + (e_{u_m}e_{v_m}^\top + e_{v_m}e_{u_m}^\top) - \mu(e_{u_m}e_{u_m}^\top + e_{v_m}e_{v_m}^\top)) \\ &= \det(M_{4(m-1)} + (e_{u_m}, e_{v_m})(e_{v_m} - \mu e_{u_m}, e_{u_m} - \mu e_{v_m})^\top) \\ &= \det M_{4(m-1)} \det(I + M_{4(m-1)}^{-1}(e_{u_m}, e_{v_m})(e_{v_m} - \mu e_{u_m}, e_{u_m} - \mu e_{v_m})^\top) \\ &= \det M_{4(m-1)} \det(I_2 + (e_{v_m} - \mu e_{u_m}, e_{u_m} - \mu e_{v_m})^\top M_{4(m-1)}^{-1}(e_{u_m}, e_{v_m})) \\ &= \det M_{4(m-1)} \\ &\quad \times \det \begin{pmatrix} 1 + e_{v_m}^\top M_{4(m-1)}^{-1} e_{u_m} - \mu e_{u_m}^\top M_{4(m-1)}^{-1} e_{u_m} & e_{v_m}^\top M_{4(m-1)}^{-1} e_{v_m} - \mu e_{u_m}^\top M_{4(m-1)}^{-1} e_{v_m} \\ e_{u_m}^\top M_{4(m-1)}^{-1} e_{u_m} - \mu e_{v_m}^\top M_{4(m-1)}^{-1} e_{u_m} & 1 + e_{u_m}^\top M_{4(m-1)}^{-1} e_{v_m} - \mu e_{v_m}^\top M_{4(m-1)}^{-1} e_{v_m} \end{pmatrix}. \end{aligned}$$

Since by induction the first factor, and by Lemma 23 each entry in the 2×2 matrix do not dependent on the choice of the clique in G nor on the ordering of vertices of H , the result follows. \square

Corollary 25. *Let G be a strongly regular graph with clique number ω and let H be any graph on at most ω vertices. Removing edges of H from cliques of G results in graphs with the same μ -polynomials, whose complements also have the same μ -polynomials.*

Proof. By Theorem 24, removing edges of H from cliques of G results in graphs with the same μ -polynomials. For the function $f(x)$ defined in Lemma 23,

$$f(A) \circ I = \alpha(t, \mu)I, \quad f(A) \circ A = \beta(t, \mu)A.$$

Furthermore, if G has parameters $(n, d; a, c)$, then $A^2 = dI + aA + c(J - I - A)$. So, there exists $\gamma(t, \mu)$ such that

$$f(A) \circ (J - I - A) = \gamma(t, \mu)(J - I - A).$$

Let G^c be the complement of G , which is also strongly regular or 1-walk regular. By a similar argument in the proof of Theorem 24, adding edges of H inside a coclique of G^c results in graphs with the same μ -polynomials. Now, deleting edges of H in a clique of G corresponds to adding edges of H in the corresponding coclique of \bar{G} . So, removing edges of H from cliques of G results in graphs whose complements also have the same μ -polynomials. \square

Corollary 26. *Let G be a strongly regular graph. Then for any two edges e and f of G , the graphs $G \setminus e$ and $G \setminus f$ have the same μ -polynomial, or they are L_μ -cospectral.*

There are exactly 15 non-isomorphic strongly regular graphs, denoted by X_i for $i = 0, 1, \dots, 14$ in [7], with parameters $(25, 12; 5, 6)$. Their adjacency matrices can be found on Spence's website: <http://www.maths.gla.ac.uk/~es/srgraphs.php>. In [7] the authors give a table that lists the number of pairwise non-isomorphic subgraphs of X_i obtained by deleting edges of 6 small graphs respectively in cliques of X_i for $i = 0, 1, \dots, 14$. For example, removing an edge from X_1 gives a family of 150 graphs, they are pairwise non-isomorphic but L_μ -cospectral.

6 Orthogonally degree-similar graphs

Motivated by the problem proposed by Wang et al. [16], we introduce orthogonally degree-similar graphs, which may be viewed as a stronger version of degree-similar graphs.

Definition 27. Two graphs G_1 and G_2 are called *orthogonally degree-similar* if there exists an orthogonal matrix Q such that Eq. (2) holds, namely,

$$Q^\top A(G_1)Q = A(G_2), \quad Q^\top D(G_1)Q = D(G_2).$$

We have some remarks for the rationality of the definition.

- (1) Two graphs G_1 and G_2 are cospectral if and only if $A(G_1)$ and $A(G_2)$ are orthogonally similar.
- (2) G_1 and G_2 are cospectral with cospectral complements if and only if $A(G_1)$ and $A(G_2)$ are similar via an orthogonal matrix Q with $Q\mathbf{1} = \mathbf{1}$ (Theorem 1).
- (3) If G_1 and G_2 are degree similar and one of them is connected, then G_1 and G_2 are cospectral with cospectral complements (Lemma 6). So we have an orthogonal matrix Q as in (2).
- (4) The invertible matrix M in Eq. (1) is not unique, since kM still satisfies Eq. (1) for any nonzero k .
- (5) As the adjacency matrices and degree matrices are symmetric, we have additional requirements for M in Eq. (1), that is,

$$M^\top A(G_1)(M^{-1})^\top = A(G_2), \quad M^\top D(G_1)(M^{-1})^\top = D(G_2).$$

- (6) The matrices M in most examples of degree similar graphs in [6] are orthogonal, e.g. Example 5.3, Example 6.3, Example 7.3, Example 8.4.

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References

- [1] A. E. Brouwer and W. H. Haemers. *Spectra of Graphs*. Springer, 2011.
- [2] E. R. van Dam and W. H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 373: 241–272, 2003.
- [3] E. R. van Dam, W. H. Haemers, and J. H. Koolen. Cospectral graphs and the generalized adjacency matrix, *Linear Algebra Appl.*, 423: 33–41, 2007.
- [4] C. Godsil and B. McKay. Some computational results on the spectra of graphs. In *Combinatorial Mathematics IV*, pages 73–92, Springer, 1976.
- [5] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, 2001.
- [6] C. Godsil and W. Sun. Degree-similar graphs. *Electron. J. Combin.*, 32(4): #P4.12, 2025.
- [7] C. Godsil, W. Sun and X. Zhang. Cospectral graphs obtained by edge deletion. *Linear Multilinear Algebra*, 73: 2781–2800, 2025.

- [8] W. H. Haemers and E. Spence. Enumeration of cospectral graphs. *European J. Combin.*, 25: 199–211, 2004.
- [9] C. R. Johnson and M. Newman. A note on cospectral graphs. *J. Combin. Theory Ser. B*, 28: 96–103, 1980.
- [10] B. McKay. On the spectral characterisation of trees. *Ars Combin.*, 3: 219–232, 1977.
- [11] S. P. Osborne. Cospectral bipartite graphs for the normalized Laplacian. Ph.D. Thesis, Iowa State University, 2013.
- [12] A. Terras. *Zeta Functions of Graphs: A Stroll through the Garden*, Cambridge University Press, 2010.
- [13] W. T. Tutte. *All the king's horses. A guide to reconstruction*. In *Graph Theory and Related Topics*, pages 15–33, 1979.
- [14] W. Wang. Generalized spectral characterization of graphs revisited. *Electron. J. Combin.*, 20(4): #P4, 2013.
- [15] W. Wang. A simple arithmetic criterion for graphs being determined by their generalized spectra. *J. Combin. Theory Ser. B*, 122: 438–451, 2017.
- [16] W. Wang, F. Li, H. Lu and Z. Xu. Graphs determined by their generalized characteristic polynomials. *Linear Algebra Appl.*, 434: 1378–1387, 2011.
- [17] W. Wang and C.-X. Xu. A sufficient condition for a family of graphs being determined by their generalized spectra. *European J. Combin.*, 27: 826–840, 2006.
- [18] W. Wang and C.-X. Xu. An excluding algorithm for testing whether a family of graphs are determined by their generalized spectra. *Linear Algebra Appl.*, 418: 62–74, 2006.