A Footnote to the KPT Theorem in Structural Ramsey Theory

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Abstract

The celebrated theorem of Kechris, Pestov and Todorčević connecting structural Ramsey theory with topological dynamics has as a consequence that the Fraïssé limit of a Ramsey class of finite relational structures has a reduct which is a total order; this implies an earlier result of Nešetřil, according to which the structures in such a class are rigid (have trivial automorphism groups). In this paper, we give an alternative proof of this fact. If $\mathcal{C}$ is a Fraïssé class of rigid structures over a finite relational language, then either the Fraïssé limit of $\mathcal{C}$ has a reduct which is a total order, or there is an explicit failure of the Ramsey property involving a pair (A, B) of structures in $\mathcal{C}$ with $|A| = 2$.

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1 Introduction

It is nearly a century since Ramsey's theorem was published. The theorem states that if a and b are positive integers, then there exists c such that, if the a-element subsets of a c-element set are coloured with two colours (red and blue, say), then there exists a b-element set which is *monochromatic* (all of its a-element subsets have the same colour).

Since its publication, the work of many mathematicians, including Paul Erdős and his collaborators, has extended it from a theorem to a subject (Ramsey theory) with links to logic, set theory, combinatorics, group theory, number theory, topological dynamics, and other areas.

In particular, Ramsey theory of finite structures deals with classes $\mathcal{C}$ of finite structures over finite relational languages. Such a structure consists of a set carrying named relations with prescribed arities. We always assume that our classes are closed under isomorphism and *hereditary* (closed under taking induced substructures), and that all structures are non-empty. We say that $\mathcal{C}$ is a *Ramsey class* if, given any two structures $A, B \in \mathcal{C}$, there exists $C \in \mathcal{C}$ such that, if the embeddings $A \to C$ are coloured red and blue, then there

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is an embedding $B \to C$ which is monochromatic, in the sense that all the embeddings whose image is contained in the image of B have the same colour.

A closely related notion is that of a *Fraïssé class*, a hereditary class $\mathcal{C}$ of finite structures satisfying

(a) the *joint embedding property*: given $B_1, B_2 \in \mathcal{C}$, there exists $C \in \mathcal{C}$ such that B_1 and B_2 are embeddable in C; and

(b) the *amalgamation property*: given $A, B_1, B_2 \in \mathcal{C}$ and embeddings $f_i : A \to B_i$ for $i = 1, 2$, there is a structure $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ such that $f_1 g_1 = f_2 g_2$.

We will also use the *strong amalgamation property*, which asserts that in (b), the embeddings can be chosen so that the images of g_1 and g_2 intersect precisely in the image of f.

The *age* of a relational structure is the class of finite structures embeddable in it. A structure M is *homogeneous* if every isomorphism between finite substructures can be extended to an automorphism of M. Fraïssé's theorem states:

Theorem 1. *A class $\mathcal{C}$ of finite structures over a finite relational language, closed under isomorphism and substructures, is the age of a finite or countable homogeneous structure M if and only if $\mathcal{C}$ satisfies the joint embedding and amalgamation properties. If it does, then M is unique up to isomorphism.*

In this situation, M is called the *Fraïssé limit* of $\mathcal{C}$.

Ramsey classes were studied by Nešetřil [8, 9], who showed:

Theorem 2. *(a) A Ramsey class over a finite language is a Fraïssé class.*

(b) The objects in $\mathcal{C}$ are rigid (they have trivial automorphism groups).

Many examples of Ramsey classes are given in [4].

The second part of this theorem was subsequently explained by the celebrated theorem of Kechris, Pestov and Todorčević [5]. First, some background. The symmetric group on a countable set has a natural topology, the topology of pointwise convergence: a basis for the topology consists of the cosets of stabilisers of finite tuples. So any subgroup has the induced subspace topology. It is known that a subgroup is closed if and only if it is the automorphism group of a relational structure.

A topological group G is *extremely amenable* if any continuous action on a compact Hausdorff space has a fixed point. Now the KPT theorem states:

Theorem 3. *Let $\mathcal{C}$ be a Ramsey class with Fraïssé limit M. Then the automorphism group of M is extremely amenable.*

Now the set of all total orders on a countable set is compact, and the symmetric group acts continuously on it. For completeness, we give the proof. Take a countable set, which we can assume to be $\mathbb{N}$. Let $\mathcal{L}$ be the set of linear (total) orders on $\mathbb{N}$. There is a topology on $\mathcal{L}$, defined as follows: A basis for the open sets consists of all the sets for which $\{0, 1, \ldots, n-1\}$ have a given ordering, for all n.

Lemma 4. *(a) The topology on $\mathcal{L}$ is compact.*

(b) The symmetric group on $\mathbb{N}$ acts continuously on $\mathcal{L}$.

Proof. (a) We can identify the topology on $\mathcal{L}$ with the product topology on $A_1 \times A_2 \times \cdots$, with $|A_i| = i$ and with the discrete topology on each set A_i.

This works as follows: if we have ordered $1, \ldots, n$, then there are n gaps into which n can be put: before the first element, between the first and second, ..., after the last. Thus choosing elements of $A_1 \times \cdots \times A_n$ corresponds to ordering $1, 2, \ldots, n$. These choices give a basis of open sets for the product topology.

Now Tychonoff's theorem shows that the product (and hence also $\mathcal{L}$) is compact.

(b) Choose a permutation f of $\mathbb{N}$. Let $m-1$ be the largest element in $f^{-1}\{0, 1, \ldots, n-1\}$. Then the inverse image of the basic open set defined by the ordering of $0, 1, \ldots, n-1$ is a union of finitely many basic open sets defined by orderings of $0, 1, \ldots, m-1$, and so is open. So f is continuous. □ □

So, if $\mathcal{C}$ is a Ramsey class with Fraïssé limit M, then $\mathrm{Aut}(M)$ preserves a total order $<$ on M. It follows that every structure A in $\mathcal{C}$ carries the total order induced from $<$, which is preserved by its automorphisms, since these automorphisms extend to automorphisms of A. Hence A is rigid.

We note here that a proof of the preservation of a linear order, not using the KPT theorem, has been given by Bodirsky [2, Proposition 2.22].

It follows that, if a Fraïssé class is made up of rigid structures, then either its elements carry invariant total orders, or the class fails to be a Ramsey class. Our purpose in the paper is to give a direct constructive proof of this theorem. In the following result, readers may interpret the term "reduct" of M to mean a structure invariant under the automorphisms of M; we will explain further in the next section.

Theorem 5. *Let $\mathcal{C}$ be a Fraïssé class of structures over a finite relational language. Assume that the elements of $\mathcal{C}$ are rigid. Then one of the following holds:*

(a) the Fraïssé limit of $\mathcal{C}$ has a reduct which is a total order;

(b) there are structures $A, B \in \mathcal{C}$ with $|A| = 2$ which demonstrate that $\mathcal{C}$ is not a Ramsey class.

(c) Moreover, if $\mathcal{C}$ has a unique 2-element structure up to isomorphism, then in the second conclusion of the theorem, we have $|A| = 2$ and $|B| = 3$.

The structure of the rest of the paper is as follows. The next section will contain some preliminaries from model theory which underlie our arguments. Then we describe a simple example of a Fraïssé class of rigid structures and give an explicit failure of the Ramsey property. Next we prove the theorem, and we conclude with some comments and further questions.

2 Logical preliminaries

We will use the theorem of Engeler, Ryll-Nardzewski and Svenonius [3, Theorem 6.3.1]. In this section we describe the theorem and its background. The context is first-order logic, though we only use a specialisation: we work over a relational language, containing no constant or function symbols, but the theorem holds in general. We refer to Wilfrid Hodges' book [3] for further background information.

A permutation group G on a set X is *oligomorphic* if its action on X^n has only finitely many orbits for every positive integer n.

A *theory* is simply a set of first-order sentences (formulae with no free variables); the theory of a structure M is the set of all first-order sentences which hold in M. Dually, a model of a theory is a structure in which all sentences of the theory hold. If T is a theory, then a *type* is a set of formulae with n free variables which is maximal with respect to being consistent with T. A type is *realised* in a model M for T if there are n elements of M which can be substituted for the free variables so as to make all the formulae true.

A theory is *ω-categorical* if it has, up to isomorphism, a unique countable model. By slight abuse of language, we say that a structure M is ω-categorical if its theory is; we treat an ω-categorical theory and its countable model as being essentially equivalent. Now the theorem of Engeler, Ryll-Nardzewski and Svenonius from 1959 says the following:

Theorem 6. *Let M be a first-order structure with theory T. Then the following are equivalent:*

(a) T is ω-categorical;

(b) for every natural number n, there are only finitely many n-types over T, all of which are realised in M;

(c) the automorphism group $\mathrm{Aut}(M)$ is oligomorphic.

If these conditions hold, then the set of n-tuples of M realising a given n-type is an orbit of $\mathrm{Aut}(M)$ on M^n, and every orbit corresponds to an n-type in this way.

It follows that a homogeneous relational structure M over a finite relational language is ω-categorical. The orbits on n-tuples of distinct elements correspond to the n-element structures in the age of M.

In general, a *reduct* of a structure M consists of a set of relations (possibly over a different language) which are definable in terms of the relations of M by first-order formula without free variables. If M is ω-categorical, then it follows from Theorem 6 that reducts of M correspond bijectively to closed overgroups of $\mathrm{Aut}(M)$ in the symmetric group.

If M is homogeneous over a finite relational language, and A is a finite substructure of M, then the restriction to A of the relations in any reduct of M are preserved by any automorphism of A, since the automorphisms of A are induced by automorphisms of M fixing A setwise. This explains why the existence of a total order as a reduct of M implies that all structures in the age of M are rigid.

3 An example

This example is taken from the second author's thesis [7].

Here is an elementary group-theoretic result we use: *Cauchy's theorem.*

Theorem 7. *If a finite group G has order divisible by a prime p, then G contains an element of order p.*

A *tournament* is a set with a binary relation $\to$ on X such that

(a) $\neg(x \to x)$ for all $x \in X$;

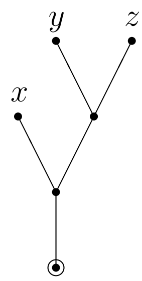
(b) for any $x, y \in X$ with $x \neq y$, exactly one of $x \to y$ and $y \to x$ holds.

Proposition 8. *(a) Finite tournaments form a Fraïssé class with the strong amalgamation property.*

(b) Finite tournaments have automorphism groups of odd order.

Proof. (a) is straightforward. For (b), if T is a tournament with $|\operatorname{Aut}(T)|$ even, then by Theorem 7 there would be an automorphism of order 2, which necessarily interchanges two points; this is not possible. □ □

A *C-structure* is a ternary relational structure defined on the set of leaves of a finite rooted binary tree as follows: given three leaves x, y, z, one of them, say x, is distinguished by the fact that the paths from the root to y and to z coincide for longer than either does with the path from the root to x. In other words, omitting vertices not on these paths and suppressing divalent vertices, the tree looks like this:



We write the relation as $R(x \mid yz)$.

Proposition 9. *(a) The finite C-relations form a Fraïssé class with the strong amalgamation property.*

(b) The automorphism group of a finite C-relation has order a power of 2.

Proof. (a) This is proved by showing that the C-relation determines the tree; we refer to Adeleke and Neumann [1] for details.

(b) In a counterexample, Theorem 7 shows that there would be an automorphism of odd prime order p. Such an automorphism preserves the tree, and fixes the root, so fixes the two children of the root; working up the tree, we see that it is the identity, a contradiction. □

Now we take $\mathcal{C}$ to be the class of structures consisting of a finite set carrying both a tournament and a C-relation, these being chosen independently. Since both tournaments and C-relations are Fraïssé classes with strong amalgamation, so is $\mathcal{C}$. Moreover, the automorphism group of a $\mathcal{C}$-structure is the intersection of the automorphism group of a tournament (of odd order) with that of a C-structure (of 2-power order), and so is trivial.

Now we show that $\mathcal{C}$ is not a Ramsey class. Since the Ramsey property has several alternations of quantifiers, we state explicitly what it means for a class $\mathcal{C}$ to fail the Ramsey property: there exist $A, B \in \mathcal{C}$ such that, for any $C \in \mathcal{C}$, we can colour the embeddings of A into C so that there is no monochromatic embedding of B.

We take A to be the unique 2-element structure in $\mathcal{C}$ (a single tournament arc $x \to y$), and B the unique 3-element structure on which the tournament is a 3-cycle. Take any structure $C \in \mathcal{C}$. We colour the embeddings of A into C (that is, the tournament arcs in C), as follows. Choose an arbitrary total order $<$ on C, and colour $u \to v$ red if $u < v$ and blue if $u > v$. Now a copy of B in C cannot be monochromatic: if all three arcs were red, it would agree with the total order $<$, and if they were blue, it would agree with the converse of $<$; both possibilities contradict the fact that B carries a 3-cycle, not a total order.

4 Proofs

Before beginning the proof, we make an observation that will be used often. A directed graph D is *acyclic* if it contains no directed cycle. Now if D is an acyclic digraph, then there is a total order $<$ on the vertices of D such that $x \to y$ implies $x < y$. To see this, take the set X_1 of *sinks* in D (the vertices with no incoming arcs) to be the minimal elements, and if $X_1, \ldots, X_i$ have been defined and their union is not D, take D_{i+1} to be the set of sinks of $D \setminus (X_1 \cup \cdots \cup X_i)$. Then construct the order by putting an arbitrary total order on each set X_i (noting that there are no arcs within X_i); and, if $x \in X_i$ and $y \in X_j$ with $i < j$, putting $x < y$.

Proof of Theorem 5 We treat first the simpler case where $\mathcal{C}$ contains a unique 2-element structure up to isomorphism; we verify that the conclusions of the Corollary hold in this case.

Let M be the Fraïssé limit of $\mathcal{C}$. Then since the language is finite, M is ω-categorical, so n-types in the theory of M coincide with orbits of the automorphism group of M on n-tuples, by the Ryll-Nardzewski theorem.

Let $\{a, b\}$ be a 2-element structure in $\mathcal{C}$. Since it is rigid, the ordered pairs (a, b) and (b, a) have different types, and so there is a first-order formulae which distinguishes them. This formula defines an ordering of the pair (a, b): we put $a \to b$ if it holds, and $b \to a$ otherwise. This is clearly a tournament which is a reduct of M.

If T is a total order, then we have the first alternative of the Theorem; so suppose not. Then T fails to satisfy transitivity, so it contains a cyclic triple $a \to b \to c \to a$. Now take A to be the unique 2-element structure in $\mathcal{C}$, and B the induced substructure on a cyclic triple of T.

Let C be any member of $\mathcal{C}$. We choose an arbitrary total order on C, and define a colouring of the 2-element substructures by the rule that $\{a,b\}$ is red if $(a \to b) \Leftrightarrow (a < b)$ (that is, the total order agrees with the tournament on $\{a,b\}$) and blue otherwise. Now C contains no monochromatic copy of B: for if all the 2-subsets of B were red, then the tournament would agree with the total order on B, while if they were blue, it would agree with its converse, and neither of these can be the case.

Now we turn to the general case, where there are finitely many 2-element structures in $\mathcal{C}$ up to isomorphism, say $A_1, A_2, \ldots, A_s$. In any structure in $\mathcal{C}$, or in its Fraïssé limit M, the pairs carrying an element of A_i form a graph Γ_i; these graphs partition the complete graph. Also, by Ryll-Nardzewski, there is a first-order formula which defines a direction on each edge of Γ_i, hence defining a directed graph D_i and its converse D_i^*.

We claim that, if any D_i contains a directed cycle, then we have a failure of the Ramsey property. For let A be the structure A_i, and B the structure induced on the vertex set of the directed cycle. Let C be any element of $\mathcal{C}$. We define a 2-colouring of the embedded copies of A in C as follows. Choose an arbitrary total order $<$ on C, and colour a pair (x,y) which is the image of an embedding of A_i red if $((x \to y) \Leftrightarrow (x < y))$ and blue otherwise. In any embedded copy of B, the pairs in the directed cycle cannot all be red, or the order $<$ would contain a directed cycle, and similarly cannot all be blue.

Now we show that we can choose one from each pair (D_i, D_i^*) and call it D_i so that, if $E_i = D_1 \cup \cdots \cup D_i$, then either the digraph E_i is acyclic, or we have a failure of the Ramsey property. This is proved by induction on i; we have already observed the case $i = 1$. So suppose that $D_1, \ldots, D_i$ are defined, and let $\{x,y\}$ be a 2-set inducing a copy of A_{i+1}.

We subdivide into three cases, according as there are directed paths in E_i joining both, just one, or neither of (x,y) and (y,x).

Case 1: There is a path from x to y and a path from y to x. Concatenating these paths, we find a directed cycle in E_i, contrary to hypothesis.

Case 2: There is a path from x to y but not from y to x. Let D_{i+1} be the digraph having an arc $x \to y$, and put $E_{i+1} = E_i \cup D_{i+1}$. Suppose that E_{i+1} contains a directed cycle. Any arc $u \to v$ of this cycle which belongs to D_{i+1} can be replaced by a directed path from u to v in E_i (since these arcs form an orbit of $\mathrm{Aut}(M)$, by Theorem 6). The result is a directed cycle in E_i, again contrary to assumption.

Case 3: There are no directed paths in E_i between x and y (in either order). Set $E_{i+1} = E_i \cup D_{i+1}$ and $E_{i+1}^* = E_i \cup D_{i+1}^*$. If either one of these digraphs is acyclic, then we change the notation if necessary so that this one is E_{i+1}, and the inductive step is complete.

Suppose that both of these digraphs contain directed cycles (which we may assume include $x \to y$ or $y \to x$ as appropriate). Note that each of these cycles must contain an arc of D_{i+1} or D_{i+1}^* other than $x \to y$ or $y \to x$, by the case assumption. Now let $A = A_{i+1}$ and let B be the structure induced on the union of these two cycles. We claim the Ramsey property fails for A and B.

So let C be any structure in $\mathcal{C}$. We know that the restriction of E_i to C is acyclic, so choose a total order $<$ on C so that $u < v$ for any arc $u \to v$ in E_i. Colour a pair (x', y') inducing A_{i+1} red if $x' < y'$ and blue otherwise. We claim that no copy of B can be monochromatic. To see this, suppose first that $x < y$. The directed cycle in E_{i+1} containing the arc $x \to y$ has all its arcs in E_i red, as well as the arc $x \to y$; but not all its arcs can be red, so there must be another arc of D_{i+1} in this cycle which is coloured blue. A similar argument holds in the other case. This concludes the discussion of Case 3.

After having dealt with all of the types of 2-element structures, either we have found a failure of Ramsey's Theorem, or we have built an acyclic directed graph on M in which every pair of vertices are connected by an arc. This digraph is a reduct of M, since each digraph D_i is defined by a first-order formula and there are only finitely many of them. So we have found a total order which is a reduct of M, and the proof of the theorem is complete. $\square$

5 Further comments

We mention here some questions that might be of interest.

(a) Our results hint that there might be a bound for $|B|$ (if the pair (A, B) fails the Ramsey property) in terms of the number of isomorphism types of 2-element structures. Is this true?

(b) The Fraïssé limit of the structure constructed in Section 3 has automorphism group which is torsion-free but not extremely amenable. This group could be investigated further. In particular, does it have any properties related to amenability? Is it simple? What are its reducts?

(c) Is there a general theory of Fraïssé classes of rigid structures?

On the last question, we have shown in the proof of the main theorem that the Fraïssé limit of such a class has a reduct which is a countably categorical tournament. These tournaments are not classified, but one could begin with the homogeneous tournaments, classified by Lachlan [6]. There are just three countable homogeneous tournaments: the rationals as ordered set; the universal homogeneous tournament, and the *circular tournamennt* or *local order*.

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References

[1] S. A. Adeleke and Peter M. Neumann, Relations related to betweenness: Their structure and automorphisms, *Memoirs Amer. Math. Soc.* **623**, 125pp. (1998). doi:https://doi.org/10.1090/memo/0623

[2] Manuel Bodirsky, Ramsey classes: Examples and constructions, *Surveys in Combinatorics 2015* (ed. Artur Czumaj *et al.*), London Math. Soc. Lecture Notes **424**, Cambridge Univ. Press, Cambridge, 2015, pp. 1–48.

[3] Wilfrid Hodges, *A Shorter Model Theory*, Cambridge University Press, Cambridge, 1997.

[4] Jan Hubička and Jaroslav Nešetřil, All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms), *Adv. Math.* **356** (2019), Article 106791, 89pp. doi:10.1016/j.aim.2019.106791

[5] A. S. Kechris, V. G. Pestov and S. Todorčević, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.* **15** (2005), 106–189. doi:10.1007/s00039-005-0503-1

[6] A. H. Lachlan, Countable homogeneous tournaments, *Trans. Amer. Math. Soc.* **284** (1984), 431–461. doi:10.1090/S0002-9947-1984-0743728-1

[7] Siavash Lashkarighouchani, *An investigation into Fraïssé classes and Ramsey theory*, Thesis, University of St Andrews, 2025.

[8] Jaroslav Nešetřil, For graphs there are only four types of hereditary Ramsey classes, *J. Combin. Theory* (B) **46** (1989), 127–132. doi:10.1016/0095-8956(89)90038-5

[9] Jaroslav Nešetřil, Ramsey classes and homogeneous structures, *Combin. Probab. Comput.* **14** (2005), 171–189. doi:10.1017/S0963548304006716