

Defective and Clustered Colouring of Graphs with Given Girth

Marcin Briański^a Robert Hickingbotham^b David R. Wood^c

Submitted: Nov 13, 2025; Accepted: Jun 4, 2026; Published: Jul 17, 2026

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

The *defective chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer d , every graph in $\mathcal{G}$ is k -colourable such that each monochromatic component has maximum degree at most d . Similarly, the *clustered chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer c , every graph in $\mathcal{G}$ is k -colourable such that each monochromatic component has at most c vertices. This paper determines or establishes bounds on the defective and clustered chromatic numbers of graphs with given girth in minor-closed classes defined by the following parameters: Hadwiger number, treewidth, pathwidth, treedepth, circumference, and feedback vertex number. One striking result is that for any integer k , for the class of triangle-free graphs with treewidth k , the defective chromatic number, clustered chromatic number and chromatic number are all equal. The same result holds for graphs with treedepth k , and generalises for graphs with no K_p subgraph.

Mathematics Subject Classifications: 05C15, 05C83

1 Introduction

This paper studies improper colourings of graphs in minor-closed classes with given girth. First, we give the essential definitions. A *colouring* of a graph¹ G is a function that assigns one colour to each vertex of G . For an integer $k \geq 1$, a *k -colouring* is a colouring using at

^aDepartment of Theoretical Computer Science, Jagiellonian University, Kraków, Poland (marcin.brianski@doctoral.uj.edu.pl). M. Briański was partially supported by a Polish National Science Center grant (BEETHOVEN; UMO-2018/31/G/ST1/03718).

^bDépartement d'Informatique, Université libre de Bruxelles, Belgium (robert.hickingbotham@ulb.be). This research was completed when R. Hickingbotham was supported by Australian Government Research Training Program Scholarship.

^cSchool of Mathematics, Monash University, Melbourne, Australia (david.wood@monash.edu). Research of Wood supported by the Australian Research Council and by NSERC.

¹We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. See latter sections and [11] for graph-theoretic definitions not given here.

most k colours. A colouring of a graph is *proper* if each pair of adjacent vertices receives distinct colours. The *chromatic number* $\chi(G)$ of a graph G is the minimum integer k such that G has a proper k -colouring. For a graph class $\mathcal{G}$, the *chromatic number* $\chi(\mathcal{G})$ is $\max\{\chi(G) : G \in \mathcal{G}\}$, and is *unbounded* if the maximum does not exist. A *monochromatic component* with respect to a colouring of a graph G is a connected component of the subgraph of G induced by all the vertices assigned a single colour. A colouring has *defect* d if every monochromatic component has maximum degree at most d . A colouring has *clustering* c if every monochromatic component has at most c vertices. Note that a colouring with defect 0 or clustering 1 is precisely a proper colouring. The *defective chromatic number* $\chi_\Delta(\mathcal{G})$ of a graph class $\mathcal{G}$ is the minimum integer k for which there exists an integer d such that every graph in $\mathcal{G}$ is k -colourable with defect d , and is *unbounded* if no such k exists. The *clustered chromatic number* $\chi_\star(\mathcal{G})$ of a graph class $\mathcal{G}$ is the minimum integer k for which there exists an integer c such that every graph in $\mathcal{G}$ is k -colourable with clustering c , and is *unbounded* if no such k exists. By definition, $\chi_\Delta(\mathcal{G}) \leq \chi_\star(\mathcal{G}) \leq \chi(\mathcal{G})$. See [76] for an extensive survey on this topic.

We consider defective and clustered colouring of graphs with given girth, where our aim is to minimise the number of colours rather than attain specific defect or clustering bounds². First note that for any integer g , the class of graphs with girth at least g has unbounded defective chromatic number. To see this, suppose that for some g and k , the class of graphs with girth at least g has defective chromatic number at most k . That is, for some d , every graph G with girth at least g is k -colourable with defect at most d . Each monochromatic component is properly $(d+1)$ -colourable, so G is properly $k(d+1)$ -colourable. This contradicts the result of Erdős [21], which states that graphs with girth g have unbounded chromatic number. Hence the class of graphs with girth at least g has unbounded defective chromatic number, and thus has unbounded clustered chromatic number. This negative result motivates studying defective and clustered colourings of graphs with given girth and satisfying some other structural assumptions. To obtain interesting results, it is not enough to just consider degeneracy, since Kostochka and Nešetřil [39, Property 6] proved that for all k, d, g there is a k -degenerate graph with girth at least g that is not k -colourable with defect d . Thus chromatic number, defective chromatic number and clustered chromatic number of the class of k -degenerate graphs with girth g equals $k+1$.

Here we focus on minor-closed graph classes. Defective and clustered colourings of graphs excluding a fixed minor are widely studied [7, 14, 15, 17, 20, 30–32, 45–50, 57–60, 72, 75]. We focus on the following minor-monotone parameters: treewidth $\text{tw}(G)$, pathwidth $\text{pw}(G)$, treedepth $\text{td}(G)$, feedback vertex number $\text{fvn}(G)$, circumference $\text{circ}(G)$, and Hadwiger number $\text{had}(G)$. See Section 2 for these well-known definitions. For now, note the following inequalities:

$$\text{had}(G) - 1 \leq \text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1 \quad (1)$$

$$\text{had}(G) - 1 \leq \text{tw}(G) \leq \text{fvn}(G) + 1 \quad (2)$$

$$\text{had}(G) - 1 \leq \text{tw}(G) \leq \text{circ}(G) - 1. \quad (3)$$

²Other papers that focus on specific defect or clustering bounds for graphs of given girth include [6, 7, 63, 73, 74].

1.1 Defective Colouring of Bounded Treewidth Graphs

Here we summarise results on defective colouring of minor-closed graph classes; refer to Tables 1 and 2. Original results in this paper are indicated by $*$.

Table 1: Defective chromatic number of bounded treewidth classes with respect to girth (ignoring lower order additive terms).

girth	fvn k	circumference k	treedepth k	pathwidth k	treewidth k
3	3^*	$\Theta(\log k)$	k	$k + 1$	$k + 1$
4	3^*	$\Omega(\sqrt{\log k})^* \dots O(\log k)$	$\sqrt{2k}^* \dots \lceil \frac{k+2}{2} \rceil^*$	$\sqrt{2k}^* \dots \lceil \frac{k+3}{2} \rceil^*$	$\lceil \frac{k+3}{2} \rceil^*$
≥ 5	2	2	2	2	2

First consider the case with no girth constraint (girth 3). Graphs with treewidth k are k -degenerate and thus properly $(k + 1)$ -colourable. This implies that graphs with pathwidth k are $(k + 1)$ -colourable, and graphs with treedepth k are properly k -colourable. Standard examples (see Section 3.1 and [76]) show that these bounds cannot be improved for defective or clustered colouring. Mohar, Reed, and Wood [54] showed that graphs with circumference k are $O(\log k)$ -colourable with clustering k , and thus with defect $k - 1$. The number of colours is tight up to the leading constant. Thus the defective and clustered chromatic number of the class of graphs with circumference k is in $\Theta(\log k)$. Every graph G with $\text{fvn}(G) \leq k$ is 3-colourable with clustering k and defect $k - 1$: just properly 2-colour the underlying forest and give a third colour to each apex vertex. Proposition 15 shows that three colours is best possible even for feedback vertex number 1. Thus for each integer $k \geq 1$, the clustered and defective chromatic number of the class of graphs G with $\text{fvn}(G) \leq k$ equals 3.

Now consider the triangle-free case (girth 4). Dvořák and Kawarabayashi [18] proved the following important result.

Theorem 1 ([18]). *For any $k \in \mathbb{N}$ the chromatic number of the class of triangle-free graphs with treewidth k equals $\lceil \frac{k+3}{2} \rceil$.*

We extend their lower bound to prove that for the class of triangle-free graphs with treewidth k , the defective chromatic number, the clustered chromatic number, and the chromatic number all equal $\lceil \frac{k+3}{2} \rceil$ (see Corollary 14 which in fact holds for any fixed excluded complete subgraph). We prove the following similar result for treedepth: for the class of triangle-free graphs with treedepth k , the defective chromatic number, the clustered chromatic number, and the proper chromatic number are all equal, and this value is between $(1 - o(1))\sqrt{2k}$ and $\lceil \frac{k+2}{2} \rceil$ (see Corollary 11). The lower bound for triangle-free graphs with treedepth k also leads to a $\Omega(\sqrt{\log k})$ lower bound for triangle-free graphs with circumference k (see Corollary 12). So the defective and clustered chromatic number of the class of triangle-free graphs with circumference k is between $\Omega(\sqrt{\log k})$ and $O(\log k)$. Finally, Proposition 15 shows that three colours is best possible even for triangle-free graphs with feedback vertex number 1. Thus for each integer $k \geq 1$, the clustered and

defective chromatic number of the class of triangle-free graphs G with $\text{fvn}(G) \leq k$ equals 3.

Now consider the girth at least 5 case. The following result of Ossona de Mendez et al. [60] is paramount. Here $\nabla(G)$ is the maximum average degree of a graph H such that the 1-subdivision of H is isomorphic to a subgraph of G .

Theorem 2 ([60]). *Every $K_{s,t}$ -subgraph-free graph G is s -colourable with defect at most some function $f(s, t, \nabla(G))$.*

We give a short proof of a result that is equivalent to Theorem 2 in Section 4.1.

Every graph with girth at least 5 contains no $K_{2,2}$ subgraph. Thus Theorem 2 with $s = t = 2$ implies that for any graph class $\mathcal{G}$ with bounded ∇ (which includes every minor-closed class), the graphs in $\mathcal{G}$ with girth at least 5 are 2-colourable with bounded defect. Girth 5 here cannot be reduced to 4, even for graphs with bounded treedepth, since we prove that the class of triangle-free graphs with treedepth k has defective chromatic number at least $\Omega(\sqrt{k})$ (see Theorem 10). By (1), the same result holds with ‘treedepth’ replaced by ‘pathwidth’, ‘treewidth’ or ‘Hadwiger number’. For each of these parameters, there is a qualitative change in behaviour from girth 4 to 5. For a class of graphs with girth at least 5 and with any of these parameters bounded, the defective chromatic number equals 2. On the other hand, for graphs with girth 4, the defective chromatic number increases with the parameter.

1.2 Defective Colouring of General Minor-Closed Classes

Now consider defective colourings of planar graphs and more general minor-closed classes; the results are summarised in Table 2.

Table 2: Defective chromatic number of minor-closed classes with respect to girth.

girth	planar graphs	Euler genus γ	(γ, k) -apex	Hadwiger number t
3	3	3	4	t
4	2 or 3	2 or 3	2, 3 or 4	$\lceil \frac{t+2}{2} \rceil^* \dots t$
≥ 5	2	2	2	2

First consider the case with no girth constraint. Cowen, Cowen, and Woodall [9] proved that planar graphs are 3-colourable with defect 2 (see [19, 61, 72] for various strengthenings). More generally, Archdeacon [2] proved that graphs of Euler genus γ are 3-colourable with defect $O(\gamma)$, and Cowen, Goddard, and Jesurum [8] improved the defect bound to $O(\sqrt{\gamma})$, which is best possible. Standard examples provide, for any integer $d \geq 0$, a planar graph that is not 2-colourable with defect d (see [76]). Thus for each integer $\gamma \geq 0$, the defective chromatic number of the class of graphs with Euler genus γ equals 3.

Now consider apex graphs. For integers $\gamma, k \geq 0$, a graph G is (γ, k) -*apex* if there exists $A \subseteq V(G)$ such that $|A| \leq k$ and $G - A$ has Euler genus at most γ . A $(0, k)$ -apex graph is called *k-apex*, in which case $G - A$ is planar. Every (γ, k) -apex graph is 4-colourable with

defect $O(\max\{\sqrt{\gamma}, k\})$: just apply the above 3-colour result for graphs of Euler genus γ , and use a fourth colour for the k apex vertices. Standard examples show that four colours is best possible, even for $(0, 1)$ -apex graphs (see [76]).

Now consider the class of graphs G with Hadwiger number $\text{had}(G) \leq t$. Hadwiger [28] famously conjectured that such graphs are properly t -colourable. The best known upper bound on the chromatic number is $O(t \log \log t)$ due to Delcourt and Postle [10]. Standard examples provide a lower bound of t on the defective (and thus clustered) chromatic number of this class. Edwards et al. [20] proved that graphs G with $\text{had}(G) \leq t$ are t -colourable with defect $O(t^2 \log t)$. Thus the defective chromatic number of the class of graphs G with $\text{had}(G) \leq t$ equals t . The defect bound in the above result of Edwards et al. [20] was improved to $O(t)$ by van den Heuvel and Wood [72].

Finally, consider the triangle-free (girth 4) case. Each of the upper bounds in the triangle-free case of Table 2 follow from the analogous upper bounds without the triangle-free assumption. The lower bounds of 2 are trivial. The lower bound of $\lceil \frac{t+2}{2} \rceil$ for graphs with Hadwiger number t follows from the fact that the defective chromatic number of the class of triangle-free graphs with treewidth k equals $\lceil \frac{k+3}{2} \rceil$ and since $\text{had}(G) \leq \text{tw}(G) + 1$ for every graph G .

1.3 Clustered Colouring of Bounded Treewidth Graphs

Here we consider clustered colourings of various bounded treewidth graph classes with respect to girth. Table 3 summarises the results.

Table 3: Clustered chromatic number of bounded treewidth classes with respect to girth (ignoring lower order additive terms).

girth	fvn k	circumference k	treedepth k	pathwidth k	treewidth k
3	3	$\Theta(\log k)$	k	$k + 1$	$k + 1$
4	3	$\Omega(\sqrt{\log k})^* \dots O(\log k)$	$\sqrt{2k}^* \dots \lceil \frac{k+2}{2} \rceil^*$	$\sqrt{2k}^* \dots \lceil \frac{k+3}{2} \rceil^*$	$\lceil \frac{k+3}{2} \rceil^*$
5 or 6	3^*	2^*	2^*	3	3
≥ 7	2^*	2^*	2^*	2 or 3	2 or 3
$\Omega(\log k)$	2^*	2^*	2^*	2^*	2^*

Liu and Wood [47] proved the following analogue of Theorem 2 for clustered colouring of bounded treewidth graphs.

Theorem 3 ([47]). *For any $s, t, k \in \mathbb{N}$, every $K_{s,t}$ -subgraph-free graph of treewidth at most k is $(s + 1)$ -colourable with clustering at most some function $f(s, t, k)$.*

As before, Theorem 3 with $s = t = 2$ implies every graph with girth at least 5 and with treewidth at most k is 3-colourable with clustering at most some function $f(k)$. In both Theorems 3 and 4 the number of colours is best possible, but the extremal examples have many triangles.

Consider the triangle-free case (girth 4). As mentioned above, we extend Theorem 1 to show that the clustered chromatic number of triangle-free graphs with treewidth k equals

$\lceil \frac{k+3}{2} \rceil$. This implies the same upper bound for triangle-free graphs with pathwidth k , and an upper bound of $\lceil \frac{k+2}{2} \rceil$ for triangle-free graphs with treedepth k . As mentioned above, Mohar et al. [54] showed that graphs with circumference k are $O(\log k)$ -colourable with clustering k . We show that this bound on the number of colours is best possible even for triangle-free graphs (see Corollary 12). Thus the clustered chromatic number of the class of triangle-free graphs with circumference k is in $\Theta(\log k)$. For triangle-free graphs with $\text{fvn}(G) \leq k$ the upper bound of 3 is best possible, even for $k = 1$ (see Proposition 15).

Now consider the case of girth at least 5. Theorem 2 implies that graphs with girth at least 5 and with bounded treedepth are 2-colourable with bounded clustering, since in a graph of bounded treedepth, every path has bounded length (see [56]). This observation in fact implies that for every graph class $\mathcal{G}$ with bounded treedepth, $\chi_*(\mathcal{G}) = \chi_\Delta(\mathcal{G})$. Similarly, we improve the number of colours from 3 to 2 in the case of graphs with girth at least 5 and circumference k (see Theorem 19), and in the case of graphs with girth at least 7 and feedback vertex number k (see Theorem 21). The upper bound of 2 for graphs with treewidth k and girth $\Omega(\log k)$ in Table 3 follows from our more general results for graphs with Hadwiger number t and with girth $\Omega(\log t)$ (see Section 5).

1.4 Clustered Colouring of General Minor-Closed Classes

Here we consider clustered colourings of various minor-closed graph classes (with unbounded treewidth) with respect to girth. Table 4 summarises the results.

Table 4: Clustered chromatic number of minor-closed classes with respect to girth.

girth	planar graphs	Euler genus γ	(γ, k) -apex	Hadwiger number t
3	4	4	5	t
4	3	3	3 or 4	$\lceil \frac{t+2}{2} \rceil^* \dots t$
5 or 6	2	2	3^*	3 or 4
7 or 8	2	2	2^* or 3^*	2, 3 or 4
≥ 9	2	2	2^*	2, 3 or 4
$\Omega(\log t)$				2^*

First consider planar graphs. The 4-Colour Theorem says that planar graphs are properly 4-colourable [62]. A much simpler proof by Cowen et al. [9] shows that planar graphs are 4-colourable with clustering 2. Standard examples show that the clustered chromatic number of the class of planar graphs equals 4 (see [76]). Similarly, Grotzsch's Theorem [26, 64, 69, 70] says that triangle-free planar graphs are properly 3-colourable. As a lower bound, for any integer $d \geq 0$, Škrekovski [73] constructed triangle-free planar graphs that are not 2-colourable with defect d ; thus the defective and clustered chromatic number of triangle-free planar graphs equals 3. (Note that Lemma 5 below implies that triangle-free planar graphs are 3-colourable with bounded clustering, without using Grotzsch's Theorem).

Now consider girth $g \geq 5$. Liu and Wood [47] proved the following analogue of Theorem 3.

Theorem 4 ([47]). *For any $s, t \in \mathbb{N}$ and for any graph X , every $K_{s,t}$ -subgraph-free X -minor-free graph is $(s + 2)$ -colourable with clustering at most some function $f(s, t, X)$.*

As before, Theorem 4 with $s = t = 2$ implies that every X -minor-free graph with girth at least 5 is 4-colourable with clustering at most some function $f(X)$.

Now consider graphs embeddable on surfaces. Esperet and Ochem [22] and Kawarabayashi and Thomassen [33] proved that graphs of bounded Euler genus are 5-colourable with bounded clustering. Improving these results, Dvořák and Norin [17] showed that every graph with Euler genus γ is 4-colourable with clustering $O(\gamma)$. Hence the clustered chromatic number of the class of graphs embeddable on any fixed surface equals 4. The proof is based on the following lemma, which is proved using the so-called island method (see [76] for a concise presentation).

Lemma 5 ([17]). *Let $\mathcal{G}$ be a hereditary graph class such that for some real numbers $c, \varepsilon, \delta > 0$ and some integers $k, n_0 \geq 2$, for all $n \geq n_0$, every n -vertex graph in $\mathcal{G}$ has a balanced separator of order at most $cn^{1-\varepsilon}$, and has at most $(k - \delta)n$ edges. Then every graph in $\mathcal{G}$ is k -colourable with clustering at most some function $f(k, c, \varepsilon, \delta, n_0)$.*

The balanced separator assumption in Lemma 5 is satisfied with $\varepsilon = \frac{1}{2}$ for planar graphs [43, 44], for graphs of bounded genus [13, 23], and for any minor-closed class [1].

The above-mentioned clustered 4-colour theorem of Dvořák and Norin [17] is implied by Lemma 5 with $\delta = \frac{1}{2}$ and $n_0 := 6(\gamma - 2)$, since then it follows from Euler's formula that every graph G with $n \geq 3$ vertices and Euler genus γ satisfies $|E(G)| \leq 3(n + \gamma - 2) \leq (4 - \delta)n$ for $n \geq n_0$. Similarly, every triangle-free graph with $n \geq 3$ vertices and Euler genus γ has at most $2(n + \gamma - 2)$ edges. Thus such graphs are 3-colourable with bounded clustering by Lemma 5. More generally, every n -vertex graph with Euler genus γ and girth g has at most $\frac{g}{g-2}(n + \gamma - 2)$ edges. In particular, if $g \geq 5$, then G has at most $\frac{5}{3}(n + \gamma - 2)$ edges, implying such graphs are 2-colourable with bounded clustering by Lemma 5. These results were mentioned by Wood [76].

Our results are summarised in Table 4. Every (γ, k) -apex graph G is 5-colourable with bounded clustering: 4-colour $G - A$ using Lemma 5 and use one new colour for A . Standard examples show that the clustered chromatic number of $(0, 1)$ -apex graphs equals 5. Similarly, triangle-free (γ, k) -apex graphs are 4-colourable with bounded clustering. Our other proofs for (γ, k) -apex graphs are in Section 4.3. For $(0, 5)$ -apex graphs with girth at least 6, Proposition 16 implies that the clustered chromatic number is at least 3.

Now consider the clustered chromatic number of the class of graphs G with $\text{had}(G) \leq t$. As mentioned above, standard examples show a lower bound of t . Kawarabayashi and Mohar [32] first proved a $O(t)$ upper bound. Their bound was $\lceil \frac{31}{2}(t + 1) \rceil$, which was improved to $\lceil \frac{7t+4}{2} \rceil$ by Wood [75]³, to $4t$ by Edwards et al. [20], to $3t$ by Liu and Oum [46], to $2t$ independently by Norin [57], van den Heuvel and Wood [72] and Dvořák and Norin [17], to $t + 1$ by Liu and Wood [49], and to t by Dujmović et al. [15] (announced earlier by Dvořák and Norin [17]). Thus the clustered chromatic number of the class of graphs G with $\text{had}(G) \leq t$ equals t .

³The result of Wood [75] depended on a result announced by Norin and Thomas [65], which has not yet been fully written.

Finally, we mention the final rows in Tables 3 and 4, where the girth is allowed to grow slowly with the excluded minor. In Section 5, we prove that every graph with Hadwiger number at most t and with girth $\Omega(\log t)$ is 2-colourable with clustering 2 (see Theorem 33). This section also explores results on proper colouring of graphs with given Hadwiger number and given girth. We show, via a result of Kühn and Osthus [42], that for every integer $g \geq 5$ there exists $c_g \in (0, 1)$ with $c_g \rightarrow 0$, such that graphs with Hadwiger number t and girth g are properly $O(t^{c_g})$ colourable, thus asymptotically improving on Hadwiger's Conjecture for girth at least 5.

2 Definitions

We consider finite simple undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For a vertex $v \in V(G)$, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ and $N_G[v] := N_G(v) \cup \{v\}$.

A *graph class* $\mathcal{G}$ is a set of graphs closed under isomorphism. A graph class $\mathcal{G}$ is *proper* if some graph is not in $\mathcal{G}$. A graph class $\mathcal{G}$ is *hereditary* if it is closed under induced subgraphs.

A graph G is *k -degenerate* if every subgraph of G has minimum degree at most k . Such graphs are properly $(k + 1)$ -colourable by a greedy algorithm.

A graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. A graph G is *H -minor-free* if H is not a minor of G . The *Hadwiger number* of a graph G , denoted $\text{had}(G)$, is the maximum integer t such that K_t is a minor of G . A graph class $\mathcal{G}$ is *minor-closed* if for every graph $G \in \mathcal{G}$ every minor of G is in $\mathcal{G}$. Similarly, a graph parameter f is *minor-monotone* if $f(H) \leq f(G)$ for every graph G and every minor H of G .

A *surface* is a compact 2-dimensional manifold. For any fixed surface Σ , the class of graphs embeddable on Σ (without crossings) is minor-closed. A surface with h handles and c cross-caps has *Euler genus* $2h + c$. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds. Euler genus is a minor-monotone graph parameter. See [55] for more about graph embeddings in surfaces.

A *tree-decomposition* of a graph G is a collection $\mathcal{T} = (B_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T , such that (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v , and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T . The *width* of $\mathcal{T}$ is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of a graph G , denoted $\text{tw}(G)$, is the minimum width of a tree-decomposition of G .

A *path-decomposition* is a tree-decomposition in which the underlying tree is a path, simply denoted by the corresponding sequence of bags $(B_1, \dots, B_n)$. The *pathwidth* of a graph G , denoted $\text{pw}(G)$, is the minimum width of a path-decomposition of G . By definition,

$$\text{tw}(G) \leq \text{pw}(G).$$

A tree T with a distinguished vertex $r \in V(T)$ is said to be *rooted* (at r). The *depth* of a vertex v in T is the number of vertices in the rv -path in T . For distinct vertices u, v

in T , if v is on the *ur*-path in T , then v is an *ancestor* of u and u is a *descendant* of v in T . For each vertex v in a rooted tree T , the subtree induced by v and all its descendants is the *subtree of T rooted at v* . A forest is *rooted* if each component tree is rooted. The *depth* of a rooted forest F is the maximum depth of a vertex in F . The *closure* of F is the graph with vertex set $V(F)$, where vw is an edge if and only if v is an ancestor of w or vice versa. The *treedepth* of a graph G , denoted $\text{td}(G)$, is the minimum depth of a rooted forest F such that G is a spanning subgraph of the closure of F . It is well-known that for every graph G ,

$$\text{pw}(G) \leq \text{td}(G) - 1.$$

To see this, introduce one bag for each root–leaf path, ordered left-to-right according to a plane drawing of F , producing a path-decomposition of G . Finally, if G is a subgraph of the closure of a rooted tree T , and v is a vertex of T , then the *subgraph of G rooted at v* is the subgraph of G induced by the vertices of the subtree of T rooted at v .

For a graph G , a set $A \subseteq V(G)$ is a *feedback vertex set* of G if $G - A$ is a forest. The *feedback vertex number* of a graph G , denoted $\text{fvn}(G)$, is the minimum number of vertices in a feedback vertex set of G . Since a forest has treewidth at most 1,

$$\text{tw}(G) \leq \text{fvn}(G) + 1.$$

The *circumference* $\text{circ}(G)$ of a graph G that is not a forest is the length of the longest cycle in G . The *circumference* of a forest is 2. Birmelé [3] showed that

$$\text{tw}(G) \leq \text{circ}(G) - 1;$$

see [4, 53] for related results.

It is well-known and easily seen that treewidth, pathwidth, treedepth, feedback vertex number and circumference are minor-monotone.

3 Lower Bounds

3.1 Generalised Standard Examples

Many authors have noted that closures of rooted trees provide lower bounds on the defective and clustered chromatic numbers [20, 29, 58, 60, 72, 76]. Such graphs have been dubbed ‘standard examples’. The following definition and theorem generalises these results.

Let H be a graph equipped with a vertex ordering $V(H) = \{v_1, \dots, v_n\}$. For an integer $d \geq 1$, let $H^{(d)}$ be the following graph. Start with a complete d -ary tree T of vertex-height n . Let $V(H^{(d)}) := V(T)$, where for each root-leaf path $(x_1, \dots, x_n)$ in T , $x_i x_j \in E(H^{(d)})$ if and only if $v_i v_j \in E(H)$. For example, $K_n^{(d)}$ is the closure of the complete d -ary tree of height n . Numerous papers have noted that $K_n^{(d+1)}$ is not $(n-1)$ -colourable with defect d (see [76]). Here we prove the following generalisation.

Theorem 6. *Let H be a graph equipped with a vertex ordering $V(H) = \{v_1, \dots, v_n\}$. Fix integers $d, k \geq 1$. Then the following are equivalent:*

- (1) $H^{(d)}$ is k -colourable with defect $d - 1$,
- (2) $H^{(d)}$ is k -colourable with clustering d ,
- (3) $H^{(d)}$ is properly k -colourable,
- (4) H is properly k -colourable.

Proof. We first prove that (4) $\implies$ (3). Consider any proper k -colouring of H . Let T be the rooted tree underlying $H^{(d)}$. For each root-leaf path $(x_1, \dots, x_n)$ of T , assign x_i the colour assigned to v_i . We obtain a proper k -colouring of $H^{(d)}$, and (2) holds.

It is immediate that (3) $\implies$ (2) $\implies$ (1).

It remains to prove that (1) $\implies$ (4). Suppose $H^{(d)}$ is k -colourable with defect $d - 1$. For each vertex x of $H^{(d)}$, let T_x be the subtree rooted at x of the complete d -ary tree underlying $H^{(d)}$. Let x_1 be the root vertex. Say x_1 is blue. If each of the subtrees of T rooted at the children of x_1 has a blue neighbour of x_1 , then x_1 has monochromatic degree at least d . Thus there is a child x_2 of x_1 such that no neighbour of x_1 in T_{x_2} is blue. Repeating this argument, we find a root-leaf path $x_1, \dots, x_n$ in $H^{(d)}$, such that for each vertex x_i , no neighbour of x_i in $T_{x_{i+1}}$ is assigned the colour of x_i . Thus the copy of H induced by $\{x_1, \dots, x_n\}$ is properly coloured with k colours, and (4) holds. $\square$

3.2 Triangle-Free Graphs

We now prove lower bounds for triangle-free graphs with given treedepth. By construction, $H^{(d+1)}$ has treedepth at most $|V(H)|$, and if H is K_p -subgraph-free, then so is $H^{(d+1)}$. Thus Theorem 6 implies:

Corollary 7. *The defective chromatic number of the class of K_p -free graphs of treedepth k is at least the chromatic number of the class of K_p -free graphs on k vertices.*

Corollary 7 applies for triangle-free graphs, but not for girth at least 5 (since $H^{(d)}$ contains a 4-cycle whenever $d \geq 2$ and H contains a path $v_i v_k v_j$ with $i < j < k$).

Kim [36] showed that the Ramsey number $R(3, t) \in \Omega(t^2 / \log t)$. That is, there exist triangle-free graphs H on $n = \Omega(t^2 / \log t)$ vertices with $\alpha(H) < t$. Thus

$$\chi(H) \geq n / \alpha(H) > n / t \geq \Omega(t / \log t) \geq \Omega(\sqrt{n / \log n}).$$

Corollary 7 with $p = 3$ and Eq. (1) thus imply:

Corollary 8. *The defective chromatic number of the class of triangle-free graphs with treedepth k (or pathwidth k or treewidth k) is in $\Omega(\sqrt{k / \log k})$.*

Theorem 9. *Fix $p \geq 3$ and $k \geq 1$. Let $\mathcal{G}$ be the class of K_p -free graphs of treedepth k . Then the defective chromatic number, the clustered chromatic number, and the chromatic number of $\mathcal{G}$ are equal.*

Proof. Obviously, the defective chromatic number of $\mathcal{G}$ is at most the clustered chromatic number of $\mathcal{G}$, which is at most the chromatic number of $\mathcal{G}$. Let c be the defective chromatic

number of $\mathcal{G}$. That is, for some d every graph in $\mathcal{G}$ is c -colourable with defect d . We now show that every graph in $\mathcal{G}$ is properly c -colourable.

Consider a graph $H \in \mathcal{G}$. So H is a spanning subgraph of the closure of some rooted tree with vertex-height k . Define $\widehat{H}$ as follows. (This construction is an extension of the definition of $H^{(d)}$ above.) Initialise $\widehat{H} := H$. We now make a sequence of changes to $\widehat{H}$, where at each step, we maintain the property that $\widehat{H}$ is a spanning subgraph of the closure of some rooted tree with vertex-height k . In particular, for $i = 2, 3, \dots, k$ and for each vertex w at depth i in $\widehat{H}$, replace the current subgraph Q_w of $\widehat{H}$ rooted at w by $d + 1$ copies of Q_w rooted at w , and for every edge xy where x is an ancestor of w and y is a descendent of w or equal to w , add an edge between x and the copy of y in each of the $d + 1$ copies of Q_w .

Consider distinct vertices $x, y \in V(\widehat{H})$. Let x_0, y_0 be the vertices in H such that x is a copy of x_0 and y is a copy of y_0 . Then $xy \in E(\widehat{H})$ if and only if x is an ancestor of y in $\widehat{H}$, and $x_0y_0 \in E(H)$ with x_0 an ancestor of y_0 . In particular, for any clique C of $\widehat{H}$, every pair of vertices in C are an ancestor–descendent pair in $\widehat{H}$, and the vertices in H that correspond to C are a clique in H . Since H is K_p -free, so is $\widehat{H}$. By construction, $\widehat{H}$ has treedepth k . Thus $\widehat{H} \in \mathcal{G}$.

By assumption, $\widehat{H}$ is c -colourable with defect d . We now show that $\widehat{H}$ contains a properly c -coloured induced subgraph isomorphic to H . Let T be the tree of vertex-height k rooted at r such that H is a subgraph of the closure of T . For each vertex v in T , we will choose exactly one vertex v' in $\widehat{H}$ to represent v . Consider w in non-decreasing order of $\text{dist}_T(r, w)$. Let r represent itself. Now assume $w \neq r$. Let v be the parent of w in T . So some vertex v' in $\widehat{H}$ is already chosen to represent v . Say v' is blue. In $\widehat{H}$ there are $d + 1$ copies $w_1, \dots, w_{d+1}$ of w with parent v' . If for each $i \in \{1, \dots, d + 1\}$, the subgraph of $\widehat{H}$ rooted at w_i has a blue neighbour of v' , then v' has monochromatic degree at least $d + 1$, which is a contradiction. Thus for some $i \in \{1, \dots, d + 1\}$, the subgraph of $\widehat{H}$ rooted at w_i has no blue neighbour of v' . Let this vertex w_i represent w . All neighbours of v' subsequently chosen by this algorithm are in the subgraph of $\widehat{H}$ rooted at w_i and are therefore not coloured blue. By the above characterisation of edges in $\widehat{H}$, the chosen vertices induce a properly c -coloured subgraph of $\widehat{H}$ isomorphic to H . Hence every graph in $\mathcal{G}$ is properly c -colourable. $\square$

We now use a variant of the approach of Dvořák and Kawarabayashi [18] to improve the lower bound on the chromatic number in Corollary 8 to $(1 - o(1))\sqrt{2k}$.

Theorem 10. *For any integer $k \geq 1$ there is a triangle-free graph with treedepth $\binom{k+1}{2}$ and chromatic number at least k .*

Proof. Let T be the tree rooted at a vertex r , and let G be the subgraph of the closure of T defined as follows. For each vertex v of T considered in non-decreasing order of $\text{dist}_T(r, v)$, if P_v is the rv -path in T , then for each independent set I of $G[V(P_v)]$ introduce one child of v with neighbourhood I .

Since the neighbourhood of each new vertex is an independent set, G is triangle-free. Consider any proper colouring of G . Let $\text{col}(v)$ be the colour assigned to each vertex v . The following claim implies the theorem.

Claim. For each integer $k \geq 1$ there is a vertex v in the first $\binom{k+1}{2}$ layers of G and there is an independent set of $G[V(P_v)]$ assigned at least k distinct colours.

We proceed by induction on k . The $k = 1$ case is true with $v = r$. Assume the claim is true for some $k \geq 1$. Thus there is a vertex w_0 in the first $\binom{k+1}{2}$ layers of G and there is an independent set I of $G[V(P_{w_0})]$ assigned at least k distinct colours.

By construction, there is a child w_1 of w_0 adjacent to no vertices in I . If no vertex in I is assigned the same colour as w_1 , then $I \cup \{w_1\}$ is an independent set of $G[V(P_{w_1})]$ assigned at least $k + 1$ distinct colours, and w_1 is in the first $\binom{k+1}{2} + 1 \leq \binom{k+2}{2}$ layers of G . Now assume $\text{col}(y_1) = \text{col}(w_1)$ for some vertex $y_1 \in I$.

We now construct a sequence of distinct vertices $w_1, \dots, w_{k+1}$ in T and a sequence of distinct vertices $y_1, \dots, y_k$ in I , such that:

- for each $j \in \{1, \dots, k\}$, $\text{col}(y_j) = \text{col}(w_j)$
- for each $j \in \{1, \dots, k + 1\}$, w_j is a child of w_{j-1}
- for each $j \in \{1, \dots, k + 1\}$, the vertex w_j is adjacent to each vertex in $\{y_1, \dots, y_{j-1}\}$ and is adjacent to no vertex in $(I \setminus \{y_1, \dots, y_{j-1}\}) \cup \{w_1, \dots, w_{j-1}\}$.

These properties hold for w_1 as shown above.

Assume that $w_1, \dots, w_j$ and $y_1, \dots, y_{j-1}$ satisfy these properties for some $j \in \{1, \dots, k\}$. Thus $\text{col}(w_j) \neq \text{col}(y_i) = \text{col}(w_i)$ for all $i \in \{1, \dots, j - 1\}$. If no vertex in $I \setminus \{y_1, \dots, y_{j-1}\}$ is assigned the same colour as w_j , then $(I \setminus \{y_1, \dots, y_{j-1}\}) \cup \{w_1, \dots, w_j\}$ is an independent set of $G[V(P_{w_j})]$ assigned $k + 1$ distinct colours, where w_j is in the first $\binom{k+1}{2} + j \leq \binom{k+2}{2}$ levels of G , and we are done. Thus we may assume that $\text{col}(y_j) = \text{col}(w_j)$ for some vertex y_j in $I \setminus \{y_1, \dots, y_{j-1}\}$. By the definition of G , there is a child w_{j+1} of w_j that is adjacent only to the independent set $\{y_1, y_2, \dots, y_j\}$. This allows us to extend our sequence.

Repeating this step constructs the desired sequences. In particular, $\{w_1, \dots, w_{k+1}\}$ is an independent set of $G[V(P_{w_{k+1}})]$ assigned at least $k + 1$ distinct colours, and w_{k+1} is in the first $\binom{k+1}{2} + k + 1 = \binom{k+2}{2}$ layers of G , as claimed. $\square$

Theorems 1, 9 and 10 imply:

Corollary 11. For any integer $k \geq 1$ for the class of triangle-free graphs with treedepth k , the chromatic number, defective chromatic number and clustered chromatic number are equal, and are at least $(1 - o(1))\sqrt{2k}$ and at most $\lceil \frac{k+2}{2} \rceil$.

We have the following corollary of Theorem 10 for graphs of given circumference, since $\text{circ}(H) < 2^{\text{td}(H)}$ for every graph H (see [56, Proposition 6.2]).

Corollary 12. The chromatic number, defective chromatic number and clustered chromatic number of the class of triangle-free graphs with circumference k is in $\Omega(\sqrt{\log k})$.

Now consider triangle-free graphs with given treewidth.

Theorem 13. Fix $p \geq 3$ and $k \geq 1$. Let $\mathcal{G}$ be the class of K_p -free graphs of treewidth k . Then the defective chromatic number, the clustered chromatic number, and the chromatic number of $\mathcal{G}$ are equal.

Proof. Obviously, the defective chromatic number of $\mathcal{G}$ is at most the clustered chromatic number of $\mathcal{G}$, which is at most the chromatic number of $\mathcal{G}$. Let c be the defective chromatic number of $\mathcal{G}$. That is, for some d every graph in $\mathcal{G}$ is c -colourable with defect d . We now show that every graph in $G \in \mathcal{G}$ is properly c -colourable.

Consider a tree-decomposition $(B_x : x \in V(T))$ of G with width at most k . Root T at a node r . Normalise the tree-decomposition so that for each edge $xy \in E(T)$ with x the parent of y , $|B_y \setminus B_x| = 1$, and for each vertex v of G there is an edge $xy \in E(T)$ with x the parent of y , such that $B_y \setminus B_x = \{v\}$. In this case, let $h(v) := y$ (the ‘home’ of v). Let $\preceq$ be the partial order on $V(G)$, where $v \prec w$ if and only if $h(v)$ is an ancestor of $h(w)$ in T . The *depth* of v is $\text{dist}_T(r, h(v))$.

We now define a sequence of K_p -free graphs $G_1, G_2, \dots, G_m$, each with a rooted normalised tree-decomposition $(B_x : x \in V(T_i))$ of width at most k , where $G_1 := G$. In each graph G_i , each vertex will be ‘processed’ or ‘unprocessed’. Initially, all vertices in G_1 are unprocessed. Suppose that $G_1, \dots, G_i$ are defined, and some vertex in G_i is unprocessed. Let v be an unprocessed vertex in G_i at minimum depth. Let $y := h(v) \in V(T_i)$. Let $U_1, \dots, U_m$ be the subtrees of T_i rooted at the children of y . Let V_j be the set of vertices $w \in V(G_i)$ where $h(w) \in V(U_j)$. Let $U := \bigcup_{j=1}^m V_j$. Let G_{i+1} be obtained from G_i by making $d+1$ copies of each vertex in U . For each edge pq of $G[U]$, add an edge to G_{i+1} between the ℓ -th copy of p and the ℓ -th copy of q , for each ℓ . For each edge $wp \in E(G_i)$ where $w \in B_y$ and $p \in U$, add an edge between w and the ℓ -th copy of p , for each ℓ . This includes the case $w = v$. Construct a normalised tree-decomposition of G_{i+1} as follows. Let T_{i+1} be obtained from T_i by taking $d+1$ copies of each U_j , where the root of each copy is a child of y . For each vertex $p \in U$, if $p \in B_x$ then place the ℓ -th copy of p in the bag corresponding to the ℓ -th copy of x in T_{i+1} . We obtain a normalised tree-decomposition $(B'_x : x \in V(T_{i+1}))$ of G_{i+1} , such that for each vertex $w \in U$, if $h(w) = x$ then the home bags of the $d+1$ copies of w are the $d+1$ copies of x . In G_{i+1} say that v is processed. We now show that G_{i+1} is K_p -free. Suppose that X is a set of vertices in G_{i+1} that induce K_p . By the Helly property, X is a subset of some bag B'_x . By construction, $G_{i+1}[B'_x]$ is isomorphic to the subgraph of G_i induced by some bag B_x , implying that G_i contains K_p . This contradiction shows that G_{i+1} is K_p -free.

This completes the definition of $G_1, G_2, \dots$. Note that for each vertex v of G the depth of each copy of v always equals the original depth of v . In G_i , let d_i be the minimum depth of an unprocessed vertex, and let n_i be the number of processed vertices in G_i at depth d_i . By construction, (d_{i+1}, n_{i+1}) is lexicographically greater than (d_i, n_i) . Thus the sequence $G_1, G_2, \dots$ is finite. Let G_n be the final graph, in which every vertex is processed. By construction, G_n has treewidth at most k and is K_p -free.

Consider a c -colouring of G_n with defect d . We now show that G_n contains a properly c -coloured induced subgraph isomorphic to G . Say v is the unprocessed vertex in G_i chosen by the above process. Use the notation above. Let $y := h(v) \in V(T_i)$. Let $U_1, \dots, U_m$ be the subtrees of T_i rooted at the children $z_1, \dots, z_m$ of y . Let V_j be the set of vertices $w \in V(G_i)$ where $h(w) \in V(U_j)$. Say v is blue. For each $j \in \{1, \dots, m\}$, if each of the $d+1$ copies of V_j in G_{i+1} has a blue neighbour of v , then v has defect $d+1$, which is a contradiction. So some copy of V_j in G_{i+1} has no blue neighbour of v . Say that this copy

of V_j and the corresponding edge of T_{i+1} is ‘chosen’ by v . This chosen edge remains in T_n . Each neighbour of v that is subsequently chosen by this algorithm is not coloured blue. Thus the union of the bags in $(B_x : x \in V(T_n))$ at nodes that are endpoints of the chosen edges in T_n contains a properly c -coloured induced subgraph isomorphic to G . Therefore every graph in $\mathcal{G}$ is properly c -colourable. $\square$

Theorems 1 and 13 imply:

Corollary 14. *Let $\mathcal{G}$ be the class of triangle-free graphs with treewidth at most k . Then the defective chromatic number of $\mathcal{G}$, the clustered chromatic number of $\mathcal{G}$, and the proper chromatic number of $\mathcal{G}$ all equal $\lceil \frac{k+3}{2} \rceil$.*

3.3 Feedback Vertex Number

We now prove two lower bounds on the clustered chromatic number of graphs with given feedback vertex number and given girth.

Proposition 15. *For each integer $d \geq 0$, there is a triangle-free graph G with $\text{fvn}(G) = 1$ that is not 2-colourable with defect d .*

Proof. As illustrated in Figure 1, let T_0 be the $(d+1)$ -leaf star. Let T be obtained from T_0 by adding, for each vertex v in T_0 , a set L_v of $2d+1$ leaves adjacent to v . Let G be obtained from T by adding one vertex α adjacent to all the leaves of T . So $\text{fvn}(G) = 1$ and G is triangle-free (since $N_G(\alpha)$ is an independent set). Suppose that G is 2-colourable with defect d . Say α is blue, and the other colour is red. For each vertex v in T_0 , at most d vertices in L_v are blue (since they are all adjacent to α). So at least $d+1$ vertices in L_v are red, implying v is blue. Thus every vertex in T_0 is blue, which is a contradiction since T_0 has a vertex of degree $d+1$. Hence G is not 2-colourable with defect d . $\square$

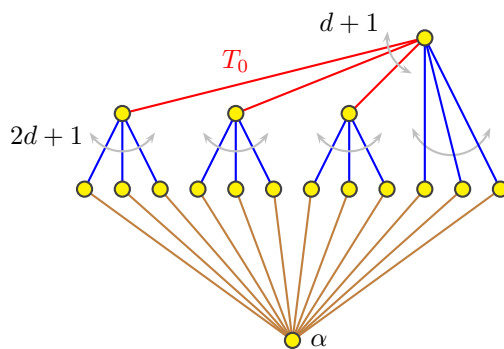


Figure 1: Construction in Proposition 15.

Proposition 16. *For any $c \geq 1$ there is a graph G with girth 6 and $\text{fvn}(G) \leq 7$, and $\text{tw}(G) \leq \text{pw}(G) \leq 8$, such that G has no 2-colouring with clustering c .*

Proof. As illustrated in Figure 2, let V be a set of 7 vertices. Let $n := 4c^2 + 4c + 1$. For each set $S \in \binom{V}{4}$, add a path P_S of length n to G (disjoint from V), where P_S and P_T are disjoint for distinct $S, T \in \binom{V}{4}$. Consider $S \in \binom{V}{4}$. For each vertex x in P_S add an edge to G from x to one vertex $v \in S$, so that for any distinct vertices $x, y \in V(P_S)$, if $xv, yv \in E(G)$ then $\text{dist}_{P_S}(x, y) \geq 4$. This is possible since $|S| = 4$. By construction, $\text{fwn}(G) \leq 7$. In fact, $G - V$ is a forest of paths. Thus $\text{tw}(G) \leq \text{pw}(G) \leq 8$.

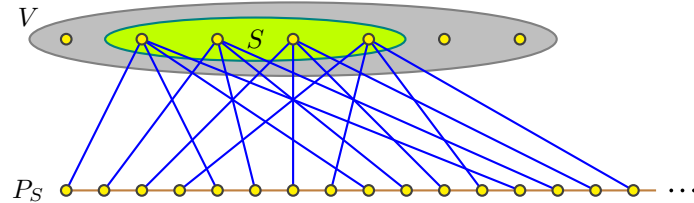


Figure 2: Construction in Proposition 16.

Suppose for the sake of contradiction that G has a cycle C of length at most 5. Since $G - V$ is acyclic, $C \cap V \neq \emptyset$. Since V is an independent set in G , $|C \cap V| \leq 2$. First suppose that $C \cap V = \{v\}$. Say $C = (v, a, b, c, d)$. Thus (a, b, c, d) is a subpath of some P_S , and $\text{dist}_{P_S}(a, d) \leq 3$, which contradicts the construction. Hence $|C \cap V| = 2$, say $C \cap V = \{v, w\}$. Since V is an independent set, v and w are not consecutive in C . Thus v and w have a common neighbour in C , which must be in some P_S , but no vertex in P_S is adjacent to two vertices in V , which is the desired contradiction. Hence G has girth at least 6. In fact, G has girth 6.

Suppose that G has a 2-colouring with clustering c . There is a monochromatic subset S of V of size 4. Say S is white. For each $v \in S$ there are less than c white vertices in P_S adjacent to v . Each vertex in P_S is adjacent to some vertex in S . Thus there are less than $4c$ white vertices in P_S . The black subgraph of P_S has at most $4c$ components. Hence P_S contains a black subpath on $(n - 4c)/4c > c$ vertices. This contradiction shows that G has no 2-colouring with clustering c . $\square$

Note the following concerning the graph G in Proposition 16. Since $G - V$ is a forest of paths, it follows that for any $v, w \in V$, $G - (V \setminus \{v\})$ is outerplanar and so $G - (V \setminus \{v, w\})$ is planar. So G is 5-apex, which shows that (γ, k) -apex graphs with girth at most 6 have clustered chromatic number at least 3 (for $\gamma \geq 0$ and $k \geq 5$).

4 Upper Bounds

4.1 Defective Colouring via Strong Colouring Numbers

Kierstead and Yang [35] introduced the following definition. For a graph G , a total order $\preceq$ of $V(G)$, and $r \in \mathbb{N}$, a vertex $w \in V(G)$ is *r -reachable* from a vertex $v \in V(G)$ if there is a path $v = w_0, w_1, \dots, w_{r'} = w$ of length $r' \in [0, r]$ such that $w \preceq v$ and $v \prec w_i$ for all $i \in [r' - 1]$. For a graph G and integer $r \in \mathbb{N}$, the *r -strong colouring number* $\text{col}_r(G)$ is

the minimum integer such that there is a total order $\preceq$ of $V(G)$ such that at most $\text{col}_r(G)$ vertices are r -reachable from each vertex of G .

An attractive aspect of strong colouring numbers is that they interpolate between degeneracy and treewidth [41]. Indeed, it follows from the definition that $\text{col}_1(G)$ equals the degeneracy of G plus 1, implying $\chi(G) \leq \text{col}_1(G)$. At the other extreme, Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos [24] showed that $\text{col}_r(G) \leq \text{tw}(G) + 1$ for all $r \in \mathbb{N}$, and indeed

$$\lim_{r \rightarrow \infty} \text{col}_r(G) = \text{tw}(G) + 1.$$

Strong colouring numbers provide upper bounds on several graph parameters of interest, including acyclic chromatic number [35], game chromatic number [34, 35], Ramsey numbers [5], oriented chromatic number [40], arrangeability [5], etc. Strong colouring numbers are important also because they characterise bounded expansion classes [77], they characterise nowhere dense classes [24], and have several algorithmic applications such as the constant-factor approximation algorithm for domination number by Dvořák [16], and the almost linear-time model-checking algorithm of Grohe, Kreutzer, and Siebertz [25].

Van den Heuvel et al. [71] proved the following linear bounds on $\text{col}_r(G)$ (always for every $r \in \mathbb{N}$): Every finite planar graph G satisfies $\text{col}_r(G) \leq 5r + 1$. More generally, every finite graph G with Euler genus γ satisfies $\text{col}_r(G) \leq (4\gamma + 5)r + 2\gamma + 1$. Even more generally, for $t \geq 4$, every K_t -minor-free graph G satisfies $\text{col}_r(G) \leq \binom{t-1}{2}(2r + 1)$.

Theorem 17. *Every $K_{s,t}$ -subgraph-free graph is s -colourable with defect at most $\text{col}_2(G) + (t - 1)\binom{\text{col}_2(G)}{s-1}$.*

Proof. Let $\preceq$ be a total ordering of $V(G)$ witnessing $\text{col}_2(G)$. Consider each $v \in V(G)$ in the order given by $\preceq$, and choose $\text{col}(v) \in \{1, \dots, s\}$ distinct from the colour assigned to the $s - 1$ leftmost neighbours of v . Suppose that v has monochromatic degree at least $\text{col}_2(G) + (t - 1)\binom{\text{col}_2(G)}{s-1} + 1$. At most $\text{col}_2(G)$ neighbours of v are to the left of v . Thus there is a set W of $(t - 1)\binom{\text{col}_2(G)}{s-1} + 1$ neighbours w of v with $\text{col}(w) = \text{col}(v)$ and $v \prec w$ for each $w \in W$. For each $w \in W$, by the choice of $\text{col}(w)$, there is a set A_w of $s - 1$ neighbours of w to the left of v . Thus $A := \bigcup_{w \in W} A_w$ is 2-reachable from v . Hence $|A| \leq \text{col}_2(G)$. For each $(s - 1)$ -subset A' of A there are at most $t - 1$ vertices $w \in W$ with $A_w = A'$, as otherwise with v there would be a $K_{s,t}$ -subgraph in G . Hence $|W| \leq (t - 1)\binom{\text{col}_2(G)}{s-1}$, which is the desired contradiction. Thus each vertex v has monochromatic degree at most $\text{col}_2(G) + (t - 1)\binom{\text{col}_2(G)}{s-1}$. Hence G is s -colourable with defect at most $\text{col}_2(G) + (t - 1)\binom{\text{col}_2(G)}{s-1}$. $\square$

Zhu [77] proved that $\nabla(G)$ and $\text{col}_2(G)$ are tied. So Theorem 17 and Theorem 2 are equivalent (up to the defect term). Note that Theorems 2 and 17 both hold in the choosability setting.

We now give an application of Theorem 17 for graphs of given circumference. We need the following tweak to Theorem 17. In a coloured graph G a vertex v is *properly coloured* if $\text{col}(v) \neq \text{col}(w)$ for each edge $vw \in E(G)$; that is, v is the only vertex in the monochromatic component containing v .

Lemma 18. *For any integer $t \geq s \geq 2$, for any $K_{s,t}$ -subgraph-free graph G and vertex $v \in V(G)$, there is an s -colouring of G with defect at most $\text{col}_2(G) + 1 + (t - 1) \binom{\text{col}_2(G) + 1}{s - 1}$, such that v is properly coloured.*

Proof. Let $\preceq$ be the total order of $V(G)$ witnessing $\text{col}_2(G)$. Let $\preceq'$ be the total order of $V(G)$ obtained from $\preceq$ by putting v first. For each vertex $x \in V(G)$, the only vertex that is 2-reachable from x in $\preceq'$ but not in $\preceq$ is v . Thus at most $\text{col}_2(G) + 1$ vertices are 2-reachable from x in $\preceq'$. Apply the colouring procedure from the proof of Theorem 17 to $\preceq'$. So v will be the first vertex coloured. If $vx \in E(G)$ then v will be one of the $s - 1$ leftmost neighbours of x , so x will be assigned a colour distinct from the colour of v . That is, v is properly coloured. The proof for the bound on the defect is analogous to the proof of Theorem 17. $\square$

Theorem 19. *For every graph G with circumference k and girth at least 5, for every vertex v of G , there is a 2-colouring of G with clustering $(4k^2)^{k^2}$ such that v is properly coloured.*

Proof. We proceed by induction on $|V(G)|$. We may assume that G is connected. If G is not 2-connected, then G has a separation (G_1, G_2) with $V(G_1 \cap G_2) = \{w\}$ for some vertex w . Without loss of generality, $v \in V(G_1)$. By induction, there is a 2-colouring of G_1 with clustering c such that v is properly coloured. By induction, there is a 2-colouring of G_2 with clustering c such that w is properly coloured. Permute the colours in G_2 so that w is assigned the same colour in G_1 and G_2 . We obtain a 2-colouring of G with clustering c such that v is properly coloured.

Now assume that G is 2-connected. So G has finite girth. As mentioned above, $\text{col}_2(G) \leq \text{tw}(G) + 1 \leq \text{circ}(G) \leq k$. By Lemma 18 with $s = t = 2$, G is 2-colourable with defect $\text{col}_2(G) + 1 + (t - 1) \binom{\text{col}_2(G) + 1}{s - 1} \leq 2k + 2$ such that v is properly coloured. Dirac [12] showed that every path in G has length less than k^2 (using 2-connectivity). (In fact, Briński et al. [4] showed that G has treedepth at most k .) Hence each monochromatic component has less than $(4k^2)^{k^2}$ vertices. $\square$

4.2 Feedback Vertex Number

We now prove upper bounds on the clustered chromatic number of graphs with given feedback vertex number and given girth.

Proposition 20. *Every graph G with $\text{fvn}(G) \leq 1$ and girth at least 5 is 2-colourable with clustering 2.*

Proof. Let v be a vertex of G such that $G - v$ is a forest. Suppose that some leaf ℓ of $G - v$ is not adjacent to v . So $\deg_G(\ell) \leq 1$. By induction, $G - \ell$ is 2-colourable with clustering 2. Assign ℓ a colour different from its neighbour. Now G is 2-coloured with clustering 2. Now assume that every leaf of $G - v$ is adjacent to v . Fix a root vertex r of $G - v$. Let ℓ be a vertex in the same component of $G - v$ as r at maximum distance from r . So ℓ is a leaf in $G - v$. Let p be the parent of ℓ in $G - v$. So $pv \notin E(G)$ as otherwise (p, v, ℓ) would

be a 3-cycle in G . If p has another child ℓ' (in addition to ℓ), then ℓ' is a leaf by the choice of ℓ , implying that (ℓ, p, ℓ', v) is a 4-cycle in G . Hence ℓ is the only child of p , and p has degree at most 2 in G . By induction, $G - \ell - p$ is 2-colourable with clustering 2. Assign ℓ a colour distinct from the colour assigned to v , and assign p a colour distinct from the colour of the neighbour of p other than ℓ . Every monochromatic component created is contained in $\{\ell, p\}$. Thus the clustering is at most 2. $\square$

Theorem 21. *Every graph G with $\text{fvn}(G) \leq k$ and girth at least 7 is 2-colourable with clustering $\binom{k}{2} + k + 1$.*

Proof. Let A be a set of at most k vertices in G such that $G - A$ is acyclic. Colour each vertex in A white. Let $N := N_G(A)$. Colour each vertex in N black. Thus each monochromatic component intersecting A is contained in A . Let $E := E(G[N])$. For each edge $vw \in E$ there is a pair $\{a, b\} \in \binom{A}{2}$ such that $va, wb \in E(G)$; charge vw to $\{a, b\}$. If distinct edges $v_1w_1, v_2w_2 \in E$ are charged to the same pair $\{a, b\} \in \binom{A}{2}$, then $G[\{v_1, w_1, v_2, w_2, a, b\}]$ contains a cycle, contradicting that G has girth at least 7. Thus no two distinct edges in E are charged to the same pair in $\binom{A}{2}$. Hence $|E| \leq \binom{|A|}{2}$.

By assumption, $G - A$ is a forest. Root each component of $G - A$. Consider each component C of $G - A - N$. Let r_C be the root of C . Colour each vertex of C at even distance from r_C in C white. Colour each vertex of C at odd distance from r_C in C black. In particular, r_C is white. Each monochromatic component of C is a single vertex. The parent of r_C in $G - A$ (if it exists) is in N , and is coloured black, which is different from the colour of r_C . No vertex in C is adjacent to a vertex in A . Thus the white vertices in C are properly coloured.

Let B be a black component of G . By construction, B is a subgraph of $G - A$. So B is a tree. Suppose that $|V(B) \setminus N| \geq 2$. Let x, y be distinct vertices in $V(B) \setminus N$ at minimum distance in B . Since $G - A - N$ is properly coloured, x and y are not adjacent. The xy -path in B must use the parent of x or the parent of y (possibly both). Without loss of generality, the parent p of x is in B . By the choice of x and y , $p \in N$. Thus x is the root of a component of $G - A - N$, implying that x is white. This contradiction shows that $|V(B) \setminus N| \leq 1$. Say $V(B) \setminus N = \{x\}$. If $\deg_B(x) \geq |A| + 1$ then there are two neighbours of x both adjacent to the same vertex in A , implying that G has a 4-cycle. Thus $\deg_B(x) \leq |A|$. Hence there at most $|A|$ edges in $B - N$. As shown above, $B[N]$ has at most $|E| \leq \binom{|A|}{2}$ edges. Thus B has at most $\binom{|A|}{2} + |A|$ edges. Since B is a tree, B has at most $\binom{|A|}{2} + |A| + 1$ vertices. Thus this 2-colouring of G has clustering at most $\binom{|A|}{2} + |A| + 1 \leq \binom{k}{2} + k + 1$. $\square$

4.3 Apex Graphs

Here we prove clustered colouring results for (γ, k) -apex graphs with given girth.

Theorem 22. *Every (γ, k) -apex graph with girth at least 9 is 2-colourable with clustering at most some function $f(\gamma, k)$.*

Proof. Let G be an n -vertex (γ, k) -apex graph, where $n \geq n_0 := 14(\frac{9}{7}\gamma + (k+2)\binom{k}{2})$. By Lemma 23 below, $|E(G)| \leq \frac{25}{14}n + \frac{9}{7}\gamma + (k+2)\binom{k}{2} \leq \frac{13}{7}n$. By a result of Alon et al. [1], G has a balanced separator of size $cn^{1/2}$ for some $c = c(\gamma, k)$. The result now follows from Lemma 5 with $\varepsilon = \frac{1}{2}$ and $\delta = \frac{1}{7}$ and $k = 2$. $\square$

Lemma 23. *For all integers $\gamma, k \geq 0$ every n -vertex (γ, k) -apex graph with girth at least 9 has at most $\frac{25}{14}n + \frac{9}{7}\gamma + (k+2)\binom{k}{2}$ edges.*

Proof. Let G be an n -vertex (γ, k) -apex graph with girth at least 9. Thus there exists $A \subseteq V(G)$ with $|A| \leq k$ such that $G - A$ has Euler genus at most γ . By Euler's formula, $G - A$ has at most $\frac{9}{7}(n - |A| + \gamma - 2)$ edges. Let N be the set of vertices in $G - A$ adjacent to at least one vertex in A . Let X be the set of vertices in N adjacent to at least two vertices in A . Thus $|X| \leq \binom{k}{2}$, as otherwise there exists two vertices in X adjacent to the same two vertices in A , implying that G contains a 4-cycle. Let Y be the set of vertices in $N \setminus X$ that are at distance at most 2 in $G - A$ from another vertex in N . For each $v \in Y$ there are vertices $a, b \in A$ and there are edges $va, wb \in E(G)$ for some vertex $w \in N$ where $\text{dist}_{G-A}(v, w) \leq 2$. In this case, charge $v \in Y$ to the set $\{a, b\}$. No two vertices in Y are charged to the same set $\{a, b\} \in \binom{A}{2}$, as otherwise G contains a cycle of length at most 8. Thus $|Y| \leq \binom{k}{2}$. Let $Z := N \setminus (X \cup Y)$. Every pair of vertices in Z are at distance at least 3 in $G - A$. For each vertex $v \in Z$, if v is isolated in $G - A$ then $\deg_G(v) \leq 1$, and by induction,

$$|E(G)| \leq |E(G - v)| + 1 \leq \frac{25}{14}(n - 1) + \frac{9}{7}\gamma + (k+1)\binom{k}{2} + 1 \leq \frac{25}{14}n + \frac{9}{7}\gamma + (k+1)\binom{k}{2}.$$

Now assume that no vertex in Z is isolated in $G - A$. Thus $|N_{G-A}[v]| \geq 2$ for each vertex $v \in Z$, and $N_{G-A}[v] \cap N_{G-A}[w] = \emptyset$ for all distinct $v, w \in Z$. Hence $|Z| \leq \frac{1}{2}(n - |A|)$. Now,

$$\begin{aligned} |E(G)| &\leq |E(G - A)| + |E(G[A])| + k|X| + |Y| + |Z| \\ &\leq \frac{9}{7}(n - |A| + \gamma - 2) + \binom{k}{2} + (k+1)\binom{k}{2} + \frac{1}{2}(n - |A|) \\ &\leq \frac{25}{14}n + \frac{9}{7}\gamma + (k+2)\binom{k}{2}, \end{aligned}$$

as desired. $\square$

The next result shows that Lemma 5 cannot prove 2-colourability for 1-apex graphs with girth 5.

Proposition 24. *For infinitely many values of n there is an n -vertex 1-apex graph with girth 5 and with $(2 - o(1))n$ edges.*

Proof. Let G be an n -vertex planar square grid graph with vertices labelled by their coordinates (x, y) . Let G' be the graph obtained from G by subdividing each edge $(x, y)(x, y + 1)$ where $x + y$ is even once. Each face of G' has size 5, and any two division vertices are at distance at least 3 in G' . Let G'' be obtained from G' by adding a vertex v

adjacent to every division vertex in G' . By construction, G'' is a 1-apex graph. Moreover, G'' has girth 5, since any two division vertices are at distance at least 3 in G' . Note that G has $(2 - o(1))n$ edges and $(1 - o(1))n$ faces. Thus G' has n original vertices, $(\frac{1}{2} - o(1))n$ division vertices, and $(\frac{5}{2} - o(1))n$ edges. Hence G'' has $(\frac{3}{2} - o(1))n$ vertices and $(3 - o(1))n$ edges. The result follows. $\square$

The next result shows that Lemma 5 cannot prove 2-colourability for k -apex graphs with girth 8.

Proposition 25. *For infinitely many values of n there is an n -vertex 23-apex graph with girth 8 and with $(2 - o(1))n$ edges.*

Proof. Let G be an n -vertex planar square grid graph. Let G' be the graph obtained from G by subdividing each edge once. Each division vertex of G' is at distance at most 5 from at most 22 other division vertices. Thus each division vertex of G' can be coloured by one element of $\{1, \dots, 23\}$ so that any two vertices of the same colour are at distance at least 6 in G' . Let G'' be obtained from G' by adding a set of vertices $A := \{v_1, \dots, v_{23}\}$ where each v_i is adjacent to every vertex in G' coloured i . By construction, G'' is a 23-apex graph. We claim it has girth 8. clearly, every cycle in G has length at least 8. Consider a cycle C that intersects A . If $C \cap A = \{v_i\}$, then $|C| \geq 8$ since vertices of G' coloured i are at distance at least 6 in G' . If $C \cap A = \{v_i, v_j\}$, then $|C| \geq 8$ since vertices of G' coloured i and j are at distance at least 2 in G' . Similarly, if $|C \cap A| \geq 3$ then $|C| \geq 12$. Hence G has girth 8. Note that G has $(2 - o(1))n$ edges. Thus G' has n original vertices, $(2 - o(1))n$ division vertices, and $(4 - o(1))n$ edges. Hence G'' has $(3 - o(1))n$ vertices and $(6 - o(1))n$ edges. The result follows. $\square$

Theorem 26. *Every (γ, k) -apex graph with girth at least 5 is 3-colourable with clustering at most some function $f(\gamma, k)$.*

Proof. Let G be an n -vertex (γ, k) -apex graph, where $n \geq n_0 := 6(\frac{5}{3}\gamma + (k+1)\binom{k}{2})$. By Lemma 27 below, $|E(G)| \leq \frac{8}{3}n + \frac{5}{3}\gamma + (k+1)\binom{k}{2} \leq \frac{17}{6}n$. By a result of Alon et al. [1], G has a balanced separator of size $cn^{1/2}$ for some $c = c(\gamma, k)$. The result now follows from Lemma 5 with $\varepsilon = \frac{1}{2}$ and $\delta = \frac{1}{6}$ and $k = 3$. $\square$

Lemma 27. *Every n -vertex (γ, k) -apex graph G with girth at least 5 has at most $\frac{8}{3}n + \frac{5}{3}\gamma + (k+1)\binom{k}{2}$ edges.*

Proof. By assumption, there exists $A \subseteq V(G)$ with $|A| \leq k$ such that $G - A$ has Euler genus at most γ . By Euler's formula, $G - A$ has at most $\frac{5}{3}(n - |A| + \gamma - 2)$ edges. Let N be the set of vertices in $G - A$ adjacent to at least one vertex in A . Let X be the set of vertices in N adjacent to at least two vertices in A . Thus $|X| \leq \binom{k}{2}$, as otherwise there exists two vertices in X adjacent to the same two vertices in A , implying that G contains a 4-cycle. Now,

$$\begin{aligned} |E(G)| &\leq |E(G - A)| + |E(G[A])| + k|X| + |N| \leq \frac{5}{3}(n - |A| + \gamma - 2) + \binom{k}{2} + k\binom{k}{2} + n \\ &\leq \frac{8}{3}n + \frac{5}{3}\gamma + (k+1)\binom{k}{2}, \end{aligned}$$

as desired. $\square$

5 Increasing Girth

Thomassen [68] first showed that graphs with minimum degree 3 and sufficiently large girth contains a minor with large minimum degree. In particular, every graph with minimum degree at least 3 and girth at least $4k - 3$ contains a minor with minimum degree k . Mader [52] improved the bounds substantially, by showing that every graph with minimum degree at least 3 and girth at least $8k + 3$ contains a minor with minimum degree 2^k . Kühn and Osthus [42] improved the constant 8 to 4 as follows:

Lemma 28 ([42]). *For any integers $k \geq 1$ and $r \geq 3$ every graph G of minimum degree r and girth at least $4k + 3$ contains a minor of minimum degree at least $\frac{1}{48}(r - 1)^{k+1}$.*

We use Lemma 28 and the following lemma independently due to Kostochka [37, 38] and Thomason [66, 67] to show that graphs excluding a fixed minor and with large girth are properly 3-colourable.

Lemma 29 ([37, 38, 66, 67]). *Every K_t -minor-free graph has minimum degree at most $ct\sqrt{\log t}$ for some absolute constant c .*

Proposition 30. *Every K_t -minor-free graph G with girth at least $(4 + o(1))\log_2 t$ is 3-colourable.*

Proof. The precise girth bound we assume is $4k + 3$ where $k := \lfloor \log_2(48ct\sqrt{\log t}) \rfloor$ and c is from Lemma 29. We claim that G is 2-degenerate. Otherwise, G has a subgraph with minimum degree at least 3. By Lemma 28, we have $\frac{1}{48}2^{k+1} \leq ct\sqrt{\log t}$, implying $k \leq \log_2(48ct\sqrt{\log t}) - 1$, which contradicts the definition of k . Hence G is 2-degenerate and properly 3-colourable. $\square$

Of course, three colours is best possible in Proposition 30 since odd cycles are K_4 -minor-free with arbitrarily large girth.

Kühn and Osthus [42] used Lemma 28 and a variant of Lemma 29 with explicit bounds on the constant c to conclude that K_t -minor-free graphs with girth at least 19 are properly $(t - 1)$ -colourable, thus satisfying Hadwiger's Conjecture. In fact, the method of Kühn and Osthus [42] gives asymptotically stronger results for graphs of girth $g \geq 5$. To see this, we use the following lemma of Kühn and Osthus [42], which is applicable for girth at least 5 whereas Lemma 28 assumes girth at least 7.

Lemma 31 ([42]). *For any integers $k \geq 1$ and $r \geq \max\{5k, 2\,000\,000\}$, every graph G of minimum degree at least $4r$ and girth at least $4k + 1$ contains a minor of minimum degree at least $\frac{1}{288}r^{(2k+1)/2}$.*

Proposition 32. *For any integer $k \geq 1$, every K_t -minor-free graph G with girth at least $4k + 1$ is properly $O((t\sqrt{\log t})^{2/(2k+1)})$ -colourable.*

Proof. We claim that G is $4r$ -degenerate where

$$r := \max\{\lfloor (288ct\sqrt{\log t})^{2/(2k+1)} \rfloor + 1, 5k, 2\,000\,000\}$$

and c is from Lemma 29. Suppose that a subgraph of G has minimum degree at least $4r$. By Lemmas 29 and 31, we have $\frac{1}{288}r^{(2k+1)/2} \leq ct\sqrt{\log t}$, implying $r \leq (288ct\sqrt{\log t})^{2/(2k+1)}$, which contradicts the definition of r . Hence G is $4r$ -degenerate and $\chi(G) \leq 4r + 1 \in O((t\sqrt{\log t})^{2/(2k+1)})$. $\square$

For example, Proposition 32 shows $\chi(G) \leq O((t\sqrt{\log t})^{2/3})$ for girth 5, and $\chi(G) \leq O((t\sqrt{\log t})^{2/5})$ for girth 9.

For clustered colouring, we show that two colours suffice if we allow the smallest possible amount of clustering.

Theorem 33. *Every K_t -minor-free graph with girth at least $(16+o(1))\log_2 t$ is 2-colourable with clustering 2.*

Proof. The precise girth lower bound we use is $16\log_2(ct\sqrt{\log t}) + 11$, where c is from Lemma 29. We proceed by induction on $|V(G)|$. First suppose that $\deg_G(v) \leq 1$ for some vertex v . By induction, $G - v$ is 2-colourable with clustering at most 2. Colour v by a colour not used by its neighbour. So G is 2-coloured with clustering at most 2. Now assume that G has minimum degree at least 2. Suppose that G has an edge vw with $\deg_G(v) = \deg_G(w) = 2$. By induction, $G - v - w$ is 2-colourable with clustering at most 2. Colour v by a colour not used by its neighbour different from w . Colour w by a colour not used by its neighbour different from v . So the monochromatic component containing v or w has at most two vertices (namely, v and w). Now G is 2-coloured with clustering at most 2. Now assume that for each edge vw of G , at least one of v and w has degree at least 3. Let g be the integer such that G has girth in $\{8g + 3, \dots, 8g + 10\}$. By Lemma 34 below, there is a minor G' of G with minimum degree at least $2^{g/2}$. By Lemma 29, we have $2^{g/2} \leq ct\sqrt{\log t}$. Hence G has girth at most $8g + 10 \leq 10 + 16\log_2(ct\sqrt{\log t})$, which contradicts the starting assumption. $\square$

It remains to prove the following lemma, which is a variant of the above-mentioned result of Mader [52].

Lemma 34. *Let G be a graph with:*

- *girth at least $8k + 3$ for some integer $k \geq 1$,*
- *minimum degree at least 2, and*
- *no edge vw with $\deg_G(v) = \deg_G(w) = 2$.*

Then G has a minor with minimum degree at least $2^{k/2}$.

Proof. Let X be a maximal set of vertices in G at pairwise distance at least $2k + 1$. For each $x \in X$, let B_x be the ball of radius k around x . So $B_x \cap B_y = \emptyset$ for all distinct $x, y \in X$. Since G has girth greater than $2k + 1$, the induced subgraph $G[B_x]$ is a tree. By assumption, G has minimum degree at least 2, and no edge vw in G has $\deg_G(v) = \deg_G(w) = 2$. Thus $G[B_x]$ has at least $2^{k/2}$ leaves. Let $V_i := \{v \in V(G) : \text{dist}_G(v, X) = i\}$. By the maximality of X , we have $V(G) = \bigcup_{i=0}^{2k} V_i$. For each $x \in X$, let B'_x be a superset of B_x obtained as follows. For $i = k + 1, k + 2, \dots, 2k$ each vertex $v \in V_i$ has a neighbour in $V_{i-1} \cap B'_x$ for some $x \in X$; add v to B'_x . Thus B'_x is a subset of the ball of radius $2k$ around x . Since G has girth greater than $4k + 1$, the induced subgraph $G[B'_x]$ is a (connected) tree. By

construction, $\bigcup_{x \in X} B'_x = V(G)$. Moreover, the number of edges between B'_x and $\bigcup_{y \neq x} B'_y$ is at least the number of leaves in $G[B_x]$ which is at least $2^{k/2}$. For all distinct $x, y \in X$ there is at most one edge between B'_x and B'_y , since otherwise G would contain a cycle of length at most $8k + 2$. Let G' be obtained from G by contracting each set B'_x to a single vertex. Thus G' has minimum degree at least $2^{k/2}$. $\square$

6 Open Problems

Closing the gaps in the lower and upper bounds in Tables 3 and 4 are natural open problems that arise from this work. In particular:

(1) For the class of triangle-free graphs with treedepth k , Theorem 9 says that the defective chromatic number, clustered chromatic number and chromatic number are equal. But what is this value? The best known lower bound is $(1 - o(1))\sqrt{2k}$ (see Theorem 10). Theorem 1 provides an upper bound of $\lceil \frac{k+1}{2} \rceil$. A $o(k)$ upper bound can be obtained as follows. Dvořák and Kawarabayashi [18] showed that the maximum chromatic number of triangle-free graphs with treewidth k equals the maximum online chromatic number of triangle-free graphs with pathwidth k . An analogous approach shows that the maximum chromatic number of triangle-free graphs of treedepth k equals the maximum online chromatic number of triangle-free graphs on k vertices. This result holds with the triangle-free condition replaced by K_p -free for any fixed p . Results of Lovász, Saks, and Trotter [51] imply⁴ that the online chromatic number of triangle-free graphs on k vertices is in $O(k/\log \log k)$. Thus, triangle-free graphs of treedepth k are $O(k/\log \log k)$ -colourable. As far as we are aware, it is possible that triangle-free graphs with treedepth k are properly $O(\sqrt{k})$ -colourable. This would strengthen the fact that triangle-free k -vertex graphs are properly $O(\sqrt{k})$ -colourable, which is easily proved as a consequence of Brooks' Theorem.

(2) Are triangle-free (γ, k) -apex graphs 3-colourable with bounded clustering? Note that $K_{k+2,n}$ is a triangle-free k -apex graph with $(k + 2)n$ edges. So Lemma 5 does not help here.

(3) Are graphs with girth at least some constant g (possibly 7) and with bounded treewidth (or bounded pathwidth) 2-colourable with bounded clustering? The answer is 'yes' for graphs G with treewidth 2 and girth at least 5, since such graphs are planar, so $|E(G)| < \frac{5}{3}|V(G)|$ by Euler's Formula, and the claim follows from Lemma 5. Perhaps Lemma 5 can be used to answer the treewidth 3 case. But for larger values of treewidth, it seems a more tailored approach is needed. Note that $g \geq 7$ by Proposition 16.

(4) Are K_t -minor-free graphs with girth at least some constant g (possibly 5) 3-colourable with bounded clustering? Note that $g \geq 5$ by Corollary 11. In fact, two colours may suffice. Are K_t -minor-free graphs with girth at least some constant g (possibly 7) 2-colourable with bounded clustering? Here $g \geq 7$ by Proposition 16.

⁴Lovász et al. [51] describe an online algorithm that partitions an n -vertex graph into $\lceil n/\log_2 \log_2 n \rceil$ independent sets and at most $4n/\log_2 \log_2 n$ 'residual' sets, where each residual set is a subset of the neighbourhood of a vertex. So for triangle-free graphs, each residual set is independent. Hence the online chromatic number of n -vertex triangle-free graphs is in $O(n/\log \log n)$. See [27] for related work.

(5) What is the clustered chromatic number of the class of triangle-free K_t -minor-free graphs. Without the triangle-free assumption, the answer is $t - 1$, as proved by Dujmović et al. [15] (and announced earlier by Dvořák and Norin [17]). The best lower bound is $\lceil \frac{t+1}{2} \rceil$, which follows from Corollary 14 since graphs of treewidth k are K_{k+2} -minor-free.

Note that for proper colourings of K_t -minor-free graphs, improved bounds are known when we additionally exclude a fixed triangle or fixed complete graph. In particular, Delcourt and Postle [10] proved a $O(t)$ upper bound for K_t -minor-free graphs excluding any fixed complete subgraph.

References

- [1] NOGA ALON, PAUL SEYMOUR, AND ROBIN THOMAS. [A separator theorem for nonplanar graphs](#). *J. Amer. Math. Soc.*, 3(4):801–808, 1990.
- [2] DAN ARCHDEACON. [A note on defective colorings of graphs in surfaces](#). *J. Graph Theory*, 11(4):517–519, 1987.
- [3] ETIENNE BIRMELÉ. [Tree-width and circumference of graphs](#). *J. Graph Theory*, 43(1):24–25, 2003.
- [4] MARCIN BRIĄŃSKI, GWENAËL JORET, KONRAD MAJEWSKI, PIOTR MICEK, MICHAŁ T. SEWERYN, AND ROOHANI SHARMA. [Treedepth vs circumference](#). *Combinatorica*, 43:659–664, 2023. arXiv:2211.11410.
- [5] GUANTAO CHEN AND RICHARD H. SCHELP. [Graphs with linearly bounded Ramsey numbers](#). *J. Combin. Theory Ser. B*, 57(1):138–149, 1993.
- [6] HOJIN CHOI, ILKYOO CHOI, JISU JEONG, AND GEEWON SUH. [\$\(1, k\)\$ -coloring of graphs with girth at least five on a surface](#). *J. Graph Theory*, 84(4):521–535, 2017.
- [7] ILKYOO CHOI AND LOUIS ESPERET. [Improper coloring of graphs on surfaces](#). *J. Graph Theory*, 91(1):16–34, 2019.
- [8] LENORE COWEN, WAYNE GODDARD, AND C. ESTHER JESURUM. [Defective coloring revisited](#). *J. Graph Theory*, 24(3):205–219, 1997.
- [9] LENORE J. COWEN, ROBERT COWEN, AND DOUGLAS R. WOODALL. [Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency](#). *J. Graph Theory*, 10(2):187–195, 1986.
- [10] MICHELLE DELCOURT AND LUKE POSTLE. [Reducing linear Hadwiger’s conjecture to coloring small graphs](#). *J. Amer. Math. Soc.*, 38(2):481–507, 2025.
- [11] REINHARD DIESTEL. Graph theory, vol. 173 of *Graduate Texts in Mathematics*. Springer, 5th edn., 2018.
- [12] GABRIEL A. DIRAC. [Some theorems on abstract graphs](#). *Proc. London Math. Soc.* (3), 2:69–81, 1952.
- [13] HRISTO N. DJIDJEV. A separator theorem for graphs of fixed genus. *Serdica*, 11(4):319–329, 1985.
- [14] VIDA DUJMOVIĆ, LOUIS ESPERET, PAT MORIN, BARTOSZ WALCZAK, AND DAVID R. WOOD. [Clustered 3-colouring graphs of bounded degree](#). *Combin. Probab. Comput.*, 31(1):123–135, 2022.

- [15] VIDA DUJMOVIĆ, LOUIS ESPERET, PAT MORIN, AND DAVID R. WOOD. [Proof of the clustered Hadwiger conjecture](#). In *Proc. 64th IEEE Annual Symp. Foundations of Comput. Sci. (FOCS '23)*, pp. 1921–1930. 2023. arXiv:2306.06224.
- [16] ZDENĚK DVOŘÁK. [Constant-factor approximation of the domination number in sparse graphs](#). *European J. Comb.*, 34(5):833–840, 2013.
- [17] ZDENĚK DVOŘÁK AND SERGEY NORIN. [Islands in minor-closed classes. I. Bounded treewidth and separators](#). 2017, arXiv:1710.02727.
- [18] ZDENĚK DVOŘÁK AND KEN-ICHI KAWARABAYASHI. [Triangle-free graphs of tree-width \$t\$ are \$\lceil \(t+3\)/2 \rceil\$ -colorable](#). *European J. Combin.*, 66:95–100, 2017.
- [19] NANCY EATON AND THOMAS HULL. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.
- [20] KATHERINE EDWARDS, DONG YEAP KANG, JAEHOON KIM, SANG-IL OUM, AND PAUL SEYMOUR. [A relative of Hadwiger’s conjecture](#). *SIAM J. Discrete Math.*, 29(4):2385–2388, 2015.
- [21] PAUL ERDŐS. [Graph theory and probability](#). *Canad. J. Math.*, 11:34–38, 1959.
- [22] LOUIS ESPERET AND PASCAL OCHEM. [Islands in graphs on surfaces](#). *SIAM J. Discrete Math.*, 30(1):206–219, 2016.
- [23] JOHN R. GILBERT, JOAN P. HUTCHINSON, AND ROBERT E. TARJAN. [A separator theorem for graphs of bounded genus](#). *J. Algorithms*, 5(3):391–407, 1984.
- [24] MARTIN GROHE, STEPHAN KREUTZER, ROMAN RABINOVICH, SEBASTIAN SIEBERTZ, AND KONSTANTINOS STAVROPOULOS. [Coloring and covering nowhere dense graphs](#). *SIAM J. Discrete Math.*, 32(4):2467–2481, 2018.
- [25] MARTIN GROHE, STEPHAN KREUTZER, AND SEBASTIAN SIEBERTZ. [Deciding first-order properties of nowhere dense graphs](#). *J. ACM*, 64(3):Art. 17, 2017.
- [26] HERBERT GRÖTZSCH. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 8:109–120, 1958/59.
- [27] GRZEGORZ GUTOWSKI, JAKUB KOZIK, PIOTR MICEK, AND XUDING ZHU. [Lower bounds for on-line graphs colorings](#). In *Proc. Int’l Symp. Algorithms and Computation (ISAAC 2014)*, vol. 8889 of *Lecture Notes in Comput. Sci.*, pp. 507–515. Springer, 2014.
- [28] HUGO HADWIGER. [Über eine Klassifikation der Streckenkomplexe](#). *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.
- [29] FRÉDÉRIC HAVET AND JEAN-SÉBASTIEN SERENI. [Improper choosability of graphs and maximum average degree](#). *J. Graph Theory*, 52(3):181–199, 2006.
- [30] DONG YEAP KANG AND SANG-IL OUM. [Improper coloring of graphs with no odd clique minor](#). *Combin. Probab. Comput.*, 28(5):740–754, 2019.
- [31] KEN-ICHI KAWARABAYASHI. [A weakening of the odd Hadwiger’s conjecture](#). *Combin. Probab. Comput.*, 17(6):815–821, 2008.
- [32] KEN-ICHI KAWARABAYASHI AND BOJAN MOHAR. [A relaxed Hadwiger’s conjecture for list colorings](#). *J. Combin. Theory Ser. B*, 97(4):647–651, 2007.

- [33] KEN-ICHI KAWARABAYASHI AND CARSTEN THOMASSEN. [From the plane to higher surfaces](#). *J. Combin. Theory Ser. B*, 102(4):852–868, 2012.
- [34] HAL A. KIERSTEAD AND WILLIAM T. TROTTER. [Planar graph coloring with an uncooperative partner](#). *J. Graph Theory*, 18(6):569–584, 1994.
- [35] HAL A. KIERSTEAD AND DAQING YANG. [Orderings on graphs and game coloring number](#). *Order*, 20(3):255–264, 2003.
- [36] JEONG HAN KIM. [The Ramsey number \$R\(3, t\)\$ has order of magnitude \$t^2/\log t\$](#) . *Random Structures Algorithms*, 7(3):173–207, 1995.
- [37] ALEXANDR V. KOSTOCHKA. [The minimum Hadwiger number for graphs with a given mean degree of vertices](#). *Metody Diskret. Analiz.*, 38:37–58, 1982.
- [38] ALEXANDR V. KOSTOCHKA. [Lower bound of the Hadwiger number of graphs by their average degree](#). *Combinatorica*, 4(4):307–316, 1984.
- [39] ALEXANDR V. KOSTOCHKA AND JAROSLAV NEŠETŘIL. [Properties of Descartes’ construction of triangle-free graphs with high chromatic number](#). *Combin. Probab. Comput.*, 8(5):467–472, 1999.
- [40] ALEXANDR V. KOSTOCHKA, ÉRIC SOPENA, AND XUDING ZHU. [Acyclic and oriented chromatic numbers of graphs](#). *J. Graph Theory*, 24(4):331–340, 1997.
- [41] STEPHAN KREUTZER, MICHAL PILIPCZUK, ROMAN RABINOVICH, AND SEBASTIAN SIEBERTZ. [The generalised colouring numbers on classes of bounded expansion](#). In PIOTR FALISZEWSKI, ANCA MUSCHOLL, AND ROLF NIEDERMEIER, eds., *Proc. 41st Int’l Symp. Math’l Foundations of Comput. Sci.* (MFCS ’16), vol. 58 of *LIPIcs*, pp. 85:1–85:13. Schloss Dagstuhl, 2016.
- [42] DANIELA KÜHN AND DERYK OSTHUS. [Minors in graphs of large girth](#). *Random Structures Algorithms*, 22(2):213–225, 2003.
- [43] RICHARD J. LIPTON AND ROBERT E. TARJAN. [A separator theorem for planar graphs](#). *SIAM J. Appl. Math.*, 36(2):177–189, 1979.
- [44] RICHARD J. LIPTON AND ROBERT E. TARJAN. [Applications of a planar separator theorem](#). *SIAM J. Comput.*, 9(3):615–627, 1980.
- [45] CHUN-HUNG LIU. [Defective coloring is perfect for minors](#). *Combinatorica*, 44:467–507, 2024.
- [46] CHUN-HUNG LIU AND SANG-IL OUM. [Partitioning \$H\$ -minor free graphs into three subgraphs with no large components](#). *J. Combin. Theory Ser. B*, 128:114–133, 2018.
- [47] CHUN-HUNG LIU AND DAVID R. WOOD. [Clustered coloring of graphs excluding a subgraph and a minor](#). 2019, arXiv:1905.09495.
- [48] CHUN-HUNG LIU AND DAVID R. WOOD. [Clustered graph coloring and layered treewidth](#). 2019, arXiv:1905.08969.
- [49] CHUN-HUNG LIU AND DAVID R. WOOD. [Clustered variants of Hajós’ conjecture](#). *J. Combin. Theory Ser. B*, 152:27–54, 2022.
- [50] CHUN-HUNG LIU AND DAVID R. WOOD. [Clustered coloring of graphs with bounded layered treewidth and bounded degree](#). *European J. Combin.*, p. 103730, 2023.

- [51] LÁSZLÓ LOVÁSZ, MICHAEL E. SAKS, AND WILLIAM T. TROTTER. [An on-line graph coloring algorithm with sublinear performance ratio](#). *Discrete Math.*, 75(1-3):319–325, 1989.
- [52] WOLFGANG MADER. [Topological subgraphs in graphs of large girth](#). *Combinatorica*, 18(3):405–412, 1998.
- [53] EMILY ABERNETHY MARSHALL AND DAVID R. WOOD. [Circumference and path-width of highly connected graphs](#). *J. Graph Theory*, 79(3):222–232, 2015.
- [54] BOJAN MOHAR, BRUCE REED, AND DAVID R. WOOD. [Colourings with bounded monochromatic components in graphs of given circumference](#). *Australas. J. Combin.*, 69(2):236–242, 2017.
- [55] BOJAN MOHAR AND CARSTEN THOMASSEN. *Graphs on surfaces*. Johns Hopkins University Press, 2001.
- [56] JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ. [Sparsity](#), vol. 28 of *Algorithms and Combinatorics*. Springer, 2012.
- [57] SERGEY NORIN. [Conquering graphs of bounded treewidth](#). 2015. Unpublished manuscript.
- [58] SERGEY NORIN, ALEX SCOTT, PAUL SEYMOUR, AND DAVID R. WOOD. [Clustered colouring in minor-closed classes](#). *Combinatorica*, 39(6):1387–1412, 2019.
- [59] SERGEY NORIN, ALEX SCOTT, AND DAVID R. WOOD. [Clustered colouring of graph classes with bounded treedepth or pathwidth](#). *Combin. Probab. Comput.*, 32:122–133, 2023.
- [60] PATRICE OSSONA DE MENDEZ, SANG-IL OUM, AND DAVID R. WOOD. [Defective colouring of graphs excluding a subgraph or minor](#). *Combinatorica*, 39(2):377–410, 2019.
- [61] ROGER K. S. POH. [On the linear vertex-arboricity of a planar graph](#). *J. Graph Theory*, 14(1):73–75, 1990.
- [62] NEIL ROBERTSON, DANIEL P. SANDERS, PAUL SEYMOUR, AND ROBIN THOMAS. [The four-colour theorem](#). *J. Combin. Theory Ser. B*, 70(1):2–44, 1997.
- [63] RISTE ŠKREKOVSKI. [A Grötzsch-type theorem for list colourings with impropriety one](#). *Combin. Probab. Comput.*, 8(5):493–507, 1999.
- [64] RICHARD STEINBERG AND D. H. YOUNGER. Grötzsch’s theorem for the projective plane. *Ars Combin.*, 28:15–31, 1989.
- [65] ROBIN THOMAS. [\$K_t\$ minors in large \$t\$ -connected graphs](#). In JIŘÍ FIALA, JAN KRATOCHVÍL, AND MIRKA MILLER, eds., *Proc. 20th Int’l Workshop on Combinatorial Algorithms (IWOC 2009)*, vol. 5874 of *Lecture Notes in Comput. Sci.*, p. 19. Springer, 2009.
- [66] ANDREW THOMASON. [An extremal function for contractions of graphs](#). *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984.
- [67] ANDREW THOMASON. [The extremal function for complete minors](#). *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.
- [68] CARSTEN THOMASSEN. [Girth in graphs](#). *J. Combin. Theory Ser. B*, 35(2):129–141, 1983.

- [69] CARSTEN THOMASSEN. [Every planar graph is 5-choosable](#). *J. Combin. Theory Ser. B*, 62(1):180–181, 1994.
- [70] CARSTEN THOMASSEN. [A short list color proof of Grötzsch’s theorem](#). *J. Combin. Theory Ser. B*, 88(1):189–192, 2003.
- [71] JAN VAN DEN HEUVEL, PATRICE OSSONA DE MENDEZ, DANIEL QUIROZ, ROMAN RABINOVICH, AND SEBASTIAN SIEBERTZ. [On the generalised colouring numbers of graphs that exclude a fixed minor](#). *European J. Combin.*, 66:129–144, 2017.
- [72] JAN VAN DEN HEUVEL AND DAVID R. WOOD. [Improper colourings inspired by Hadwiger’s conjecture](#). *J. London Math. Soc.*, 98:129–148, 2018. arXiv:1704.06536.
- [73] RISTE ŠKREKOVSKI. [List improper colourings of planar graphs](#). *Combin. Probab. Comput.*, 8(3):293–299, 1999.
- [74] RISTE ŠKREKOVSKI. [List improper colorings of planar graphs with prescribed girth](#). *Discrete Math.*, 214(1-3):221–233, 2000.
- [75] DAVID R. WOOD. [Contractibility and the Hadwiger conjecture](#). *European J. Combin.*, 31(8):2102–2109, 2010.
- [76] DAVID R. WOOD. [Defective and clustered graph colouring](#). *Electron. J. Combin.*, DS23, 2018. Version 1.
- [77] XUDING ZHU. [Colouring graphs with bounded generalized colouring number](#). *Discrete Math.*, 309(18):5562–5568, 2009.