A Combinatorial Proof of a Cubic Congruence for the q-Secant Inversion Enumerator

Ji-Cai Liu[a]

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Abstract

Let $\mathcal{A}(2n)$ be the set of up–down alternating permutations

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots < \sigma_{2n},$$

and let

$$E_{2n}(q) = \sum_{\sigma \in \mathcal{A}(2n)} q^{\mathrm{inv}(\sigma)}.$$

We give a self-contained combinatorial proof of the congruence

$$E_{2n}(q) \equiv q^{2n(n-1)} - \binom{n}{2}(1+q)^2 \pmod{(1+q)^3}.$$

This refines the Andrews–Foata congruence modulo $(1+q)^2$ for the q-secant numbers. The proof decomposes alternating permutations into orbits generated by switches of natural pairs $\{2i-1, 2i\}$. Three free natural pairs produce an orbit of type $(\mathbb{Z}/2\mathbb{Z})^3$, and hence a factor $(1+q)^3$. The remaining contribution comes from permutations with exactly two free natural pairs; after forced outside blocks are removed, a sign-reversing sliding involution leaves one unpaired core, giving the correction term $-\binom{n}{2}(1+q)^2$.

Mathematics Subject Classifications: 05A05, 05A15, 05A19

1 Introduction

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ is called *up–down alternating* if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots .$$

[a]Department of Mathematics, Wenzhou University, Wenzhou 325035, PR China (jcliu2016@gmail.com).

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We write $\mathcal{A}(m)$ for the set of such permutations of $\{1, 2, \ldots, m\}$, and we use the convention that $\mathcal{A}(0)$ consists of the empty permutation. The entries in odd positions are troughs and the entries in even positions are peaks. The inversion number of σ is

$$\mathrm{inv}(\sigma) = \#\{(i, j) : 1 \leqslant i < j \leqslant m,\ \sigma_i > \sigma_j\}.$$

Throughout this paper we define

$$E_{2n}(q) = \sum_{\sigma \in \mathcal{A}(2n)} q^{\mathrm{inv}(\sigma)}. \tag{1.1}$$

A congruence modulo $(1+q)^r$ always means divisibility by $(1+q)^r$ in $\mathbb{Z}[q]$.

Alternating permutations were introduced by André [1], who showed that the numbers $|\mathcal{A}(2n)|$ are the Euler secant numbers and that $|\mathcal{A}(2n+1)|$ are the Euler tangent numbers. Thus

$$\sum_{n \geqslant 0} |\mathcal{A}(2n)| \frac{x^{2n}}{(2n)!} = \sec x, \qquad \sum_{n \geqslant 0} |\mathcal{A}(2n+1)| \frac{x^{2n+1}}{(2n+1)!} = \tan x.$$

For background and many later appearances of alternating permutations, see Stanley's survey [9].

The polynomials in (1.1) are the usual q-secant numbers. In their analytic form they go back to Jackson's basic trigonometric functions [6]. We use standard q-series notation, including the q-shifted factorial, as in Andrews's monograph [2]. If $(a;q)_r = (1-a)(1-aq)\cdots(1-aq^{r-1})$, then the analytic q-secant numbers are introduced by

$$\sum_{n \geqslant 0} E_{2n}^{\mathrm{an}}(q) \frac{u^{2n}}{(q;q)_{2n}} = \left(\sum_{n \geqslant 0} (-1)^n \frac{u^{2n}}{(q;q)_{2n}} \right)^{-1}. \tag{1.2}$$

A standard interpretation, proved in several settings by Gessel [5], Stanley [8], and others [3], identifies $E_{2n}^{\mathrm{an}}(q)$ with the inversion enumerator (1.1). We therefore use the combinatorial form (1.1) as the definition in this paper. Andrews and Gessel [4] obtained useful recurrences for the related q-secant and q-tangent numbers, and Andrews and Foata [3] used both analytic and combinatorial methods to prove

$$E_{2n}(q) \equiv q^{2n(n-1)} \pmod{(1+q)^2}. \tag{1.3}$$

The exponent $2n(n-1)$ is the largest possible inversion number on $\mathcal{A}(2n)$, and it is attained by the unique permutation

$$(2n-1, 2n,\ 2n-3, 2n-2,\ \ldots,\ 3, 4,\ 1, 2).$$

Here is the short verification, included to fix the normalization. For every $\sigma \in \mathcal{A}(2n)$, the n adjacent pairs of positions $(1,2), (3,4), \ldots, (2n-1, 2n)$ are forced non-inversions, so

$$\mathrm{inv}(\sigma) \leqslant \binom{2n}{2} - n = 2n(n-1).$$

Equality can hold only if these are the only non-inversions. Hence each adjacent position-pair must be increasing, and every entry in an earlier position-pair must be larger than every entry in a later position-pair. This forces the displayed permutation and proves uniqueness. Thus (1.3) says that, near $q = -1$, the entire inversion enumerator is congruent to its highest-degree term up to second order.

The purpose of this paper is to determine the next term in this $(1+q)$-adic expansion. Our main theorem is the following.

Theorem 1.1. For every integer $n \geqslant 0$,

$$E_{2n}(q) \equiv q^{2n(n-1)} - \binom{n}{2}(1+q)^2 \pmod{(1+q)^3}. \tag{1.4}$$

This is a genuine refinement of (1.3). It gives the exact quadratic correction at $q = -1$, and it explains why the Andrews–Foata congruence does not usually lift unchanged to modulus $(1+q)^3$. At $q = 1$, Theorem 1.1 also implies the congruence

$$|\mathcal{A}(2n)| \equiv 1 - 4\binom{n}{2} \pmod 8,$$

which refines the classical congruence $|\mathcal{A}(2n)| \equiv 1 \pmod 4$, often attributed to Sylvester; see, for instance, Nielsen [7] and the discussion in [3].

The proof is entirely combinatorial. The central observation is that a non-adjacent natural pair $\{2i-1, 2i\}$ may be switched without destroying alternation, and the switch changes the inversion number by exactly one. Consequently, three independent free pairs produce an eight-element orbit whose weight is divisible by $(1+q)^3$. Only the permutations with fewer than three free natural pairs survive. The rank-zero part gives the highest-degree monomial, there is no rank-one part, and the rank-two part is evaluated by a sliding sign-reversing involution on a family of cores. The examples included below are meant to make this orbit structure transparent.

2 Natural pairs and preliminary lemmas

For $1 \leqslant i \leqslant n$, let

$$P_i = \{2i-1, 2i\}$$

be the i-th *natural pair*. If the two entries of P_i occur in adjacent positions of an alternating permutation, we call P_i *locked*; otherwise we call it *free*.

Lemma 2.1 (Switching a free pair). Let $\sigma \in \mathcal{A}(2n)$, and suppose that $P_i = \{2i-1, 2i\}$ is free in σ. Let $\rho_i(\sigma)$ be obtained from σ by interchanging the two values $2i-1$ and $2i$. Then $\rho_i(\sigma) \in \mathcal{A}(2n)$, ρ_i^2 is the identity, and

$$\mathrm{inv}(\rho_i(\sigma)) = \mathrm{inv}(\sigma) \pm 1.$$

More precisely, all inversion comparisons except the mutual comparison of $2i-1$ and $2i$ have the same status before and after the switch.

Proof. Put $a = 2i - 1$ and $b = 2i$. We first check the alternating inequalities. Since P_i is free, the two positions occupied by a and b are not adjacent. Hence any adjacent comparison involving one of them is a comparison with an outside letter $x \notin \{a, b\}$. Because a and b are consecutive values, every such x satisfies either $x < a$ or $x > b$. Thus x has exactly the same order relation with a as it has with b:

$$x < a \iff x < b, \qquad x > a \iff x > b.$$

After the switch, each adjacent comparison that formerly involved a or b therefore has the same direction as before. All other adjacent comparisons are untouched, so $\rho_i(\sigma)$ is still alternating.

The inversion statement follows from the same observation, but it is useful to spell out why no hidden comparison changes. Let $x \notin \{a, b\}$. If $x < a$, then both a and b are larger than x; if $x > b$, then both a and b are smaller than x. Thus replacing a by b at one position and b by a at the other position does not change the inversion status of any comparison involving x. The only comparison whose status can change is the mutual comparison of a and b. If a was to the left of b, the switch creates exactly one inversion; if b was to the left of a, it removes exactly one inversion. Finally, switching the same two values a second time gives back σ, so ρ_i^2 is the identity. □

If P_i and P_j are distinct free pairs, then ρ_i and ρ_j commute, because they act on disjoint sets of values. Moreover, switching one natural pair does not change the positions of any other natural pair; hence the set of free pairs is constant on the whole switching orbit.

For $\sigma \in \mathcal{A}(2n)$, put

$$F(\sigma) = \{i : P_i \text{ is free in } \sigma\}, \qquad f(\sigma) = |F(\sigma)|.$$

In each switching orbit choose the unique representative ω in which, for every free pair P_i, the smaller value $2i-1$ occurs to the left of $2i$. Let $\mathcal{R}_n$ be the set of these representatives.

Lemma 2.2 (Orbit factorization). With the notation above,

$$E_{2n}(q) = \sum_{\omega \in \mathcal{R}_n} q^{\mathrm{inv}(\omega)}(1+q)^{f(\omega)}. \tag{2.1}$$

Proof. Fix $\omega \in \mathcal{R}_n$, and let $F(\omega) = \{i_1, \ldots, i_s\}$. Switching an arbitrary subset $S \subseteq F(\omega)$ gives every element of the orbit exactly once. By Lemma 2.1, switching the pairs in S adds exactly $|S|$ inversions to the canonical representative ω, because each switched pair contributes precisely its own internal inversion. Thus the orbit has total weight

$$q^{\mathrm{inv}(\omega)} \sum_{S \subseteq F(\omega)} q^{|S|} = q^{\mathrm{inv}(\omega)}(1+q)^{f(\omega)}.$$

Summing over all switching orbits proves (2.1). □

Example 2.3 (A rank-three orbit). For $n = 3$, the canonical representative

$$\omega = 132546$$

has three free natural pairs. Switching subsets of P_1, P_2, P_3 gives

switched pairs	word	inv
$\varnothing$	132546	2
P_1	231546	3
P_2	142536	3
P_3	132645	3
P_1, P_2	241536	4
P_1, P_3	231645	4
P_2, P_3	142635	4
P_1, P_2, P_3	241635	5

The total contribution is

$$q^2 + 3q^3 + 3q^4 + q^5 = q^2(1+q)^3.$$

This example illustrates the general cancellation of all representatives with at least three free pairs.

We next record the elementary forcing property of locked extremal pairs.

Lemma 2.4 (Locked extremal pairs). Consider an up–down alternating word of even length whose values are the union of consecutive natural pairs $P_a, P_{a+1}, \ldots, P_b$. If the largest pair P_b is locked, then the word begins with the adjacent block $(2b-1, 2b)$. If the smallest pair P_a is locked, then the word ends with the adjacent block $(2a-1, 2a)$.

Proof. We first prove the assertion for the largest pair P_b. The value $2b$ is the largest letter in the word, so it must occur in a peak position. If the locked block were ordered $(2b, 2b-1)$, then $2b-1$ would be a trough. Since the word has even length, that trough is not the last position, and its right neighbour would have to be larger than $2b-1$. The only available larger value is $2b$, already on its left, a contradiction. Thus the locked block is ordered $(2b-1, 2b)$. If this increasing block did not occupy the first two positions, then $2b-1$ would have a left neighbour, and the alternating inequality at that trough would require the left neighbour to be larger than $2b-1$. Again the only possible value is $2b$, already on the right. Therefore P_b is forced to be the first block $(2b-1, 2b)$.

The proof for the smallest pair is the right-end analogue. The value $2a-1$ is the smallest letter, so it must occur in a trough position. The order $(2a, 2a-1)$ is impossible: the letter $2a$ would need a left neighbour smaller than $2a$, but the only smaller value is $2a-1$, already on its right. Hence the locked block is ordered $(2a-1, 2a)$. If this block were not the last two positions, then the peak $2a$ would have a right neighbour smaller than $2a$, again impossible because $2a-1$ is already on its left. Thus P_a is forced to be the final block $(2a-1, 2a)$. □

Lemma 2.5 (Zero and one free pair). For $n \geqslant 1$, the only representative in $\mathcal{R}_n$ with no free natural pair is

$$\Omega_n = (2n-1, 2n,\ 2n-3, 2n-2,\ \ldots,\ 3, 4,\ 1, 2),$$

and

$$\mathrm{inv}(\Omega_n) = 2n(n-1).$$

There is no representative in $\mathcal{R}_n$ with exactly one free natural pair.

Proof. Assume first that no natural pair is free. Then all natural pairs are locked. By Lemma 2.4, the largest locked pair P_n is forced to be the first block $(2n-1, 2n)$. After this block is removed, the remaining word is still alternating and uses the consecutive pairs $P_1, \ldots, P_{n-1}$, all of which are locked. Repeating the same argument gives

$$\Omega_n = (2n-1, 2n,\ 2n-3, 2n-2,\ \ldots,\ 3, 4,\ 1, 2).$$

Conversely, this word is clearly alternating and has every natural pair locked, so it is the unique zero-free representative. Each pair of distinct locked blocks contributes four inversions, and there are $\binom{n}{2}$ such pairs of blocks. Hence

$$\mathrm{inv}(\Omega_n) = 4\binom{n}{2} = 2n(n-1).$$

Now suppose, for a contradiction, that exactly one natural pair P_i is free. All larger pairs $P_n, P_{n-1}, \ldots, P_{i+1}$, if any, are locked; applying Lemma 2.4 from the left forces them to appear, in this order, as the left outside blocks. Similarly, all smaller pairs $P_{i-1}, P_{i-2}, \ldots, P_1$, if any, are locked and are forced from the right. After these forced outside blocks have been removed, only the two letters of P_i remain in the unresolved interval. They are therefore adjacent, contradicting the assumption that P_i is free. Thus no canonical representative has exactly one free natural pair. □

It remains to evaluate the representatives with exactly two free pairs. We first reduce this to a standardized core.

For $m \geqslant 2$, let $\mathcal{C}_m$ be the set of alternating permutations of $\{1, 2, \ldots, 2m\}$ with the following properties:

- the endpoint pairs P_1 and P_m are free;
- the free endpoint pairs are in canonical order, namely 1 is to the left of 2, and $2m-1$ is to the left of $2m$;
- every middle pair $P_2, \ldots, P_{m-1}$ is locked.

For $m = 2$, there are no middle pairs.

Lemma 2.6 (Reduction to cores). Let $1 \leqslant i < j \leqslant n$. The parity-weighted contribution of canonical representatives with free-pair set $\{i, j\}$ equals the parity-weighted contribution of $\mathcal{C}_{j-i+1}$:

$$\sum_{\substack{\omega \in \mathcal{R}_n \\ F(\omega) = \{i,j\}}} (-1)^{\mathrm{inv}(\omega)} = \sum_{\eta \in \mathcal{C}_{j-i+1}} (-1)^{\mathrm{inv}(\eta)}. \tag{2.2}$$

Proof. Fix $\omega \in \mathcal{R}_n$ with $F(\omega) = \{i, j\}$. All natural pairs outside the interval $[i, j]$ are locked. Applying Lemma 2.4 repeatedly shows that the larger outside pairs

$$P_n, P_{n-1}, \ldots, P_{j+1}$$

are forced to occur as increasing locked blocks on the far left, while the smaller outside pairs

$$P_{i-1}, P_{i-2}, \ldots, P_1$$

are forced to occur as increasing locked blocks on the far right. Deleting these forced outside blocks leaves an alternating word on the values in $P_i \cup P_{i+1} \cup \cdots \cup P_j$. After standardizing these values by subtracting $2i - 2$, the two endpoint pairs become P_1 and P_{j-i+1}, both remain free and in canonical order, and all middle pairs are locked. Hence we obtain an element of $\mathcal{C}_{j-i+1}$.

The construction is reversible: given a core in $\mathcal{C}_{j-i+1}$, undo the standardization and reinstall the forced larger blocks on the left and the forced smaller blocks on the right. Lemma 2.4 ensures that no other placement of the outside locked blocks is possible. Thus we have a bijection between the representatives with free-pair set $\{i, j\}$ and the cores in $\mathcal{C}_{j-i+1}$.

It remains to check that this bijection preserves inversion parity. The standardization step does not change any relative order inside the remaining interval. Reinstalling the forced outside blocks changes only inversions that involve at least one outside block. Each outside block has two letters, and each natural pair in the core has two letters. For one outside block and one core pair, all four cross-comparisons have the same inversion status: either the outside block is larger and lies to the left, or the outside block is smaller and lies to the right. Thus the contribution is a multiple of four. The same "two letters versus two letters" argument applies to any two outside blocks. Therefore all inversions added or removed by deleting the forced outside blocks have even total number, and the signed sums on the two sides of (2.2) are equal. □

Example 2.7 (Forced outside blocks). For $n = 5$, consider

$$\omega = (9, 10,\ 3, 6,\ 5, 7,\ 4, 8,\ 1, 2).$$

Then $F(\omega) = \{2, 4\}$. The outside locked blocks $P_5 = (9, 10)$ and $P_1 = (1, 2)$ are forced to the two ends. Removing them leaves

$$(3, 6, 5, 7, 4, 8),$$

which standardizes to the core $143526 \in \mathcal{C}_3$. The full representative has inversion number 32, while the core has inversion number 4; the difference is even, as in Lemma 2.6.

Lemma 2.8 (The signed core sum). For every $m \geqslant 2$,

$$\sum_{\eta \in \mathcal{C}_m} (-1)^{\mathrm{inv}(\eta)} = -1. \tag{2.3}$$

Consequently, for every $1 \leqslant i < j \leqslant n$,

$$\sum_{\substack{\omega \in \mathcal{R}_n \\ F(\omega) = \{i,j\}}} (-1)^{\mathrm{inv}(\omega)} = -1. \tag{2.4}$$

Proof. The second assertion follows immediately from the first assertion and Lemma 2.6. We prove (2.3) by a sign-reversing involution on $\mathcal{C}_m$ with one unpaired core.

For $m = 2$, the core set is

$$\mathcal{C}_2 = \{(1, 3, 2, 4)\},$$

and this single core has inversion number 1. Hence the signed sum is -1. Assume from now on that $m \geqslant 3$.

For a locked middle pair P_r, $2 \leqslant r \leqslant m-1$, let B_r denote the adjacent ordered block occupied by its two values in the current word. Thus, if the block is written locally as b_1b_2, then $\{b_1, b_2\} = \{2r-1, 2r\}$. Define a map Φ on $\mathcal{C}_m$ as follows. First inspect the letter $2m$. If some middle block is adjacent to $2m$, choose among all such blocks the one with largest index r. Slide this whole block across $2m$, and at the same time reverse the order inside the block:

$$b_1b_2a \longmapsto ab_2b_1, \qquad ab_1b_2 \longmapsto b_2b_1a, \qquad (a = 2m).$$

If no middle block is adjacent to $2m$, apply the same rule with $a = 2m-1$. If neither $2m$ nor $2m-1$ is adjacent to a middle block, set $\Phi(\eta) = \eta$.

We first check that a non-fixed move stays inside $\mathcal{C}_m$. Only the three displayed positions and their two possible outside neighbours can be affected. Consider first a local factor b_1b_2a, where $a \in \{2m-1, 2m\}$, and write the optional outside letters as

$$x\, b_1b_2a\, y.$$

Because both entries of B_r are smaller than a, the original alternating pattern is

$$x < b_1 > b_2 < a > y,$$

with the inequalities involving x or y omitted at an end of the word. After the move, the local factor is ab_2b_1, and the required pattern is

$$x < a > b_2 < b_1 > y.$$

The inequalities $a > b_2$ and $b_2 < b_1$ are already forced by the original pattern. If x exists, then $x < a$: this is automatic for $a = 2m$; for $a = 2m-1$, the only possible obstruction would be $x = 2m$, but then $2m$ would have been adjacent to the middle block B_r, contrary to the fact that the rule has moved on from $2m$ to $2m-1$. If y exists and belongs to a middle block, then that block was also adjacent to a before the move; by the maximal choice of r, its index is smaller than r, so all of its entries are smaller than b_1. If y is an endpoint letter, then it is one of $1, 2$, because the other large endpoint would be adjacent to a and would lock P_m. Thus $b_1 > y$ whenever y exists.

The other local factor is ab_1b_2. The original and desired local patterns are

$$x < a > b_1 < b_2 > y, \qquad x < b_2 > b_1 < a > y.$$

Again $b_2 > b_1$ and $b_1 < a$ are immediate. If x exists and belongs to a middle block, that block was adjacent to a before the move and, by maximality of r, has smaller index

than r; hence $x < b_2$. If x is an endpoint letter, then it is one of $1, 2$, for otherwise P_m would be locked. Finally, $a > y$ is automatic for $a = 2m$. For $a = 2m-1$, the only possible obstruction is $y = 2m$, but then $2m$ was adjacent to B_r before the move, again contradicting the rule that no middle block was adjacent to $2m$. Thus alternation is preserved in both local forms.

The endpoint conditions in the definition of $\mathcal{C}_m$ are preserved as well. The slide moves neither 1 nor 2, so P_1 remains free and in canonical order. It also does not reverse the left-to-right order of $2m-1$ and $2m$. We only have to exclude a newly created adjacency between them. If the move is made around $2m$, such an adjacency would force, in the two local forms above, one of the impossible inequalities $2m-1 < b_1$ or $b_2 > 2m-1$. If the move is made around $2m-1$, a new adjacency with $2m$ would mean that $2m$ was adjacent to the moved middle block before the move, and then the rule would have acted at $2m$ first. Hence P_m remains free and canonical. All middle pairs are still locked, so $\Phi(\eta) \in \mathcal{C}_m$.

Next we prove that Φ is an involution on the non-fixed cores. After B_r is slid across the chosen letter a, the same block B_r, now reversed, is still adjacent to the same letter a on the opposite side. No middle block of larger index becomes newly adjacent to a. In the local form b_1b_2a, the only possible new neighbour of a apart from B_r is the outside letter x, and the inequality $x < b_1$ forces any middle block containing x to have index smaller than r. In the local form ab_1b_2, the only possible new neighbour apart from B_r is y, and the inequality $b_2 > y$ gives the same conclusion. Therefore the next application of the rule chooses B_r again and slides it back. If the first move was made at $2m$, the second move is plainly detected at $2m$. If the first move was made at $2m-1$, no middle block can become adjacent to $2m$ during the move. Any such new adjacency would have to involve the moved block and an outside letter equal to $2m$. In the two local forms, this would mean that before the move $2m$ was adjacent either to $2m-1$, contradicting the freeness of P_m, or to the same middle block B_r, contradicting the fact that the rule found no middle block adjacent to $2m$. Hence the second move is again detected at $2m-1$, and Φ^2 is the identity away from fixed points.

A non-fixed move reverses the sign $(-1)^{\mathrm{inv}}$. The two entries of B_r, both smaller than a, cross the single letter a, changing the inversion number by $+2$ or -2. The relative order of every other pair of letters outside the displayed three-letter factor is unchanged. The order inside B_r is reversed, so exactly one internal inversion of B_r is created or removed. Consequently

$$\mathrm{inv}(\Phi(\eta)) \equiv \mathrm{inv}(\eta) + 1 \pmod 2$$

for every non-fixed core η. All non-fixed cores therefore cancel in pairs in the signed sum.

It remains to identify the fixed cores. A fixed core has no middle block adjacent to $2m$ or to $2m-1$. Consider the largest middle pair $P_{m-1} = \{2m-3, 2m-2\}$. Its locked block cannot be decreasing. If it were $(2m-2, 2m-3)$, then $2m-3$ would be a trough; since the word has even length, that trough has a right neighbour, and the right neighbour must be larger than $2m-3$. The only possible larger values are $2m-2$, already on the left, and the two large endpoint letters $2m-1, 2m$, which would give a forbidden adjacency to a large endpoint. Thus the block is increasing, $(2m-3, 2m-2)$. If this increasing block

were not the first two positions, then the left neighbour of $2m-3$ would again have to be one of $2m-1, 2m$, another forbidden adjacency. Hence $(2m-3, 2m-2)$ is forced to be the first block.

Remove this first block and standardize the remaining values by the order-preserving bijection onto $\{1, 2, \ldots, 2m-2\}$. The result is a fixed core in $\mathcal{C}_{m-1}$: the small endpoint pair remains the small endpoint pair, the old pair P_m becomes the new large endpoint pair after standardization, and no new adjacency is created because a first block was removed. Repeating the previous paragraph forces all middle blocks, from largest to smallest, to the front. Therefore the only fixed core is

$$\eta_m = (2m-3, 2m-2,\ 2m-5, 2m-4,\ \ldots,\ 3, 4,\ 1, 2m-1, 2, 2m),$$

where the initial string of middle blocks is empty when $m = 2$.

Finally, η_m has odd inversion number. The forced middle blocks contribute $4\binom{m-2}{2}$ inversions among themselves. Their interactions with the two small endpoint letters 1 and 2 contribute $4(m-2)$ more inversions, and they contribute none with the two large endpoint letters. The final four-letter pattern $(1, 2m-1, 2, 2m)$ contributes exactly one inversion, namely $(2m-1, 2)$. Hence

$$\operatorname{inv}(\eta_m) = 4\binom{m-2}{2} + 4(m-2) + 1 \equiv 1 \pmod 2.$$

The unique fixed core therefore contributes -1, and all other cores cancel. This proves (2.3). □

Example 2.9 (Two slides in $\mathcal{C}_5$). First consider

$$\eta = (1, 4, 3, 9, 2, 6, 5, 10, 7, 8) \in \mathcal{C}_5.$$

The letter 10 is adjacent to two middle blocks, $P_3 = (6, 5)$ on the left and $P_4 = (7, 8)$ on the right. The rule chooses the larger-index block P_4, slides it across 10, and reverses it:

$$(1, 4, 3, 9, 2, 6, 5, \boxed{10, 7, 8}) \longmapsto (1, 4, 3, 9, 2, 6, 5, \boxed{8, 7, 10}).$$

The inversion numbers are 11 and 10, so the signs are opposite.

As a second example, take

$$\eta' = (1, 4, 3, 8, 7, 9, 5, 6, 2, 10) \in \mathcal{C}_5.$$

No middle block is adjacent to 10, so the rule inspects 9. The adjacent middle blocks are $P_4 = (8, 7)$ and $P_3 = (5, 6)$; the rule chooses P_4:

$$(1, 4, 3, \boxed{8, 7, 9}, 5, 6, 2, 10) \longmapsto (1, 4, 3, \boxed{9, 7, 8}, 5, 6, 2, 10).$$

The inversion numbers are 15 and 16. Applying Φ once more chooses the same block and returns to the original core.

Example 2.10 (The unpaired core). For $m = 6$, the unique core not moved by the sliding involution is

$$\eta_6 = (9, 10,\ 7, 8,\ 5, 6,\ 3, 4,\ 1, 11, 2, 12).$$

No middle block is adjacent to 11 or 12, and the final four active endpoint letters have the forced alternating pattern $1 < 11 > 2 < 12$. Its inversion number is 41, which is odd, so this single fixed point contributes -1 to the signed sum over $\mathcal{C}_6$.

Example 2.11 (A complete core calculation for $m = 3$). The five elements of $\mathcal{C}_3$ are

core	inv	$(-1)^{\mathrm{inv}}$
143526	4	$+1$
152436	4	$+1$
152634	5	-1
153426	5	-1
341526	5	-1

Thus

$$\sum_{\eta \in \mathcal{C}_3} (-1)^{\mathrm{inv}(\eta)} = 1 + 1 - 1 - 1 - 1 = -1.$$

The sliding involution pairs four of these cores and leaves the fixed core $\eta_3 = (3, 4, 1, 5, 2, 6)$ unpaired. This small case shows explicitly how the correction term in Theorem 1.1 arises from the rank-two representatives.

3 Proof of the main theorem

We now prove Theorem 1.1. For $n = 0$, both sides of (1.4) are equal to 1. Assume henceforth that $n \geqslant 1$. By the orbit factorization (2.1), every representative $\omega \in \mathcal{R}_n$ with $f(\omega) \geqslant 3$ contributes a multiple of $(1+q)^3$. Hence, modulo $(1+q)^3$, only representatives with zero, one, or two free natural pairs can contribute. Lemma 2.5 gives the unique zero-free contribution

$$q^{\mathrm{inv}(\Omega_n)} = q^{2n(n-1)},$$

and also shows that there is no one-free contribution. Therefore

$$E_{2n}(q) \equiv q^{2n(n-1)} + (1+q)^2 \sum_{1 \leqslant i < j \leqslant n} \sum_{\substack{\omega \in \mathcal{R}_n \\ F(\omega) = \{i,j\}}} q^{\mathrm{inv}(\omega)} \pmod{(1+q)^3}.$$

Since the double sum is multiplied by $(1+q)^2$, it is enough to know each inner sum modulo $(1+q)$, that is, after substituting $q = -1$. By Lemma 2.8, for each fixed $\{i, j\}$,

$$\sum_{\substack{\omega \in \mathcal{R}_n \\ F(\omega) = \{i,j\}}} q^{\mathrm{inv}(\omega)} \equiv \sum_{\substack{\omega \in \mathcal{R}_n \\ F(\omega) = \{i,j\}}} (-1)^{\mathrm{inv}(\omega)} = -1 \pmod{1+q}.$$

There are $\binom{n}{2}$ choices of $\{i, j\}$. Thus

$$E_{2n}(q) \equiv q^{2n(n-1)} - \binom{n}{2}(1+q)^2 \pmod{(1+q)^3},$$

which proves the theorem.

Example 3.1 (The case $n = 3$). For a small verification of the formula, direct enumeration gives

$$E_6(q) = q^{12} + 2q^{11} + 5q^{10} + 7q^9 + 9q^8 + 10q^7 + 10q^6 + 8q^5 + 5q^4 + 3q^3 + q^2.$$

Theorem 1.1 says that

$$E_6(q) \equiv q^{12} - 3(1+q)^2 \pmod{(1+q)^3}.$$

In the orbit proof, the unique zero-free representative is 563412, with inversion number 12. The three residual two-free choices $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ each have signed contribution -1. All representatives with three free pairs occur in eight-element orbits of the type shown in Example 2.3.

4 Concluding remarks

Theorem 1.1 gives a purely orbit-theoretic explanation of the first obstruction to lifting the Andrews–Foata congruence from modulus $(1+q)^2$ to modulus $(1+q)^3$. The factor $(1+q)^3$ comes from actual orbits under three commuting involutions, not from an algebraic manipulation of the generating function. The only surviving second-order term is caused by the two-free cores, where the sliding involution has exactly one unpaired object of odd inversion parity.

It would be natural to ask for higher $(1+q)$-adic refinements of $E_{2n}(q)$. The present proof suggests that such refinements should require a systematic description of canonical representatives with a prescribed small number of free natural pairs, together with higher-rank analogues of the sliding involution. The difficulty is that, beyond rank two, several endpoint intervals can interact, so the clean single-core reduction used here is no longer sufficient by itself.

References

[1] D. André. Sur les permutations alternées. *J. Math. Pures Appl.*, 7: 167–184, 1881.

[2] G. E. Andrews. The Theory of Partitions. Encyclopedia of Mathematics and its Applications, Vol. 2, Addison–Wesley, Reading, MA, 1976.

[3] G. E. Andrews and D. Foata. Congruences for the q-secant numbers. *European J. Combin.*, 1: 283–287, 1980.

[4] G. E. Andrews and I. Gessel. Divisibility properties of the q-tangent numbers. *Proc. Amer. Math. Soc.*, 68: 380–384, 1978.

[5] I. Gessel. Generating Functions and Enumeration of Sequences, Ph.D. thesis, Massachusetts Institute of Technology, 1977.

[6] F. H. Jackson. A basic-sine and cosine with symbolical solutions of certain differential equations. *Proc. Edinburgh Math. Soc.*, 22: 28–39, 1904.

[7] N. Nielsen. Traité èlémentaire des nombres de Bernoulli, Gauthier–Villars, Paris, 1923.

[8] R. P. Stanley. Binomial posets, Möbius inversion and permutation enumeration. *J. Combin. Theory Ser. A*, 20: 336–356, 1976.

[9] R. P. Stanley. A survey of alternating permutations. In *Combinatorics and Graphs*, Contemp. Math. 531, pages 165–196. Amer. Math. Soc., Providence, RI, 2010.