Moments for Restricted Sum of Reciprocal Parts of Partitions

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Abstract

We study the distribution of a partition statistic $\widetilde{\mathrm{srp}}(\lambda)$, which is the sum of reciprocals of the different parts of the partition λ via its moments. The generating functions for the moments are related to polylogarithms, and using Wright's circle method, we derive asymptotic formulas for the moments. In particular, we show that as $n \to \infty$,

$$\mathbb{E}\left[\widetilde{\mathrm{srp}}(\lambda) : \lambda \in \mathcal{P}_n\right] = \left(\log \frac{\sqrt{6n}}{\pi}\right)\left(1 + O\left(n^{-\frac{1}{4}}\right)\right)$$

and

$$\mathrm{Var}\left[\widetilde{\mathrm{srp}}(\lambda) : \lambda \in \mathcal{P}_n\right] = O\left(n^{-\frac{1}{4}}(\log n)^2\right),$$

where $\mathcal{P}_n$ is the set of ordinary partitions of n.

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1 Introduction

An *integer partition* λ of n is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ whose sum is equal to n, and we call each integer λ_i appearing in the partition λ a *part* of the partition λ. In a recent paper [4], the authors initiated the study of the distribution of the partition statistic $\mathrm{srp}(\lambda)$, which is the sum of reciprocals of parts of the partition λ. In particular, the authors proved that

$$\mathbb{E}\left[\mathrm{srp}(\lambda) : \lambda \in \mathcal{D}_n\right] \sim \frac{\log(3n)}{4} \quad \text{and} \quad \mathrm{Var}\left[\mathrm{srp}(\lambda) : \lambda \in \mathcal{D}_n\right] \sim \frac{\pi^2}{24},$$

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where $\mathcal{D}_n$ is the set of partitions of n into distinct parts. This is surprising, as the variance tends to a constant, while the range of the partition statistic becomes wider and the mean of the statistic is increasing. The behaviors of $\mathrm{srp}(\lambda)$ for $\mathcal{P}_n$, the set of ordinary partitions of n,

$$\mathbb{E}\left[\mathrm{srp}(\lambda) : \lambda \in \mathcal{P}_n\right] \sim \pi\sqrt{\frac{n}{6}} \quad \text{and} \quad \mathrm{Var}\left[\mathrm{srp}(\lambda) : \lambda \in \mathcal{P}_n\right] \sim \frac{\pi^2}{15} n$$

support that the asymptotic constant variance is an exceptional phenomenon. The behavior of $\mathrm{srp}(\lambda)$ and the shape of the generating function have motivated follow-up research on sharper asymptotics for the mean and variance [2], a probability measure for $\mathrm{srp}(\lambda)$ on $\mathcal{D}_n$ [1], the modularity and structure of the generating function [3, 5].

As the allowance of the repetition of the part seems to affect the variance, it is natural to ask how the variance is changed if there is non-allowance of the part in the summation of reciprocals of the parts in the ordinary partitions. To this end, we define a new partition statistic $\widetilde{\mathrm{srp}}(\lambda)$. Let $\widetilde{\mathrm{srp}}(\lambda)$ be *the sum of the reciprocals of the different parts of the partition* λ. Then, we can find that

$$\sum_{\lambda \in \mathcal{P}} \zeta^{\widetilde{\mathrm{srp}}(\lambda)} q^{|\lambda|} = \prod_{n \geqslant 1} \left(1 + \frac{\zeta^{\frac{1}{n}} q^n}{1-q^n}\right),$$

where $\mathcal{P}$ is the set of ordinary partitions. For example, Table 1 shows $\mathrm{srp}(\lambda)$ and $\widetilde{\mathrm{srp}}(\lambda)$ for the partitions of 4.

λ	4	3+1	2+2	2+1+1	1+1+1+1
$\mathrm{srp}(\lambda)$	$\frac{1}{4}$	$\frac{1}{3}+1$	$\frac{1}{2}+\frac{1}{2}$	$\frac{1}{2}+1+1$	$1+1+1+1$
$\widetilde{\mathrm{srp}}(\lambda)$	$\frac{1}{4}$	$\frac{1}{3}+1$	$\frac{1}{2}$	$\frac{1}{2}+1$	1

Table 1: The values of $\mathrm{srp}(\lambda)$ and $\widetilde{\mathrm{srp}}(\lambda)$ for partitions λ of 4

To study the distribution of $\widetilde{\mathrm{srp}}(\lambda)$ along the ordinary partitions of n, we investigate the k-th moment of $\widetilde{\mathrm{srp}}(\lambda)$ via the generating function, and it turns out that the generating functions are connected to polylogarithm functions.

Let $m_k(n)$ be *the k-th moment of* $\widetilde{\mathrm{srp}}(\lambda)$ *on* $\mathcal{P}_n$:

$$m_k(n) := \sum_{\lambda \in \mathcal{P}_n} \widetilde{\mathrm{srp}}(\lambda)^k.$$

We also let $\mathcal{S}_k(q)$ be the generating function for $m_k(n)$:

$$\mathcal{S}_k(q) := \sum_{n \geqslant 1} m_k(n) q^n = \lim_{\zeta \to 1} \left(\zeta \frac{d}{d\zeta}\right)^k \prod_{n \geqslant 1} \left(1 + \frac{\zeta^{\frac{1}{n}} q^n}{1-q^n}\right).$$

Then, it turns out that $(q;q)_\infty \mathcal{S}_k(q)$ can be explicitly expressed in terms of polylogarithm functions, where here and throughout the paper we use the standard q-series notation $(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1})$.

Theorem 1. *Let $k \in \mathbb{N}_0$. Then the generating function of $m_k(n)$ is*

$$\mathcal{S}_k(q) = \frac{1}{(q;q)_\infty} \mathcal{B}_k\left(g_1(q), \ldots, g_k(q)\right),$$

where $\mathcal{B}_k(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$ is the complete Bell polynomial,

$$g_\ell(q) = \sum_{j=1}^{\ell} (-1)^{j-1}(j-1)! S(\ell, j) \operatorname{Li}_\ell(q^j), \tag{1}$$

and $S(\ell, j)$ is the Stirling number of the second kind and $\operatorname{Li}_\ell(x)$ is the polylogarithm function of order ℓ.

For example, we have the first, second, and third moment generating functions as follows:

$$\begin{aligned}
\mathcal{S}_1(q) &= \frac{1}{(q;q)_\infty} \operatorname{Li}_1(q), \\
\mathcal{S}_2(q) &= \frac{1}{(q;q)_\infty} \left(\operatorname{Li}_2(q) - \operatorname{Li}_2(q^2) + \operatorname{Li}_1(q)^2\right), \\
\mathcal{S}_3(q) &= \frac{1}{(q;q)_\infty} \left(\operatorname{Li}_3(q) - 3\operatorname{Li}_3(q^2) + 2\operatorname{Li}_3(q^3) + 3\operatorname{Li}_2(q)\operatorname{Li}_1(q)\right. \\
&\qquad\qquad\qquad\qquad \left. -3\operatorname{Li}_2(q^2)\operatorname{Li}_1(q) + \operatorname{Li}_1(q)^3\right).
\end{aligned}$$

By employing Wright's circle method, we derive asymptotic formulas for the k-th moments.

Theorem 2. *Suppose $k \in \mathbb{N}_0$. Then we have that as $n \to \infty$,*

$$m_k(n) = \left(\log \frac{\sqrt{6n}}{\pi}\right)^k \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \left(1 + O\left(n^{-\frac{1}{4}}\right)\right).$$

From Theorem 2, we can immediately derive the asymptotic mean and variance of $\widetilde{\operatorname{srp}}(\lambda)$ on $\mathcal{P}_n$. While the values of $\widetilde{\operatorname{srp}}(\lambda)$ scatter more as n grows and the variance of $\operatorname{srp}(\lambda)$ grows linearly, the variance of $\widetilde{\operatorname{srp}}(\lambda)$ surprisingly tends to zero.

Corollary 3. *As $n \to \infty$, the expected value and the variance of $\widetilde{\operatorname{srp}}(\lambda)$ on $\mathcal{P}_n$ are*

$$\mathbb{E}\left[\widetilde{\operatorname{srp}}(\lambda) : \lambda \in \mathcal{P}_n\right] = \left(\log \frac{\sqrt{6n}}{\pi}\right) \left(1 + O\left(n^{-\frac{1}{4}}\right)\right)$$

and

$$\operatorname{Var}\left[\widetilde{\operatorname{srp}}(\lambda) : \lambda \in \mathcal{P}_n\right] = O\left(n^{-\frac{1}{4}} (\log n)^2\right).$$

As we derive the asymptotic formulas for the k-th moments for all positive integers k, we can conclude the limiting distribution of normalized $\widetilde{\operatorname{srp}}(\lambda)$.

Corollary 4. *The normalized $\widetilde{\mathrm{srp}}(\lambda)$ on $\mathcal{P}_n$ is asymptotically distributed according to a point mass distribution at one. In particular,*

$$\lim_{n\to\infty} \frac{1}{p(n)} \left| \left\{ \lambda \in \mathcal{P}_n : \frac{\widetilde{\mathrm{srp}}(\lambda)}{\log \frac{\sqrt{6n}}{\pi}} \leqslant x \right\} \right| = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x \geqslant 1, \end{cases}$$

where $p(n)$ is the number of ordinary partitions of n.

The paper is organized as follows. In Section 2, we recall Faà di Bruno's formula, which we employ to express the generating functions in terms of polylogarithms. We also give a general asymptotic formula involving powers of logarithms, which generalizes [4, Proposition 3.3], based on Wright's circle method. In Section 3, we find an explicit expression for the generating function $\mathcal{S}_k(q)$ by using q-series manipulation and Faà di Bruno's formula. Finally, in Section 4, we derive asymptotic formulas for the k-th moments $m_k(n)$ and its corollaries.

2 Preliminary

Matsusaka [5] noticed that several number theoretic generating functions involving a sum of divisors and the k-th moments of $\mathrm{srp}(\lambda)$ can be explicitly expressed by employing the classical Faà di Bruno's formula. Here we follow Matsusaka's notation to employ Faà di Bruno's formula to derive an explicit expression for $\mathcal{S}_k(q)$.

Let $\phi : \mathcal{P} \to \mathbb{C}$ be a map on the set $\mathcal{P}$ of all partitions. For each positive integer k, the *partition trace* of ϕ is defined by

$$\mathrm{Tr}_k\left(\phi; X_1, \ldots, X_k\right) := \sum_{\lambda \in \mathcal{P}_k} \phi(\lambda) X_\lambda,$$

where

$$X_\lambda := \prod_{j=1}^{k} X_j^{m_j}$$

for each $\lambda = (1^{m_1}, \ldots, k^{m_k}) \in \mathcal{P}_k$. Here, a partition is expressed in frequency notation, where each $m_j \geqslant 0$ denotes the multiplicity of the part j.

The *complete Bell polynomial* $\mathcal{B}_k(X_1, \ldots, X_k) \in \mathbb{Z}[X_1, \ldots, X_k]$ is defined by the generating function

$$\sum_{k \geqslant 0} \mathcal{B}_k(X_1, \ldots, X_k) \frac{t^k}{k!} = \prod_{j \geqslant 1} \exp\left(X_j \frac{t^j}{j!} \right),$$

and its exact expression is

$$\mathcal{B}_k(X_1, \ldots, X_k) = k! \sum_{\substack{m_1 + 2m_2 + \cdots + km_k = k \\ m_1, \ldots, m_k \geqslant 0}} \prod_{j=1}^{k} \frac{X_j^{m_j}}{m_j!(j!)^{m_j}}. \tag{2}$$

Then it can be represented via partition traces:

$$\mathcal{B}_k(X_1, \ldots, X_k) = \mathrm{Tr}_k\left(\phi_B; X_1, \ldots, X_k\right),$$

where

$$\phi_B(\lambda) = k! \prod_{j=1}^{k} \frac{1}{m_j!(j!)^{m_j}}.$$

Theorem 5 (Faà di Bruno's formula)**.** *Let k be a positive integer. Suppose $f(x), g(x)$, and $h(x)$ are functions with all necessary derivatives defined. For partitions $\lambda = (1^{m_1}, \ldots, k^{m_k}) \in \mathcal{P}_k$, we define*

$$\phi_F(\lambda) = k! f^{(\ell(\lambda))}\left(g(x)\right) \prod_{j=1}^{k} \frac{1}{m_j!(j!)^{m_j}},$$

where $\ell(\lambda) = m_1 + \cdots + m_k$. Then we have

$$\frac{d^k}{dx^k} f\left(g(x)\right) = \mathrm{Tr}_k\left(\phi_F; g^{(1)}(x), g^{(2)}(x), \ldots, g^{(k)}(x)\right).$$

In particular, applying this formula with $f(x) = e^x$ and $g(x) = \log h(x)$, we obtain

$$\frac{h^{(k)}(x)}{h(x)} = \mathcal{B}_k\left((\log h)^{(1)}(x), (\log h)^{(2)}(x), \ldots, (\log h)^{(k)}(x)\right). \tag{3}$$

Next, employing Wright's circle method, we give a general asymptotic formula involving powers of logarithms, which is a generalization of [4, Proposition 3.3]. Note that it can be obtained from [9, Section 6] for $k = 0$, and [7, Proposition 1.9] for $k = 1$. We also remark that Hypothesis 4 in [7, Proposition 1.9] can be weakened, as stated in the following proposition. Throughout this paper, Log denotes the principal branch of the logarithm, whereas log is used for the real-valued logarithm.

Proposition 6. *Suppose that $F(q)$ is an analytic function for $q = e^{-z}$ where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$, and suppose that $F(q)$ has an expansion $F(q) = \sum_{n\geqslant 0} a(n)q^n$ near $q = 1$. Furthermore, for $k \in \mathbb{N}_0$, assume the following hypotheses for a fixed positive constant Δ:*

1. *As $z \to 0$ in the bounded cone $|y| \leqslant \Delta x$ (major arc),*

$$F(e^{-z}) = \alpha z^{\beta} \left(\mathrm{Log}\,\frac{1}{z}\right)^k e^{\frac{\gamma^2}{z}} \left(1 + O_\Delta(z)\right),$$

where $\alpha \in \mathbb{C}$ and $\beta, \gamma \in \mathbb{R}$ with $\gamma > 0$.

2. *As $z \to 0$ in the bounded cone $\Delta x < |y| < \pi$ (minor arc),*

$$\left|F(e^{-z})\right| \ll_\Delta e^{\frac{\gamma^2 - \varepsilon}{x}},$$

where $\varepsilon = \varepsilon(\Delta)$ is a positive real constant.

Then, as $n \to \infty$, we have

$$a(n) = \frac{\alpha\gamma^{\beta+\frac{1}{2}}}{2\sqrt{\pi}} e^{2\gamma\sqrt{n}} n^{-\frac{2\beta+3}{4}} \left(\log \frac{\sqrt{n}}{\gamma}\right)^k \left(1 + O\left(n^{-\frac{1}{4}}\right)\right).$$

Proof. Let $q = e^{-z}$ where $z = \frac{\gamma}{\sqrt{n}}(1 + iu)$. By Cauchy's integral formula and separating the integral, we have that

$$a(n) = \frac{1}{2\pi i} \int_{|q|=e^{-\frac{\gamma}{\sqrt{n}}}} \frac{F(q)}{q^{n+1}} \, dq = M + E_1 + E_2,$$

where

$$\begin{aligned} M &:= \frac{\alpha\gamma}{2\pi\sqrt{n}} \int_{|u|\leqslant\Delta} z^\beta \left(\operatorname{Log} \frac{1}{z}\right)^k \exp\left(\frac{\gamma^2}{z} + nz\right) du, \\ E_1 &:= \frac{\gamma}{2\pi\sqrt{n}} \int_{|u|\leqslant\Delta} \left(F(e^{-z}) - \alpha z^\beta \left(\operatorname{Log} \frac{1}{z}\right)^k e^{\frac{\gamma^2}{z}}\right) e^{nz} \, du, \\ E_2 &:= \frac{\gamma}{2\pi\sqrt{n}} \int_{\Delta<|u|<\frac{\pi}{\gamma}\sqrt{n}} F(e^{-z}) e^{nz} \, du. \end{aligned}$$

First, we evaluate the main contribution as

$$\begin{aligned} M &= \frac{\alpha}{2\pi} \left(\frac{\gamma}{\sqrt{n}}\right)^{\beta+1} \int_{-\Delta}^{\Delta} (1+iu)^\beta \left(\log \frac{\sqrt{n}}{\gamma} - \operatorname{Log}(1+iu)\right)^k \\ &\qquad\qquad \times \exp\left(\gamma\sqrt{n}\left(\frac{1}{1+iu} + 1 + iu\right)\right) du \\ &= \frac{\alpha}{2\pi} \left(\frac{\gamma}{\sqrt{n}}\right)^{\beta+1} \left[\left(\log \frac{\sqrt{n}}{\gamma}\right)^k \int_{-\Delta}^{\Delta} (1+iu)^\beta \exp\left(\gamma\sqrt{n}\left(\frac{1}{1+iu} + 1 + iu\right)\right) du \right. \\ &\qquad + \sum_{j=1}^{k} \binom{k}{j} \left(\log \frac{\sqrt{n}}{\gamma}\right)^{k-j} \int_{-\Delta}^{\Delta} (1+iu)^\beta \left(-\operatorname{Log}(1+iu)\right)^j \\ &\qquad\qquad \left. \times \exp\left(\gamma\sqrt{n}\left(\frac{1}{1+iu} + 1 + iu\right)\right) du \right]. \end{aligned}$$

For the first integral above, we set $h(u) = \frac{1}{1+iu} + 1 + iu$. Then $\operatorname{Re}(h(u)) = \frac{1}{1+u^2} + 1 \leqslant 2 = h(0)$ for $|u| \leqslant \Delta$, $h'(0) = 0$, and $h''(0) = -2 < 0$. Thus, by the saddle point method [6, Section 3.1 eq. (3.29)], we obtain

$$\int_{-\Delta}^{\Delta} (1+iu)^\beta \exp\left(\gamma\sqrt{n}\left(\frac{1}{1+iu} + 1 + iu\right)\right) du = \sqrt{\frac{\pi}{\gamma\sqrt{n}}} e^{2\gamma\sqrt{n}} \left(1 + O\left(n^{-\frac{1}{4}}\right)\right).$$

Since $|\mathrm{Log}(1+iu)| \leqslant |u|$ and $\mathrm{Re}\,(h(u)) \leqslant 2 - \frac{u^2}{1+\Delta^2}$ for $|u| \leqslant \Delta$, we have that for $j = 1, \dots, k$,

$$\begin{aligned}
&\int_{-\Delta}^{\Delta} (1+iu)^{\beta} \left(-\mathrm{Log}(1+iu)\right)^j \exp\left(\gamma\sqrt{n}\left(\frac{1}{1+iu} + 1 + iu\right)\right) du \\
&\ll_{\Delta} e^{2\gamma\sqrt{n}} \int_{-\Delta}^{\Delta} |u|^j \exp\left(-\frac{\gamma\sqrt{n}u^2}{1+\Delta^2}\right) du \\
&\ll_{\Delta} e^{2\gamma\sqrt{n}} n^{-\frac{j+1}{4}} \int_{-\infty}^{\infty} |u|^j \exp\left(-\frac{\gamma u^2}{1+\Delta^2}\right) du \ll_{\Delta} e^{2\gamma\sqrt{n}} n^{-\frac{j+1}{4}}.
\end{aligned}$$

Next, from hypothesis (1), the following bound for E_1 holds:

$$|E_1| \ll_{\Delta} \frac{1}{\sqrt{n}} \int_{|u|\leqslant\Delta} \left| z^{\beta+1} \left(\mathrm{Log}\,\frac{1}{z}\right)^k \exp\left(\frac{\gamma^2}{z} + nz\right)\right| du \ll_{\Delta} n^{-\frac{\beta}{2}-1} (\log n)^k e^{2\gamma\sqrt{n}}.$$

Hence, the contribution on the major arc is

$$M + E_1 = \frac{\alpha\gamma^{\beta+\frac{1}{2}}}{2\sqrt{\pi}} e^{2\gamma\sqrt{n}} n^{-\frac{2\beta+3}{4}} \left(\log\frac{\sqrt{n}}{\gamma}\right)^k \left(1 + O\left(n^{-\frac{1}{4}}\right)\right).$$

By hypothesis (2), we have the bound for the minor arc

$$|E_2| \ll_{\Delta} \frac{1}{\sqrt{n}} \int_{\Delta<|u|<\frac{\pi}{\gamma}\sqrt{n}} \exp\left(n\,\mathrm{Re}(z) + \frac{\gamma^2 - \varepsilon}{\mathrm{Re}(z)}\right) du \ll \exp\left(2\gamma\sqrt{n} - \frac{\varepsilon}{\gamma}\sqrt{n}\right)$$

for some $\varepsilon > 0$, which completes the proof. □

3 Generating Functions

In this section, we obtain the explicit expression of the generating function $\mathcal{S}_k(q)$ in terms of polylogarithm functions.

Proof of Theorem 1. The case $k = 0$ is immediate as $m_0(n) = p(n)$. Now we assume k is a positive integer. Recall that

$$\mathcal{S}_k(q) = \left[\left(\zeta\frac{d}{d\zeta}\right)^k \prod_{n\geqslant1}\left(1 + \frac{\zeta^{\frac{1}{n}} q^n}{1-q^n}\right)\right]_{\zeta=1},$$

and let

$$g_\ell(q) := \left[\left(\zeta\frac{d}{d\zeta}\right)^\ell \sum_{n\geqslant1} \log\left(1 + \frac{\zeta^{\frac{1}{n}} q^n}{1-q^n}\right)\right]_{\zeta=1}.$$

Then, by Faà di Bruno's formula, (3) in Theorem 5, with $\zeta = e^z$, $\zeta\frac{d}{d\zeta} = \frac{d}{dz}$, and

$$h(z) = \prod_{n\geqslant1}\left(1 + \frac{\zeta^{\frac{1}{n}} q^n}{1-q^n}\right),$$

we have

$$\prod_{n\geqslant 1}(1-q^n)\sum_{n\geqslant 1} m_k(n)q^n = \mathcal{B}_k\left(g_1(q),\ldots,g_k(q)\right).$$

Next, we show that $g_\ell(q)$ can be expressed as in (1). By a change of variable,

$$\left[\left(\zeta\frac{d}{d\zeta}\right)^{\ell}\log\left(1+\frac{\zeta^{\frac{1}{n}}q^n}{1-q^n}\right)\right]_{\zeta=1} = \left[\frac{1}{n^\ell}\left(\frac{d}{dz}\right)^{\ell}\log\left(1+e^z\right)\right]_{e^z=\frac{q^n}{1-q^n}}.$$

We claim that

$$\left(\frac{d}{dz}\right)^{\ell}\log\left(1+e^z\right) = \left(\frac{d}{dz}\right)^{\ell-1}\frac{1}{1+e^{-z}} = \sum_{j=1}^{\ell}(-1)^{j-1}(j-1)!S(\ell,j)\left(\frac{1}{1+e^{-z}}\right)^{j},$$

It can be shown by induction. When $\ell=1$, it is clear. For $\ell>1$, by letting $u=\frac{1}{1+e^{-z}}$, i.e. $z=\log(u)-\log(1-u)$, we get

$$\frac{dz}{du}=\frac{1}{u}+\frac{1}{1-u}=\frac{1}{u(1-u)}, \qquad i.e.\ \frac{du}{dz}=u(1-u).$$

Thus, it follows that

$$\begin{aligned}\left(\frac{d}{dz}\right)^{\ell}\frac{1}{1+e^{-z}} = \frac{d}{dz}\sum_{j=1}^{\ell}(-1)^{j-1}(j-1)!S(\ell,j)u^j &= \sum_{j=1}^{\ell}(-1)^{j-1}j!S(\ell,j)u^{j-1}\frac{du}{dz}\\ &= \sum_{j=1}^{\ell+1}(-1)^{j-1}(j-1)!S(\ell+1,j)u^j\end{aligned}$$

from the fact that

$$S(\ell+1,j)=jS(\ell,j)+S(\ell,j-1) \quad \text{and} \quad S(\ell,1)=S(\ell,\ell)=1. \tag{4}$$

Since $u=q^n$ at $e^z=\frac{q^n}{1-q^n}$, we arrive at

$$g_\ell(q)=\sum_{n\geqslant 1}\frac{1}{n^\ell}\sum_{j=1}^{\ell}(-1)^{j-1}(j-1)!S(\ell,j)q^{jn} = \sum_{j=1}^{\ell}(-1)^{j-1}(j-1)!S(\ell,j)\,\mathrm{Li}_\ell\left(q^j\right). \qquad \square$$

4 Asymptotics

In this section, we prove the asymptotics of $m_k(n)$ and the limiting distribution of normalized $\widetilde{\mathrm{srp}}(\lambda)$ on $\mathcal{P}_n$.

Proof of Theorem 2. Let $q = e^{-z}$, where $z = x + iy$ with $x, y \in \mathbb{R}$ and $x > 0$. By [8, equation (9.5)], for $n \in \mathbb{N}$ and $|z| < 2\pi$

$$\operatorname{Li}_n\left(e^{-z}\right) = \left(H_{n-1} - \operatorname{Log} z\right) \frac{(-z)^{n-1}}{(n-1)!} + \sum_{\substack{m \geqslant 0 \\ m \neq n-1}} \zeta(n-m) \frac{(-z)^m}{m!}, \tag{5}$$

where $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for $n \in \mathbb{N}$, $H_0 := 0$ and $\zeta(s)$ is the Riemann zeta function. Since

$$g_\ell(q) = \sum_{j=1}^{\ell} (-1)^{j-1} (j-1)! S(\ell, j) \operatorname{Li}_\ell(q^j)$$

and $\sum_{j=1}^{\ell} (-1)^{j-1}(j-1)! S(\ell, j) = 0$ for $\ell \geqslant 2$ from (4), we find that there is no constant term in the z-expansion of $g_\ell(e^{-z})$. Therefore, (5) implies that as $z \to 0$ in $|z| < \frac{2\pi}{k}$,

$$g_1\left(e^{-z}\right) = \operatorname{Li}_1\left(e^{-z}\right) = -\operatorname{Log} z + O(z) \qquad \text{and} \qquad g_\ell\left(e^{-z}\right) = O\left(z \operatorname{Log} z\right) \text{ for } \ell = 2, \ldots, k.$$

By (2), we have the explicit expression

$$\mathcal{B}_k\left(g_1(q), \ldots, g_k(q)\right) = k! \sum_{\substack{m_1 + 2m_2 + \cdots + k m_k = k \\ m_1, \ldots, m_k \geqslant 0}} \prod_{j=1}^{k} \frac{g_j(q)^{m_j}}{m_j! (j!)^{m_j}},$$

from which it is clear that as $z \to 0$ in $|z| < \frac{2\pi}{k}$,

$$\mathcal{B}_k\left(g_1(q), \ldots, g_k(q)\right) = g_1(q)^k + O\left(g_1(q)^{k-2} g_2(q)\right) = (-\operatorname{Log} z)^k + O\left(z\, (-\operatorname{Log} z)^{k-1}\right).$$

For a positive integer $\ell \geqslant 2$,

$$\left|\operatorname{Li}_\ell\left(e^{-z}\right)\right| = \left|\sum_{n \geqslant 1} \frac{e^{-nz}}{n^\ell}\right| \leqslant \sum_{n \geqslant 1} \frac{1}{n^\ell} = \zeta(\ell).$$

Also, as $z = x + iy \to 0$ with $x, y \in \mathbb{R}$ and $x > 0$,

$$\left|\operatorname{Li}_1\left(e^{-z}\right)\right| \leqslant \operatorname{Li}_1(e^{-x}) \ll |\log x|.$$

Suppose that $\Delta > 0$. By [9, pp.112–113] or [7, Theorem 4.1], we have that as $z \to 0$ in the bounded cone $|y| \leqslant \Delta x$,

$$\frac{1}{(e^{-z}; e^{-z})_\infty} = \sqrt{\frac{z}{2\pi}} \exp\left(\frac{\pi^2}{6z} - \frac{z}{24}\right) \left(1 + O\left(e^{-\frac{4\pi^2}{z}}\right)\right),$$

and that as $z \to 0$ in the bounded cone $\Delta x < |y| < \pi$,

$$\frac{1}{(e^{-z}; e^{-z})_\infty} \ll_\Delta \frac{1}{(e^{-x}; e^{-x})_\infty} e^{-\frac{\varepsilon}{x}},$$

for some $\varepsilon > 0$.

Thus, we obtain that as $z \to 0$ in the bounded cone $|y| \leqslant \Delta x$,

$$\mathcal{S}_k(q) = \frac{1}{(q;q)_\infty}\mathcal{B}_k\left(g_1(q), \ldots, g_k(q)\right) = \sqrt{\frac{z}{2\pi}}\left(\operatorname{Log}\frac{1}{z}\right)^k e^{\frac{\pi^2}{6z}}\left(1 + O(z)\right),$$

and that as $z \to 0$ in the bounded cone $\Delta x < |y| < \pi$,

$$\mathcal{S}_k(q) = \frac{1}{(q;q)_\infty}\mathcal{B}_k\left(g_1(q), \ldots, g_k(q)\right) \ll_\Delta x^{\frac{1}{2}}\left|\log x\right|^k e^{\frac{\pi^2}{6x} - \frac{\varepsilon}{x}}.$$

Therefore, applying Proposition 6 yields the desired asymptotic formula. □

Employing the asymptotics of moments and the uniqueness theorem of the moment generating function, we derive the asymptotic distribution of $\widetilde{\mathrm{srp}}(\lambda)$ on the set $\mathcal{P}_n$ as $n \to \infty$.

Proof of Corollary 3. Theorem 2 deduces the asymptotics of the mean

$$\mathbb{E}\left[\widetilde{\mathrm{srp}}(\lambda) : \lambda \in \mathcal{P}_n\right] = \frac{m_1(n)}{p(n)} = \left(\log\frac{\sqrt{6n}}{\pi}\right)\left(1 + O\left(n^{-\frac{1}{4}}\right)\right)$$

and the asymptotics of the variance

$$\begin{aligned}
\operatorname{Var}[\widetilde{\mathrm{srp}}(\lambda) : \lambda \in \mathcal{P}_n] &= \frac{m_2(n)}{p(n)} - \left(\frac{m_1(n)}{p(n)}\right)^2 \\
&= \left(\log\frac{\sqrt{6n}}{\pi}\right)^2\left(1 + \mathcal{O}\left(n^{-\frac{1}{4}}\right)\right) - \left[\log\frac{\sqrt{6n}}{\pi}\left(1 + \mathcal{O}\left(n^{-\frac{1}{2}}\right)\right)\right]^2 \\
&= O\left(n^{-\frac{1}{4}}(\log n)^2\right)
\end{aligned}$$

as $n \to \infty$. □

Proof of Corollary 4. By Theorem 2, the normalized ratio of moments is

$$\frac{m_k(n)}{\left(\log\frac{\sqrt{6n}}{\pi}\right)^k p(n)} \sim 1 \qquad \text{as } n \to \infty.$$

Since the only distribution whose moments are identically 1 comes from the point mass probability function that satisfies $p(x = 1) = 1$ and $p(x = a) = 0$ for all $a \neq 1$, this gives the proof. □

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