

Rowmotion on the Chain of V 's Poset and Whirling Dynamics

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Submitted: Oct 1, 2025; Accepted: Apr 24, 2026; Published: Jul 17, 2026
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Abstract

Given a finite poset P , we study the *whirling* action on vertex-labelings of P with the elements $\{0, 1, 2, \dots, k\}$. When such labelings are (weakly) order-reversing, we call them *k-bounded P-partitions*. We give a general equivariant bijection between *k-bounded P-partitions* and order ideals of the poset $P \times [k]$ which conveys whirling to the well-studied *rowmotion* operator. As an application, we derive periodicity and homomesy results for rowmotion acting on the *chain of V's* poset $V \times [k]$. We are able to generalize some of these results to the more complicated dynamics of rowmotion on $C_n \times [k]$, where C_n is the **claw poset** with n unrelated elements each covering $\hat{0}$.

Mathematics Subject Classifications: 05E18

1 Introduction

We connect the well-studied operation of *rowmotion* on the order ideals of a finite poset with the less familiar *whirling* action on P -partitions with bounded labels. One of our main results is an equivariant bijection that carries one to the other for any finite poset P . Here we are following in the footsteps of Haddadan [6], who noticed this connection when P itself is a chain. We then leverage this to study the rowmotion action on the “chain of V 's” poset $V_k := V \times [k]$ (a 3-element V -shaped poset cross a finite chain, see Figure 3), which has surprisingly good dynamical properties. We also generalize this to the case where we replace V with a **n -claw**, C_n , a poset with a single minimal element covered by exactly n incomparable elements. In both cases we obtain both periodicity results and homomesy.

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1.1 Basic definitions and background in combinatorial dynamics

Let P be a finite poset, and $\mathcal{J}(P)$ be the set of order ideals of P . (For basic poset definitions, we refer the reader to Stanley [15, Ch. 3].) *Combinatorial rowmotion* is an invertible map $\rho : \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ which takes each ideal $I \in \mathcal{J}(P)$ to the order ideal generated by the minimal elements of the complement of I in P . The periodicity of this map on products of chains was first studied by Brouwer and Schrijver [3], and Cameron and Fon-der-Flaass [4]. Later Striker and Williams [17] considered it as one element of the “toggle group” of a poset and related it to a kind of “promotion” operator on order ideals. Around the same time, Armstrong, Stump, and Thomas [2] studied rowmotion on *root posets*, relating it to “Kreweras complementation” on noncrossing partitions, and used this to prove a conjecture of Panyushev about the equality of the average cardinality of antichains for each rowmotion orbit.

Propp and Roby [11] noticed that this conjecture was merely one instance of a much broader phenomenon which they dubbed *homomesy*. Given a finite set S , a “statistic” $f : S \rightarrow \mathbb{C}$, and an invertible map φ on S , we call f *homomesic* if the average value of f is the same for every φ -orbit $\mathcal{R}$, i.e., $\frac{1}{\#\mathcal{R}} \sum_{x \in \mathcal{R}} f(x) = c$, where c is a constant not dependent on the choice of orbit $\mathcal{R}$. Over the past decade, many authors have proved homomesy results as one tool to understand various combinatorial dynamical systems.

Along with the work initiated by Reiner, Stanton, and White on the *Cyclic Sieving Phenomenon* [12, 13,], the confluence of all this work was the beginning of *dynamical algebraic combinatorics* as a distinct area within algebraic combinatorics (with antecedents going back to the Robinson–Schensted–Knuth correspondence and related operations on Young tableaux such as promotion, evacuation, and cyclage). In the past decade, the subfield has grown in a number of directions, and the study of rowmotion has been of continuing interest. For more background information, see the survey articles of Hopkins [8], Roby [14], and Striker [16].

Cameron and Fon-der-Flaass [4] were the first to describe rowmotion as a product of involutions called *toggles*, as detailed in Section 1.3. A natural generalization of toggling at a poset element x is “*whirling* at x ,” which cycles the label at x among j possible values. (Toggles are the case when $j = 2$.) Joseph, Propp, and Roby defined these and the operation of *whirling* on sets of functions between finite sets, obtaining various homomesy results for different classes of functions (injective, surjective, etc.) [9]. This is described in Section 2.

A bijective function $f : P \rightarrow [p]$ (with $\#P = p$) such that $f(x) < f(y)$ whenever $x <_P y$ is called a *linear extension*. We denote by $\mathcal{L}(P)$ the set of *all linear extensions* of P ; its cardinality, $e(P)$, is an important numerical invariant of a poset. Its refinement, the *order polynomial* $\Omega_P(k)$, counts the number of k -bounded P -partitions. For some special posets P , mainly ones connected with Lie theory (root and minuscule posets) and those of partition or shifted shapes, product formulae for $\Omega_P(k)$ are known. Hopkins surveys these posets, the formulae, and gives the heuristic: *Posets with order-polynomial product formulae are the same as the posets with good dynamical behavior*. The one poset in his

list whose rowmotion dynamics were relatively unexplored is $V \times [k]$, a gap this paper fills. In separate work Hopkins and Rubey study the dynamics of Schützenberger promotion on linear extensions of $V \times [k]$, which also exhibits unusually good behavior [7]. In his doctoral thesis, Ben Adenbaum provides a combinatorial approach (along with a density argument) to proving that this same periodicity extends to *piecewise-linear rowmotion* on the order polytope $\mathcal{O}(P)$ [1, Prop. 3.23]. Since most dynamical results in the piecewise-linear realm have been proven first at the (higher) birational level, then tropicalized to the PL realm, this is a notable methodology. The authors are unaware of any successful attempts to lift periodicity of the rowmotion action on $V \times [k]$ (even conjecturally) to the birational realm.

1.2 Organization of this paper

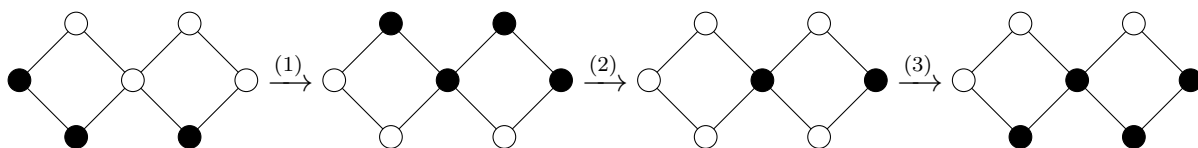
In Section 1 after the introduction, we review the toggling definition of rowmotion. Section 2 describes whirling, and includes the equivariant bijection which allows us to study rowmotion on V_k as whirling on k -bounded P -partitions. Section 3 contains our main periodicity and homomesy results for rowmotion on V_k , which use decompositions of the “orbit board” of the corresponding whirling action into “whorms”. Finally, Section 4 contains the periodicity and homomesy results which generalize to rowmotion on the “chain of claws” graph, $C_n \times [k]$, and extends our proofs to this situation.

1.3 Rowmotion as a product of toggles

Writing rowmotion as a product of toggles has been quite useful for understanding its properties and generalizing it to other situations, e.g., lifting to the piece-wise linear and birational level [5,]. Thomas and Williams call this “Rowmotion in Slowmotion” [18].

Definition 1. We define the (order-ideal) rowmotion map, $\rho : \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ as follows: For any $I \in \mathcal{J}(P)$, $\rho(I)$ is the order ideal generated by the minimal elements of the complement of I , as in the example below.

Example 2. Here is one iteration of ρ on an order ideal with the action broken down into its three steps: (1) complement, (2) take minimal elements, (3) saturate down.



Cameron and Fon-der-Flaass [4] showed that for any finite poset P , rowmotion can be realized as “toggling once at each element of P along any linear extension (from top to bottom)”. Other toggling orders also lead to interesting maps, such as Striker–Williams “promotion” (of order ideals) of a poset, which is toggling from left-to-right along “files” of a poset [17]. To *toggle* an order ideal I at a poset element x , means to add x to I if it is not included, or delete x if it is, but only if the result would also be an order ideal. Formally:

Definition 3. For each fixed $x \in P$ define the (*order-ideal*) toggle $\tau_x : \mathcal{J}(P) \rightarrow \mathcal{J}(P)$ by

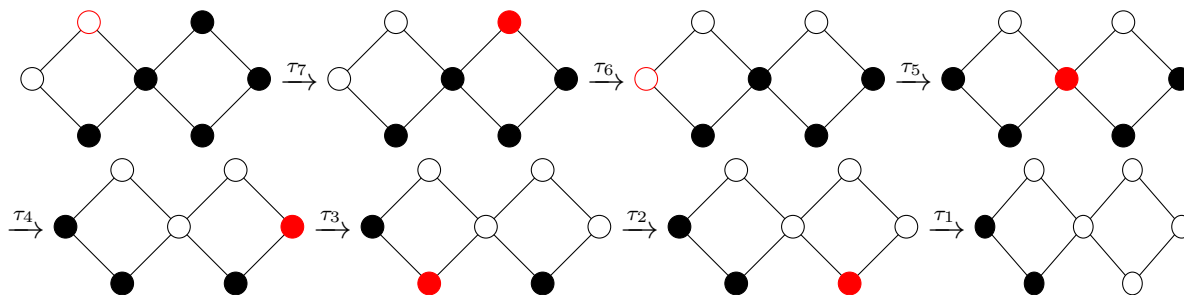
$$\tau_x(I) = \begin{cases} I \setminus \{x\} & \text{if } x \in I \text{ and } I \setminus \{x\} \in \mathcal{J}(P) \\ I \cup \{x\} & \text{if } x \notin I \text{ and } I \cup \{x\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that order-ideal toggles [4, §2] are involutions, and that toggles at incomparable elements commute (a special case of Prop 12).

Example 4. We will toggle node-by-node down the following fixed linear extension: at

each step we consider whether or not to toggle the red node in or out.
$$\begin{array}{ccccc} & & 7 & & 6 \\ & \swarrow & & \searrow & \\ 5 & & 4 & & 3 \\ & \swarrow & & \searrow & \\ & & 2 & & 1 \end{array}$$

For this linear extension we toggle the elements row-by-row from top-to-bottom, left-to-right within each row (although the order within rows is irrelevant, since toggles at incomparable elements commute, hence the name “rowmotion”).



Proposition 5 ([4, Lemma 1]). Let $x_1, x_2, \dots, x_p$ be any linear extension (i.e., any order-preserving listing of the elements) of a finite poset P with p elements. Then the composite map $\tau_{x_1} \tau_{x_2} \cdots \tau_{x_p}$ coincides with the rowmotion operation ρ .

2 The whirling map

An order ideal in any poset can be considered as a binary labeling of the poset with 0 (indicating elements outside the order ideal) and 1 (indicating those inside of it). Equivalently, this represents an order-reversing map $f : P \rightarrow \{0, 1\}$. Then toggling at an element, simply switches the labels at that element (or leaves them alone if the result would not be order-reversing). Similarly, we can define whirling at a poset element to cycle through a larger set of possible labels until it arrives at one that gives a legitimate order-reversing map. To whirl the entire labeling, one whirls once at each element along a linear extension.

A notion equivalent to whirling in this context was first defined by James Propp (dubbed *winching* by Peter Winkler) and was used by S. Haddadan[6], as an aid to proving certain homomesies for rowmotion on order ideals of type A root and minuscule

posets (triangles and rectangles). Joseph, Propp, and Roby [9] later defined whirling in the context of functions between finite sets. We review this first, before defining whirling of “ k -bounded P -partitions” in Section 2.2. We prove a general equivariant bijection between rowmotion on $P \times [k]$, where P is *any* finite poset, which is similar to the equivariant bijections Haddadan used in her work with triangular and rectangular posets.

2.1 Whirling functions between finite sets

Let $\mathcal{F} \subseteq [k]^{[n]}$ be a family of functions $f : [n] \rightarrow [k]$. For the rest of section 2.1, we use $\{1, \dots, k\} = [k]$ to represent the congruence classes of $\mathbb{Z}/k\mathbb{Z}$, as opposed to the usual $\{0, 1, \dots, k-1\}$. For fixed values of k and n , we represent such functions in *one-line* notation, e.g., $f = 21344$ represents the function $f \in [4]^{[5]}$ with $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, $f(4) = 4$, and $f(5) = 4$.

Definition 6 ([9, Definition 2.3]). For $f \in \mathcal{F}$ we define the *whirl* $w_i : \mathcal{F} \rightarrow \mathcal{F}$ at index i as follows: repeatedly add 1 (modulo k) to the value of $f(i)$ until we get a function in $\mathcal{F}$.

Example 7. Let $\mathcal{F} = \{f \in [4]^{[5]} : f(1) \neq f(2)\}$. If we apply w_2 to $f = 21344$, adding 1 in the second position gives 22344, but this is not in $\mathcal{F}$. Adding 1 again in this position gives the result: $w_2(f) = 23344$.

We will now highlight some specific results from the paper where whirling was first introduced. Let $\text{Inj}_m(n, k)$ be the set of m -*injective functions*, that is, functions $f : [n] \rightarrow [k]$ such that $\#f^{-1}(t) \leq m$ for all $t \in [k]$. Similarly, let $\text{Sur}_m(n, k)$ be the set of m -*surjective functions*, that is, $f : [n] \rightarrow [k]$ such that $\#f^{-1}(t) \geq m$ for all $t \in [k]$. Note that injective functions are 1-injections and surjective functions are 1-surjections. We also define the statistic $\eta_j(f) = \#f^{-1}(\{j\})$.

Theorem 8 ([9, Theorem 2.11]). *Fix $\mathcal{F}$ to be either $\text{Inj}_m(n, k)$ or $\text{Sur}_1(n, k)$ for given $n, k, m \in \mathbb{P}$. Then under the action of $\mathbf{w} = w_n \circ w_{n-1} \circ \dots \circ w_1$ on $\mathcal{F}$, η_j is $\frac{n}{k}$ -mesic for any $j \in [k]$.*

This result is conjectured to hold for $\text{Sur}_m(n, k)$, but is still open for $m > 1$. Proof details can be found in Sections 2.2–2.4 of [9].

Example 9. Here is the orbit of $\mathbf{w}$ on $\text{Inj}_1(3, 6)$ containing $f = 415$.

$$415 \xrightarrow{\mathbf{w}} 621 \xrightarrow{\mathbf{w}} 342 \xrightarrow{\mathbf{w}} 563 \xrightarrow{\mathbf{w}} 124 \xrightarrow{\mathbf{w}} 356 \xrightarrow{\mathbf{w}} 412 \xrightarrow{\mathbf{w}} 534 \xrightarrow{\mathbf{w}} 651 \xrightarrow{\mathbf{w}} 263 \xrightarrow{\mathbf{w}} \downarrow$$

Figure 1 shows the corresponding *orbit board* (a matrix whose rows are the successive orbit elements) partitioned into “chunks,” each of which contains exactly the set of numbers $\{1, 2, \dots, k\}$. Notice that each value $1, 2, \dots, 6$ appear exactly 5 times in this orbit of size 10, in accordance with the $1/2$ -mesy of Theorem 8. See [9, Section 2] for precise definitions.

4	1	5
6	2	1
3	4	2
5	6	3
1	2	4
3	5	6
4	1	2
5	3	4
6	5	1
2	6	3

Figure 1: The orbit board of $\mathbf{w}$ on $\text{Inj}_1(3, 6)$ containing $f = 415$, partitioned into “chunks”.

2.2 Whirling k -bounded P -partitions and rowmotion

Now we extend the definition of whirling to k -bounded P -partitions, and prove our main general result relating whirling and rowmotion. Throughout the rest of the paper, P will denote a finite poset. Define $[0, k] := \{0, 1, 2, \dots, k\}$. A P -partition is a map σ from P to $\mathbb{N}$ such that if $x <_P y$, then $\sigma(x) \geq \sigma(y)$ [15, Ch. 3].

Definition 10. A k -bounded P -partition is a function $f : P \rightarrow [0, k]$ such that if $x \leq_P y$, then $f(x) \geq f(y)$. Let $\mathcal{F}_k(P)$ be the set of all such functions.

Throughout the rest of the paper we use $\{0, 1, \dots, k\}$ to represent the congruence classes of $\mathbb{Z}/(k+1)\mathbb{Z}$, as usual.

Definition 11. For $f \in \mathcal{F}_k(P)$ and $x \in P$, define $w_x : \mathcal{F}_k(P) \rightarrow \mathcal{F}_k(P)$, called the *whirl at x* , as follows: repeatedly add 1 (mod $k+1$) to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

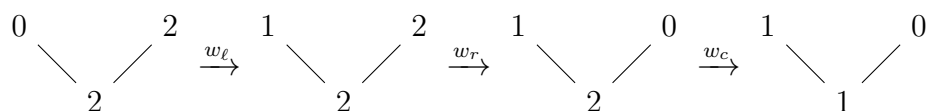
The case $k = 1$ of the above definition recovers toggling of order ideals (Def. 3).

Proposition 12. If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.

Proof. Since x and y are incomparable, there are no inequalities constraining the relationship between $f(x)$ and $f(y)$. So $w_x w_y = w_y w_x$. $\square$

Definition 13. Let $(x_1, x_2, \dots, x_p)$ be a linear extension of P . Define $w : \mathcal{F}_k(P) \rightarrow \mathcal{F}_k(P)$ by $w := w_{x_1} w_{x_2} \dots w_{x_p}$. The above proposition shows that this is well-defined, since one can get from any linear extension to any other by a sequence of interchanges of incomparable elements.

Example 14. Let P be the V poset with labels $\begin{smallmatrix} \ell & & r \\ & \searrow & \nearrow \\ & c & \end{smallmatrix}$, $k = 2$, and $w = w_c w_r w_\ell$.

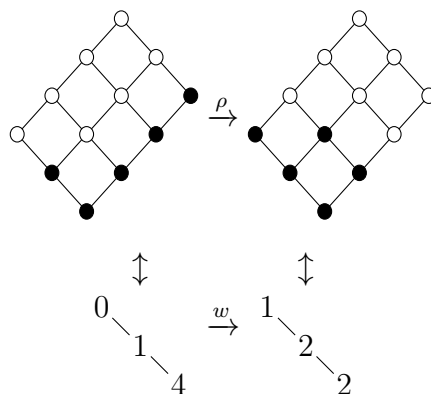


There is a natural bijection between order ideals of a poset P and 1-bounded P -partitions in $\mathcal{F}_1(P)$. Specifically, a 1-bounded P -partition in $\mathcal{F}_1(P)$ is simply the indicator function of an order ideal $I \in J(P)$. We extend this to an equivariant bijection $\mathcal{F}_k(P) \rightarrow \mathcal{J}(P \times [k])$ which sends w to ρ , meaning the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}_k(P) & \xrightarrow{w} & \mathcal{F}_k(P) \\ \updownarrow & & \updownarrow \\ \mathcal{J}(P \times [k]) & \xrightarrow{\rho} & \mathcal{J}(P \times [k]) \end{array}$$

We will call the chains $\{(x, 1), (x, 2), \dots, (x, k)\} \subseteq P \times [k]$, for $x \in P$, the *fibers* of $P \times [k]$, and construct an equivariant bijection that first sends w_x to order-ideal toggling down the fiber $\{(x, 1), (x, 2), \dots, (x, k)\}$.

Example 15. Here is an example of the equivariant bijection between rowmotion on order ideals of $[3] \times [4]$ and whirling on $\mathcal{F}_4([3])$.



Lemma 16. *There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w_x to the toggle product $\tau_{(x,1)}\tau_{(x,2)} \dots \tau_{(x,k)}$.*

Proof. The bijection ϕ between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ will be a natural one, where $\phi(f) = I$ implies $f(x)$ counts the number of elements in the intersection of I and the fiber at x in $P \times [k]$. That is, the intersection of the fiber at x in $P \times [k]$, $\{(x, 1), (x, 2), \dots, (x, k)\}$, and an order ideal I is $\{(x, 1), (x, 2), \dots, (x, d)\}$ for some $d \in [k]$, if and only if $f(x) = d$. Otherwise, the intersection is empty if and only if $f(x) = 0$.

We will now complete the proof by fixing f and x and showing that

$$\phi(w_x(f)) = \tau_{(x,1)}\tau_{(x,2)} \dots \tau_{(x,k)}(\phi(f)).$$

Let $m = \max(\{f(y) : y \leq x, \text{ for } y \in P\} \cup \{0\})$ and $M = \min(\{f(y) : x \leq y, \text{ for } y \in P\} \cup \{k\})$, so $m \leq f(x) \leq M$. There are two cases:

- *Case 1.* If $f(x) < M$, then $(w_x(f))(x) = f(x) + 1$.

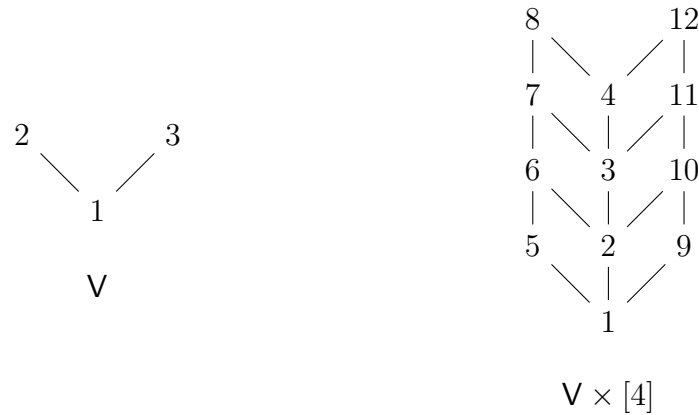


Figure 2: A linear extension of V on the left and the linear extension for $V \times [4]$ constructed from it on the right, as in the proof of Theorem 17.

- *Case 2.* If $f(x) = M$, then $f(x) + 1$ is greater than M so we increment through values modulo $k + 1$ until we get to $(w_x(f))(x) = m$.

On the other hand, we consider the action of $\tau = \tau_{(x,1)}\tau_{(x,2)} \dots \tau_{(x,k)}$ on $\phi(f)$. Recall that $\tau_{(x,i)}$ does not change the order ideal whenever (x, i) is not maximal in the order ideal or minimal in the complement of the order ideal. Because of this, there are only two outcomes of $\tau(S(f))$, either one element was added to order ideal, or elements were removed until removing anymore would not result in an order ideal. This reflects the two cases above, which completes the proof. $\square$

The following general theorem is extremely useful for studying the dynamics of rowmotion for certain kinds of posets. Such an equivariant bijection was used by Haddadan [6, Lemma 3.1] in the specific situation she studied of rowmotion and promotion on triangular and rectangular posets, but has not been stated in anything like this generality in literature we have seen.

Theorem 17. Fix any linear extension $(x_1, x_2, \dots, x_p) \in \mathcal{L}(P)$. There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends whirling, $w = w_{x_1}w_{x_2} \dots w_{x_p}$ on $\mathcal{F}_k(P)$ to rowmotion on $\mathcal{J}(P \times [k])$.

Proof. The bijection is based on the one from Lemma 16. We will construct the linear extension that traverses $P \times [k]$ fiber by fiber, i.e., $(x_i, j) \rightarrow x_{k(i-1)+j}$. Then the composite map $\tau_{x_1}\tau_{x_2} \dots \tau_{x_p}$ on $P \times [k]$ coincides with rowmotion by Proposition 5. By Lemma 16, the toggles $\tau_{x_{k(i-1)+1}}\tau_{x_{k(i-1)+2}} \dots \tau_{x_{k(i-1)+k}}$ coincide with w_{x_i} . Together these give us that $w = w_{x_1}w_{x_2} \dots w_{x_p}$, is exactly rowmotion. $\square$

Example 18. Figure 2 shows a linear extension of the poset, V and the corresponding linear extension for $V \times [4]$ on the right, as in the proof of Theorem 17, of $V \times [4]$.

The following definitions will allow us to partition orbit boards of whirling into subsets called *whorms*. The partitioning of orbit boards of combinatorial actions into some special

class of disjoint subsets is a useful tool, used by several authors, including different kinds of “snakes” [6, 10], and “chunks” [9,].

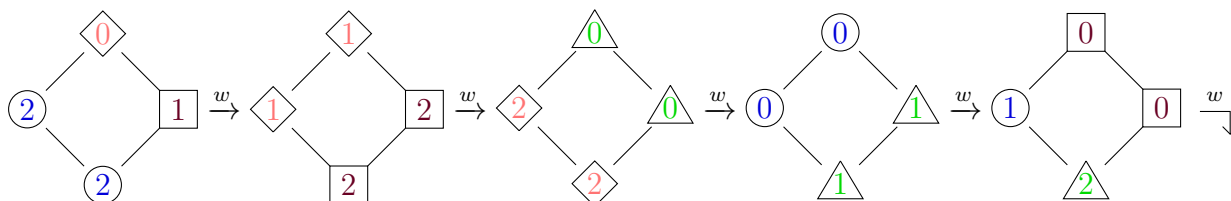
Definition 19. For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* to (x, f) if either:

1. $y = x$ and $g(y) = w(f)(x) = f(x) + 1$, or
2. x covers y , $f = g$, and $f(x) = g(y)$.

We consider whirl-successive elements to be whirl elements which are one step away from each other, either by moving one covering relation down the poset or by whirling the function at the element, and ending one label greater. *While we must consider the entire P -partitions f and g to check whether two whirl elements are whorm-connected, we think of whirl elements as being simply $(x, f(x))$, the location and its label, and indicate them in this way in the examples that follow.*

Definition 20. Two whirl elements (x, f) and (y, g) are *whorm-connected* if there exists a sequence of whirl-successive elements $\{(x, f) = (x_0, f_0), (x_1, f_1), \dots, (x_p, f_p) = (y, g)\}$. A *whorm* is a maximal set of whorm-connected whirl elements, that is, if (x, f) is in a whorm and (x, f) is whorm-connected to (y, g) , then (y, g) is in the whorm.

Example 21. An orbit of whirling P -partitions (for $P = [2] \times [2]$) with its four whorms indicated by the same color and (redundantly) node-shape.



3 Dynamics for rowmotion on $\mathbf{V} \times [k]$

In this section we consider the dynamics of rowmotion acting on the order ideals of the chain of $\mathbf{V}$'s poset $\mathbf{V}_k$, establishing its periodicity and finding interesting examples of homomesy.

3.1 The chain of $\mathbf{V}$'s poset

While most posets currently known to enjoy nice dynamical properties under rowmotion seem to be the root and minuscule posets associated with various types of Lie algebras, the current poset seems to be *sui generis*.

Definition 22. Let $\mathbf{V}$ be the 3-element poset with Hasse diagram , and define $\mathbf{V}_k = \mathbf{V} \times [k]$, where $[k]$ is the chain poset. We call $\mathbf{V}_k$ the *chain of $\mathbf{V}$'s poset*. We will use the labeling convention of Figure 3 throughout this section.

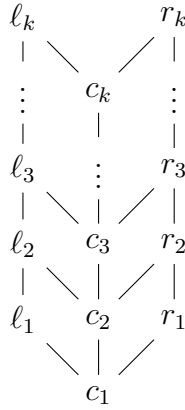


Figure 3: Hasse diagram of V_k with our vertex-labeling convention.

Our main goals for this section are the following theorems. We will leverage the equivariant bijection (Theorem 17) and the notion of whorms from Section 2.2.

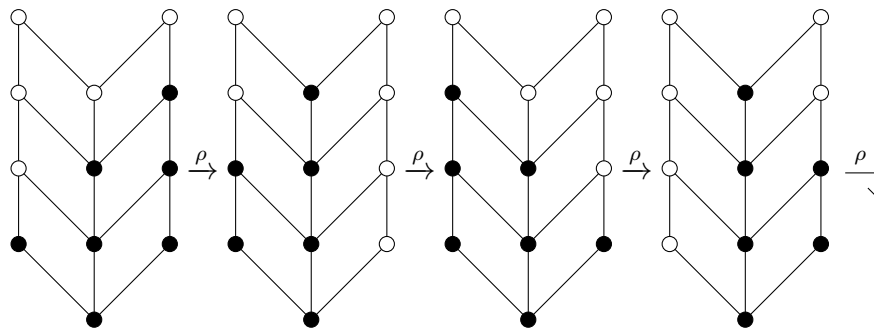
Theorem 23. *The order of rowmotion on $\mathcal{J}(V_k)$ is $2(k+2)$.*

Theorem 24. *Let χ_s be the indicator function for $s \in V_k$. We have the following homomies for the action of ρ on $\mathcal{J}(V_k)$*

1. *The statistic $\chi_{\ell_i} - \chi_{r_i}$ is 0-mesic for all $i \in [k]$.*
2. *The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$ -mesic.*

Example 25. Figure 4 shows the two ρ -orbits, of sizes 5 and 10, on $\mathcal{J}(V_4)$, confirming Theorem 23. It is also easy to check the homomies of Theorem 24.

Example 26. Here is ρ -orbit on $\mathcal{J}(V_4)$ of length 4, which divides the order $2(4+2) = 12$. Across the orbit the total number of elements at rank 1 in the side fibers is 6, minus the two at the top of the center fiber, for an average of $\frac{6-2}{4} = 1 = \frac{2(4-1)}{4+2}$, agreeing with Theorem 24(2).



To prove these theorems we utilize our equivariant bijection (Theorem 17) from $\mathcal{J}(V_k)$ to $\mathcal{F}_k(V)$, then represent the latter by triples $f = (\ell, c, r)$ with $\ell \leq c$ and $r \leq c$.

This bijection ϕ sends an order ideal I to a triple (ℓ, c, r) , counting the number of elements of the order ideal in the left, center, and right fibers respectively.

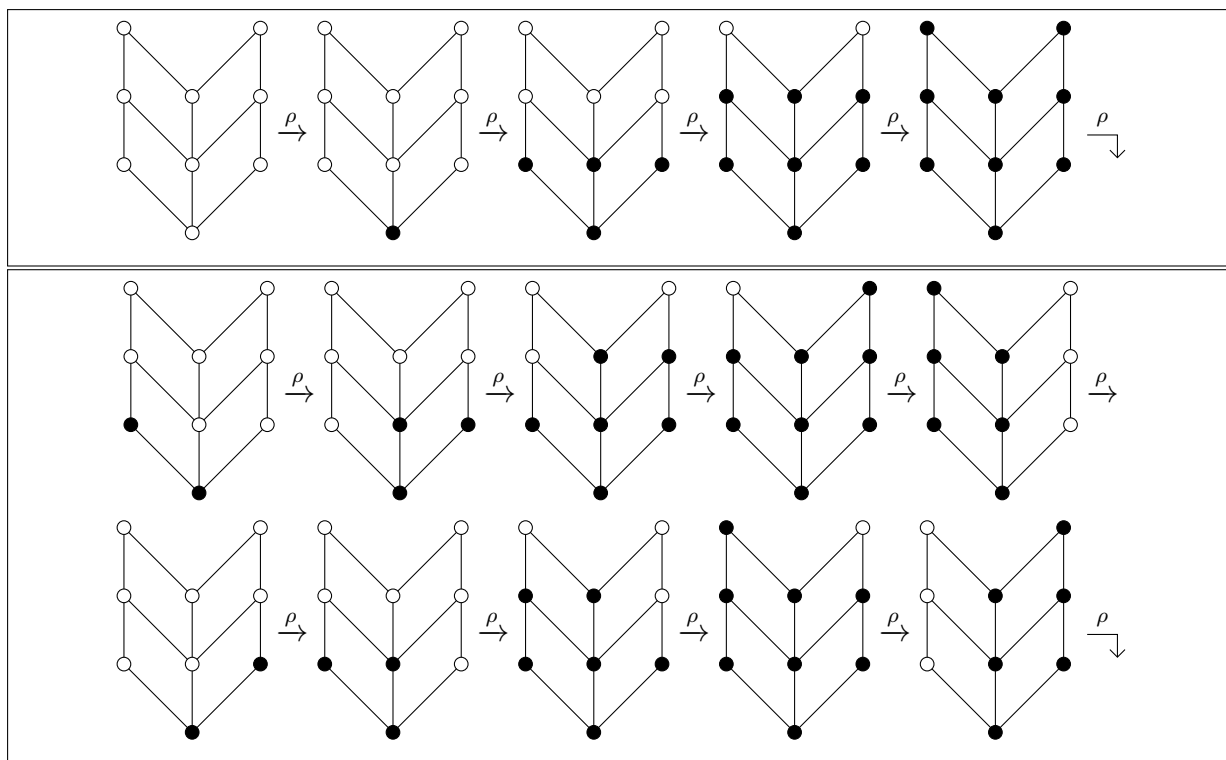


Figure 4: The two orbits of rowmotion on order ideals of $\mathbf{V} \times [3]$

Example 27. Here is the orbit of $\mathcal{F}_4(\mathbf{V})$ corresponding to Example 26.

$$(1, 3, 3) \xrightarrow{w} (2, 4, 0) \xrightarrow{w} (3, 3, 1) \xrightarrow{w} (0, 4, 2) \xrightarrow{w} \downarrow$$

Figure 5 shows an orbit of rowmotion on order ideals of $P = \mathbf{V} \times [4]$. Below each ideal is the corresponding P -partition in $\mathcal{F}_4(P)$, and these are being whirled equivariantly with ρ .

Proposition 28. *The number of order ideals of $\mathbf{V}_k$ is given by*

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Proof. Every order ideal of $\mathbf{V}_k$ is associated with a triple of labels of the elements of $\mathbf{V}$. Let $j \in [0, k]$ denote the label of c . The possible labels for ℓ and r are precisely the elements of $[0, j]$, yielding a total of $(j+1)^2$ labelings. Summing over all possible values of j produces the desired result. $\square$

3.2 Periodicity and homomesy via center-seeking whorms

To show that the order of ρ on $\mathcal{J}(\mathbf{V}_k)$ is $2(k+2)$ we end up proving something stronger, namely that $\rho^{k+2}(I)$ is the reflection of I across the center chain. Our method is to investigate the whorms that arise from repeatedly whirling a k -bounded P -partition.

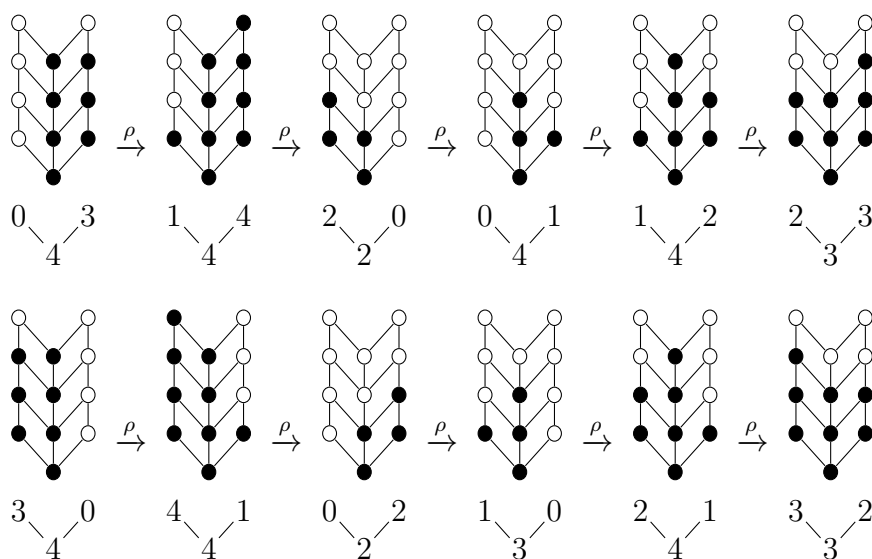


Figure 5: An orbit of rowmotion on $P = \mathbf{V} \times [4]$, with corresponding P -partitions (which are being whirled), as in Theorem 17.

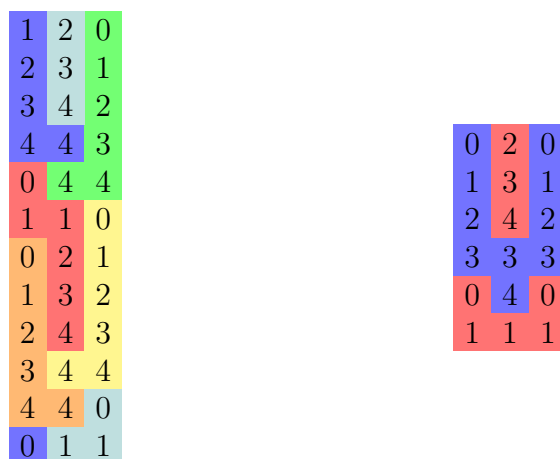


Figure 6: Two orbit boards of $\mathcal{F}_4(\mathbf{V})$, one with six whorms and one with two whorms.

Recall from Definition 20 that, given a whirling orbit board, $\mathcal{R} = \{f, w(f), w^2(f), \dots\}$ of w on $\mathcal{F}_k(\mathbf{V})$, a whorm ξ is a maximal set of whorm-connected elements. Figure 6 shows two orbit boards of $\mathcal{F}_4(\mathbf{V})$, one with six whorms and one with two whorms. Notice that each whorm in the second orbit has two “starting” positions.

Each whorm in an orbit board of $\mathbf{V} \times [k]$ starts on the left, or the right, or both left and right; we call the former *one-tailed* (specifically *right-tailed* or *left-tailed*) and the latter *two-tailed*. We will also use the terms *left-whorm* and *right-whorm* for short in the one-tailed case.

Since these whorms move down the orbit board at every step, except for one move to

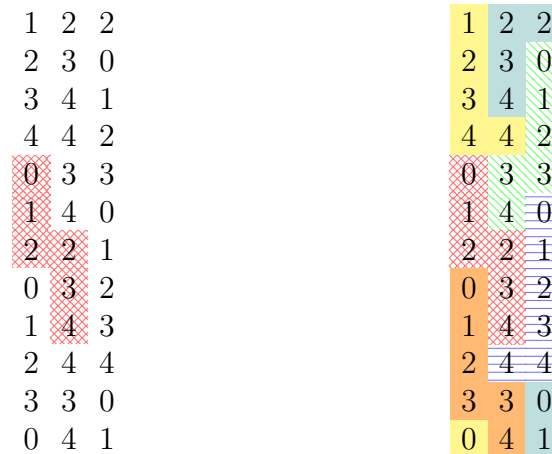


Figure 7: The same orbit board twice, the one on the left highlights a single whorm, the one on the right highlights all whorms.

the center, we consider them as a sequence of function values in the orbit board which start at 0 and end at k , where one value is repeated when moving into the center. We call these *center-seeking whorms*. (Since an orbit board is actually a cylinder, we have a “can of whorms” to deal with.)

In the left orbit of Figure 7 we isolate one example of a left whorm:

$$\xi = \{(\ell, (0, 3, 3)), (\ell, (1, 4, 0)), (\ell, (2, 2, 1)), (c, (2, 2, 1)), (c, (0, 3, 2)), (c, (1, 4, 3))\},$$

visualized within an orbit board of $\mathcal{F}_4(\mathbf{V})$.

It is easy to see that an orbit board is tiled either entirely by one-tailed whorms or entirely by two-tailed whorms. In fact, if $f(\ell) = f(r)$ in any row, then $w^d f(\ell) = w^d f(r)$ for all $d \in \mathbb{N}$ and all $f \in \mathcal{F}_k(\mathbf{V})$. This is because the labels of ℓ and r represent the result of whirling at *incomparable* elements of the poset $\mathbf{V}$. Furthermore, this forces the whorms to go to the middle from both sides in the same row, which makes the elements of the two-tails whirl-connected, so happens exactly when we have a two-tailed whorms. In the situation where $f(\ell) \neq f(r)$ in any row, this also persists, so none of the whorms can be two-tailed.

We first observe that all one-tailed whorms have $k + 2$ elements, since each contains the $k + 1$ elements $0, \dots, k$, exactly one of which is doubled. Define $\mathbf{t}(\xi) := 1 + \min\{f(c) : (c, f) \in \xi\}$, the number of elements in an outer column (the “tail length”) and $\mathbf{h}(\xi) := k + 2 - \mathbf{t}(\xi)$, the number of elements in the center column (the “head length”). For the red whorm in the orbit on the left of Figure 7, $\mathbf{t}(\xi) = 3$ and $\mathbf{h}(\xi) = 3$. For the blue two-tailed worm on the right of Figure 6, $\mathbf{t}(\xi) = 4$ and $\mathbf{h}(\xi) = 2$. Note that in a one-tailed orbit board, each whorm has the same cardinality, namely $k + 2$; but in a two-tailed orbit board, the whorms can have different cardinalities (e.g., 10 and 8 in Figure 6).

Example 29. The right orbit board in Figure 7 is the previous example with all the whorms colored. The number of elements in the left column of the yellow, red, and orange whorms are 5, 3, 4 respectively, and the orbit board is of length 12.

It follows that the order of whirling divides the sum of $\mathbf{t}(\xi)$ over all whorms $\xi \in S$. In the setting of $\mathcal{F}_k(\mathbf{V})$, as long as we know $\mathbf{t}(\xi)$ and whether $f(\ell) = 0$, $f(r) = 0$, or both, then we can recover the entire whorm.

The center of any row intersects with exactly one whorm. A natural cyclic ordering on whorms in an orbit board arises from scanning down its center fiber.

Definition 30. We will place a circular order on the whorms. Let ξ_1 and ξ_2 be whorms in an orbit board of $\mathcal{F}_k(\mathbf{V})$. If there exists $(c, f) \in \xi_1$ with $f(c) = k$ such that $(c, w(f)) \in \xi_2$, then we say ξ_2 is *in front of* ξ_1 . We call a sequence of whorms *consecutive* if each is in front of the next.

Example 31. In Figure 7 the blue (horizontal lines) whorm is in front of the red (crosshatch) whorm, which is in front of the green (northwest lines) whorm.

It is not hard to show that within a one-tailed orbit board, consecutive whorms alternate starting from the left and from the right. For when a whorm goes into the center from one side, say the left without loss of generality, then the next value in the left column will be 0, while in the right column it will be greater than 0. This persists until the next whorm moves to the center (possibly during the same step). At that point, by definition of whirling, the value in the center column will change from k to $1 + \max\{f(\ell), f(r)\} = 1 + f(r)$, meaning that the values in the right column and in the center will be equal. See the one-tailed examples in Figures 6 and 7. We use this in the next lemma, which is key to proving our main results.

Lemma 32. *Given an orbit board $\mathcal{R}$ of w on $\mathcal{F}_k(\mathbf{V})$, let ξ_1, ξ_2 , and ξ_3 be three consecutive whorms (not necessarily all distinct), that is, ξ_3 is in front of ξ_2 which is in front of ξ_1 in $\mathcal{R}$.*

1. *If $\mathcal{R}$ is tiled entirely by one-tailed whorms, then*

$$\mathbf{t}(\xi_1) + \mathbf{t}(\xi_2) + \mathbf{t}(\xi_3) = 2(k + 2).$$

2. *Otherwise, if $\mathcal{R}$ is tiled entirely by two-tailed whorms, then*

$$\mathbf{t}(\xi_1) + \mathbf{t}(\xi_2) = k + 2.$$

Proof. (1) For the case of one-tailed whorms the key observation is that $\mathbf{t}(\xi_3) = \mathbf{h}(\xi_1) + \mathbf{h}(\xi_2)$ for consecutive whorms. This holds because the tail of ξ_3 begins one row below where the head of ξ_1 starts, and ends one row below where ξ_2 ends. But since $\mathbf{t}(\xi) = k + 2 - \mathbf{h}(\xi)$ for any ξ we get

$$\mathbf{t}(\xi_3) = \mathbf{h}(\xi_1) + \mathbf{h}(\xi_2) = (k + 2 - \mathbf{t}(\xi_1)) + (k + 2 - \mathbf{t}(\xi_2)),$$

which implies the result. See Figure 6 for examples.

(2) For the case of two-tailed whorms a similar argument shows that $\mathbf{t}(\xi_2) = \mathbf{h}(\xi_1)$ for two consecutive whorms. The result then follows as before. $\square$

In fact, the entire orbit board can be reconstructed simply from knowing the values of $t(\xi_1)$ and $t(\xi_2)$ for two consecutive whorms in the one-tailed case, and from a single $t(\xi_1)$ in the two-tailed case.

Example 33. In Figure 7 we have $k = 4$, $t(\text{green}) = 4$, $t(\text{red}) = 3$, and $t(\text{blue}) = 5$, which sum to $12 = 2(4 + 2)$.

Lemma 34. *Given an orbit board with one-tailed whorms, let ξ_1, ξ_2, ξ_3 , and ξ_4 be consecutive, then*

$$t(\xi_4) = t(\xi_1).$$

Otherwise, if the orbit board contains two-tailed whorms, then $t(\xi_1) = t(\xi_3)$.

Proof. First assume the orbit board is one-tailed. By Lemma 32, the difference of the equations,

$$t(\xi_1) + t(\xi_2) + t(\xi_3) = 2(k + 2), \text{ and,}$$

$$t(\xi_2) + t(\xi_3) + t(\xi_4) = 2(k + 2),$$

give the result. The proof follows similarly when the orbit board has two-tailed whorms. $\square$

Lemma 35. *Let $\mathcal{R}$ be an orbit board of w on $\mathcal{F}_k(\mathbf{V})$.*

1. *If $\mathcal{R}$ is tiled entirely by one-tailed whorms, then there are at most six distinct whorms.*
2. *Otherwise, if $\mathcal{R}$ is tiled entirely by two-tailed whorms, then there are at most two distinct whorms.*

Proof. For orbit boards with one-tailed whorms, we are *not* claiming the board starts to repeat after three steps, since ξ_4 will start on the opposite side from ξ_1 . If we keep applying the previous Lemma to even more consecutive whorms, we see $t(\xi_5) = t(\xi_2)$ and $t(\xi_6) = t(\xi_3)$. Finally we get $t(\xi_k) = t(\xi_{k-6})$ for $k \geq 7$. Since a whorm is completely characterized by its tail length and sidedness, this means that the orbit board must repeat by the row in which ξ_7 begins. The proof in the two-tailed case is similar. $\square$

Theorem 36. *Let $(x, y, z) \in \mathcal{F}_k(\mathbf{V})$, then $w^{k+2}(x, y, z) = (z, y, x)$.*

Proof. Let $\mathcal{R}$ be an orbit board of w . The proof involves a careful analysis of when whorms move into the center of the orbit board. For the one-tailed case, we use Lemma 32 and that $t(\xi) = k + 2 - h(\xi)$ for any ξ to conclude that

$$h(\xi_1) + h(\xi_2) + h(\xi_3) = (k + 2).$$

At this point, ξ_4 arrives from the side opposite ξ_1 , but with the same tail length. Therefore these two whorms must be reflections of one another (around the vertical middle of the $\mathcal{R}$, though translated vertically as well). This forces the values in row $k + 2 + 1$ to be the mirror reflection of those in row 1 (where ξ_1 starts).

In the two-tailed case, that is, when $x = z$, by Lemma 34 we know $t(\xi_1) = t(\xi_3)$. This implies ξ_1 and ξ_3 are the same whorm, since they are two-tailed whorms. Therefore, the orbit board repeats after $t(\xi_1) + t(\xi_2) = k + 2$ steps, by Lemma 32. $\square$

			0	2	2
			1	3	0
			2	2	1
			0	3	2
			1	3	3
			2	2	0
			0	3	1
			1	2	2
			2	3	0
			3	3	1
0	1	1			
1	2	0			
2	2	1			
0	2	2			
1	1	0			
0	2	1			
1	2	2			
2	2	0			

Figure 8: Two orbits of whirling, the left one of length 8 for $k = 2$ and the right one of length 10 for $k = 3$, each, confirming the whirling order in these cases.

0	0	0			
0	1	0			
1	1	1			
			0	1	1
			1	1	0

Figure 9: Both orbits of whirling when $k = 1$, confirming the order is 6 in this case.

Corollary 37. *The order of w on $\mathcal{F}_k(\mathbf{V})$ divides $2(k + 2)$.*

We are now ready to finish off the proofs of our main results for this section.

Proof of Theorem 23. Easy computations shows that order of w is *exactly* $2(k + 2)$ for $k = 1, 2$, or 3 . (See Figures 8–9.) Now for $k \geq 4$ we can pick any three distinct values b_1, b_2, b_3 that sum to $k + 2$ and construct an orbit board from six consecutive whorms with three distinct tail lengths and $\mathbf{t}(\xi_i) = b_i = \mathbf{t}(\xi_{i+3})$ for $i \in [3]$. These whorms will all be distinct, so we get no repeats until after $2(k + 2)$ steps. It is also straightforward to directly construct an order ideal in $\mathbf{V} \times [k]$ whose orbit is of this size. Hence, The order of w on $\mathcal{F}_k(\mathbf{V})$ is *exactly* $2(k + 2)$. $\square$

In our proofs the following definition will be handy.

Definition 38. A *super orbit* of an action is an orbit repeated one or more times. The length of a super orbit is always a multiple of the orbit size. Similarly a *super orbit board* will be the array representing a super orbit.

Proof of Theorem 24. The symmetry homomesy (1) is immediate from Theorem 36.

For (2) if I is an order ideal of $\mathbf{V} \times [k]$, then there is a P -partition $f = (x, y, z) \in \mathcal{F}$ in bijection with I given by Theorem 17. We know that

- $\ell_1 \in I \iff x > 0$,
- $r_1 \in I \iff z > 0$, and
- $c_k \in I \iff y = k$.

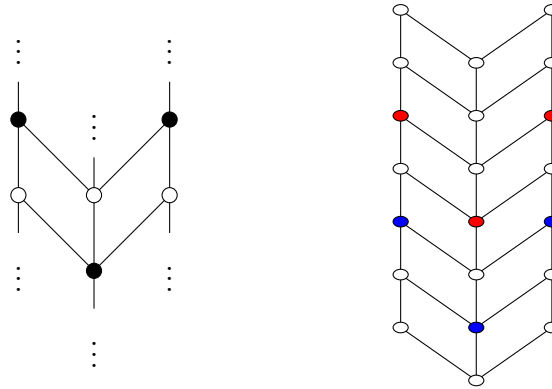


Figure 10: On the left, a section of $\mathbf{V}_k$ with elements of a flux capacitor configuration, ℓ_i, r_i and c_{i-1} , shaded black. On the right, a section of $\mathbf{V}_7$ with elements counted by $F_3 - F_5$, shaded black.

In the one-tailed case, let $\xi_1, \xi_2, \dots, \xi_6$ be six consecutive whorms partitioning the (possibly super) orbit board by Lemma 35. This gives a total of $4(k+2)$ entries in the left and right columns, six of which are 0, leaving $4(k+2) - 6$ nonzero entries, each of which contributes to either χ_{r_1} or χ_{ℓ_1} . The number of entries equal to k in the center column is 6 because there are 6 whorms that end at k in the center. Thus, we get the average

$$\frac{\sum_{f \in \mathcal{R}} \chi_{\ell_1}(f) + \chi_{r_1}(f) - \chi_{c_k}(f)}{\#\mathcal{R}} = \frac{4(k+2) - 12}{2(k+2)} = \frac{2(k-1)}{k+2}.$$

The two-tailed case follows analogously. $\square$

Using the whorm decomposition of the orbit board also gives a very clean proof of the following “flux capacitor” homomesy for rowmotion on $\mathcal{J}(\mathbf{V}_k)$. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, as shown in Figure 10.

Lemma 39. *Let $\mathcal{R}$ be a **one-tailed** orbit board for the action of rowmotion on order ideals of $\mathcal{J}(\mathbf{V}_k)$. The the average value of the difference of successive flux-capacitor indicator functions, $F_i - F_{i+1}$, is $\frac{3}{k+2}$ for $i \in [2, k-1]$.*

Proof. From the whirling orbit-board point of view, the statistic $F_i - F_{i+1}$ is nonzero exactly when i appears in the left or right column or $i-1$ appears in the center column. (Otherwise, e.g., both left-column indicator functions are zero, or both are one, canceling out when we take the difference $F_i - F_{i+1}$.)

In the one-tailed case, if i appears in the tail of a given whorm then $i-1$ will not appear in the head, and vice versa. So each of the six whorms in a super orbit board (Lemma 35) will contribute exactly 1 to the statistic $F_i - F_{i+1}$, depending on whether i appears in a side column or $i-1$ appears in the center column. This gives a total contribution of 6 to the statistic. Divide this by the size of the super orbit, $2(k+2)$, to get the result. $\square$

6	6	6
0	7	0
1	8	1
2	9	2
3	3	3
0	4	0
1	5	1
2	6	2
3	7	3
4	8	4
5	9	5

Figure 11: An orbit board of whirling in $\mathcal{F}_9(\mathbf{V})$, tiled by 2 two-tailed worms

Unfortunately in the two-tailed (symmetric) case, this lemma fails to hold, as the following counterexample shows. Earlier versions of this work incorrectly asserted that the above lemma held in **all** cases, which would have led to a stronger version of Theorem 41 which omitted the “symmetrically placed” hypothesis.

Example 40. Consider the orbit board from whirling $\mathcal{F}_9(\mathbf{V})$ starting at 666, as shown in Figure 11. we see that $F_2 - F_3$ has average $\frac{4}{k+2}$ (since there are four 2s in outer column and no 1s in the center column). Meanwhile, $F_7 - F_8$ has average $\frac{2}{k+2}$ (since there are zero 7s in outer column and two 6s in the middle).

Theorem 41. For $k > 1$. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$ be the indicator function of the flux capacitor at height i . Under the action of rowmotion on order ideals of $\mathcal{J}(\mathbf{V}_k)$, the difference of **symmetrically-placed** flux-capacitor statistics, $F_i - F_{k+2-i}$, is $\frac{3(k+2-2i)}{k+2}$ -mesic.

Proof. For the one-tailed case, we can use a simple telescoping sum to get a more general result. For any $i < j$ we may write

$$F_i - F_j = (F_i - F_{i+1}) + (F_{i+1} - F_{i+2}) + \cdots + (F_{j-1} - F_j).$$

Grouping consecutive pairs and using Lemma 39, we get the average of $F_i - F_j$ is $\frac{3(j-i)}{k+2}$. This specializes to the stated homomesy when $j = k+2-i$ for symmetrically-placed flux-capacitor statistics. A similar argument works for $j > i$, when the homomesic average will be non-positive.

For the two-tailed case we need a separate argument, since Lemma 39 no longer applies. Fix $j = k+2-i$, and assume again for now that $i < j$. Within any two-tailed whorm, the multiset of labels is the set of all integers in $[0, k]$ with integers in $[0, \mathbf{t}(\xi) - 2]$ appearing twice, and $\mathbf{t}(\xi) - 1$ appearing thrice (where the whorm moves to the center). Call the two whorms tiling the length $k+2$ (super) orbit board ξ_1 and ξ_2 . By Lemma 32 we have

$$(\mathbf{t}(\xi_1) - 1) + (\mathbf{t}(\xi_2) - 1) = k.$$

For example, in the length 11 orbit board in Figure 11, the labels on the red and blue whorms are 6 and 3, which sum to $k = 9$.

Now assume that $t(\xi_1) - 1 = m$, so $t(\xi_2) - 1 = k - m$. From the whirling whorms point of view, the contribution of a given label λ in the tiling to the statistic $F_i - F_{k+2-i}$ is given as follows:

- If λ lies in a side column and $i \leq \lambda \leq k + 1 - i$, then that label contributes $+1$ to the statistic.
- If λ lies in the center column and $i - 1 \leq \lambda \leq k - i$, then that label contributes $+1$ to the statistic.

Outside of the stated intervals, the statistics cancel each other out, so λ contributes zero. This all follows directly from the definition of F_i .

Example 42. Here are the two whorms from the orbit shown in Figure 11 written horizontally in one line. The tails are highlighted in blue and the heads highlighted in green. The labels that contribute to $F_2 - F_9$ are circled. Here $m = 6$ and $k - m = 3$, both of which lie between $i = 2$ and $k + 2 - i = 9$.

$$\begin{array}{l} \xi_1 \quad 0 \quad 1 \quad \textcircled{2} \textcircled{3} \textcircled{4} \textcircled{5} \textcircled{6} \textcircled{6} \textcircled{7} \quad 8 \quad 9 \\ \xi_2 \quad 0 \quad 1 \quad \textcircled{2} \textcircled{3} \textcircled{3} \textcircled{4} \textcircled{5} \textcircled{6} \textcircled{7} \quad 8 \quad 9 \end{array}$$

The number of circled labels in the whorms will give us the total value of the statistic $F_2 - F_9$ provided the circled labels in the tail section (blue part) of the whorm are counted twice since the whorm is two-tailed. Divide by the length of the orbit board, 11 to get $\frac{21}{11} = \frac{3(k+2-2i)}{k+2}$.

In general for $i \leq m \leq k + 2 - i$ the “symmetrically-placed” condition forces the number of contributing labels in the *head* of ξ_1 to equal the number of contributing labels in the *tail* of ξ_2 . Hence, the total number of $+1$ ’s in the orbit will be thrice the number of contributing labels in the tail of ξ_1 plus thrice the number of contributing labels in the tail of ξ_2 . But this is just thrice the total number of contributing labels, since the number in the tail of ξ_2 equals the number in the head of ξ_1 . Hence, we get a total contribution of $3((k + 2 - i) - i) = 3(k + 2 - 2i)$. The homomesy follows by dividing by the length, $k + 2$, of the orbit board. A symmetrical argument, where we count contributing -1 ’s, handles the case where $i > j = k + 2 - i$. $\square$

4 Dynamics for rowmotion on $\mathbf{C}_n \times [k]$

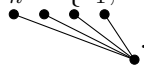
In this section we extend our results for the chain of V’s poset to a “chain of claws” poset, defined below. The equivariant bijection (Theorem 17) and techniques from Section 3 extend with only limited difficulty to this new setting.

columns: b_1 b_2 b_3 b_4 b_5 $\widehat{0}$

1	1	0	1	2	3
2	2	1	2	3	3
3	3	2	3	0	3
0	0	3	0	1	3
1	1	0	1	2	2
2	2	1	2	0	3
3	3	2	3	1	3
0	0	3	0	2	3
1	1	0	1	3	3
2	2	1	2	0	2
0	0	2	0	1	3
1	1	3	1	2	3
2	2	0	2	3	3
3	3	1	3	0	3
0	0	2	0	1	2

Figure 12: An orbit board of whirling on $C_5 \times [3]$.

4.1 The chain of claws poset

Definition 43. We define the *claw poset* $C_n = \{b_1, \dots, b_n, \widehat{0}\}$ where each b_i covers $\widehat{0}$. For example, the Hasse diagram of C_4 would be . The *chain of claws poset* is defined to be $C_n \times [k]$.

Using the established equivariant bijection (Theorem 17) between $\mathcal{J}(C_n \times [k])$ and k -bounded P -partitions $\mathcal{F}_k(C_n)$ that sends rowmotion to whirling, we can prove homomesy and periodicity results similar to those for $C_2 = V$. Now instead of *triples* of numbers, we will consider orbit boards of $(n+1)$ -tuples on $[0, k]$, $(f(b_1), f(b_2), \dots, f(b_n), f(\widehat{0}))$, satisfying $f(b_i) \leq f(\widehat{0})$ for each $i \in [n]$. Notice that for the case $n = 2$, we have moved the column formerly in the middle (corresponding to $\widehat{0}$) to be the **rightmost** one. The following proposition is an analogue of Proposition 28.

Proposition 44. The number of order ideals of $C_n \times [k]$ is given by $\sum_{i=1}^{k+1} i^n$.

Proof. Every order ideal of $C_n \times [k]$ is associated with a tuple of labels of the elements of C_n . Let $j \in [0, k]$ denote the label of $\widehat{0}$. The possible labels for $b_1, \dots, b_{n-1}$, and b_n are precisely the elements of $[0, j]$, yielding a total of $(j+1)^n$ labelings. Summing over all possible values of j produces the desired result. $\square$

The orbit board in Figure 12 has been decomposed (canonically) into a mixture of one-tailed and multi-tailed whorms. Since now we list the labels at $\widehat{0}$ in the rightmost column, the whorms are “(right) edge-seeking” rather than “center-seeking”. So each

whorm starts with one or more tails lying within the first k columns, and then jumps to the rightmost column at some point.

Note that the first, second, and fourth columns in Figure 12 are identical. This is no accident. If two (or more) entries among the first n in a given row are the *same*, then those positions (columns) remain the same throughout the *entire* orbit board. This is because the entries $b_1, \dots, b_n$ represent the result of whirling at *incomparable* elements of the poset C_n , so the whirls at those elements commute with one another. Furthermore, these two entries must belong to the same whorm, because each will be whorm-connected via $\widehat{0}$ exactly when their value matches the value of the last entry. So orbit boards for whirling on C_n can decompose into multi-tailed whorms with up to n tails. (The n -tails case entails near-constant rows and exactly two whorms decomposing the super-orbit board.)

These observations will allow us to generalize our periodicity and homomesy results from V to C_n . First we define a map that will be equivalent to whirling $k+2$ times. This generalizes the “reflection” $(x, y, z) \mapsto (z, y, x)$, which is the result of whirling $k+2$ times in the V_k case.

Definition 45. For any $A \subseteq [0, k]$, define the family of order-reversing maps $\mathcal{F}_k^A(C_n) = \{f : f \in \mathcal{F}_k(C_n) \text{ and } f(b_j) \in A \text{ for all } j \in [n]\}$. Then set $\overline{w}_A : \mathcal{F}_k^A(C_n)$ to be the map which whirls (cyclically increments *within the subset* A) each label on the non- $\widehat{0}$ elements of C_n . By definition of whirling (repeatedly incrementing until a valid labeling is reached), this preserves the condition that $f(b_i) \leq f(\widehat{0})$ for all $i \in [n]$, so is well-defined. Since there are no relations between the leaves of C_n , one can update the labels in any order.

This definition becomes particularly useful, when A is precisely the set of current labels at nonzero elements of f .

Definition 46. Given $f \in \mathcal{F}_k(C_n)$, set $A(f) = \{a : f(b_j) = a \text{ for some } j \in [n]\}$, i.e., $A(f)$ is the set of values that the P -partition f attains on the non- $\widehat{0}$ elements of C_n . Set $\alpha = \#A$ and $\alpha(f) = \#A(f)$. For any $f, g \in \mathcal{F}_k(C_n)$, if $g = w^j(f)$ for some $j \in \mathbb{N}$, then $\alpha(f) = \alpha(g)$. (Whirling preserves the number of *distinct* labels.) So we may sometimes write just α when an orbit is fixed. For the rest of this section, we will consider the map $\overline{w}_{A(f)}$, where whirling takes place within the set of possible labels given by the current function. (This set remains constant throughout an orbit of $\overline{w}_{A(f)}$.)

Example 47. Consider $f = (1, 3, 3, 0, 4, 1, 6) \in \mathcal{F}_9(C_6)$. We see $A(f) = \{0, 1, 3, 4\}$ so

$$\overline{w}_{A(f)}(1, 3, 3, 0, 4, 1, 6) = (3, 4, 4, 1, 0, 3, 6).$$

The last entry remains unchanged, and the earlier entries are increasing cyclically within the set $A(f) = \{0, 1, 3, 4\}$, with $\alpha = 4$. As another example, the reader can compare rows five apart in the orbit board of Figure 12.

In the special case of $V (= C_2)$ our set $A = A(f)$ within any orbit will have at most two elements, hence $\overline{w}_{A(f)}$ will just toggle between those two values at the left and the right. This means that $\overline{w}_{A(f)}$ is the same as reflecting values across the center of the orbit board, which we already saw was the effect of w^{k+2} . Our next theorem will generalize this to the case of C_n , except now the center column has moved to be the rightmost one.

4.2 Periodicity and homomesy via edge-seeking whorms

Our goal now is to generalize the periodicity and homomesy results for $V \times [k]$ to $C_n \times [k]$. Our first result shows that the sum of the tail-lengths of the first $\alpha(f) + 1$ whorms (counting the length only once for multi-tailed whorms) is constant. If $f \in \mathcal{F}^k(C_n)$ satisfies $f(\hat{0}) \notin A(f)$, then f will contain entries from $\alpha + 1$ distinct whorms. Otherwise, f will intersect exactly α whorms.

Define $t(\xi) = 1 + \min\{f(\hat{0}) : (\hat{0}, f) \in \xi\}$ and $h(\xi) := k + 2 - t(\xi)$. If there exists $(\hat{0}, f) \in \xi_1$ with $f(\hat{0}) = k$ such that $(\hat{0}, w(f)) \in \xi_2$, then we say ξ_2 is *in front of* ξ_1 . (In other words, if their heads are consecutive in the rightmost column of the orbit board.) In Figure 12, the yellow whorm is in front of the red whorm. We call a sequence of whorms *consecutive* if each is in front of the next. The next result generalizes Lemma 32.

Lemma 48. *Fix a whirling orbit of $\mathcal{F}_k(C_n)$, decomposed into whorms, with $\alpha = \alpha(f)$ for any f in the orbit. If $\xi_1, \dots, \xi_{\alpha+1}$ are $\alpha + 1$ consecutive whorms, then*

$$t(\xi_1) + \dots + t(\xi_{\alpha+1}) = \alpha(k + 2). \quad (1)$$

Using $h(\xi) + t(\xi) = k + 2$, this can be written equivalently as

$$h(\xi_1) + \dots + h(\xi_{\alpha+1}) = k + 2. \quad (2)$$

or as

$$t(\xi_{\alpha+1}) = h(\xi_1) + \dots + h(\xi_\alpha). \quad (3)$$

Proof. The proof follows similarly to that of Lemma 32. The key observation is that $t(\xi_{\alpha+1}) = h(\xi_1) + \dots + h(\xi_\alpha)$ for consecutive whorms. This holds because the tail of $\xi_{\alpha+1}$ begins one row below where the head of ξ_1 starts, and ends one row below where ξ_α ends. But since $t(\xi) = k + 2 - h(\xi)$ for any ξ we get

$$t(\xi_{\alpha+1}) = h(\xi_1) + \dots + h(\xi_\alpha) = (k + 2 - t(\xi_1)) + \dots + (k + 2 - t(\xi_\alpha)),$$

which implies the result. See Figure 12 for examples. $\square$

Corollary 49. *Fix a whirling orbit of $\mathcal{F}_k(C_n)$, decomposed into whorms, with $\alpha = \alpha(f)$ for any f in the orbit. If $\xi_1, \dots, \xi_{\alpha+2}$ are consecutive whorms, then $t(\xi_1) = t(\xi_{\alpha+2})$.*

Proof. Applying Lemma 48 to two consecutive collections of whorms, we obtain

$$t(\xi_1) + \dots + t(\xi_{\alpha+1}) = \alpha(k + 2) = t(\xi_2) + \dots + t(\xi_{\alpha+2}),$$

which reduces to $t(\xi_1) = t(\xi_{\alpha+2})$. $\square$

The next theorem, which generalizes Theorem 36, gives a simple way of describing the $(k + 2)$ nd iteration of whirling as the much simpler action of whirling (in any order) at each leaf of C_n within the set of allowable labels $A(f)$.

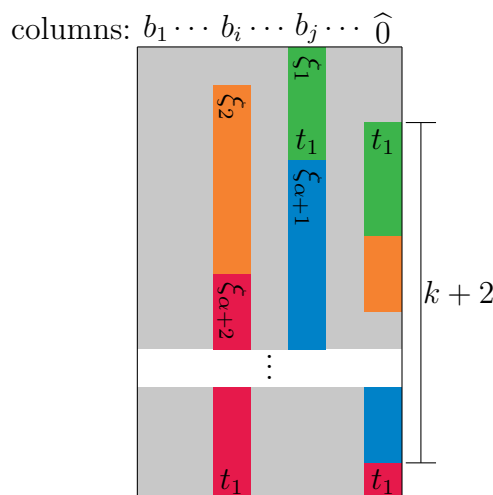


Figure 13: Sketch for the proof of Theorem 50. Consecutive whorms are denoted by $\xi_1, \dots, \xi_{\alpha+2}$. The last label in the tail of ξ_1 is t_1 . Note that the tail lengths of $\xi_{\alpha+2}$ and ξ_1 , are the same, so the labels in **column** b_j reproduce those in **column** b_i exactly $k+2$ rows further along the orbit.

Theorem 50. *Let w be the whirling action on k -bounded P -partitions in $\mathcal{F}_k(\mathbf{C}_n)$. For any $f \in \mathcal{F}_k(\mathbf{C}_n)$ with $A = A(f)$ and $\alpha = \alpha(f) = \#A(f)$, we have $w^{k+2}f = \overline{w}_{A(f)}f$. Thus, $w^{\alpha(k+2)}f = f$.*

Proof. Let $\mathcal{R}$ be an orbit board of w , decomposed into whorms, with $A = A(f)$ for some (row) f in the orbit. Let $\xi_1, \dots, \xi_\alpha$ be the consecutive whorms whose tails intersect f . Because the whorms are consecutive, in row f the largest value in A belong to ξ_1 , the next largest to ξ_2 , and so on. In this proof we will show that the labels in the tail(s) of ξ_1 will appear in the tail(s) of $\xi_{\alpha+2}$ exactly $k+2$ rows later. This process can be iterated to include all the whorms, proving that the labels change as described by $\overline{w}_{A(f)}f$.

By Lemma 48 (3), the sum of the head lengths of the whorms $\xi_1, \xi_2, \dots, \xi_{\alpha+1}$ is exactly $k+2$. Thus, the head of $\xi_{\alpha+2}$ begins exactly $k+2$ rows below the row where the head of ξ_1 begins. Equivalently, the tail of $\xi_{\alpha+2}$ ends in the row exactly $k+2$ below where the tail of ξ_1 ends. Since these two whorms have the *same* tail length by Corollary 49, we get that each entry within each tail of $\xi_{\alpha+2}$ is the same as each entry within each tail of ξ_1 , cyclically shifted down $k+2$ rows and appearing in different column(s). Note also that the tail column(s) of $\xi_{\alpha+2}$ are the *same* as the tail column(s) of ξ_2 . So in any of these (ξ_2) columns, the label $k+2$ rows later has become the label from **the previous whorm** ξ_1 in row f , which is the cyclically next larger value within $A(f)$. Thus, $w^{k+2}f = \overline{w}_{A(f)}f$, proving the theorem. $\square$

Example 51. Here is the orbit board generated by $(0, 0, 2, 5, 0, 6) \in \mathcal{F}_6(\mathbf{C}_5)$, with $\alpha = 3$ and divided into three sections of length $k+2$. In each section one of the whorms ξ_1, ξ_5 , and ξ_9 is highlighted. Each whorm starts in the first row and ends in the penultimate row of its section. The tail-lengths of ξ_1 and $\xi_{\alpha+2}$ are the same, just as in Corollary 49. Reading

3	1	3	3
0	2	0	4
1	3	1	3
2	0	2	4

Figure 14: A short orbit board of whirling on $C_3 \times [4]$. In this example $\alpha = 2$.

across from one section to the next (cyclically) in each row, illustrates Theorem 50: $w^{k+2} = \overline{w}_A$.

0	0	2	5	0	6	2	2	5	0	2	6	5	5	0	2	5	6
1	1	3	6	1	6	3	3	6	1	3	6	6	6	1	3	6	6
2	2	4	0	2	4	4	4	0	2	4	4	0	0	2	4	0	4
3	3	0	1	3	5	0	0	1	3	0	5	1	1	3	0	1	5
4	4	1	2	4	6	1	1	2	4	1	6	2	2	4	1	2	6
5	5	2	3	5	5	2	2	3	5	2	5	3	3	5	2	3	5
0	0	3	4	0	6	3	3	4	0	3	6	4	4	0	3	4	6
1	1	4	5	1	5	4	4	5	1	4	5	5	5	1	4	5	5

Porism 52. In the above situation, each **column** of the super-orbit board of length $\alpha(k+2)$ intersects exactly $\alpha + 1$ whorms.

Proof. Let $\mathcal{R}$ be the super-orbit board of w with $\alpha(k+2)$ rows. Decompose this into whorms, with $A = A(f)$ for some f in the orbit. Let $\xi_1, \dots, \xi_\alpha$ be the consecutive whorms whose tails intersect f . By definition of *consecutive*, the whorms ξ_i and $\xi_{\alpha+i}$ occupy the same columns, so the whorms $\xi_1, \xi_{\alpha+1}, \xi_{2\alpha+1}, \dots$ intersect this column. By Corollary 49, we have the equalities

$$\begin{aligned}
t(\xi_{2\alpha+1}) &= t(\xi_\alpha), \\
t(\xi_{3\alpha+1}) &= t(\xi_{2\alpha}) = t(\xi_{\alpha-1}), \\
t(\xi_{4\alpha+1}) &= t(\xi_{3\alpha}) = t(\xi_{2\alpha-1}) = t(\xi_{\alpha-2}), \\
&\vdots \\
t(\xi_{\alpha\alpha+1}) &= t(\xi_{(\alpha-1)\alpha}) = t(\xi_{(\alpha-2)\alpha-1}) = \dots = t(\xi_2).
\end{aligned}$$

This gives the sum

$$t(\xi_1) + t(\xi_{\alpha+1}) + \dots + t(\xi_{\alpha\alpha+1}) = t(\xi_1) + t(\xi_2) + \dots + t(\xi_{\alpha+1}) = \alpha(k+2)$$

by Lemma 48. This is also the length of the super-orbit board; hence, the total number of whorms intersecting any given column is $\alpha + 1$, since the sum of tail lengths of $\alpha + 1$ whorms is the total number of rows in the super-orbit board. $\square$

By Porism 52, we will have at most $\alpha(\alpha + 1)$ total distinct whorms in an orbit board. (Indeed for equality to be achieved, the orbit board has to have maximal length $\alpha(k+2)$ before repeating.) On the other hand, consider the orbit board of $\mathcal{F}_4(C_3)$ in Figure 14

with $\alpha = 2$. Here $w^4(f) = f$ so the orbit is only 4 rows long with 2 distinct whorms. We can extend this to a super orbit board with 12 rows and 6 whorms (in general with $\alpha(k+2)$ rows and $\alpha(\alpha+1)$ whorms).

Theorem 53. *Let $m = \min(k+1, n)$. The order of rowmotion on the chain of claws poset $\mathcal{J}(\mathbf{C}_n \times [k])$ is $(k+2) \operatorname{LCM}(1, 2, \dots, m)$.*

Earlier versions of this work reported the weaker statement that the order divides $m!(k+2)$.

Proof. By Theorem 50, the whirling order of any $f \in \mathcal{F}_k(\mathbf{C}_n)$ (i.e., the size of the orbit containing f) divides $\alpha(f)(k+2)$, where $\alpha(f) \leq m = \min(k+1, n)$. Thus the order of the entire map divides the LCM of these values.

To prove that this order is exact, we claim that for every $p \in [m]$, there exists an (actual) orbit of length $p(k+2)$, except when $p = k+1$, in which case the orbit has length $k+1$. Our strategy is similar to the construction used to prove Theorem 23. However, here we make use of Theorem 50, that $w^{k+2}f = \overline{w}_{A(f)}f$, i.e., that after $k+2$ steps, w simply cyclically rotates the possible values of $A(f)$.

Set $f_1 := (0, 0, 0, \dots, 0)$, and for each integer $p \in [2, m]$, set

$$f_p = (0, 1, \dots, p-2, p-1, p-1, \dots, p-1, p-1).$$

Let $\mathcal{R}_p$ be the orbit board of w containing f_p . Here $\alpha(f_p) = p$, and the action $\overline{w}_{A(f_p)}$ partitions $\mathcal{R}_p$ into cycles of length p . Thus p divides the length of $\mathcal{R}_p$. See Figure 15 for an example of this construction.

We will now show the following:

- If $p < k+1$, then the length of $\mathcal{R}_p$ is $p(k+2)$.
- If $p = k+1$, then the length of $\mathcal{R}_p$ is $k+1$.

In the first case, it suffices to show there is no positive integer $q < k+2$ such that $w^q(f_p) = \overline{w}_{A(f_p)}f_p$. We demonstrate this by showing that $w^q(f_p)(\widehat{0}) \neq f_p(\widehat{0})$ for all positive $q < k+2$, noting that $\overline{w}_{A(f_p)}$ leaves the rightmost label unchanged.

By definition, the rightmost label of f_p is $p-1$, which is less than k by assumption. Since $\widehat{0}$ is the minimal element of the poset, by definition of whirling its (rightmost) label will increase by one at each step until it reaches the maximum value k at step $k-(p-1)$. At this step, the other values in the row (in some order) are $k-1, k-2, \dots, k-p$, which have all increased from their initial values of $0, 1, \dots, p$ as the constraining label at $\widehat{0}$ has increased.

From this point onward, each step will involve exactly one (possibly multi-tailed) whorm moving to the rightmost column with value k . There are p whorms, so the rightmost label remains k for p additional steps, and thus cannot reset to $p-1$ for a total of $k-(p-1)+p = k+1$ steps. Thus, $w^q(f_p) \neq \overline{w}_{A(f_p)}f_p$ for all $q < k+2$, and length of $\mathcal{R}_p$ is $p(k+2)$, finishing this case.

1	3	5	5	5
2	4	0	0	6
3	5	1	1	5
4	0	2	2	6
5	1	3	3	5
0	2	4	4	6

Figure 16: An orbit board of whirling on $C_4 \times [6]$. The average of the statistic $B_3 - B_2$ is $2/3$ which does not agree with the statement of the generalized flux capacitor in Remark 56. Notice if the labels in repeated column are ignored, then the average of the statistic $B_3 - B_2$ is $1/2$ which agrees with the generalization.

I satisfies $f(b_i) = 0$. (See the proof of Lemma 16.) But this is counted by the number of whorm beginnings (considering each tail to be separate); by Porism 52 this is $n(\alpha + 1)$. Furthermore, $\chi_{(\widehat{0}, k)}(I) = 1$ if and only if the f corresponding to I satisfies $f(\widehat{0}) = k$, which is counted by the number of whorm endings, that is $\alpha(\alpha + 1)$. Thus, the numerator counts 1 for each of the $n\alpha(k + 2) - n(\alpha + 1)$ entries that are not at the start of a tail, and -1 for the $\alpha(\alpha + 1)$ whorm heads in the rightmost column. Divide by the size of the super-orbit board to get the average. In the $n = 2$ case the only possibilities are $\alpha = 1$ or $\alpha = n$, for both of which (4) reduces to $\frac{nk-2}{k+2} = \frac{2k-2}{k+2}$; this explains why we get the homomesy Theorem 24(2) for $V \times [k]$.

Remark 56. The “flux-capacitor” homomesy of Theorem 41 also fails to generalize to the claw-graph setting. Earlier versions of this work incorrectly asserted the following analogous “flux-capacitor” homomesy: “Let $B_i = \chi_{(i-1, \widehat{0})} + \sum_{\ell=1}^n \chi_{(i, \ell)}$. Then for the action of rowmotion on $\mathcal{J}(C_n \times [k])$, $B_i - B_j$ is $\frac{(j-i)(n+1)}{k+2}$ -mesic for all $i, j \in [n]$.” Unfortunately we only obtain this average in general for the subset of orbit boards which are tiled entirely by one-tailed whorms. See Figure 16 for a counterexample.

Acknowledgments

We are grateful to Banff International Research Station for hosting online and hybrid workshops on dynamical algebraic combinatorics in October 2020 and November 2021. Discussions with the following colleagues were particularly helpful: Ben Adenbaum, Chinmay Dharmendra, Darij Grinberg, Sam Hopkins, Michael Joseph, James Propp, Bruce Sagan, and Jessica Striker. We also thank two anonymous reviewers of an earlier (extended abstract) version of this work.

References

- [1] Benjamin Adenbaum. *On Pattern Avoidance and Dynamical Algebraic Combinatorics*. Ph.D. Dissertation, Dartmouth College, 2024. Dartmouth College Ph.D. Dissertations, 251. <https://digitalcommons.dartmouth.edu/dissertations/251>

- [2] Drew Armstrong, Christian Stump, and Hugh Thomas. A uniform bijection between nonnesting and noncrossing partitions. *Trans. Amer. Math. Soc.*, 365(8):4121–4151, March 2013.
- [3] Andries Brouwer and Alexander Schrijver. On the period of an operator, defined on antichains. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, no. ZW 24/74, 1974.
- [4] Peter J. Cameron and Dmitry G. Fon-Der-Flaass. Orbits of antichains revisited. *European Journal of Combinatorics*, 16(6):545–554, 1995.
- [5] David Einstein and James Propp. Combinatorial, piecewise-linear, and birational homomesy for products of two chains. *Algebraic Combinatorics*, 4(2):201–224, 2021.
- [6] Shahrzad Haddadan. Some instances of homomesy among ideals of posets. *The Electronic Journal of Combinatorics*, 28(1):#P1.60, 2021.
- [7] Sam Hopkins and Martin Rubey. Promotion of Kreweras words. *Sém. Lothar. Combin.*, 86B:Art. 77, 12, 2022.
- [8] Sam Hopkins. Order polynomial product formulas and poset dynamics, 2024. [doi: 10.1090/pspum/110/02006](https://doi.org/10.1090/pspum/110/02006).
- [9] Michael Joseph, James Propp, and Tom Roby. Whirling injections, surjections, and other functions between finite sets. *Discrete Mathematics & Theoretical Computer Science*, 27:3, 2025.
- [10] Michael Joseph and Tom Roby. Toggling independent sets of a path graph. *The Electronic Journal of Combinatorics*, 25(1):#P1.18, 2018.
- [11] James Propp and Tom Roby. Homomesy in products of two chains. *The Electronic Journal of Combinatorics*, 22(3):#P3.4, 2015.
- [12] Victor Reiner, Dennis Stanton, and Dennis White. The cyclic sieving phenomenon. *Journal of Combinatorial Theory, Series A*, 108(1):17–50, 2004.
- [13] Victor Reiner, Dennis Stanton, and Dennis White. What is... cyclic sieving. *Notices Amer. Math. Soc.*, 61(2):169–171, 2014.
- [14] Tom Roby. Dynamical algebraic combinatorics and the homomesy phenomenon. In *Recent Trends in Combinatorics*, pages 619–652. Springer, 2016.
- [15] Richard P. Stanley. *Enumerative Combinatorics: Volume 1, 2nd Edition*. no. 49 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2011.
- [16] Jessica Striker. Dynamical algebraic combinatorics: promotion, rowmotion, and resonance. *Notices Amer. Math. Soc.*, 64(6):543–549, 2017.
- [17] Jessica Striker and Nathan Williams. Promotion and rowmotion. *European Journal of Combinatorics*, 33(8):1919–1942, 2012.
- [18] Hugh Thomas and Nathan Williams. Rowmotion in slow motion. *Proceedings of the London Mathematical Society*, 119(5):1149–1178, May 2019.