

The chromatic symmetric function of graphs glued at a single vertex

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Abstract

We describe how the chromatic symmetric function of two graphs glued at a single vertex can be expressed as a matrix multiplication using certain information of the two individual graphs. We then prove new e -positivity results by using a connection between forest triples, defined by the first author, and Hikita's probabilities associated to standard Young tableaux. Specifically, we prove that gluing a sequence of unit interval graphs and cycles results in an e -positive graph. We also prove e -positivity for a graph obtained by gluing the first and last vertices of such a sequence. This generalizes e -positivity of cycle-chord graphs and supports Ellzey's conjectured e -positivity for proper circular arc digraphs.

Mathematics Subject Classifications: 05E05, 05E10, 0C15

1 Introduction

The Stanley–Stembridge conjecture [15, 16] is a monumental problem in algebraic combinatorics. It states that if G is the incomparability graph of a $(\mathbf{3} + \mathbf{1})$ -free poset, then its chromatic symmetric function $X_G(\mathbf{x})$ is e -positive. By a reduction due to Guay-Paquet [8], it suffices to prove e -positivity whenever G is a unit interval graph. Shareshian and Wachs [13] refined this conjecture by introducing the chromatic quasisymmetric function $X_G(\mathbf{x}; q)$, showing that it is symmetric whenever G is a unit interval graph, and conjecturing that in this case, its e -coefficients are positive polynomials in q . Hikita [11] proved the Stanley–Stembridge conjecture by giving a combinatorial formula for $X_G(\mathbf{x}; q)$ in terms of probabilities associated to standard Young tableaux, whose summands are all nonnegative when $q = 1$. The Shareshian–Wachs conjecture is still open and has seen keen interest because of its connection to Hessenberg varieties [1, 3, 4, 9, 10, 12] and Hecke algebras [5, 14]. Ellzey [6] generalized further by defining a chromatic quasisymmetric function

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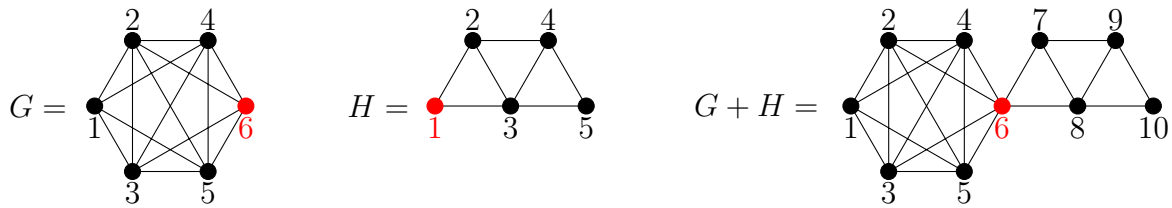


Figure 1: Graphs G and H glued at a single vertex.

$X_{\vec{G}}(\mathbf{x}; q)$ for any directed graph \vec{G} , showing symmetry whenever \vec{G} is a proper circular arc digraph, and conjecturing e -positivity in this case.

In this paper, we consider the operation of gluing graphs G and H together at a single vertex. In Section 3, we show that the resulting chromatic symmetric function can be expressed as a matrix multiplication using certain information about G and H . In Section 4, we prove a surprising connection between forest triples, defined by the first author [17], and Hikita’s probabilities on standard Young tableaux (Theorem 52). From this connection and Hikita’s remarkable breakthrough, we prove e -positivity of graphs obtained from gluing a sequence of unit interval graphs and cycles (Corollary 54). In Section 5, we prove that if G° is obtained by gluing two vertices of a graph G , then $X_{G^\circ}(\mathbf{x})$ is equal to the trace of the matrix corresponding to G (Theorem 60). In particular, we prove e -positivity for graphs obtained by gluing the first and last vertices of a unit interval graph (Corollary 63). These graphs are proper circular arc graphs, lending support to Ellzey’s conjecture. This also generalizes e -positivity of cycle-chord graphs (Corollary 68).

2 Background

All graphs in this paper will have vertex set $[n] = \{1, \dots, n\}$ for some n . For graphs $G = ([n], E)$ and $H = ([n'], E')$, we define the *sum*

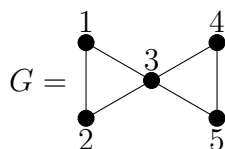
$$G + H = ([n + n' - 1], \{E \cup \{i + n - 1, j + n - 1\} : \{i, j\} \in E'\}).$$

Informally, we glue vertex n of G to vertex 1 of H . The main focus of this paper is to study how the chromatic symmetric function behaves under this operation.

2.1 Chromatic symmetric functions and unit interval graphs

A *proper colouring* of $G = ([n], E)$ is a function $\kappa : [n] \rightarrow \mathbb{P} = \{1, 2, 3, \dots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $\{u, v\} \in E$. The *chromatic symmetric function* of G is [15, Definition 2.1]

$$X_G(\mathbf{x}) = \sum_{\kappa \text{ proper colouring}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$



$$\begin{aligned}
 X_G(\mathbf{x}; q) &= (q^2 + 2q^3 + q^4)e_{32} + (q + 3q^2 + 4q^3 + 3q^4 + q^5)e_{41} \\
 &\quad + (1 + 3q + 4q^2 + 4q^3 + 4q^4 + 3q^5 + q^6)e_5 \\
 &= q^2[2]_q[2]_q e_{32} + q[3]_q[2]_q[2]_q e_{41} + [5]_q[2]_q[2]_q e_5.
 \end{aligned}$$

Figure 2: The bowtie graph G and the chromatic quasisymmetric function $X_G(\mathbf{x}; q)$.

We say that $X_G(\mathbf{x})$ is *e-positive* if $X_G(\mathbf{x})$ has nonnegative coefficients when expanded in the basis of *elementary symmetric functions* e_λ , defined by $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where $e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$. We say that G is a *natural unit interval graph* (or *NUIG*) if there are unit intervals $I_1 = [a_1, a_1 + 1], \dots, I_n = [a_n, a_n + 1] \subset \mathbb{R}$ with $a_1 < \cdots < a_n$ and $\{i, j\} \in E(G)$ if and only if $I_i \cap I_j \neq \emptyset$. Equivalently, for every $1 \leq i < j < k \leq n$, if $\{i, k\} \in E(G)$, then $\{i, j\} \in E(G)$ and $\{j, k\} \in E(G)$. These are also equivalent to *proper interval graphs*, where the intervals I_i are allowed to be of different lengths but they cannot contain each other. Hikita proved the famous Stanley–Stembridge conjecture.

Theorem 1. [15, Conjecture 5.5], [11, Corollary 1.8] *If G is an NUIG, then $X_G(\mathbf{x})$ is e-positive.*

An *ascent* of a proper colouring κ of $G = ([n], E)$ is an edge $\{u, v\} \in E$ with $u < v$ and $\kappa(u) < \kappa(v)$. The *chromatic quasisymmetric function* of G is [13, Definition 1.2]

$$X_G(\mathbf{x}; q) = \sum_{\kappa \text{ proper colouring}} q^{\text{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

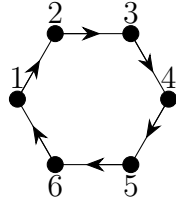
Note that if we set $q = 1$, we get $X_G(\mathbf{x}; 1) = X_G(\mathbf{x})$. Shareshian and Wachs showed that if G is an NUIG, then $X_G(\mathbf{x}; q)$ is in fact symmetric [13, Theorem 4.5]. They conjectured that in this case, $X_G(\mathbf{x}; q)$ is *e-positive*, meaning that its *e-coefficients*, which are polynomials in the variable q , have positive coefficients. This refined conjecture is still open.

Conjecture 2. [13, Conjecture 5.1] *If G is an NUIG, then $X_G(\mathbf{x}; q)$ is e-positive.*

Example 3. The complete graph K_n is an NUIG and has $X_{K_n}(\mathbf{x}; q) = [n]_q! e_n$, where

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q \text{ and } [k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}.$$

Example 4. The bowtie graph G in Figure 2 is an NUIG and $X_G(\mathbf{x}; q)$ is *e-positive*.



$$X_{\vec{C}_6}(\mathbf{x}; q) = 2q^3 e_{222} + 4q^2 [3]_q e_{42} + 2q^2 [3]_q e_{42} + 3q^2 [3]_q [2]_q e_{33} + 6q [5]_q e_6$$

$$\alpha = \quad 222 \quad \quad 42 \quad \quad 24 \quad \quad 33 \quad \quad 6$$

Figure 3: The directed cycle \vec{C}_6 and the chromatic quasisymmetric function $X_{\vec{C}_6}(\mathbf{x}; q)$.

Example 5. The path P_n is an NUIG and has [13, Section 5]

$$X_{P_n}(\mathbf{x}; q) = \sum_{\alpha \models n} [\alpha_1]_q ([\alpha_2]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\text{sort}(\alpha)},$$

so in particular, $X_{P_n}(\mathbf{x}; q)$ is e -positive.

More generally, for a directed graph $\vec{G} = ([n], \vec{E})$, Ellzey defined the *chromatic quasisymmetric function* of \vec{G} to be [6, Definition 1.3]

$$X_{\vec{G}}(\mathbf{x}; q) = \sum_{\kappa \text{ proper colouring}} q^{\text{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)},$$

where $\text{asc}(\kappa)$ is the number of directed edges $(u, v) \in \vec{E}$ with $\kappa(u) < \kappa(v)$. Note that if $G = ([n], E)$ is a graph and we direct edges from the smaller vertex to the larger vertex, then we recover the earlier definition of Shareshian and Wachs. We say that \vec{G} is a *proper circular arc digraph* if there are arcs A_1, \dots, A_n on some circle with left endpoints a_1, \dots, a_n in clockwise order, that do not contain each other, such that $(i, j) \in \vec{E}$ if and only if $a_j \in A_i$. Ellzey showed that $X_{\vec{G}}(\mathbf{x}; q)$ is symmetric for such graphs [6, Corollary 5.7] and conjectured e -positivity. This extension of the refined Stanley–Stembridge conjecture is open.

Conjecture 6. [6, Conjecture 1.4] If \vec{G} is a proper circular arc digraph, then $X_{\vec{G}}(\mathbf{x}; q)$ is e -positive.

Example 7. The directed cycle \vec{C}_n is a proper circular arc digraph and [6, Corollary 6.2]

$$X_{\vec{C}_n}(\mathbf{x}; q) = \sum_{\alpha \models n} \alpha_1 ([\alpha_1]_q - 1) ([\alpha_2]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\text{sort}(\alpha)}, \quad (1)$$

so $X_{\vec{C}_n}(\mathbf{x}; q)$ is e -positive. The example of \vec{C}_6 is given in Figure 3, along with the corresponding compositions. It turns out that if we change the directed edge $(n, 1)$ to $(1, n)$, the resulting graph \vec{C}'_n also has that $X_{\vec{C}'_n}(\mathbf{x}; q)$ is symmetric and e -positive [7, Section 3].

2.2 Forest triples

The first author proved a signed combinatorial formula for the e -expansion of $X_G(\mathbf{x})$. In the case that G is an NUIG, a similar formula is given for $X_G(\mathbf{x}; q)$. Let $G = ([n], E)$ be a graph and fix a total ordering \prec on E .

Definition 8. A *no broken circuit (NBC) tree* of G is a subtree T whose edges do not contain a subset of the form $B = C \setminus \{\max(C)\}$, where C is a set of edges that form a cycle and $\max(C)$ is the largest edge of C under \prec .

Definition 9. A *tree triple* of G is an object $\mathcal{T} = (T, \alpha, r)$ consisting of the following data.

- T is an NBC tree of G .
- α is a composition of size $|T|$.
- r is an integer with $1 \leq r \leq \alpha_1$, the first part of α .

A *forest triple* of G is a sequence $\mathcal{F} = (\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1), \dots, \mathcal{T}_m = (T_m, \alpha^{(m)}, r_m))$ of tree triples such that each vertex of G is in exactly one tree and we have $\min(T_1) < \dots < \min(T_m)$. Let $\text{FT}(G)$ denote the set of forest triples of G . The *type* of \mathcal{F} is the partition

$$\text{type}(\mathcal{F}) = \text{sort}(\alpha^{(1)} \dots \alpha^{(m)}),$$

given by concatenating the compositions and sorting to make a partition. The *sign* of \mathcal{F} is the integer

$$\text{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^m (\ell(\alpha^{(i)}) - 1)} = (-1)^{\ell(\text{type}(\mathcal{F})) - m}.$$

We will often be interested in $\alpha_1^{(1)}$, the first part of the composition associated to the tree containing vertex 1. We define the *reduced type* of \mathcal{F} , denoted $\text{type}'(\mathcal{F})$, to be the partition $\text{type}(\mathcal{F})$ with an instance of $\alpha_1^{(1)}$ removed.

Now we can use forest triples to calculate the chromatic symmetric function.

Theorem 10. [17, Theorem 5.10] *The chromatic symmetric function of G satisfies*

$$X_G(\mathbf{x}) = \sum_{\mathcal{F} \in \text{FT}(G)} \text{sign}(\mathcal{F}) e_{\text{type}(\mathcal{F})}.$$

In the case that G is an NUIG, we will order edges lexicographically, so that if $\{i, j\}$ and $\{i', j'\}$ are edges of G with $i < j$ and $i' < j'$, we define $\{i, j\} \prec \{i', j'\}$ if either $i < i'$, or $i = i'$ and $j < j'$. Then T is an NBC tree of G precisely when no vertex of T has two larger neighbours. We also define a permutation of $V(T)$, denoted $\text{list}(T)$, as follows. We first read the smallest vertex of T . Then at each step, we read the smallest unread vertex of T that is adjacent to a read vertex of T . Now for a forest triple $\mathcal{F} = (\mathcal{T}_i = (T_i, \alpha^{(i)}, r_i))$ of G , we define the integer

$$\text{weight}(\mathcal{F}) = \text{inv}_G(\text{list}(T_1) \cdots \text{list}(T_m)) + \sum_{i=1}^m (r_i - 1),$$

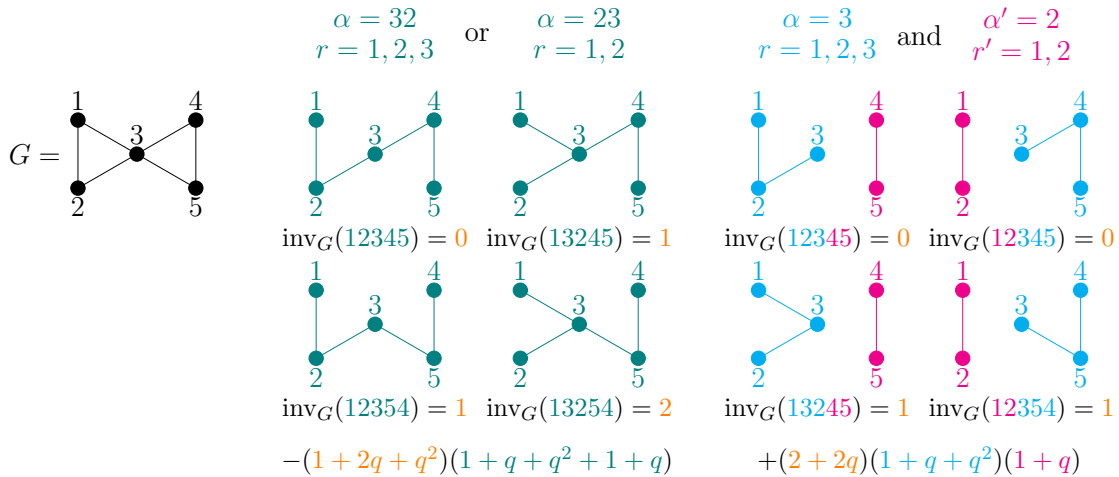


Figure 4: The bowtie graph G and the forest triples of G of type 32.

where $\text{inv}_G(\sigma) = |\{(i, j) : i < j, \sigma_i > \sigma_j, \{\sigma_j, \sigma_i\} \in E(G)\}|$ is the number of G -inversions of the permutation σ . We can similarly use forest triples to calculate the chromatic quasisymmetric function.

Theorem 11. [17, Theorem 3.4] *The chromatic quasisymmetric function of an NUIG G satisfies*

$$X_G(\mathbf{x}; q) = \sum_{\mathcal{F} \in \text{FT}(G)} \text{sign}(\mathcal{F}) q^{\text{weight}(\mathcal{F})} e_{\text{type}(\mathcal{F})}.$$

Example 12. Figure 4 shows the forest triples of type 32 for the bowtie graph G from Figure 2. We could have a single tree triple $\mathcal{T} = (T, \alpha, r)$, where either $\alpha = 32$ and $1 \leq r \leq 3$, or $\alpha = 23$ and $1 \leq r \leq 2$. Or we could have two tree triples of the form $\mathcal{T} = (T, 3, r)$ and $\mathcal{T}' = (T', 2, r')$, where $1 \leq r \leq 3$ and $1 \leq r' \leq 2$. By calculating the weights, we see that the coefficient of e_{32} in $X_G(\mathbf{x}; q)$ is

$$-(1 + 2q + q^2)(1 + q + q^2 + 1 + q) + (2 + 2q)(1 + q + q^2)(1 + q) = q^2(1 + q)^2 = q^2[2]_q[2]_q.$$

2.3 Standard Young tableaux

Hikita proved a formula for the e -expansion of $X_G(\mathbf{x}; q)$ in terms of probabilities associated to standard Young tableaux.

Definition 13. The *Young diagram* of a partition λ is the left-justified array of boxes with λ_i boxes in the i -th row (from the bottom). A *standard Young tableau (SYT)* of shape λ is a filling T of these boxes with the integers 1 through n , using each exactly once, so that rows are increasing from left to right and columns are increasing from bottom to top. We let SYT_λ denote the set of SYT of shape λ and we let $\text{SYT}_n = \bigsqcup_{\lambda \vdash n} \text{SYT}_\lambda$. If $T \in \text{SYT}_\lambda$, we write $\text{shape}(T) = \lambda$.

We can imagine constructing T by successively dropping boxes labelled $1, \dots, n$ on the top of some column. If box k is dropped in column b_k , then the condition that T has increasing rows is precisely that the sequence $b(T) = (b_1, \dots, b_n)$ is a *ballot sequence*, meaning that every initial string (b_1, \dots, b_j) contains at least as many i 's as $(i + 1)$'s for every i .

We will often be interested in $b_n(T)$, the column of T that contains the largest entry n , or in other words, the length of the row of T that contains n . We define the *reduced shape* of T , denoted $\text{shape}'(T)$, to be the partition $\text{shape}(T)$ with an instance of $b_n(T)$ removed.

Now given an NUIG G , Hikita described a certain subset of standard Young tableaux $\text{SYT}(G) \subseteq \text{SYT}_n$. In his formulation, he reads the vertices of G in reverse order, from n to 1, so we denote by \bar{G} the reverse graph

$$\bar{G} = ([n], \{\{n + 1 - j, n + 1 - i\} : \{i, j\} \in E(G)\}).$$

In other words, i is a \bar{G} -neighbour of j if and only if $(n + 1 - i)$ is a G -neighbour of $(n + 1 - j)$.

Definition 14. Let G be an NUIG, let $T \in \text{SYT}_n$, and let $1 \leq k \leq n$. Let $T|_{<k}$ the tableau obtained by restricting T to the entries less than k and let cols_k be the number of columns of $T|_{<k}$. We now define the sequence $\delta^{(k)}(T) = (\delta_0^{(k)}, \delta_1^{(k)}, \dots, \delta_{\text{cols}_k}^{(k)}, \delta_{\text{cols}_k+1}^{(k)})$ by $\delta_0^{(k)} = 1$, $\delta_{\text{cols}_k+1}^{(k)} = 0$, and

$$\delta_i^{(k)} = \begin{cases} 1 & \text{if the top entry of column } i \text{ of } T|_{<k} \text{ is a } \bar{G}\text{-neighbour of } k, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq \text{cols}_k$. Following Hikita's notation [11, Definition 1.2], we define the sets of column indices

$$R_k(T) = \{1 \leq i \leq \text{cols}_k : \delta_i^{(k)} = 1, \delta_{i-1}^{(k)} = 0\} \text{ and} \\ W_k(T) = \{1 \leq i \leq \text{cols}_k + 1 : \delta_i^{(k)} = 0, \delta_{i-1}^{(k)} = 1\}.$$

Finally, we define $\text{SYT}(G)$ to be the subset of those $T \in \text{SYT}_n$ such that for every $1 \leq k \leq n$, we have $b_k(T) \in W_k(T)$, where $b_k(T)$ denotes the column that contains the entry k .

Informally, a standard Young tableau $T \in \text{SYT}(G)$ is constructed as follows. We start with the empty tableau. Then for $1 \leq k \leq n$, after we have dropped $(k - 1)$ boxes, some columns of our partial tableau $T|_{<k}$ will have a \bar{G} -neighbour of k as the top cell and some will not. We have $\delta_i^{(k)} = 1$ if i is a "neighbour column" and $\delta_i^{(k)} = 0$ if i is a "non-neighbour column". Then $R_k(T)$ is the set of the "leftmost neighbour columns" in a string and $W_k(T)$ is the set of the "leftmost non-neighbour columns" in a string. The condition that $b_k(T) \in W_k(T)$ is that the box k must be dropped in a "leftmost non-neighbour column".

Note that if $i \in W_k(T)$, this means that the top entry of column $(i - 1)$ of $T|_{<k}$ is a \bar{G} -neighbour of k and the top entry of column i is a \bar{G} -non-neighbour of k . Therefore, column $(i - 1)$ received a box more recently than column i , so by induction, the $(i - 1)$ -th column was strictly taller than the i -th column and we can indeed drop box k in column i .

We now define factors that we will associate to a standard Young tableau $T \in \text{SYT}(G)$.

Definition 15. Let G be an NUIG and let $T \in \text{SYT}(G)$. We define the rational function

$$c_T(G) = \prod_{k=1}^n c_T^{(k)}(G), \text{ where } c_T^{(k)}(G) = q^{|\{j>b_k: \delta_j=1\}|} [b_k]_q \frac{\prod_{i \in R_k(T)} [i - b_k]_q}{\prod_{i \in W_k(T): i \neq b_k} [i - b_k]_q}.$$

Now Hikita proved the following formula for $X_G(\mathbf{x}; q)$.

Theorem 16. [11, Theorem 1.6] *If G is an NUIG, then we have*

$$X_G(\mathbf{x}; q) = \sum_{T \in \text{SYT}(G)} c_T(G) e_{\text{shape}(T)}.$$

In particular, if we evaluate at $q = 1$, we have that $X_G(\mathbf{x})$ is e -positive.

Remark 17. We can think of the factor

$$p(b_k, q) = \frac{\prod_{i \in R_k(T)} [i - b_k]_q}{\prod_{i \in W_k(T): i \neq b_k} [i - b_k]_q}$$

as the *probability* that box k is dropped in column b_k because for every $q \in \mathbb{R}_{\geq 0}$, we have $0 \leq p(b_k, q) \leq 1$ and $\sum_{b_k \in W_k(T)} p(b_k, q) = 1$.

Proposition 18. *Let G be an NUIG with $\{n - k + 1, n\} \in E(G)$, in other words, the vertices $\{1, \dots, k\}$ are all adjacent in \bar{G} . Then every tableau $T \in \text{SYT}(G)$ must have all of the entries $\{1, \dots, k\}$ in the first row.*

Proof. Because the vertices $\{1, \dots, k\}$ are all adjacent in G , we have for every $1 \leq k' \leq k$ that $\delta^{(k')}(T) = (1, \dots, 1, 0)$, so the next entry k' must also be placed in the first row. \square

Example 19. Figure 5 shows the tableaux $T \in \text{SYT}(G)$ for the bowtie graph G . By Proposition 18, because $\{3, 5\} \in E(G)$, the entries 1, 2, 3 must be in the first row. Now for $k = 4$, the only \bar{G} -neighbour of 4 is 3, which we have coloured in red, so $\delta^{(4)} = (1, 0, 0, 1, 0)$. Note that we index $\delta^{(4)}$ starting at 0. Then, $R_4(T) = \{3\}$ and $W_4(T) = \{1, 4\}$, so the box 4 can be dropped in the first or the fourth column. If $b_4(T) = 1$, then $c_T^{(4)}(G) = q \frac{[2]_q}{[3]_q}$, and if $b_4(T) = 4$, then $c_T^{(4)}(G) = [4]_q \cdot \frac{1}{[3]_q}$. Next, when $k = 5$, the \bar{G} -neighbours are 3 and 4, and we have given the corresponding factors $c_T^{(5)}(G)$ and the final coefficients $c_T(G)$. We have

$$X_G(\mathbf{x}; q) = q^2 [2]_q [2]_q e_{32} + (q [4]_q [2]_q + q^2 [2]_q [2]_q) e_{41} + [5]_q [2]_q [2]_q e_5.$$

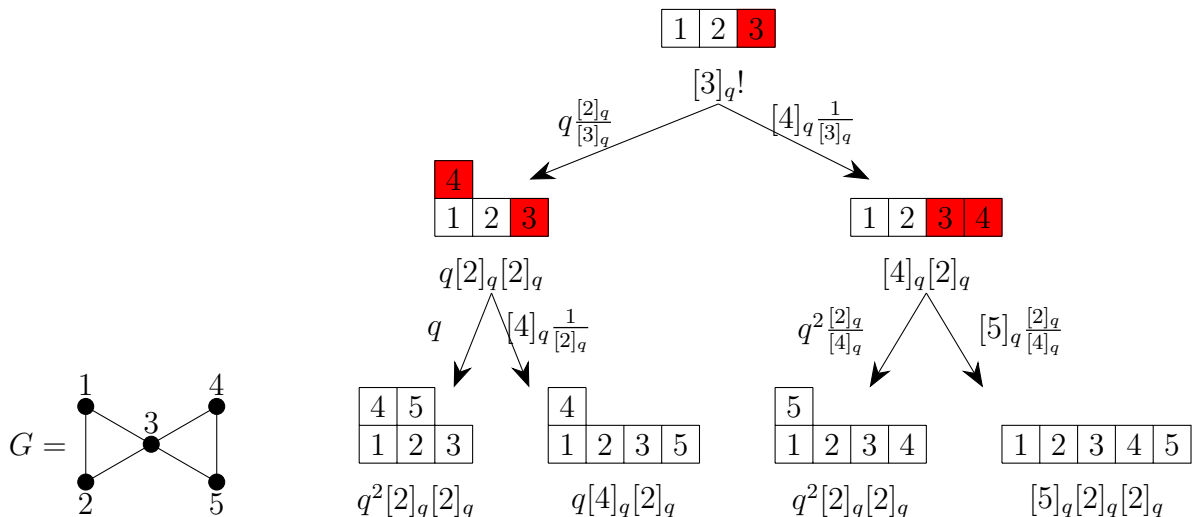


Figure 5: The bowtie graph G and the standard Young tableaux in $\text{SYT}(G)$.

3 Forest triple matrix

In this section, we use forest triples to define a matrix that will allow us to calculate a chromatic symmetric function $X_{G+H}(\mathbf{x})$ in terms of information about G and H . Informally, the idea is that a forest triple of $G + H$ essentially consists of a forest triple of $G + P_j$ for some j , along with a forest triple of H .

Definition 20. Let $G = ([n], E)$ be a graph and consider the subset of $\text{FT}(G + P_j)$

$$\text{FT}^{(i)}(G + P_j) = \{\mathcal{F} \in \text{FT}(G + P_j) : \alpha_1 = i, r = 1, \{n, n+1, \dots, n+j-1\} \subseteq T', \alpha'_\ell \geq j\},$$

where $\mathcal{T} = (T, \alpha, r)$ is the tree triple of \mathcal{F} with $1 \in T$ and $\mathcal{T}' = (T', \alpha', r')$ is the tree triple of \mathcal{F} with $n \in T'$. In other words, the tree containing vertex 1 must have a composition with first part i , the tree containing vertex n must contain the entire path P_j , and its composition must have last part at least j . Note that we could have $\mathcal{T} = \mathcal{T}'$.

Definition 21. We now define the infinite matrix F_G by

$$(F_G)_{i,j} = \sum_{\mathcal{F} \in \text{FT}^{(i)}(G + P_j)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})}.$$

If G is an NUIG, we also define the infinite matrix $F_G(q)$ by

$$(F_G(q))_{i,j} = \sum_{\mathcal{F} \in \text{FT}^{(i)}(G + P_j)} \text{sign}(\mathcal{F}) q^{\text{weight}(\mathcal{F})} e_{\text{type}'(\mathcal{F})}.$$

We first calculate some examples.

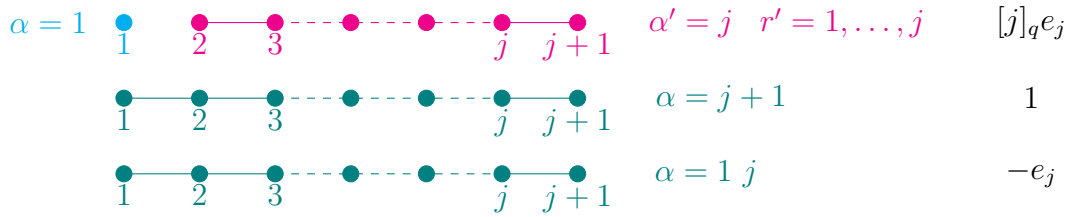


Figure 6: The forest triples used to calculate $F_{P_2}(q)$.

Proposition 22. For the graph P_1 with one vertex, $F_{P_1}(q)$ is the infinite identity matrix.

Proof. We are considering forest triples of $P_1 + P_j = P_j$. Since $n = 1$, then $\mathcal{T} = \mathcal{T}'$ and the only possible forest triple \mathcal{F} is the single tree triple $\mathcal{T} = (P_j, j, 1)$. We have that $\text{sign}(\mathcal{F}) = (-1)^{1-1} = 1$, $\text{weight}(\mathcal{F}) = \text{inv}_G(1 \cdots j) + (1 - 1) = 0$, and $\text{type}'(\mathcal{F}) = \emptyset$. \square

Proposition 23. For the two-vertex path P_2 , we have

$$(F_{P_2}(q))_{i,j} = \begin{cases} q[j-1]_q e_j & \text{if } i = 1, \\ 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases} \quad F_{P_2}(q) = \begin{bmatrix} 0 & qe_2 & q[2]_q e_3 & q[3]_q e_4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. We are considering forest triples of $P_2 + P_j = P_{j+1}$. If the vertices 1 and 2 are in different tree triples $\mathcal{T} = (T, \alpha, r)$ and $\mathcal{T}' = (T', \alpha', r')$, then we must have $\alpha = 1$ and $r = 1$. Because $\alpha'_\ell \geq j$, we must have $\alpha' = j$ and there are j possibilities corresponding to the choices of $1 \leq r' \leq j$. If the vertices 1 and 2 are in the same tree triple $\mathcal{T} = (T, \alpha, r)$, then T must be the entire path P_{j+1} . Because $\alpha_\ell \geq j$, the only possibilities are $\alpha = (j+1)$ and $\alpha = 1 \ j$, and we must have $r = 1$. These forest triples are given in Figure 6. \square

Example 24. Let us calculate the entry $(F_G(q))_{4,3}$ for the bowtie graph G . The forest triples of $\text{FT}^{(4)}(G + P_3)$ are given in Figure 7. By calculating the signed weights, we have $(F_G(q))_{4,3} = q^2[2]_q e_3$. As an illustration, when $q = 1$, we have

$$F_G = \begin{bmatrix} 4e_4 & 2e_{41} + 6e_5 & e_{33} + 3e_{51} + 8e_6 & 2e_{43} + 4e_{61} + 10e_7 & 3e_{53} + 5e_{71} + 12e_8 & \cdots \\ 2e_3 & e_{31} + 4e_4 & 2e_{41} + 6e_5 & 3e_{51} + 8e_6 & 4e_{61} + 10e_7 & \cdots \\ 0 & 2e_3 & e_{31} + 4e_4 & 2e_{41} + 6e_5 & 3e_{51} + 8e_6 & \cdots \\ 2e_1 & e_{11} & 2e_3 & 4e_4 & 6e_5 & \cdots \\ 4 & 4e_1 & e_{11} & 0 & 0 & \cdots \\ 0 & 4 & 4e_1 & e_{11} & 0 & \cdots \\ 0 & 0 & 4 & 4e_1 & e_{11} & \cdots \\ 0 & 0 & 0 & 4 & 4e_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

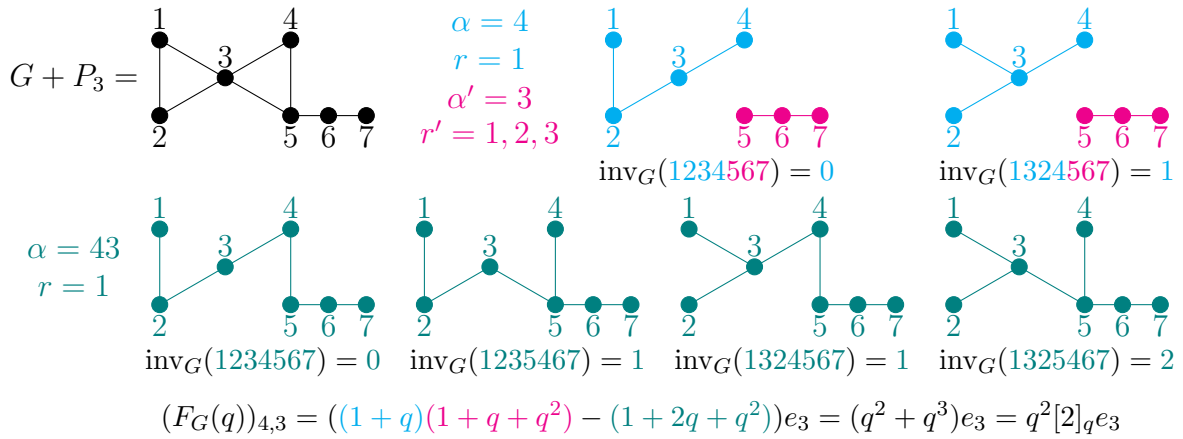


Figure 7: The forest triples used to calculate $(F_G(q))_{4,3}$.

Although F_G is an infinite matrix, we now show that every column contains finitely many nonzero entries.

Proposition 25. *Let $G = ([n], E)$ be a graph. If $i \geq n + j$, or if $j \geq i \geq n \geq 2$, then $FT^{(i)}(G + P_j)$ is empty, so $(F_G)_{i,j} = 0$.*

Proof. If $i \geq n + j$, then we cannot have $\alpha_1 = i$ because $G + P_j$ has only $(n + j - 1)$ vertices. If $j \geq i \geq n \geq 2$, then since $|T| = |\alpha| \geq \alpha_1 = i \geq n$, we must have $n \in T$ and $\mathcal{T} = \mathcal{T}'$. Now the tree T must contain the $(j + 1)$ vertices $1, n, n + 1, \dots, n + j - 1$ and we must have $\alpha_1 = i$ and $\alpha_\ell \geq j$. But now if $\ell(\alpha) \geq 2$, we have $|\alpha| \geq i + j \geq n + j$, while $G + P_j$ has only $(n + j - 1)$ vertices, and if $\ell(\alpha) = 1$, then $|\alpha| = i \leq j$, but T has at least $(j + 1)$ vertices. \square

We will often be interested in the first column of F_G , so we introduce the following notation.

Definition 26. We define the infinite row vectors

$$\vec{v} = [e_1 \quad 2e_2 \quad 3e_3 \quad \cdots], \quad \vec{v}(q) = [e_1 \quad [2]_q e_2 \quad [3]_q e_3 \quad \cdots], \quad \text{and} \quad \vec{w} = [1 \quad 0 \quad 0 \quad \cdots].$$

For a graph $G = ([n], E)$, we define the symmetric function

$$X_G^{(i)}(\mathbf{x}) = (F_G)_{i,1} = \sum_{\substack{\mathcal{F} \in \text{FT}(G) \\ \alpha_1^{(1)} = i, r_1 = 1}} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})}.$$

If G is an NUIG, we define $X_G^{(i)}(\mathbf{x}; q) = (F_G(q))_{i,1}$.

Now we can calculate $X_G(\mathbf{x})$ as a matrix multiplication. Because the columns of F_G have finitely many nonzero entries by Proposition 25, the matrix multiplications make sense.

Proposition 27. *We have*

$$X_G(\mathbf{x}) = \vec{v}F_G\vec{w}^T. \tag{2}$$

Similarly, if G is an NUIG, then $X_G(\mathbf{x}; q) = \vec{v}(q)F_G(q)\vec{w}^T$.

Proof. We have

$$X_G(\mathbf{x}) = \sum_{i \geq 1} i e_i \left(\sum_{\mathcal{F} \in \text{FT}^{(i)}(G+P_1)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})} \right) = \sum_{i \geq 1} i e_i (F_G)_{i,1} = \vec{v}F_G\vec{w}^T,$$

where the factor of i accounts for the possible choices of $1 \leq r_i \leq \alpha_1^{(1)} = i$. The proof of the second statement is similar. \square

In other words, $X_G(\mathbf{x})$ is the dot product between \vec{v} and the first column of F_G , consisting of the $X_G^{(i)}(\mathbf{x})$. In particular, if F_G (respectively $F_G(q)$ if G is an NUIG) has all e -positive entries, then $X_G(\mathbf{x})$ (respectively $X_G(\mathbf{x}; q)$) is e -positive.

Example 28. For the bowtie graph G , we can calculate $X_G(\mathbf{x}; q)$ as

$$\begin{aligned} X_G(\mathbf{x}; q) &= \vec{v}(q)F_G(q)\vec{w}^T \\ &= e_1 \cdot q^2 [2]_q [2]_q e_4 + [2]_q e_2 \cdot q^2 [2]_q e_3 + [4]_q e_4 \cdot q [2]_q e_1 + [5]_q e_5 \cdot [2]_q [2]_q \\ &= q^2 [2]_q [2]_q e_{32} + q [3]_q [2]_q e_{41} + [5]_q [2]_q [2]_q e_5. \end{aligned}$$

We now prove that the operation of gluing graphs corresponds to multiplying matrices.

Proposition 29. *For graphs G and H , we have*

$$F_{G+H} = F_G F_H.$$

Moreover, if G and H are NUIGs, then $F_{G+H}(q) = F_G(q)F_H(q)$.

Proof. We will find a bijection

$$\text{break} : \text{FT}^{(i)}(G + H + P_j) \rightarrow \bigsqcup_{k \geq 1} \text{FT}^{(i)}(G + P_k) \times \text{FT}^{(k)}(H + P_j)$$

such that if $\text{break}(\mathcal{F}) = (\mathcal{F}|_G, \mathcal{F}|_H)$, then

$$\text{sign}(\mathcal{F}) = \text{sign}(\mathcal{F}|_G)\text{sign}(\mathcal{F}|_H) \text{ and } e_{\text{type}'(\mathcal{F})} = e_{\text{type}'(\mathcal{F}|_G)}e_{\text{type}'(\mathcal{F}|_H)}. \tag{3}$$

Then we will have, as desired, that

$$\begin{aligned} (F_{G+H})_{i,j} &= \sum_{\mathcal{F} \in \text{FT}^{(i)}(G+H+P_j)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})} \\ &= \sum_{k \geq 1} \sum_{\mathcal{F}|_G \in \text{FT}^{(i)}(G+P_k)} \text{sign}(\mathcal{F}|_G) e_{\text{type}'(\mathcal{F}|_G)} \sum_{\mathcal{F}|_H \in \text{FT}^{(k)}(H+P_j)} \text{sign}(\mathcal{F}|_H) e_{\text{type}'(\mathcal{F}|_H)} \\ &= \sum_{k \geq 1} (F_G)_{i,k} (F_H)_{k,j} = (F_G F_H)_{i,j}. \end{aligned}$$

We will also show that if G and H are NUIGs, then

$$\text{weight}(\mathcal{F}) = \text{weight}(\mathcal{F}|_G) + \text{weight}(\mathcal{F}|_H),$$

from which the second statement will follow.

Let $\mathcal{F} \in \text{FT}^{(i)}(G + H + P_j)$. Let $n = |G|$, $n' = |H|$, and let $\mathcal{T} = (T, \alpha, r)$ be the tree triple of \mathcal{F} with $n \in T$. Let $T|_G$ and $T|_H$ be the trees obtained by restricting T to the vertices at most n and at least n respectively. Because $|\alpha| = |T| \geq |T|_G$, there is some minimal t with $\alpha_1 + \cdots + \alpha_t \geq |T|_G$; let

$$k = \alpha_1 + \cdots + \alpha_t - |T|_G + 1.$$

We now define the tree triples

$$\mathcal{T}|_G = (T|_G \cup P_k, \alpha_1 \cdots \alpha_t, r) \text{ and } \mathcal{T}|_H = (T|_H, k \alpha_{t+1} \cdots \alpha_\ell, 1),$$

where $T|_G \cup P_k$ is obtained by attaching the path on vertices $n, \dots, n+k-1$ to $T|_G$. We now define $\mathcal{F}|_G$ to be the forest triple consisting of $\mathcal{T}|_G$ and the tree triples of \mathcal{F} whose trees contain a vertex less than n . We define $\mathcal{F}|_H$ to be the forest triple consisting of $\mathcal{T}|_H$ and the tree triples of \mathcal{F} whose trees contain a vertex greater than n , but we subtract $(n-1)$ from all of the vertices so that they lie in $[n'+k-1]$. Some examples are given in Figure 8.

We first note that the first part of $\alpha^{(1)}$ has not been affected. By minimality of t , we have $\alpha_1 + \cdots + \alpha_{t-1} < |T|_G$ and therefore $\alpha_t \geq k$, so we indeed have $\mathcal{F}|_G \in \text{FT}^{(i)}(G + P_k)$. By construction, we have $\mathcal{F}|_H \in \text{FT}^{(k)}(G + P_j)$. When splitting \mathcal{F} into $\mathcal{F}|_G$ and $\mathcal{F}|_H$, the total number of trees and the total number of composition parts both increase by one, so $\text{sign}(\mathcal{F}) = \text{sign}(\mathcal{F}|_G)\text{sign}(\mathcal{F}|_H)$. When considering $\text{type}'(\mathcal{F}|_H)$, the extra part k in $\mathcal{T}|_H$ is removed, so (3) holds. Given $\mathcal{F}|_G \in \text{FT}^{(i)}(G + P_k)$ and $\mathcal{F}|_H \in \text{FT}^{(k)}(G + P_j)$, the inverse map is given as follows. We first add $(n-1)$ to all of the vertices of trees in $\mathcal{F}|_H$ so that they lie in $[n, n+n'+k-2]$. Then letting $\mathcal{T} = (T, \alpha, r) \in \mathcal{F}|_G$ be the tree triple with $n \in T$ and letting $\mathcal{T}' = (T', \alpha', 1) \in \mathcal{F}|_H$ be the tree triple with $n \in T'$, we define the tree triple

$$\mathcal{T} \cup \mathcal{T}' = ((T \setminus P_k) \cup T', \alpha_1 \cdots \alpha_\ell \cdot \alpha'_2 \cdots \alpha'_\ell, r), \tag{4}$$

where $(T \setminus P_k) \cup T'$ is obtained by removing the path on vertices $n, \dots, n+k-1$ from T , and then gluing to T' . Then \mathcal{F} is the forest triple consisting of $\mathcal{T} \cup \mathcal{T}'$ and the other tree triples in $\mathcal{F}|_G$ and $\mathcal{F}|_H$. Finally, if G and H are NUIGs, then

$$\begin{aligned} \text{inv}_{G+H+P_j}(\text{list}(T)) &= \text{inv}_G(\text{list}(T|_G)) + \text{inv}_{H+P_j}(\text{list}(T|_H)) \\ &= \text{inv}_{G+P_k}(\text{list}(T|_G \cup P_k)) + \text{inv}_{H+P_j}(\text{list}(T|_H)) \end{aligned}$$

because $(G + H + P_j)$ -inversions in $\text{list}(T)$ appear either in $\text{list}(T|_G)$ or $\text{list}(T|_H)$ and the path adds no new inversions. So we have $\text{weight}(\mathcal{F}) = \text{weight}(\mathcal{F}|_G) + \text{weight}(\mathcal{F}|_H)$. \square

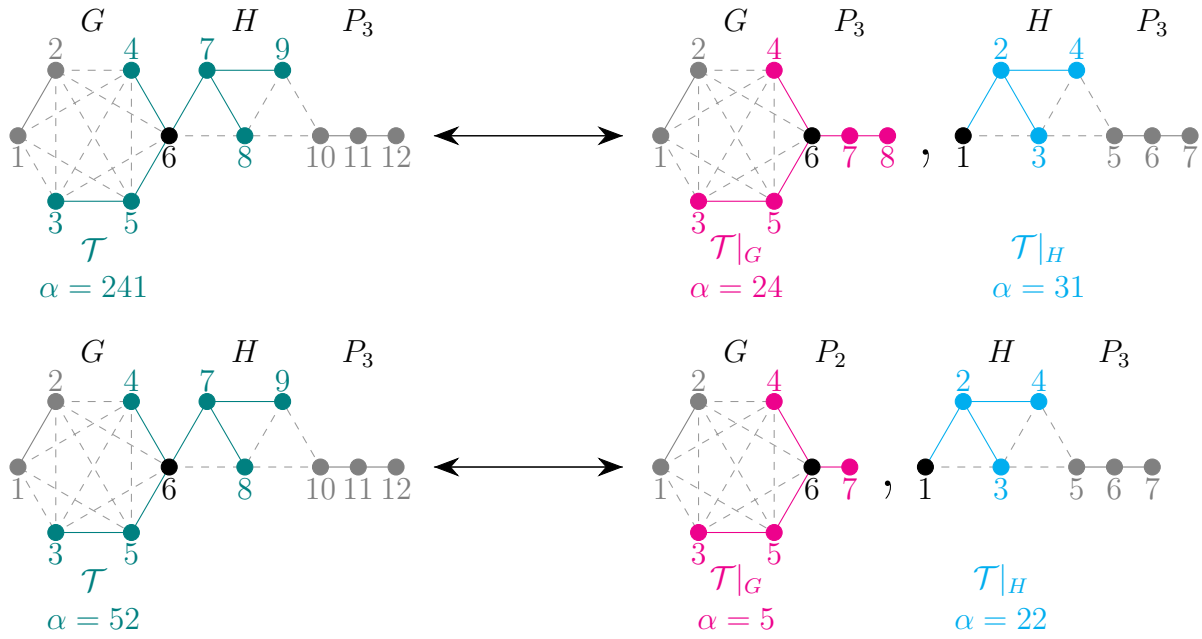


Figure 8: Some examples of the map break : $\text{FT}^{(2)}(G + H + P_3) \rightarrow \bigsqcup_{k \geq 1} \text{FT}^{(2)}(G + P_k) \times \text{FT}^{(k)}(H + P_3)$.

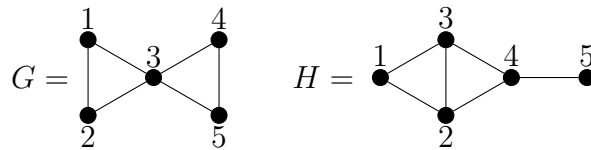


Figure 9: The bowtie graph G and the kite graph H .

Example 30. Because the bowtie graph G from Figure 7 is the sum of two complete graphs $G = K_3 + K_3$, Proposition 29 tells us that $F_G(q) = F_{K_3}(q)^2$.

Remark 31. Proposition 22 and Proposition 29 tell us that the forest triple matrix is a *monoid homomorphism* on graphs under the gluing operation.

Remark 32. By Proposition 27, we can now calculate the chromatic symmetric function $X_{G+H}(\mathbf{x})$ as

$$X_{G+H}(\mathbf{x}) = \vec{v}F_{G+H}\vec{w}^T = \vec{v}F_GF_H\vec{w}^T,$$

that is, $X_{G+H}(\mathbf{x})$ is the dot product of the row vector $\vec{v}F_G$ and the column vector $F_H\vec{w}^T$, whose i -th entry is $X_H^{(i)}(\mathbf{x})$. So we can calculate $X_{G+H}(\mathbf{x})$ if we know $\vec{v}F_G$ and $X_H^{(i)}(\mathbf{x})$ for every i .

Remark 33. Note $X_G(\mathbf{x})$ and $X_H(\mathbf{x})$ are not enough to calculate $X_{G+H}(\mathbf{x})$. For example, the bowtie graph G and the kite graph H in Figure 9 satisfy $X_G(\mathbf{x}) = X_H(\mathbf{x})$, but

$$X_{G+H}(\mathbf{x}) \neq X_{H+H}(\mathbf{x}).$$

Proposition 34. For the path P_n , we have

$$(F_{P_n}(q))_{i,j} = \sum_{\alpha \models n+j-i-1, \alpha_\ell \geq j} ([\alpha_1]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\text{sort}(\alpha)}. \quad (5)$$

Note that if $i = n + j - 1$, we are including the empty composition in our sum. The summand is the empty product, so $(F_{P_n}(q))_{n+j-1,j} = 1$.

Proof. If $n \leq 2$, then the result follows from Proposition 22 and Proposition 23. For $n \geq 3$, because $P_n = P_{n-1} + P_2$, we can use induction and Proposition 29. Indeed, defining the polynomial $w_\alpha(q) = ([\alpha_1]_q - 1) \cdots ([\alpha_\ell]_q - 1)$, we have

$$\begin{aligned} (F_{P_n}(q))_{i,j} &= (F_{P_{n-1}}(q)F_{P_2}(q))_{i,j} = (F_{P_{n-1}}(q))_{i,1} q[j-1]_q e_j + (F_{P_{n-1}}(q))_{i,j+1} \\ &= \sum_{\alpha \models n-1-i} w_\alpha(q) e_{\text{sort}(\alpha)} ([j]_q - 1) e_j + \sum_{\alpha \models n+j-i-1, \alpha_\ell \geq j+1} w_\alpha(q) \\ &= \sum_{\substack{\alpha \models n+j-i-1 \\ \alpha_\ell = j}} w_\alpha(q) e_{\text{sort}(\alpha)} + \sum_{\substack{\alpha \models n+j-i-1 \\ \alpha_\ell \geq j+1}} w_\alpha(q) e_{\text{sort}(\alpha)} = \sum_{\substack{\alpha \models n+j-i-1 \\ \alpha_\ell \geq j}} w_\alpha(q) e_{\text{sort}(\alpha)}. \end{aligned}$$

□

Remark 35. We could also calculate $F_{P_n}(q)$ by using the sign-reversing involution in [17, Proposition 3.7] and enumerating the fixed points.

Example 36. For the path P_5 , we have

$$F_{P_5}(q) = \begin{bmatrix} q^2 e_{22} + q[3]_q e_4 & 2q^2 [2]_q e_{32} + q[4]_q e_5 & q^2 [2]_q [2]_q e_{33} + q[3]_q e_{42} + q[5]_q e_6 & \cdots \\ q[2]_q e_3 & q^2 e_{22} + q[3]_q e_4 & q^2 [2]_q e_{32} + q[4]_q e_5 & \cdots \\ qe_2 & q[2]_q e_3 & q[3]_q e_4 & \cdots \\ 0 & qe_2 & q[2]_q e_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In previous work, the authors found sign-reversing involutions on forest triples. This allows us to calculate some more forest triple matrices.

Proposition 37. [17, Theorem 4.10] For the complete graph K_n , we have

$$(F_{K_n}(q))_{i,j} = \begin{cases} q^{n-1} [n-2]_q! [j-i]_q e_{n-i+j-1} & \text{if } 1 \leq i \leq \min\{n-1, j-1\}, \\ q^{n-i+j-1} [n-2]_q! [i-j]_q e_{n-i+j-1} & \text{if } \max\{n, j+1\} \leq i \leq n+j-1, \\ 0 & \text{otherwise.} \end{cases}$$

For the almost-complete graph K'_n obtained from K_n by removing the edge $(1, n)$, we have

$$(F_{K'_n}(q))_{i,j} = \begin{cases} q^{n-2}[n-2]_q! [j - (i-1)]_q e_{n-i+j-1} & \text{if } 1 \leq i \leq \min\{n-1, j\}, \\ q^{n-i+j-1}[n-2]_q! [(i-1) - j]_q e_{n-i+j-1} & \text{if } \max\{n, j+1\} \leq i \leq n+j-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the entries of $F_{K_n}(q)$ and $F_{K'_n}(q)$ are all e -positive.

Example 38. For the complete graph K_3 , we have

$$F_{K_3}(q) = \begin{bmatrix} 0 & q^2 e_3 & q^2 [2]_q e_4 & q^2 [3]_q e_5 & q^2 [4]_q e_6 & q^2 [5]_q e_7 & \cdots \\ 0 & 0 & q^2 e_3 & q^2 [2]_q e_4 & q^2 [3]_q e_5 & q^2 [4]_q e_6 & \cdots \\ [2]_q & q e_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & q e_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & [2]_q & q e_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & [2]_q & q e_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & [2]_q & q e_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & [2]_q & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proposition 39. [18, Lemma 6.1] For the cycle graph

$$C_n = ([n], \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}),$$

we have

$$(F_{C_n})_{i,j} = \sum_{\substack{\alpha \models n-i+j-1 \\ \alpha_1 \geq j \geq i+1 \text{ or } 0 \leq \alpha_1 \leq j-1 \leq i-1}} |j-i| (\alpha_2 - 1) \cdots (\alpha_\ell - 1) e_{\text{sort}(\alpha)},$$

where we sum over nonempty weak compositions where only α_1 is allowed to be 0. Note that the entries of F_{C_n} are all e -positive.

By Proposition 27 and Proposition 29, we immediately recover some known e -positivity results for graphs obtained by gluing a sequence of these graphs at single vertices.

Corollary 40.

1. [17, Corollary 4.12] If G is a sum of complete graphs and almost-complete graphs, then $X_G(\mathbf{x}; q)$ is e -positive.
2. [18, Theorem 6.5] If G is a sum of complete graphs, almost-complete graphs, and cycles, then $X_G(\mathbf{x})$ is e -positive.

Note that in Part (2), such graphs G are not NUIGs, so $X_G(\mathbf{x}; q)$ may fail to be symmetric.

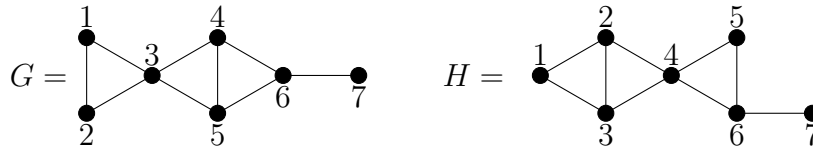


Figure 10: The graphs $G = K_3 + K'_4 + K_2$ and $H = K'_4 + K_3 + K_2$ have $F_G(q) = F_H(q)$.

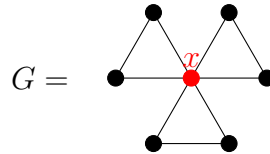


Figure 11: The graph obtained by gluing three triangles at a common vertex x .

Proof. This follows from Proposition 27, Proposition 29, Proposition 37, and Proposition 39. \square

Remark 41. We can ask whether non-isomorphic graphs G and H can have $F_G(q) = F_H(q)$. It turns out that the graphs $G = K_3 + K'_4 + K_2$ and $H = K'_4 + K_3 + K_2$ in Figure 10 in fact have $F_G(q) = F_H(q)$.

We now give an alternative way of calculating the forest triple matrix. This will help us compare the forest triple matrix to the tableau matrix that we will define in the next section.

Proposition 42. *The forest triple matrix of G satisfies*

$$(F_G)_{i,j} = \frac{1}{(j-1)!} (F_{G+K_j})_{i,1} = \frac{1}{(j-1)!} X_{G+K_j}^{(i)}(\mathbf{x}) = \frac{1}{(j-1)!} \sum_{\substack{\mathcal{F} \in FT(G+K_j) \\ \alpha_1^{(1)}=i, r_1=1}} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})}.$$

If G is an NUIG, then we have $(F_G(q))_{i,j} = \frac{1}{[j-1]_q!} (F_{G+K_j}(q))_{i,1} = \frac{1}{[j-1]_q!} X_{G+K_j}^{(i)}(\mathbf{x}; q)$.

Proof. We need only prove the first equality because the others follow by definition. In Proposition 37, we saw that the first column of F_{K_j} has a $(j-1)!$ in the j -th position and is 0 elsewhere. Therefore, by Proposition 29, we have

$$(F_G)_{i,j} = \frac{1}{(j-1)!} (F_G F_{K_j})_{i,1} = \frac{1}{(j-1)!} (F_{G+K_j})_{i,1}.$$

The proof of the second statement is similar. \square

Remark 43. We could also consider the operation of gluing several graphs at a fixed vertex x of G . We can still express the chromatic symmetric function as a matrix multiplication, although it is usually not e -positive. We can define the subset of forest triples

$$\text{FT}^{(i)}((G, x) + P_j) = \{\mathcal{F} \in \text{FT}((G, x) + P_j) : \alpha_1 = i, r = 1, P_j \subseteq T, \alpha_\ell \geq j\},$$

where $(G, x) + P_j$ is obtained by attaching the path of length j at vertex x of G and $\mathcal{T} = (T, \alpha, r)$ is the tree triple of \mathcal{F} with $x \in T$. Now if we define the infinite matrix

$$(F_{(G,x)})_{i,j} = \sum_{\mathcal{F} \in \text{FT}^{(i)}((G,x)+P_j)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})},$$

and we define $(G, x) + (H, y)$ by gluing vertices x and y together, then the same arguments will give us that

$$X_G(\mathbf{x}) = \vec{v} F_{(G,x)} \vec{w}^T \text{ and } F_{(G,x)+(H,y)} = F_{(G,x)} F_{(H,y)}.$$

For example, the graph G in Figure 11 is obtained by gluing three triangles at a common vertex x , so we can calculate its chromatic symmetric function as $X_G(\mathbf{x}) = \vec{v} F_{(K_3,1)}^3 \vec{w}^T$.

4 Tableau matrix

In this section, we use Hikita's probabilities on standard Young tableaux to define a matrix. We will show that this matrix in fact coincides with the forest triple matrix. This connection will allow us to better understand both Hikita's tableaux and forest triples.

Definition 44. Let $G = ([n], E)$ be an NUIG and consider the subset of $\text{SYT}(G + K_j)$

$$\text{SYT}^{(i)}(G + K_j) = \{T \in \text{SYT}(G + K_j) : \text{the entry } n + j - 1 \text{ is in column } i\}.$$

We now define the infinite matrix $T_G(q)$ by

$$(T_G(q))_{i,j} = \frac{1}{[i]_q [j-1]_q!} \sum_{T \in \text{SYT}^{(i)}(G+K_j)} c_T(G + K_j) e_{\text{shape}'(T)}.$$

Note that if we evaluate at $q = 1$, the entries of $T_G(1)$ are all e -positive.

We first calculate some examples.

Proposition 45. *For the graph P_1 with one vertex, $T_{P_1}(q)$ is the infinite identity matrix.*

Proof. We are considering standard Young tableaux $T \in \text{SYT}(P_1 + K_j)$. By Proposition 18, there is only one such tableau T , with all j entries on the first row and $c_T(K_j) = [j]_q!$. \square

Proposition 46. For the two-vertex path P_2 , we have

$$(T_{P_2}(q))_{i,j} = \begin{cases} q[j-1]_q e_j & \text{if } i = 1, \\ 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Note that $T_{P_2}(q) = F_{P_2}(q)$.

Proof. We consider standard Young tableaux $T \in \text{SYT}(P_2 + K_j)$. By Proposition 18, T must have the entries $1, \dots, j$ on the first row. Then $\delta^{(j+1)}(T) = (1, 0, \dots, 0, 1, 0)$, so the entry $(j+1)$ can be dropped in the first column or the $(j+1)$ -th column. Now we have

$$c_T(P_2 + K_j) = \begin{cases} [j]_q! \cdot q \frac{[j-1]_q}{[j]_q} & \text{if } b_{j+1}(T) = 1, \\ [j]_q! \cdot [j+1]_q \frac{1}{[j]_q} & \text{if } b_{j+1}(T) = j + 1, \end{cases}$$

which gives us (6). □

Example 47. For the bowtie graph G , only the tableau $T \in \text{SYT}(G + K_3)$ below has the entry 7 in column 4. It has

$$c_T(G + K_3) = [2]_q \cdot [3]_q \cdot q \frac{[2]_q}{[3]_q} \cdot q [2]_q \frac{1}{[2]_q} \cdot [3]_q \frac{1}{[2]_q} \cdot [4]_q \frac{[2]_q}{[3]_q} = q^2 [4]_q [2]_q [2]_q,$$

so $(T_G(q))_{4,3} = q^2 [2]_q e_3 = (F_G(q))_{4,3}$.

$$T = \begin{array}{|c|c|c|c|} \hline 4 & 5 & 6 & \\ \hline 1 & 2 & 3 & 7 \\ \hline \end{array}$$

We now show that every column of $T_G(q)$ contains finitely many nonzero entries.

Proposition 48. Let $G = ([n], E)$ be an NUIG. If $i \geq n + j$, or if $j \geq i \geq n \geq 2$, then $\text{SYT}^{(i)}(G + K_j)$ is empty, so $(T_G(q))_{i,j} = 0$.

Proof. If $i \geq n + j$, then we cannot have any box in column i because there are only $(n + j - 1)$ boxes. If $j \geq i \geq n \geq 2$, then by Proposition 18, the boxes $1, \dots, i, \dots, j$ are in the first row. Now, if the entry $(n + j - 1)$ is in column i , it must be in a different row. But this would give us at least $i + j \geq n + j$ boxes and there are only $(n + j - 1)$ boxes. □

We can similarly calculate $X_G(\mathbf{x}; q)$ as a matrix multiplication using the tableau matrix.

Proposition 49. We have

$$X_G(\mathbf{x}; q) = \vec{v}(q) T_G(q) \vec{w}^T.$$

Proof. We have

$$\begin{aligned} X_G(\mathbf{x}; q) &= \sum_{i \geq 1} e_i \left(\sum_{T \in \text{SYT}^{(i)}(G+K_1)} c_T(G+K_1) e_{\text{shape}'(T)} \right) \\ &= \sum_{i \geq 1} [i]_q e_i(T_G(q))_{i,1} = \vec{v}(q) T_G(q) \vec{w}^T. \end{aligned}$$

□

We now prove that the operation of gluing graphs also corresponds to multiplying tableau matrices. Informally, the idea is that a tableau for $G + H + K_j$ is essentially a tableau for $H + K_j$ and a tableau for $G + K_k$, for the right choice of k .

Proposition 50. *For NUIGs G and H , we have*

$$T_{G+H}(q) = T_G(q) T_H(q). \quad (7)$$

Proof. We will find a bijection

$$\text{break} : \text{SYT}^{(i)}(G + H + K_j) \rightarrow \bigsqcup_{k \geq 1} \text{SYT}^{(i)}(G + K_k) \times \text{SYT}^{(k)}(H + K_j)$$

such that if $\text{break}(T) = (T|_G, T|_H) \in \text{SYT}^{(i)}(G + K_k) \times \text{SYT}^{(k)}(H + K_j)$, then

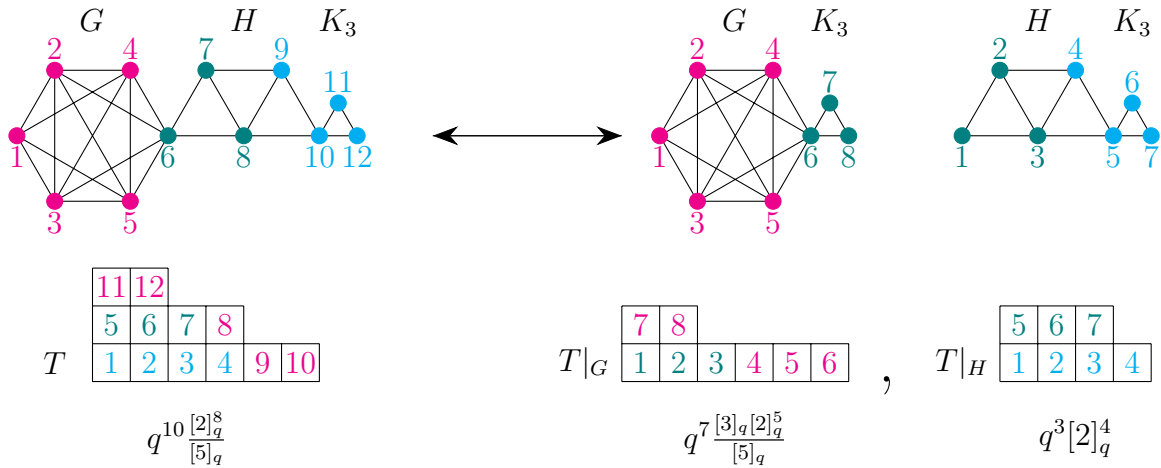
$$c_T(G + H + K_j) = \frac{c_{T|_G}(G + K_k) c_{T|_H}(H + K_j)}{[k]_q!} \text{ and } e_{\text{shape}'(T)} = e_{\text{shape}'(T|_G)} e_{\text{shape}'(T|_H)}. \quad (8)$$

Then we will have, as desired, that

$$\begin{aligned} (T_{G+H}(q))_{i,j} &= \frac{1}{[i]_q [j-1]_q!} \sum_{T \in \text{SYT}^{(i)}(G+H+K_j)} c_T(G+H+K_j) e_{\text{shape}'(T)} \\ &= \sum_{k \geq 1} \left(\frac{1}{[i]_q [k-1]_q!} \sum_{T|_G \in \text{SYT}^{(i)}(G+K_k)} c_{T|_G}(G+K_k) e_{\text{shape}'(T|_G)} \right) \\ &\quad \left(\frac{1}{[k]_q [j-1]_q!} \sum_{T|_H \in \text{SYT}^{(k)}(H+K_j)} c_{T|_H}(H+K_j) e_{\text{shape}'(T|_H)} \right) \\ &= \sum_{k \geq 1} (T_G(q))_{i,k} (T_H(q))_{k,j} = (T_G(q) T_H(q))_{i,j}. \end{aligned}$$

Let $T \in \text{SYT}^{(i)}(G+H+K_j)$ and let $b = b(T)$. Let $n = |G|$, $n' = |H|$, and let $k = b_{n'+j-1}(T)$ be the column of T that contains the entry $(n' + j - 1)$. We now define the tableaux $T|_G \in \text{SYT}^{(i)}(G + K_k)$ and $T|_H \in \text{SYT}^{(k)}(H + K_j)$ by setting

$$b(T|_G) = (1, 2, \dots, k, b_{n'+j}(T), \dots, b_{n+n'+j-2}(T)) \text{ and } b(T|_H) = (b_1(T), \dots, b_{n'+j-1}(T)).$$



$$\begin{aligned}
 c_T &= [1]_q \cdot [2]_q \cdot [3]_q \cdot \frac{[4]_q}{[3]_q} \cdot q^2 \frac{[2]_q}{[4]_q} \cdot q \frac{[2]_q^2}{[3]_q} \cdot [3]_q \cdot \frac{[4]_q}{[3]_q} \cdot \frac{[5]_q [2]_q}{[4]_q} \cdot \frac{[6]_q [3]_q}{[5]_q} \cdot q^4 \frac{[2]_q}{[6]_q} \cdot q^3 \frac{[2]_q^2}{[5]_q} \\
 c_{T|_H} &= [1]_q \cdot [2]_q \cdot [3]_q \cdot \frac{[4]_q}{[3]_q} \cdot q^2 \frac{[2]_q}{[4]_q} \cdot q \frac{[2]_q^2}{[3]_q} \cdot [3]_q \\
 c_{T|_G} &= [1]_q \cdot [2]_q \cdot [3]_q \cdot \frac{[4]_q}{[3]_q} \cdot \frac{[5]_q [2]_q}{[4]_q} \cdot \frac{[6]_q [3]_q}{[5]_q} \cdot q^4 \frac{[2]_q}{[6]_q} \cdot q^3 \frac{[2]_q^2}{[5]_q}
 \end{aligned}$$

Figure 12: An example of the map

$$\text{break} : \text{SYT}^{(2)}(G + H + K_3) \rightarrow \bigsqcup_{k \geq 1} \text{SYT}^{(2)}(G + K_k) \times \text{SYT}^{(k)}(H + K_3).$$

An example is given in Figure 12.

For every $1 \leq t \leq n' + j - 1$, we have $b_t(T|_H) = b_t(T)$ and $\delta^{(t)}(T|_H) = \delta^{(t)}(T)$, which means that $W_t(T|_H) = W_t(T)$, $T|_H \in \text{SYT}^{(k)}(H + K_j)$, and $c_{T|_H}^{(t)}(H + K_j) = c_T^{(t)}(G + H + K_j)$. Now we have

$$\delta^{(n'+j)}(T) = (1, 0, \dots, 0, 1, 0) = \delta^{(k+1)}(T|_G),$$

so we have the same possible choices of columns in which to drop the subsequent boxes in T and $T|_G$. In other words, for $1 \leq t \leq n - 1$, we have $\delta^{(n'+j-1+t)}(T) = \delta^{(k+t)}(T|_G)$, $R_{n'+j-1+t}(T) = R_{k+t}(T|_G)$, $W_{n'+j-1+t}(T) = W_{k+t}(T|_G)$, and $c_T^{(n'+j-1+t)}(G + H + K_j) = c_{T|_G}^{(k+t)}(G + K_k)$. Because $T \in \text{SYT}(G + H + P_j)$, we have

$$b_{k+t}(T|_G) = b_{n'+j-1+t}(T) \in W_{n'+j-1+t}(T) = W_{k+t}(T|_G),$$

and since $b_{n+k-1}(T|_G) = b_{n+n'+j-2}(T) = i$, then $T|_G \in \text{SYT}^{(i)}(G + K_k)$. The inverse map

is constructed as follows. Given

$$T|_G \in \text{SYT}^{(i)}(G + K_k) \text{ and } T|_H \in \text{SYT}^{(k)}(H + K_j),$$

we must have $b(T|_G)_t = t$ for $1 \leq t \leq k$ by Proposition 18, so we can recover the tableau $T \in \text{SYT}^{(i)}(G + H + P_j)$ by setting

$$b(T) = (b_1(T|_H), \dots, b_{n'+j-1}(T|_H), b_{k+1}(T|_G), \dots, b_{n+k-1}(T|_G)).$$

Finally, we check that (8) holds. We have

$$\begin{aligned} c_{T|_H}(H + K_j)c_{T|_G}(G + K_k) &= \prod_{t=1}^{n'+j-1} c_{T|_H}^{(t)}(H + K_j) \prod_{t=1}^k c_{T|_G}^{(t)}(G + K_k) \prod_{t=k+1}^{n+k-1} c_{T|_G}^{(t)}(G + K_k) \\ &= \prod_{t=1}^{n'+j-1} c_T^{(t)}(G + H + K_j)[k]_q! \prod_{t=n'+j}^{n+n'+j-2} c_T^{(t)}(G + H + K_k) \\ &= [k]_q! c_T(G + H + K_k). \end{aligned}$$

We have $e_{\text{shape}(T)}e_k = e_{\text{shape}(T|_G)}e_{\text{shape}(T|_H)}$ because $T|_H$ contains an additional row of length k . Thus we have $e_{\text{shape}'(T)} = e_{\text{shape}'(T|_G)}e_{\text{shape}'(T|_H)}$ because we remove an instance of k . \square

Proposition 51. *For the path P_n , we have $T_{P_n}(q) = F_{P_n}(q)$.*

Proof. If $n \leq 2$, then the result follows from Proposition 45 and Proposition 46. For $n \geq 3$, we have by Proposition 29 and Proposition 50 that

$$T_{P_n}(q) = (T_{P_2}(q))^{n-1} = (F_{P_2}(q))^{n-1} = F_{P_n}(q).$$

\square

We now show that the tableau matrix and the forest triple matrix agree for every NUIG.

Theorem 52. *Let $G = ([n], E)$ be an NUIG. Then $F_G(q) = T_G(q)$.*

Proof. We first show that $F_G(q)$ and $T_G(q)$ have the same first column, or in other words, that $F_G(q)\vec{w}^T = T_G(q)\vec{w}^T$. By Proposition 27 and Proposition 49, we can calculate the chromatic quasisymmetric function $X_{P_i+G}(\mathbf{x}; q)$ for any i by using either the forest triple matrix or the tableau matrix, that is,

$$\begin{aligned} X_{P_i+G}(\mathbf{x}; q) &= \vec{v}(q)F_{P_i+G}(q)\vec{w}^T = \vec{v}(q)F_{P_i}(q)F_G(q)\vec{w}^T \text{ and} \\ X_{P_i+G}(\mathbf{x}; q) &= \vec{v}(q)T_{P_i+G}(q)\vec{w}^T = \vec{v}(q)T_{P_i}(q)T_G(q)\vec{w}^T. \end{aligned}$$

By Proposition 51, we have that $F_{P_i}(q) = T_{P_i}(q)$, so let M be the infinite matrix whose i -th row is the row vector $\vec{v}(q)F_{P_i}(q) = \vec{v}(q)T_{P_i}(q)$. Then we have that

$$M(F_G(q)\vec{w}^T - T_G(q)\vec{w}^T) = 0.$$

By Proposition 25 and Proposition 48, the first columns of $F_G(q)$ and $T_G(q)$ satisfy

$$(F_G(q))_{i,1} = (T_G(q))_{i,1} = 0$$

for $i \geq n + 1$, so we can restrict M to an n -by- n matrix $M(n)$ and restrict $F_G(q)\vec{w}^T - T_G(q)\vec{w}^T$ to a column vector of size n . We now show that this restricted matrix $M(n)$ has $\det(M(n)) \neq 0$, which will establish our first claim.

By Proposition 34, we have

$$M_{i,j} = \sum_{k \geq 1} [k]_q e_k (F_{P_i}(q))_{k,j} = \sum_{k \geq 1} [k]_q e_k \sum_{\substack{\alpha \models i+j-k-1 \\ \alpha_\ell \geq j}} ([\alpha_1]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\text{sort}(\alpha)}.$$

In particular, $M_{i,j}$ is a symmetric function of degree $(i + j - 1)$, so $M_{n,n}$ is the only entry of $M(n)$ with degree $(2n - 1)$. Furthermore, $M_{n,n}$ contains the term $[2n - 1]_q e_{2n-1}$, corresponding to the summand where $k = 2n - 1$ and α is the empty composition. Therefore, we have

$$\det M(n) = [2n - 1]_q e_{2n-1} \det M(n - 1) + g(n),$$

where $g(n)$ is a symmetric function in which e_{2n-1} does not appear. Note that $\det M(1) = M_{1,1} = e_1$. By induction, we have $\det M(n - 1) \neq 0$, so the factor e_{2n-1} appears with nonzero coefficient in $\det M(n)$, and in particular, $\det M(n) \neq 0$. This proves that for every NUIG G , the matrices $F_G(q)$ and $T_G(q)$ have the same first column.

Finally, by Proposition 42, the j -th column of $F_G(q)$ is simply the first column of $F_{G+K_j}(q)$, divided by $[j - 1]_q!$. Similarly, by definition, the j -th column of $T_G(q)$ is the first column of $T_{G+K_j}(q)$, divided by $[j - 1]_q!$. Because $G + K_j$ is an NUIG, we have shown above that $F_{G+K_j}(q)$ and $T_{G+K_j}(q)$ have the same first column. Therefore,

$$(F_G(q))_{i,j} = \frac{1}{[j - 1]_q!} (F_{G+K_j}(q))_{i,1} = \frac{1}{[j - 1]_q!} (T_{G+K_j}(q))_{i,1} = (T_G(q))_{i,j},$$

so we have $F_G(q) = T_G(q)$. □

This connection between $F_G(q)$ and $T_G(q)$ immediately gives us new e -positivity results.

Corollary 53. *If G is an NUIG, then the entries of F_G are all e -positive.*

Proof. This follows because the rational functions $c_T(G)$ satisfy $c_T(1) \geq 0$. □

Corollary 54. *If G is a sum of NUIGs and cycles, then $X_G(\mathbf{x})$ is e -positive.*

Proof. This follows from Proposition 27, Proposition 39, and Corollary 53. □

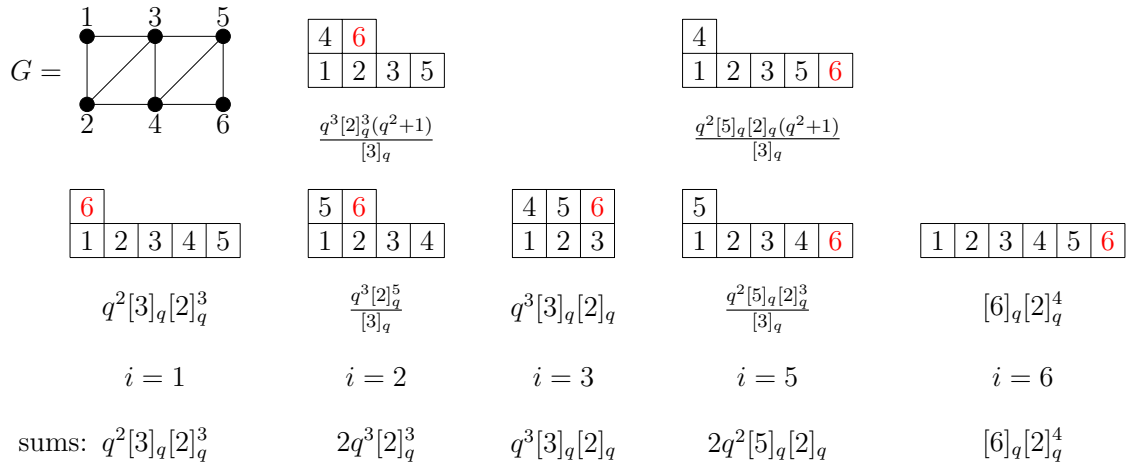


Figure 13: The tableaux of $\text{SYT}(G)$ organized by the column i of entry 6.

We also gain new insight on Hikita's standard Young tableaux. Recall that the factors $c_T(q)$ are rational functions in q and may not be polynomials. If we sum over all tableaux $T \in \text{SYT}(G)$ of a fixed shape λ , we must get a polynomial because it is the coefficient of e_λ . We can now show that certain smaller partial sums must also be polynomials.

Corollary 55. *Let G be an NUIG and fix an integer $1 \leq i \leq n$. We have*

$$\sum_{\substack{\mathcal{F} \in \text{FT}(G) \\ \alpha_1^{(1)} = i, r_1 = 1}} \text{sign}(\mathcal{F}) q^{\text{weight}(\mathcal{F})} e_{\text{type}(\mathcal{F})} = \frac{1}{[i]_q} \sum_{\substack{T \in \text{SYT}(G) \\ n \text{ in column } i}} c_T(G) e_{\text{shape}(T)}. \quad (9)$$

Proof. The left hand side is $e_i(F_G(q))_{i,1}$ and the right hand side is $e_i(T_G(q))_{i,1}$. \square

Corollary 56. *Let $G = ([n], E)$ be an NUIG. Then for every partition $\lambda \vdash n$ and integer $1 \leq i \leq n$, the sum*

$$c_\lambda^{(i)}(G) = \sum_{\substack{T \in \text{SYT}(G) \\ \text{shape}(T) = \lambda \\ n \text{ in column } i}} c_T(G)$$

is a polynomial in the variable q and it is divisible by $[i]_q$.

Proof. Take the coefficient of e_λ in (9). \square

Example 57. Figure 13 shows an NUIG G and the standard Young tableaux of $\text{SYT}(G)$, organized by the column i in which the largest entry 6 appears. Although some of the $c_T(G)$ are not polynomials, the sums for each i are polynomials and are divisible by $[i]_q$.

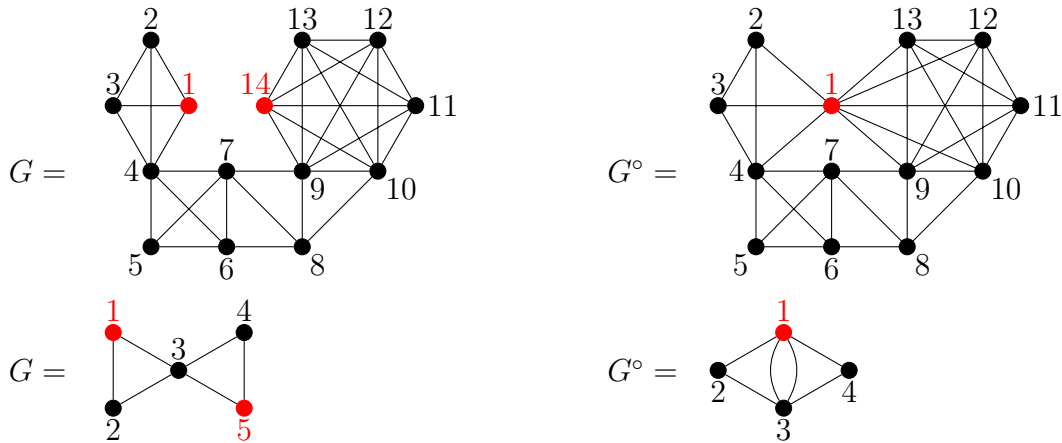


Figure 14: The graph G° obtained by gluing the first and last vertices of G . In the second example, these vertices have a common neighbour in G , so we get the multiple edge $\{1, 3\}$ in G° .

5 Gluing the first and last vertices

In this section, we show that we can use our matrix F_G to calculate the chromatic symmetric function $X_{G^\circ}(\mathbf{x})$ of the graph G° obtained by gluing vertices 1 and n of G together.

Definition 58. Let $G = ([n], E)$ be a graph with $n \geq 2$ vertices. We define the graph

$$G^\circ = ([n-1], \{\{i, j\} \in E : 1 \leq i < j \leq n-1\} \cup \{\{i, 1\} : \{i, n\} \in E\}).$$

Example 59. Figure 14 shows some examples of this graph G° . It will be convenient for our proof to allow G° to have multiple edges between the same vertices and loops. Note that multiple edges do not affect the chromatic symmetric function because they do not affect the notion of a proper colouring. A loop occurs exactly when vertices 1 and n are adjacent, in which case $X_{G^\circ}(\mathbf{x}) = 0$.

We now state the main result of this section. Note that by Proposition 25, we have $(F_G)_{k,k} = 0$ for $k \geq n$, so it makes sense to take the trace of F_G .

Theorem 60. Let $G = ([n], E)$ be a graph with $n \geq 2$ vertices. Then we have

$$X_{G^\circ}(\mathbf{x}) = \text{trace}(F_G) = \sum_{k=1}^{n-1} (F_G)_{k,k}.$$

Before we prove Theorem 60, we will discuss some examples and applications.

Example 61. For the bowtie graph G , we calculated the matrix F_G in Example 24, so we can calculate the chromatic symmetric function of G° , shown in the bottom right of Figure 14, as

$$X_{G^\circ}(\mathbf{x}) = \text{trace}(F_G) = 2e_{31} + 16e_4.$$

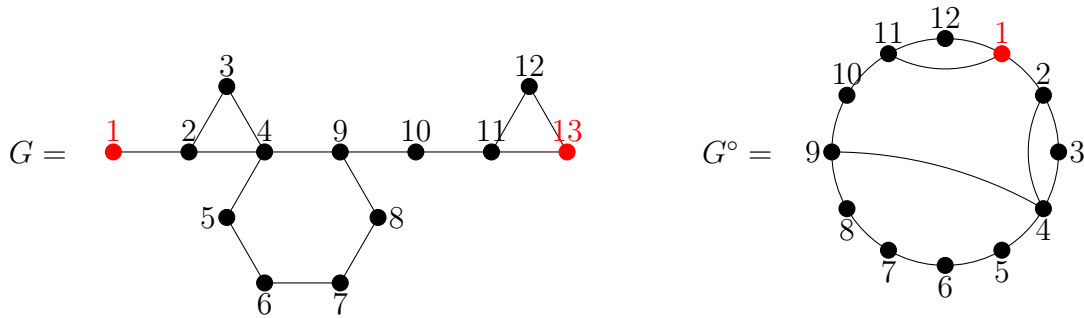


Figure 15: The graph $(P_2 + C_3 + P_1 + C_6 + P_3 + C_3)^\circ$ is a cycle C_{12} with three sequential chords.

Example 62. For the complete graph $G = K_n$, the vertices 1 and n are adjacent, so $X_{G^\circ}(\mathbf{x}) = 0$. Indeed, we saw in Proposition 37 that $F_{K_n}(q)$ has all 0's on the main diagonal, so $\text{trace}(F_{K_n}) = 0$. The examples of $n = 2$ and $n = 3$ are also given in Proposition 23 and Example 38.

Corollary 63. *If G is an NUIG, then $X_{G^\circ}(\mathbf{x})$ is e -positive.*

Proof. This follows from Corollary 53 and Theorem 60. □

Remark 64. The graph G° arising from an NUIG G in this way will be a *proper circular arc graph*, so Corollary 63 lends support to Conjecture 6.

Corollary 65. *If G is a sum of NUIGs and cycles, then $X_{G^\circ}(\mathbf{x})$ is e -positive.*

Proof. This follows from Corollary 54 and Theorem 60. □

We can also recover some known e -positivity results. The *twinned cycle graph* \tilde{C}_n is the graph obtained by duplicating a vertex of the cycle C_n , namely

$$\tilde{C}_n = ([n + 1], \{\{1, 2\}, \{2, 3\}, \dots, \{n - 1, n\}, \{n, 1\}, \{n - 1, n + 1\}, \{n, n + 1\}, \{n + 1, 1\}\}).$$

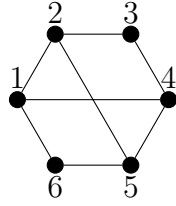
Corollary 66. [2, Theorem 3.29] *The twinned cycle graph \tilde{C}_n is e -positive.*

Proof. We have $\tilde{C}_n = (P_{n-2} + K'_4)^\circ$, so the result follows from Corollary 63. □

A *cycle-chord graph* is a graph of the form $C_n \cup \{a, 1\}$ obtained by adding some edge $\{a, 1\}$ to the cycle C_n .

Corollary 67. [19, Corollary 3.4] *Cycle-chord graphs are e -positive.*

Proof. We have $C_n \cup \{a, 1\} = (P_a + C_{n-a+2})^\circ$, so the result follows from Corollary 65. □



$$X_G(\mathbf{x}) = 2e_{222} - 6e_{33} + 26e_{42} + 28e_{51} + 102e_6$$

Figure 16: The cycle C_6 with crossing chords $\{1, 4\}$ and $\{2, 5\}$ is not e -positive.

More generally, given $1 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_m \leq n$, we can consider the *sequential cycle-chord graph* G obtained by adding edges $\{a_1, b_1\}, \dots, \{a_{m-1}, b_{m-1}\}, \{a_m, 1\}$ to C_n . Informally, these chords can intersect at endpoints but do not cross or nest, meaning the chords occur sequentially. An example is given in Figure 15. We obtain the following new e -positivity result.

Corollary 68. *Sequential cycle-chord graphs are e -positive.*

Proof. We have

$$G = (P_{a_1} + C_{b_1-a_1+1} + P_{a_2-b_1+1} + \dots + P_{a_m-b_{m-1}+1} + C_{n-a_m+2})^\circ,$$

so the result follows from Corollary 65. □

Remark 69. Figure 16 shows that some cycle graphs with non-sequential chords are not e -positive.

To prove Theorem 60, it will be convenient to consider more general objects than forest triples. We will show that they can also be used to calculate the chromatic symmetric function and they will be easier to use in our bijection.

Definition 70. A *connected subgraph triple of G* is an object $\mathcal{C} = (H, \alpha, r)$ consisting of the following data.

- H is a connected subgraph of G .
- α is a composition of size $|H|$.
- r is an integer with $1 \leq r \leq \alpha_1$.

A *subgraph triple of G* is a sequence $\mathcal{S} = (\mathcal{C}_1 = (H_1, \alpha^{(1)}, r_1), \dots, \mathcal{C}_m = (H_m, \alpha^{(m)}, r_m))$ of connected subgraph triples such that each vertex of G is in exactly one H_i and we have $\min(H_1) < \dots < \min(H_m)$. We define the *type* and *sign* of \mathcal{S} as

$$\text{type}(\mathcal{S}) = \text{sort}(\alpha^{(1)} \dots \alpha^{(m)}), \text{ and } \text{sign}(\mathcal{S}) = (-1)^{\sum_{i=1}^m (\ell(\alpha^{(i)}) + |H_i| - |E(H_i)|)}.$$

We define the *reduced type* of \mathcal{S} , denoted $\text{type}'(\mathcal{S})$, to be the partition $\text{type}(\mathcal{S})$ with an instance of $\alpha_1^{(1)}$ removed. Let $\text{ST}(G)$ denote the set of subgraph triples of G , and as in Definition 20, we define the subset

$$\text{ST}^{(i)}(G + P_j) = \{\mathcal{S} \in \text{ST}(G + P_j) : \alpha_1 = i, r = 1, \{n, n + 1, \dots, n + j - 1\} \subseteq H', \alpha'_\ell \geq j\},$$

where $\mathcal{C} = (H, \alpha, r)$ and $\mathcal{C}' = (H', \alpha', r')$ are the connected subgraph triples of \mathcal{S} with $1 \in H$ and $n \in H'$. Note that we could have $\mathcal{C} = \mathcal{C}'$.

We now show that sums of subgraph triples and forest triples agree.

Proposition 71. *Let $G = ([n], E)$ be a graph. Then we have*

$$\sum_{\mathcal{S} \in \text{ST}^{(i)}(G + P_j)} \text{sign}(\mathcal{S}) e_{\text{type}'(\mathcal{S})} = \sum_{\mathcal{F} \in \text{FT}^{(i)}(G + P_j)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})}. \quad (10)$$

Proof. Let $A = \text{ST}^{(i)}(G + P_j) \setminus \text{FT}^{(i)}(G + P_j)$. We will find a bijection $\psi : A \rightarrow A$ such that

$$\text{type}'(\psi(\mathcal{S})) = \text{type}'(\mathcal{S}) \text{ and } \text{sign}(\psi(\mathcal{S})) = -\text{sign}(\mathcal{S}).$$

Then the result will follow because

$$\sum_{\mathcal{S} \in \text{ST}^{(i)}(G + P_j)} \text{sign}(\mathcal{S}) e_{\text{type}'(\mathcal{S})} - \sum_{\mathcal{F} \in \text{FT}^{(i)}(G + P_j)} \text{sign}(\mathcal{F}) e_{\text{type}'(\mathcal{F})} = \sum_{\mathcal{S} \in A} \text{sign}(\mathcal{S}) e_{\text{type}'(\mathcal{S})} = 0.$$

Let $\mathcal{S} = (\mathcal{C}_i = (H_i, \alpha^{(i)}, r_i)) \in A$. By definition, some connected subgraph will contain a broken circuit. Let e be the largest edge under our fixed total ordering \prec such that some H_i contains a broken circuit of the form $B = C \setminus \{e\}$. We then define the subgraph triple

$$\psi(\mathcal{S}) = \begin{cases} \mathcal{S} \setminus \mathcal{C}_i \cup \{(H_i \setminus e, \alpha^{(i)}, r_i)\} & \text{if } e \in H_i, \\ \mathcal{S} \setminus \mathcal{C}_i \cup \{(H_i \cup e, \alpha^{(i)}, r_i)\} & \text{if } e \notin H_i. \end{cases}$$

In other words, we either add or remove the edge e from \mathcal{S} . The map ψ is its own inverse. We have $\text{type}'(\psi(\mathcal{S})) = \text{type}'(\mathcal{S})$ because the compositions have not changed, and we have $\text{sign}(\psi(\mathcal{S})) = -\text{sign}(\mathcal{S})$ because we have changed the number of edges by one. This completes the proof. \square

Now we can also calculate the chromatic symmetric function using subgraph triples.

Corollary 72. *We have $X_G(\mathbf{x}) = \sum_{\mathcal{S} \in \text{ST}(G)} \text{sign}(\mathcal{S}) e_{\text{type}(\mathcal{S})}$.*

Proof. Using Proposition 27 and Proposition 71, we get

$$X_G(\mathbf{x}) = \sum_{i=1}^n i e_i \left(\sum_{\mathcal{S} \in \text{ST}^{(i)}(G + P_1)} \text{sign}(\mathcal{S}) e_{\text{type}'(\mathcal{S})} \right) = \sum_{\mathcal{S} \in \text{ST}(G)} \text{sign}(\mathcal{S}) e_{\text{type}(\mathcal{S})}.$$

\square

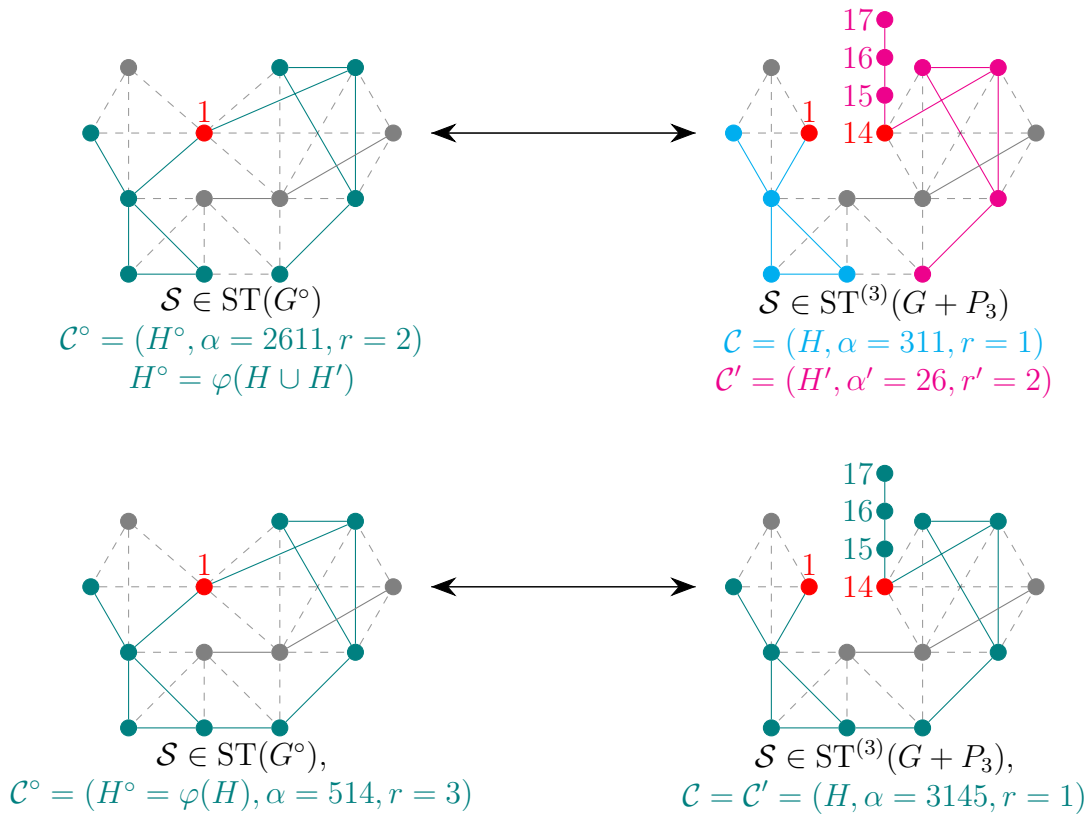


Figure 17: Some examples of the map $\psi : \text{ST}(G^\circ) \rightarrow A$.

We now prove Theorem 60 by finding a bijection on subgraph triples.

Proof of Theorem 60. Let $A = \bigsqcup_k \text{ST}^{(k)}(G + P_k)$. We will find a bijection $\psi : \text{ST}(G^\circ) \rightarrow A$ such that

$$\text{type}'(\psi(\mathcal{S})) = \text{type}(\mathcal{S}) \text{ and } \text{sign}(\psi(\mathcal{S})) = \text{sign}(\mathcal{S}). \quad (11)$$

Then the result will follow because

$$X_{G^\circ}(\mathbf{x}) = \sum_{\mathcal{S} \in \text{ST}(G^\circ)} \text{sign}(\mathcal{S}) e_{\text{type}(\mathcal{S})} = \sum_{\mathcal{S} \in A} \text{sign}(\mathcal{S}) e_{\text{type}'(\mathcal{S})} = \sum_{k \geq 1} (F_G)_{k,k} = \text{trace}(F_G).$$

Recall that the graphs G and G° have the same number of edges because we are allowing multiple edges in G° . Therefore, there is a bijection on the edge multisets $\varphi : E(G) \rightarrow E(G^\circ)$ where for $u < v$, we have

$$\varphi(\{u, v\}) = \begin{cases} \{u, 1\} & \text{if } v = n, \\ \{u, v\} & \text{otherwise.} \end{cases}$$

Now for a subgraph $H = (V(H), E(H))$ of G , we define the subgraph of G°

$$\varphi(H) = (V(H) \setminus \{n\}, \varphi(E(H))),$$

and for a subgraph $H^\circ = (V(H^\circ), E(H^\circ))$ of G° , we define the subgraph of G

$$\varphi^{-1}(H^\circ) = (V^*, \varphi^{-1}(E(H^\circ))), \text{ where } V^* = \begin{cases} V(H^\circ) \cup \{n\} & \text{if } 1 \in V(H^\circ), \\ V(H^\circ) & \text{otherwise.} \end{cases}$$

We now define our bijection ψ . Some examples are given in Figure 17. For $\mathcal{S} \in \text{ST}(G^\circ)$, let $\mathcal{C}^\circ = (H^\circ, \alpha, r)$ be the connected subgraph triple of \mathcal{S} with $1 \in H^\circ$. Note that $\varphi^{-1}(H^\circ)$ has either 1 or 2 connected components. First suppose that $\varphi^{-1}(H^\circ)$ contains 2 connected components H and H' with $1 \in H$ and $n \in H'$. Because $|\alpha| = |H^\circ| \geq |H'|$, there is some minimal t with $\alpha_1 + \dots + \alpha_t \geq |H'|$; let

$$k = \alpha_1 + \dots + \alpha_t - |H'| + 1.$$

Then we define $\psi(\mathcal{S})$ by replacing \mathcal{C}° with

$$\mathcal{C}' = (H' \cup P_k, \alpha_1 \dots \alpha_t, r) \text{ and } \mathcal{C} = (H, k \alpha_{t+1} \dots \alpha_\ell, 1),$$

where $H' \cup P_k$ is obtained by attaching the path on vertices $n, \dots, n+k-1$ to H' . By minimality of t , we have $\alpha_1 + \dots + \alpha_{t-1} < |H'|$ and therefore $\alpha_t \geq k$, so we indeed have $\psi(\mathcal{S}) \in \text{ST}^{(k)}(G + P_k)$. If instead $\varphi^{-1}(H^\circ)$ has only 1 connected component H , then letting $k = r$, we define $\psi(\mathcal{S})$ by replacing \mathcal{C}° with

$$\mathcal{C} = (H, k \alpha_2 \dots \alpha_\ell \cdot \alpha_1, 1).$$

Note that $\alpha_1 \geq r = k$, so we indeed have $\psi(\mathcal{S}) \in \text{ST}^{(k)}(G + P_k)$. When applying ψ , we add $(k+1)$ vertices, k edges, and one more composition part, so $\text{sign}(\mathcal{S}) = \text{sign}(\psi(\mathcal{S}))$. When considering $\text{type}'(\psi(\mathcal{S}))$, the extra part k in \mathcal{C} is removed, so (11) holds. We now verify that ψ is a bijection by constructing the inverse. Let $\mathcal{S} \in A$ and let $\mathcal{C} = (H, \alpha, r)$ and $\mathcal{C}' = (H', \alpha', r')$ be the connected subgraph triples of \mathcal{S} with $1 \in H$ and $n \in H'$. If $\mathcal{C} \neq \mathcal{C}'$, then $\psi^{-1}(\mathcal{S})$ is given by replacing \mathcal{C} and \mathcal{C}' with

$$\mathcal{C}^\circ = (\varphi(H \cup H'), \alpha' \cdot \alpha_2 \dots \alpha_\ell, r').$$

If instead $\mathcal{C} = \mathcal{C}'$, then $\psi^{-1}(\mathcal{S})$ is given by replacing \mathcal{C} with

$$\mathcal{C}^\circ = (\varphi(H), \alpha_\ell \cdot \alpha_2 \dots \alpha_{\ell-1}, \alpha_1).$$

Note that in this case, $\alpha_1 = k \leq \alpha_\ell$, so \mathcal{C}° is indeed a connected subgraph triple of G° . This completes the proof. \square

We can use the formula for F_{P_n} to provide an alternative proof for the chromatic symmetric function of cycles.

Corollary 73. [6, Corollary 6.2] *The chromatic symmetric function of the cycle C_n is*

$$X_{C_n}(\mathbf{x}) = \sum_{\alpha \vdash n} \alpha_1(\alpha_1 - 1) \dots (\alpha_\ell - 1) e_{\text{sort}(\alpha)}.$$

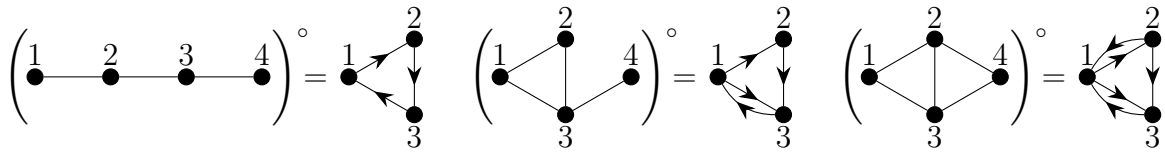


Figure 18: The directed multigraphs obtained from P_4 , $K_3 + P_2$, and K'_4 .

Proof. Using Proposition 34 and Theorem 60, we have

$$\begin{aligned} X_{C_n}(\mathbf{x}) &= \text{trace}(F_{P_{n+1}}) = \sum_{k \geq 1} \left(\sum_{\alpha \models n, \alpha_\ell \geq k} (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\text{sort}(\alpha)} \right) \\ &= \sum_{\alpha \models n} \alpha_\ell (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\text{sort}(\alpha)} = \sum_{\alpha \models n} \alpha_1 (\alpha_1 - 1) \cdots (\alpha_\ell - 1) e_{\text{sort}(\alpha)}. \end{aligned}$$

□

Curiously, if we include the parameter q , we find by Proposition 34 and (1) that the trace of $F_{P_{n+1}}(q)$ is

$$\text{trace}(F_{P_{n+1}}(q)) = \sum_{\alpha \models n} \alpha_1 ([\alpha_1]_q - 1) \cdots ([\alpha_\ell]_q - 1) e_{\text{sort}(\alpha)} = X_{\vec{C}_n}(\mathbf{x}; q).$$

This suggests the following q -analogue of Theorem 60. Note that we need to carefully keep track of our multiple directed edges in \vec{G}° .

Conjecture 74. Let $G = ([n], E)$ be an NUIG and let \vec{G}° be the directed multigraph obtained by directing the edges of G from the smaller vertex to the larger vertex, and then gluing vertices 1 and n . Then we have

$$X_{\vec{G}^\circ}(\mathbf{x}; q) = \text{trace}(F_G(q)).$$

Example 75. Figure 18 shows this construction of \vec{G}° for the NUIGs $G = P_4$, $G = K_3 + P_2$, and $G = K'_4$. In each case, the underlying simple graph of \vec{G}° is K_3 , and indeed

$$\text{trace}(F_{P_4}) = \text{trace}(F_{K_3+P_2}) = \text{trace}(F_{K'_4}) = 6e_3 = X_{K_3}(\mathbf{x}).$$

When we calculate the q -analogues, multiple edges contribute to the number of ascents, so the $X_{\vec{G}^\circ}(\mathbf{x}; q)$ are no longer the same. It turns out that

$$\begin{aligned} X_{(\vec{P}_4)^\circ}(\mathbf{x}; q) &= 3q[2]_q e_3 = \text{trace}(F_{P_4}(q)), \\ X_{(\vec{K}_3+\vec{P}_2)^\circ}(\mathbf{x}; q) &= (q^3 + 4q^2 + q)e_3 = \text{trace}(F_{K_3+P_2}(q)), \text{ and} \\ X_{(\vec{K}'_4)^\circ}(\mathbf{x}; q) &= 3q^2[2]_q e_3 = \text{trace}(F_{K'_4}(q)), \end{aligned}$$

which is consistent with Conjecture 74. We have checked Conjecture 74 by computer for all NUIGs with at most 8 vertices.

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